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A roughly smooth optimal consumption path: smoothing the rough annuity puzzle

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Orientador: Luis Henrique Bertolino Braido

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A ROUGHLY SMOOTH OPTIMAL CONSUMPTION PATH: SMOOTHING THE ROUGH ANNUITY PUZZLE

Tese apresentada ao Curso de Doutorado em Economia da Escola de Pós-Graduação em Economia para obtenção do grau de Doutor em Economia.

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"Veja, não diga que a canção está perdida.
Tenha fé em Deus, tenha fé na vida.
Tente outra vez.
Beba, pois a água viva ainda está na fonte.
Você tem dois pés pra cruzar a ponte.
Tente, levante sua mão sedenta e recomece a andar.
Não pense que a cabeça aguenta se você parar.
Queira, basta ser sincero e desejar profundo.
Você será capaz de sacudir o mundo.
Tente, e não diga que a vitória está perdida,
se é de batalhas que se vive a vida.
Tente outra vez."
Àqueles por quem tentar outra vez sempre vale a pena.
Para meu filho, minha esposa, minha família.

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Acabou. Graças a Deus.

Resumo

A inconsistência entre a teoria e o comportamento empírico dos agentes no que tange ao mercado privado de pensões tem se mostrado um dos mais resistentes puzzles presentes na literatura econômica. Em modelos de otimização intertemporal de consumo e poupança sob incerteza em relação ao tempo de vida dos agentes, anuidades são ativos dominantes, anulando ou restringindo fortemente a demanda por ativos cujos retornos não estão relacionados à probabilidade de sobrevivência. Na prática, entretanto, consumidores são extremamente céticos em relação às anuidades. Em oposição ao seguro contra longevidade oferecido pelas anuidades, direitos sobre esses ativos - essencialmente ilíquidos - cessam no caso de morte do titular. Nesse sentido, choques não seguráveis de liquidez e a presença de bequest motives foram consideravelmente explorados como possíveis determinantes da baixa demanda verificada. Apesar dos esforços, o puzzle persiste. Este trabalho amplia a dominância teórica das anuidades sobre ativos não contingentes em mercados incompletos; total na ausência de bequest motives, e parcial, quando os agentes se preocupam com possíveis herdeiros. Em linha com a literatura, simulações numéricas atestam que uma parcela considerável do portfólio ótimo dos agentes seria constituída de anuidades mesmo diante de choques de liquidez, bequest motives, e preços não atuarialmente justos. Em relação a um aspecto relativamente negligenciado pela academia, mostramos que o tempo ótimo de conversão de poupança em anuidades está diretamente relacionado à curva salarial dos agentes. Finalmente, indicamos que, caso as preferências dos agentes sejam tais que o nível de consumo ótimo decaia com a idade, a demanda por anuidades torna-se bastante sensível ao sobrepreço (em relação àquele atuarialmente justo) praticado pela indústria, chegando a níveis bem mais compatíveis com a realidade empírica.

Palavras-chave: Investimentos, Alocação de ativos, Anuidade

Abstract

This thesis extend the theoretical dominance of annuities over non-contingent discount notes; under standard assumptions, we show that *full annuitization* is optimal even in incomplete annuity markets. Through numerical simulations, we scrutinize factors affecting annuitization decision, consolidating and extending previous research by taking into account unfair prices, bequest motives, and out-of-pocket medical expenses. We also take into consideration the insurer's risk of default, and relax an implicit assumption in most past models and detach annuitization from retirement, i.e.: we do not presume that consumers are already retired from work when they decide whether or not to annuitize. In line with previous literature, our results originate very high levels of annuitization. Yet, we show that the demand for annuities drops sharply, if preferences are such that the implied optimal consumption path decays with age. We also show that optimal annuitization timing is closely related to the endowments pattern.

Keywords: Asset allocation, Annuitization timing, Annuity puzzle

Contents

1 Introduction

The question of why agents do not voluntarily hold a considerable fraction of their portfolios on annuities, despite their high theoretical value, is open in the literature on consumption/saving and asset choice for quite a while. Yaari (1965) is the cornerstone of the life-cycle theory where consumers face an uncertain time of death. His most rousing and widely cited result is the dominance of annuities over regular discount notes: in the absence of bequest motives, all saving should be held in the form of annuities, an outcome the literature regards as *full annuitization.* Yet, in practice, consumers just do not buy annuities, at least not as much as theory prescribes, an incongruence known as *annuity puzzle.*

Annuities are financial instruments that provide a regular stream of life-contingent payments against an upfront premium. Returns associated to these actuarial notes, because of the contingency clause, are higher than those of regular assets. Higher yield, together with the longevity insurance it provide, mark the attractiveness of an annuity and makes the puzzle hard to solve. In fact, Davidoff et al. (2005) under less restrictive conditions, extended and reasserted the full annuitization theoretical result of Yaari (1965). Numerically, Davidoff et al. (2005) do try to reconcile the puzzle accounting for a series of issues that could squeeze the annuity demand, bequest motives, a major drawback to life-contingent assets, included. Their simulations, however, even relying on strong mismatch between consumers' preferences of consumption over time and the cash flow of payments provided by annuities, still originate high levels of annuitization.

There are indeed some results showing that low annuitization levels may be optimal, such as those of Inkmann et al. (2011) and Lockwood (2011). Nevertheless, skepticism remains. Fully rational explanations to the puzzle are challenged, and behavioral analysis is of growing importance also in this field.[1] Complexity, framing, and loss aversion, for example, are frequently named as issues influencing the annuity demand.[2] Regardless the puzzle explanation, though, academic research on this theme has a relevant influence on public regulation. If average Americans typically save less than what theory recommends, if higher annuitization increases welfare, and if it would also help to alleviate the pressure on Social Security, why not provide incentives for people to annuitize? Davidoff et al. (2005) actually suggest that "some mandatory annuitization may be welfare increasing". According to Benartzi et al. (2011), following *The 2006 Pension Protection Act*, lump sum distributions would receive a 10 percent penalty starting in 2012. The same Act made auto-enrollment the default option in 401(k) plans, a successful attempting to overcome the acquiescence bias (Hurd, 1999) and push upward the participation rate in private pension schemes. In a recent interview to Bloomberg, Mark Iwry, a senior adviser to the Secretary of the Treasury and deputy assistant secretary for retirement and health policy, reinforces the importance of retirement planning and suggests that federal government in the U.S. is trying to make the annuitization choice an easier option in private pension plans, particularly through deferred annuities as proposed by Milevsky (2005), Brown (2007), and Gong and Webb (2010).

Acknowledging that annuitization issues addressed in academic research have a real impact on people's life, and that the behavioral perspective may indeed contribute to explain the puzzle,

[1]Behavioral perspectives on the puzzle (as well as rational) are summarized in Brown (2007).

[2]See, for instance, Brown et al. (2008), and Benartzi et al. (2011)

this thesis is a fully rational approach that attempts to bring further understanding to the annuitization problem. Our framework follows closely those in Fischer (1973) and in Friedman and Warshawsky (1990), which somehow are a particular case of that in Yaari (1965). However, we analyze consumers' choices when agents are entitled to endowments in every period, not just the initial one, and under incomplete annuity markets, where only (irreversible) life annuities are available, as oppose to the fully-liquid one-period annuity. These assumptions represent a significant loss of tractability, and analytical results do not come out easily, even in the simple three-period model with CRRA utility function we first present. Simplicity, though, pays off on precise outcomes and, thus, direct sensitivity analysis and fairly strong intuition to the n-period case we analyze later. Despite the appeal of the actuarial investment, optimal asset allocation will strongly depend on the endowments. As result, in the three-period model, we show that a portfolio composed only by the non-contingent bond, no annuities at all, is a possible optimal saving decision even under the assumption of actuarially fair annuities, when nonfinancial income have a v-like shape. Nevertheless, if future endowments are constant, full annuitization is quite often optimal. In fact, within a n-period scope, we extend the full annuitization result of Yaari (1965) and Davidoff et al. (2005) to an incomplete annuity market set, under fairly standard assumptions. In particular, we need the interest rate to equal the inverse of the discount factor in every period, constant endowments, as if agents are retired from work, and fair annuities. Yaari (1965) does not need any restriction on the relation of interest rates and discount factor, or on endowments, and essentially even fair prices are not necessary; Davidoff et al. (2005) do not need exponential discounting, fairness, or even separable utility, but there is only initial wealth in their model, no future endowments. However, the completeness of annuity markets, an assumption in both papers, is such a strong restriction that its relaxation is indeed a worthy achievement.

Through numerical simulations we scrutinize factors affecting annuitization decision, consolidating and extending previous researches by taking into account unfair prices, bequest motives, and out-of-pocket medical expenses. We also take into consideration the insurer's risk of default, and relax an implicit assumption in most past models and detach annuitization from retirement, i.e.: we do not presume that consumers are already retired from work when they decide whether or not to annuitize. In fact, initial age in our analysis is twenty years. In general, we contemplate no uncertainty other than survivability. Thus, we assume that agents face deterministic endowments throughout life and provide the optimal consumption/saving choice through time. Based on empirical data whenever possible, our results suggest that the puzzle dies hard, very hard, if so. Full annuitization, or high levels of partial annuitization, relies on the fact that an optimizer agent seeks higher and flat consumption, which is exactly what a constant-benefit immediate life annuity provides. Thus, based on empirical evidence that consumption diminishes with retirement, we propose preferences such that the desire for present consumption is accentuated.[3] We do so by relating the discount factor to survival probabilities. In this way, it is finally possible to obtain a mismatch between annuities' stream of benefits and consumers needs strong enough to induce saving in the risk-free asset, with very significant reduction on annuitization rates.

[3]With respect to consumption over time, see Banks et al. (1998); Bernheim et al. (2001); Hurd and Rohwedder (2003) and Thaler and Benartzi (2004).

We also address an issue usually disregarded in the literature: the optimal timing to convert savings into annuities. As in Milevsky and Young (2007), then, we restrict annuitization to occur, at most, once in an agent's life. In the market, there is nothing preventing an investor to buy immediate life annuities multiple times. Even those retirees who hold company-sponsored pension plans may roll the balance over to Individual retirement accounts (IRAs) and, thus, annuitize with annuities providers of their choice as many times as convenient. Despite that, we regard annuitization as an one-time decision. In fact, people are not buying annuities even once. In this sense, the onetime purchase assumption is not totally unreasonable. The outcomes of our numerical simulations are in line with the analytical results from the three-period model. If there is uncertainty only regarding the length of life, all else equal, the wealthier the agent, the earlier annuitization occurs. Modeling choices when initial wealth equals the first period income from work, as a thumb-rule, consumers will not annuitize until their income start falling. In our basic set we suppose that an employee leaves work on his/her 65th birthday and simultaneously claim Social Security benefits. In such way, optimal age to annuitize is quite often in the early 60's. If retirement occurs at the age of 75, optimal timing also moves ahead to the early 70's. In general then, annuitization decision is indeed closely related to retirement from work, with no gaining being extracted from postponement, if the overprice of available annuities, in comparison to actuarially fair prices, is constant across consumers of all ages.

This thesis is organized in correlated chapters and is not composed by independent papers. That said, in the following section, we summarize the private pension market in the United States and briefly review the literature on the puzzle, presenting theoretical and numerical results regarding optimal annuitization decision, highlighting preferences and market imperfections that usually depress the annuity demand; we also present our own time-series and cross-ages estimates of the *money's worth* of immediate annuities; section 3 builds the foundation on which rests our reasoning, through analytical development of a three-period model; in section 4, within a n-period scope and incomplete annuity markets, we extend the optimality of full annuitization, show extensive simulations exhibiting the puzzle resilience, and finally suggest consumers' preferences that imply as result a tiny fraction of the contingent note in optimal portfolio; section 5 concludes.

2 Foundations

2.1 Literature review

Yaari (1965) main focus was on the identification of the optimal consumption plan c^* over time when consumers face an uncertain time of death and may invest in annuities as well as in regular discount notes (or equivalently, in this paper, risk-free bonds). On this matter, the author shows that in the absence of bequest motives, the presence of actuarially fair annuities in the economy would result in a consumption path c^* under uncertain lifetime equal to that obtained in the *Fisher problem*, as Yaari (1965) called the problem in an environment without uncertainty and annuities.[4] Trivial as it is under complete annuity markets and the absence of bequest motives, the full dominance of annuities over discount notes is a plain reasoning in

[4] Fisher (1930).

Yaari (1965). Despite the many observed types - usually related to index-linking, certain-period guarantees, or joint-and-survivor clauses -, an annuity is generally understood as a longevity insurance, a contingent financial instrument that pays a regular stream of benefits in exchange for a lump sum premium paid by a consumer.[5] In its simplest form, benefit payments start immediately after the purchase, are constant through time and lasts as long as the beneficiary is alive. This contingency to life, the most characteristic feature of annuities, is the reason why these instruments pay a higher rate of return compared to a non-contingent benchmark asset(s), and also - to avoid moral hazard issues - the reason why annuities can not be sold (by an agent). In Yaari (1965) there are no restrictions regarding the selling of annuities. More importantly, though, the market is complete. If trades can occur any time, a complete annuity market is equivalent to annuities being available in every period t and paying off entirely in $t+1$.[6] Assuming that $R_t = (1+r_t)$ is the one-period gross rate of return related to the bond, an actuarially fair annuity under the completeness hypothesis would yield R_t/p_t, where $p_t < 1$ is the survival probability from period t to $t+1$. Then, if bequests are not considered, a consumer would have two saving instruments equal - for what matters - in every aspects but the rate of return. Naturally, the asset with lower return, the bond in this case, would never be bought. Saving would be composed only of annuities, thus full annuitization.

Although the result in Yaari (1965) is commonly attributed to the fairness of the annuities - because only fair annuities were considered - it actually comes in response to its greater rate of return. Any rate R_t^a such that $R_t < R_t^a < R_t/p_t$ would do the job. This is explored in Davidoff et al. (2005), which extends the full annuitization result of Yaari (1965) to a much less restrictive set of conditions, particularly without supposing additive separability, expected utility, exponential discounting and, of course, fairness. The completeness of the annuity market, however, plays a fairly important role in the great level of generality achieved in Davidoff et al. (2005). In fact, as stated above, when annuity markets are complete and bequests does not matter, the consumer's demand for bonds is trivially null and it is not hard come up with an economy that sustains this result and in which the preferences can not even be represented by a utility function, regardless the fairness assumption. The only requirements would be that more (consumption) should be always better than less and that $R_t^a > R_t$. Results of Davidoff et al. (2005), however, are not restricted to the complete market environment, and the authors do state conditions for full initial annuitization - i.e.: only the annuity is bought in the first-period - in a three-period model where trade occurs any time, the annuity is available only in the initial period, and there is no income other than financial in periods two and three. Davidoff et al. (2005) also show the optimality of some annuitization, within a n-period scope and when all trades occur in the first period (other restrictions as just cited), but due to the level of generalization of the preferences they assume, this is as far as the authors can go.

2.2 The annuity market

In a field dominated buy numerical simulation due to the lack of tractability of models involving incomplete annuity markets, Yaari (1965) and Davidoff et al. (2005) are the two

[5]To a good survey on the annuity markets and annuities types in the United Kingdom see Cannon and Tonks (2006).

[6]In Yaari (1965) time is continuous, but the reasoning applies as well.

analytical pillars of the literature. Both clearly state the dominance of annuities over discount notes. In the real world, nevertheless, annuities apparently do not rule. Immediate annuities sales, data from LIMRA International in Drinkwater (2007), amounted to inexpressive \$6.2 billion in 2006. As comparison, deferred annuities sales totaled \$224.2 billion in the same year. To put these number into perspective, it should be noted that deferred annuities are much more commonly used as a tax-deferred saving instrument during the accumulation phase of pension plans than as a retirement plan tool (Brown et al., 2000; Brown and Poterba, 2006). In 2005, still following Drinkwater (2007), \$5.7 billion of immediate annuities were sold and another \$10.1 billion were annuitized, representing a annuitization rate of 0.6 percent of the average stock of deferred assets. This rate, annuitization over the stock of deferred annuities, is hardly a good proxy regarding consumer's preferences, though. A better measure would be the fraction of total sales of immediate annuities over the total lump sum distributions from retirement plans. Even if they are not so small as they seem, however, the value of immediate sales overestimates the size of the life annuities market since non-contingent products are included, i.e.: sales of annuities that guarantee benefit payments over a certain number of years are also considered in the total figure (Brown, 2007).

Immediate annuities sales in the U.S. as well as voluntary annuitization in the U.K. are just the tip of the iceberg in the private pension market.[7] In 2007, there were almost \$6.1 trillion in assets of both defined benefit (DB) and defined contribution (DC) pension plans in the United States. DB plans accounted for \$2.64 trillion, and DC's for \$3.44 trillion, mostly in 401(k) accounts.[8] Assets in Individual Retirement Accounts (IRAs) - traditionally a vehicle for rollovers from employer-sponsored plans (Investment Company Institute, 2010) - amounted to another couple of trillion dollars. The puzzle, in this matter, would arise from the shift from DB to DC plans - in 1975 there were over 27.2 million active participants in DB plans and over 11.2 million on DC plans; in 2007 these numbers changed to 19.4 million (DB) and 66.8 million (DC) - associated with the low fraction of DC plans that offer an annuity option and the even lower fraction of participants that indeed elect annuities as the distribution form.[9] Investment Company Institute (2008), a survey with employees that were supposed to be retiring between 2002 and 2007, shows that 70 percent of retirees in DC plans, from 608 respondents, had multiple distribution options; from those, 69 percent had the possibility of choosing an annuity. Considering only 401(k) plans (368 respondents), about 45 percent had the option to annuitize. Regarding selected choices, 4 percent had chosen partial and 16 percent full annuitization of their respective balances. According to Hewitt Associates, however, in a survey with more than 300 employers covering about 3.8 million participants, the fraction of 401(k) plans offering annuities as choice option was 0.31 in 1999 and 0.17 in 2003. In 2009, it was equal to 0.14, with only 1 percent of those who had the annuity option actually taking it. The low rate of 401(k) plans

[7] In the U.K., the government requires that three-quarters of balances in tax-privileged retirement plans must be annuitized. The voluntary market is that not related to tax-privileged investments. See Cannon and Tonks (2006).

[8] See Private Pension Plan Bulletin Historical Tables and Graphs, U.S. Department of Labor.

[9] The move from DB to DC plans has been explained by changes in the workforce (the shift from manufacturing to the service industry, from large to small companies, union to non-union position) as well as by the easier regulation on DC plans. See Gustman and Steinmeier (1992), Ippolito (1995), Kruse (1995), and Papke et al. (1996). About the percentage of DC plans offering annuities see Brown (2007), Benartzi et al. (2011) and Hewitt Associates in: http//www.dol.gov/ebsa/pdf/1210-AB33-669.pdf.

offering annuity as a distribution option is also pointed by Benartzi et al. (2011), for whom virtually no 401(k) plan offer the annuitization choice.

Defined benefit plans, on the other hand, typically distribute account balances in the form of annuities. According to Brown (2000), the vast majority does not even permit lump sum procedures. When employees have the "cash" option, having IBM workers as reference, Benartzi et al. (2011) report that 88 percent still choose to annuitize. This fraction goes to 0.53 when the authors average other 75 DB plans, and to 0.66 when it is based on Hurd and Panis (2006) analysis of data from Health and Retirement Surveys. Bütler and Teppa (2007) research DB plans in Switzerland and find that, on average among seven companies where annuities were the default option, 69 percent of the employees chose annuitization over a lump sum. In a evidence of acquiescence bias (Hurd, 1999), the authors report that nearly 89 percent - average of two firms - of employees opted for a lump sum distribution, a choice that exhibited a v-shape dependence on the size of the accounts. Finally, Benartzi et al. (2011) state that for one large DB plan, 13 percent of all participants chose annuities over lump sum; when accounts with procedures bellow $5,000 were not considered, the annuitants share rose to 96 percent. Thus, as the authors explicit: "the common view that there is little demand for annuities even in defined benefit plans is largely driven by looking at the overall population of participants...".

It may be the case that the myopia stressed by Benartzi et al. (2011) regarding the level of account balances is also biasing the rate of participants that select annuities in defined contribution plans. Indeed, the authors alert that many DC accounts have low balances and that this is not always taking into consideration. However, it seems less controversial the fact that 401(k) schemes, by far the most popular type of DC plan, rarely offer an annuity option. Thus, assuming that life annuities sales, compared to contemporary lump sum distributions, are as small as they appear to be, and that possible differences between employees of firms offering DB and DC plans are irrelevant, the superior rate of annuitization in DB plans suggests that choices from DC retirees might not reflect their underlying preferences (Benartzi et al., 2011), supporting the puzzle.[10]

2.3 Possible puzzle explanations

2.3.1 Annuity prices

Annuities do face some drawbacks - bequest motives, unfair prices, lack of liquidity, lack of inflation protection, the possibility of higher returns in the stock market -, and the literature have been exploring them in an attempt to find explanations to the low demand.[11] Prices, of course, are not actuarially fair and there has been quite a few researches on this matter.

Warshawsky (1988) found that from 1919 to 1984 loads factors on annuities varied from 10 to 29 cents per dollar of fair actuarial present value. From 8 to 16 cents of those could be linked to adverse selection costs: annuitants tend to live more than an average individual. More recently, Friedman and Warshawsky (1990), Mitchell et al. (1999) and Poterba and Warshawsky (1999) have all shown the adverse selection evidenced in the lower mortality of annuitants. Finkelstein

[10] Prices could explain the discrepancy between annuitization levels in DB and DC plans, but Benartzi et al. (2011) affirm that the price of annuities in DB plans are similar to prices in the broader market.

[11] Brown (2007) provides a good review on these attempts.

and Poterba (2004) explore the adverse selection in several dimensions in the U.K. market and their findings show that, on average, the expected discounted present value of payouts are somewhere from 80 to 85 percent of the fair value, with about half of this value being related to selection costs. Highlighting the wide variation among prices from different providers and having U.S. treasuries as yield reference, Mitchell et al. (1999) found in 1995 an average *money's worth* - the fraction given by dividing the expected discounted present value of the benefits from a immediate life annuity by the actual premium charged by insurers - varying from 0.814 (men) to 0.854 (women), considering a random 65-year-old individual from the population, and equal to 0.927 (both men and women) among 65-year-old individuals in the pool of annuitants.[12] When the corporate yield curve was the reference, these values were, of course, lower: 0.756 (men) and 0.785 (women), among population; 0.853 (men) and 0.847 (women), among annuitants. In general, the money's worth in Mitchell et al. (1999) is lower for men when the population mortality table is used, and quite even between genders when only annuitants are considered; regardless the table used, money's worth values were higher for younger consumers of both sexes when treasuries were the yield reference. When the corporate yield curve was used together with the annuitant mortality table, the measure was proportional to age, in both genders.

Brown et al. (2000) find values of money's worth from 1995 to 1999 similar to Mitchell et al. (1999), so that the load of about 15 percent have became a popular reference in the literature. Justifying the use of treasuries rates, Brown et al. (2000) declare that "insurance regulation makes the default risk for annuity providers very low". The authors, however, recognize that annuities are not riskless saving, and also that insurers invest in risky bonds. According to Benartzi et al. (2011), all states in the U.S. provide some guarantee to annuities, although limited on average to $200,000.[13] However, the authors say, "the money for the bailout comes from other insurance companies that operate in the state, not the state itself". Finkelstein and Poterba (2004) reported that, among the firms in their survey, about one-quarter of the insurers' assets were hold in corporate bonds, while the rest were allocated in nominal government issuances. Then, it is somehow odd that numerical simulations conducted usually do not account for insurers' risk. It is also odd that simulations use loads on nominal annuities and compare the results to theoretical findings on real annuities. Friedman and Warshawsky (1990) actually called attention to this fact, stating that most of the analysis held in the literature, theirs included, was in real terms while available annuities, used as reference for loads estimation, specified nominal payments. No real annuity was available in the market at that time. A decade later, Brown et al. (2000) mention one unique provider of inflation-linked annuities in the States with respective money's worth (65-year-old men, based on population mortality and treasuries rates) equal to 0.749, which is equivalent to a load (one minus the money's worth) of 0.251, or an actual overprice (one divided by the money's worth) of 33.5 percent.[14] Even though large loadings as above do diminish the optimal rate of conversion of wealth into annuities, they alone are far from completely explain the low levels of annuitization. Analyzed together with other assumptions,

[12]The load factor is usually given by the difference of the money's worth to one (Brown, 2007). Thus, it is an underestimated measure of the (usually) overprice - given by the inverse of the money's worth - charged by insurers having the fair price as reference.

[13]Annuities in the U.S. are regulated in the state level.

[14]Interestingly, at the time the paper of Brown et al. (2000) was written, the inflation-linked annuity were available in the market for two years without a single selling.

as bequest motives for example, the combined reduction may be substantial, though. Assuming that agents compulsorily annuitize half of their wealth through (actuarially fair annuities provided by the) Social Security, Friedman and Warshawsky (1990) show that low annuitization levels (of the remaining wealth) were optimal either with high loads and low bequest motives or with lower loads and higher motives. It may be argued that Friedman and Warshawsky (1990) results rely on the Social Security benefits. However, Social Security coverage is mandatory in the United States and it can not be disregarded when trying to reproduce empirical facts. It may be the case that loads in Friedman and Warshawsky (1990) were too high. In fact, the literature constantly evokes Mitchell et al. (1999) to affirm that "available pricing on annuities does not seem to be sufficient to render annuitization unattractive" (Davidoff et al., 2005).

Table 1: Money's worth of single premium immediate life annuities (constant and escalating benefits) available to 65 and 70-year-old male consumers from 1991 to 2011.

	Constant 65Y		Cola 3% 65Y		Constant 70Y		Cola 3% 70Y	
Data	**Treasuries**	**Agencies**	**Treasuries**	**Agencies**	**Treasuries**	**Agencies**	**Treasuries**	**Agencies**
Jun-91	0.845	-	-	-	0.827	-	-	-
Jan-92	0.857	-	-	-	0.838	-	-	-
Jun-92	0.855	-	-	-	0.855	-	-	-
Dec-92	0.843	-	-	-	0.828	-	-	-
Jun-93	0.848	-	-	-	0.831	-	-	-
Oct-93	0.845	-	-	-	0.828	-	-	-
May-94	0.791	-	-	-	0.779	-	-	-
Oct-94	0.796	-	-	-	0.786	-	-	-
May-95	0.846	-	-	-	0.828	-	-	-
Nov-95	0.854	-	-	-	0.834	-	-	-
May-96	0.826	-	-	-	0.809	-	-	-
Nov-96	0.847	-	-	-	0.828	-	-	-
May-97	0.838	-	-	-	0.823	-	-	-
Nov-97	0.838	-	-	-	0.819	-	-	-
May-98	0.835	-	-	-	0.810	-	-	-
Nov-98	0.878	-	-	-	0.850	-	-	-
May-99	0.842	-	-	-	0.816	-	-	-
Nov-99	0.862	-	-	-	0.832	-	-	-
May-00	0.861	-	-	-	0.834	-	-	-
Oct-00	0.901	-	-	-	0.868	-	-	-
May-01	0.907	-	-	-	0.874	-	-	-
Oct-01	0.920	-	-	-	0.881	-	-	-
May-02	0.892	-	-	-	0.859	-	-	-
Dec-02	0.890	0.867	-	-	0.858	0.840	-	-
Jun-03	0.863	0.842	-	-	0.855	0.838	-	-
Nov-03	0.860	0.841	-	-	0.830	0.815	-	-
May-04	0.855	0.836	-	-	0.826	0.811	-	-
Nov-04	0.851	0.831	-	-	0.818	0.803	-	-
Jun-05	0.854	0.838	-	-	0.820	0.806	-	-
Nov-05	0.837	0.821	-	-	0.804	0.791	-	-
May-06	0.839	0.823	-	-	0.808	0.796	-	-
Nov-06	0.869	0.852	-	-	0.829	0.816	-	-
May-07	0.879	0.861	0.851	0.830	0.838	0.824	0.810	0.794
Nov-07	0.900	0.881	0.869	0.847	0.859	0.845	0.827	0.811
Jul-08	0.946	0.918	0.929	0.896	0.899	0.878	0.878	0.854
Nov-08	0.957	0.904	0.923	0.863	0.916	0.872	0.880	0.833
May-09	0.950	0.892	0.930	0.864	0.904	0.857	0.878	0.827
Nov-09	0.888	0.842	0.858	0.806	0.849	0.812	0.817	0.777
Jul-10	0.900	0.841	0.867	0.800	0.860	0.813	0.820	0.769
Jan-11	0.864	0.822	0.821	0.775	0.830	0.796	0.782	0.745
Jul-11	0.862	0.827	0.821	0.783	0.828	0.800	0.785	0.754
Dec-11	0.895	0.838	0.865	0.799	0.854	0.810	0.816	0.767
Average: All	0.866				0.839			
Average: 2002-11	0.882	0.851			0.847	0.822		
Average: 2007-11	0.904	0.863	0.873	0.826	0.864	0.831	0.829	0.793

To provide a contemporary estimate of money's worth as well as a good overview on its evolution over time, we use data from several editions of the Annuity Shopper, a web based publication, published since 1986, that was also used as reference in Brown et al. (2000). To calculate the expected discounted present value, we use zero yields as in the U.S. Treasury Strips Curve provided by Bloomberg. Maturities available are 3 and 6 months, one year (1Y), 2Y, 3Y, 4Y, 5Y, 7Y, 8Y, 9Y, 10Y, 15Y, 20Y, 25Y and 30Y. Annuity Shopper provides the monthly benefit of several firms. We average all and discount them using the appropriate year rate adjusted for the appropriated number of months.[15] For years not provided by Bloomberg, e.g. 6Y, we linear interpolate, using the closest available data, to obtain the proper rate. We assume that the curve is flat for maturities above 30 years. Depending on the availability of data, we calculate the money's worth since 1991 for 50, 55, 60, 65, 70, 75, 80, 85, and 90-year-old male individuals using cohort mortality tables from the population. Social Security provides cohort tables for every 10-year-period from 1900 to 2100, i.e.: for individuals that were born, or will be born, in 1900, 1910, ..., 2090, and 2100 (Bell and Miller, 2005). To calculate the money's worth from 1991 to 2011, as we do, it would be needed yearly cohort tables from 1901 to 1961. Whenever available, we use the provided data. When it does not match our needs, we linear interpolate from the closest available tables. We are aware that it may affect the results. Nevertheless, this was the best we could do, and the results for matching years are actually quite close to those of Brown et al. (2000). We repeat the described procedure to calculate the money's worth having corporate yields as reference. In fact, we are very conservative on this matter, and use a zero curve basically comprised of U.S. government agency and similar issuers, also provided by Bloomberg. Calculations are made for immediate life annuities with constant benefits and also for, as a proxy for real annuities, a 3 percent escalating annuity, a immediate life annuity with benefits that increase 3 percent on every policy anniversary date.

Table 1 shows the money's worth based on average benefits, taken from Annuity Shopper's publications, offered in exchange to a $100,000 premium for 65-year-old and 70-year-old consumers since 1991.[16] Despite some unusual extreme values in 2008 and in early 2009, the money's worth for a 65-year-old consumer generally does not move much away from the 0.85 figure. The whole period average is 0.866, when U.S. treasuries are taken as reference. Naturally, because yields are higher, results based on agencies rates are lower. Using data only after 2002, the average is 0.882 (UST), and 0.851 (agencies). In line with Mitchell et al. (1999), numbers for a 70-year-old consumer are lower, in a pattern that is also observed when other ages are considered.[17] Indeed, table 2 shows that the money's worth sharply decreases as consumers get older. Also, because the spread over treasuries are usually higher for longer terms, and because the life horizon of older individuals is lower, the difference between money's worth using UST and agencies is smaller for elder agents.

As stated above, annuities are not riskless instruments, and insurers do hold corporate issues in their portfolios. Besides that, particularly for 50 and 55-year-old consumers, the money's worth calculated using treasuries were above unit in 2008 and 2009. Since it is very unlikely

[15] We disregard the 3 and 6 months rates.

[16] Annuity Shopper data for COLA annuities start in 2007; Bloomberg zero-coupon yield data of agencies-like issuances start in 2002.

[17] Money's worth in Mitchell et al. (1999) are lower than those in table 1 possibly because differences in the cohort probabilities. We use cohort survival probabilities published in 2005 (Bell and Miller, 2005).

Table 2: Average money's worth of single premium immediate life annuities (constant and escalating benefits) available to selected ages for male consumers. Data for all ages end in 2011; start dates are shown between brackets.

		Constant Benefits				Cola 3%	
		Treasuries		Agencies		Treasuries	Agencies
Age	Average: All	Avg: 2002-11	Avg: 2007-11	Avg: 2002-11	Avg: 2007-11	Avg: 2007-11	Avg: 2007-11
50Y (1998)	0.966	0.971	0.999	0.915	0.922	0.986	0.885
55Y (1998)	0.941	0.945	0.971	0.898	0.907	0.951	0.870
60Y (1992)	0.893	0.914	0.940	0.876	0.888	0.912	0.850
65Y (1991)	0.866	0.882	0.904	0.851	0.863	0.873	0.826
70Y (1991)	0.839	0.847	0.864	0.822	0.831	0.829	0.793
75Y (1992)	0.800	0.801	0.814	0.782	0.788	0.777	0.749
80Y (1998)	0.739	0.736	0.738	0.722	0.720	0.702	0.682
85Y (1998)	0.665	0.657	0.655	0.648	0.642	0.619	0.607
90Y (1998*)	0.575	0.572	0.565	0.565	0.557	0.539	0.531

* No data from 2001 to 2003

that insurance companies were offering annuities bellow the actuarially fair price, it seems that U.S. treasuries are not the most appropriate yield reference to calculate money's values. In addition, probably for the reason that insurers bear more risk when benefits escalate along time, money's worth of COLA annuities are always lower than those of constant (in nominal terms) benefit annuities. All in, it seems that a money's worth of 0.85 for real annuities is a very conservative estimate.

2.3.2 Bequest motives

Together with unfair prices, as in Friedman and Warshawsky (1990), bequest motives naturally arise as plausible explanation for low annuitization rates. The extent to which bequests prevent purchases of annuities depends on its strength. Furthermore, because annuitization does depend on the level of wealth if future endowments are not zero - as if an agent receives Social Security benefits for example -, to assure low annuity demand equally for poorer and richer individuals the bequest strength would have to be somehow proportional to the wealth level. This is made clear in Davidoff et al. (2005), who show that an wealthy enough agent would "saturate" his/her bequest motives and then annuitize all the remaining resources.

Even the existence of bequest motives, at least regarding substantial planned ones, is not a consensus in the literature. Hurd (1987) finds no support for bequest motives. A couple of years later, assuming that marginal utility of leaving inheritance is zero for agents without children, Hurd (1989) finds that most bequests are accidental, with the desired ones being small. Bernheim (1991), on contrary, assuming that childless individuals may also extract utility from bequeathed wealth, presents "empirical evidence in support of the view that private saving is strongly influenced by the desire to live bequests". More recently, Brown (2000) obtains no evidence that bequest motives have significant effect in the decision to annuitize, while Hurd and Smith (2002) estimate the distribution of bequests expected from the elderly and conclude that households on their early seventies would bequeath about two-fifths of their wealth. Using data on single households and not ruling out these motives for childless individuals, Kopczuk and Lupton (2007) estimate that three-quarters of the elderly single population have bequests

motives and are expected to leave almost four-fifths of their net wealth, half of that due to bequest motives. The authors call attention to the fact that their estimation strategy avoid them to distinguish between bequests and precautionary motives for holding wealth, but they doubt that expenses related to uncertain medical costs have a "large influence on the estimated presence and magnitude of the bequest motive". On this matter, Ameriks et al. (2011) find that "public care aversion", the desire to avoid public sponsored long-term care, and bequest motives are both relevant to explain the low rate of spending among retirees.

Regarding the functional form, Hurd and Smith (2002), and Kopczuk and Lupton (2007) model utility as being linear on bequests. De Nardi (2004), De Nardi et al. (2010), and Ameriks et al. (2011) assume that the functional form follows utility on consumption, usually with a couple of added parameters to account for the strength and luxury of bequests, resembling quite closely, with the exception of the luxury parameter, Fischer (1973) and Friedman and Warshawsky (1990). As shown in Lockwood (2011), who formulates and uses himself a dynasty-kind of altruistic bequests with similar effects, all these formulations adopted with the respective parameters but the one in De Nardi et al. (2010) have strong impact on the demand for annuities. A formulation as in Ameriks et al. (2011), in particular, even with small loads, is so strong that makes annuitization completely unattractive. To strong bequest motives, nevertheless, correspond very high saving rates, which may be one of the reasons why Brown (2007) states that some "explanations solve one puzzle at expense of creating new ones".

Besides prices and bequest motives, one important disadvantage of annuities in incomplete markets is the irreversibility of the trade. It is generally prohibited to borrow against future benefits, and the schedule of payments can not be altered (Brown, 2007). Then, actuarial notes usually offers no sort of protection against a liquidity shock. Typically, concerns in this matter regard uninsurable health costs, or out-of-pocket medical expenses, normally related to long-term medical care. Again, the effect on annuitization will depend on the size of the shock; it will also depend on its timing, as shown Davidoff et al. (2005). When occurring late in life, such shocks have little impact or even boost annuitization, since an agent may annuitize and then take advantage of the higher returns to enhance savings in the non-contingent bond to account for future expenses. The need for liquidity early in life, however, may depress annuitization levels in favor of saving in the fully liquid non-contingent bond.

Contrast annuities with better investment opportunities, as in a risky asset with superior expected rate of return, is successfully taken into consideration by Inkmann et al. (2011), who are able to reproduce the low levels of annuitization observed in data from the United Kingdom. Nevertheless, despite the fact that they are not a very common vehicle of retirement protection, variable annuities, in particular equity-linked annuities, are available in market. As for the theoretical perspective, the fact that an agent would like to invest in a risky asset does not change the dominance status of the annuities. The return of whatever basket of assets an investor wants to buy will always be greater if the payoffs are contingent to life.

Although equity-linked annuities may offer some protection against the rise in prices, because annuities most frequently offer only nominal benefits in the U.S., inflation concerns is often cited as a factor lowering annuity demand. The typical answer is always that, although there are not many, providers of inflation-linked annuities in the U.S exist. In addition, these instruments

are long and widely available in the U.K. market and their demand is equally low (Benartzi et al., 2011). Moreover, it may be the case that unpredicted economic shocks would have to be strong enough to provoke large changes in inflation expectations such that the return on the non-contingent asset could more than compensate the mortality cross subsidy from the annuities.

3 Act I: A life in three scenes

3.1 The three-period model

Conditional on living, agents facing survival probabilities maximize the summation of the discounted value of utility from consumption in each period. The probability of surviving from t until $t+1$ is given by p_t, and no individual lives more than three periods, so that: $1 > p_1 > p_2 > p_3 = 0$. If alive, agents are entitled in each period to a finite and positive endowment: $e_t \in \mathbb{R}_{++}$. There are two assets in the economy: an one-period certificate of deposit that yields $r_t > 0$ per period and that we loosely denominate risk-free bond; and actuarially fair *single-premium immediate life annuities.*[18] *Immediate* means that benefits start in the period following the date of purchase, that occurs against a *single premium.*[19] *Life* means that the benefits, assumed to be constant in real terms, will be paid as long as the annuitant is alive.[20] The risk-free bond pays-off regardless the status of the investor. Cumulative savings on the bond are certainly possible; annuities, however, can be bought only once in life. In our basic environment, there are no bequest motives and no source of uncertainty other than survival probabilities.

In each period, a consumer faces the standard decision of balancing expenditures between present and future consumption, thus deciding the fraction of the disposable wealth to be invested in each of the available assets. To avoid the risk that some agent dies in debt, we assume that there is no short selling of the risk-free bond and, to avoid moral hazard issues, that a consumer can not sell annuities under any circumstances. The discount factor β, survival probabilities p_t, risk-free interest rates r_t, and endowments e_t, are assumed to be exogenously given. Because $p_3 = 0$, we anticipate that there is no optimal saving in the last period. Then, an agent

[18] Actuarially fair annuities may also arise as result of a zero profit hypothesis in the longevity insurance market. To see that and gain insight about this financial instrument, assume for simplicity that life lasts at most two periods. The probability of being alive in $t = 2$ is given by p_i, where i indicates that p is related to some individual characteristic, say, age. Assume that there is a continuum of individuals of each type i, with mass 1. Assume further that agent-i optimal saving decision in $t = 1$ is given by s_i. There are two assets in the economy: discount notes offered by a bank with unit price in $t = 1$ given by $\frac{1}{(1+r_b)}$, r_b exogenously given; and annuities offered by insurance companies in perfect competition. Both firms are enduring. An annuity is a financial contract that entitles its holder to receive regular benefits (in fact, *annuities*) as long as he/she lives. Then, if type-i agents buy s_i units of the bond in $t = 1$, $(1 + r_b)s_i$ units will be delivered in $t = 2$, whether an agent-i is alive or not. Now suppose that type i agents hand their savings to the insurance company agreeing to be paid back only if alive in $t = 2$. The insurer will buy the discount note, receive $(1 + r_b)s_i$ in $t = 2$ and redistribute this sum among the mass of p_i type-i individuals that are alive. The annuity price in $t = 1$ is, then, $\frac{s_i}{\frac{(1+r_b)s_i}{p_i}} = \frac{p_i}{(1+r_b)}$, which is defined as the actuarially fair price. The fairness eases notation and does help on algebra, but results in general do not depend on this assumption.

[19] As oppose to immediate, deferred annuities start payments somewhere in the future, according to the purchase contract. Because of this, an agent could increase the value of the contract through the disbursement of extra premium(s). In a three-period model, it does not make much sense talking about deferred annuities. Anyway, even in the n-period model we next present, we do not consider deferred annuities, only immediate.

[20] *Joint-and-survivor* annuities extend the payments to the surviving spouse; some annuities also guarantee the payment for a certain numbers of periods. We consider only life annuities with no certain-period guarantee and with no joint-and-survivor clause.

maximizes the expected utility of consumption over an uncertain life span, solving the following problem:

$$\max_{a_1,a_2,b_1,b_2} \left[u(c_1) + \beta p_1 u(c_2) + \beta^2 p_1 p_2 u(c_3)\right]$$
$$s.t. : a_1, a_2, b_1, b_2, a_1 + b_1, a_2 + b_2 \in [0,1] \text{ and } a_1 a_2 = 0 \tag{RP}$$

where consumption in each period is always equal to the not saved fraction of the correspondent disposable wealth:

$$c_1 = (1 - a_1 - b_1)w_1;\ c_2 = (1 - a_2 - b_2)w_2;\ c_3 = w_3$$

Since there is no credit market, and as variables are percentages invested in annuities, a_t, and/or in the risk-free bond, b_t, choices must lie in the interval $[0, 1]$; similarly, total saving can not exceed the disposable wealth in each period, so that $a_t + b_t \in [0, 1]$. To assure that annuities are bought only in one period, the constraint $a_1 a_2 = 0$ is necessary. The initial wealth equals the first endowment and the subsequent disposable resources are composed by the respective endowment plus previous savings with the correspondent rate of return:

$$\begin{aligned}
&w_1 = e_1;\\
&w_2 = e_2 + R_1 b_1 w_1 + R_1 R_2[p_1(p_2 + R_2)]^{-1} a_1 w_1;\ \text{and}\\
&w_3 = e_3 + R_2 b_2 w_2 + R_2 p_2^{-1} a_2 w_2 + R_1 R_2[p_1(p_2 + R_2)]^{-1} a_1 w_1
\end{aligned}$$

where $R_t = 1 + r_t$ is the gross return on bond investments $b_t w_t$; $R_1 R_2[p_1(p_2 + R_2)]^{-1}$ and $R_2 p_2^{-1}$ are the inverse of the actuarially fair price per unit of benefit A_t of an annuity bought, respectively, in the first or in the second period, so that:

$$\frac{R_1 R_2}{p_1(p_2 + R_2)} a_1 w_1 = A_1;\ \text{and}\ \frac{R_2}{p_2} a_2 w_2 = A_2$$

The proposed problem, RP, can not be solved through standard Lagrange-Karush-Kuhn-Tucker multipliers because of the non-concave $a_1 a_2 = 0$ restriction. We thus disregard this constraint and solve the "unrestricted" problem UP. Every solution to UP so that the annuity is bought only in one period is also a solution to RP. If this is the case, we are done; however, if $a_1^* a_2^* \neq 0$, the solution to RP must be obtained through the comparison of optima when annuities are available only once in life: in $t = 1$, or in $t = 2$. The problem underlying the first (latter) situation is equivalent to UP with a_2 (a_1) always equal to zero, and we represent it by T1 (T2). In fact, RP is equivalent to problem max{T1,T2}. Nevertheless, solving UP and restrict the comparative analysis of T1 and T2 to the subset of the parameter space P where $a_1^* a_2^* > 0$ is much easier than solving max{T1,T2} for all parameters in P. Then, without restriction $a_1 a_2 = 0$ and assuming that $u(\cdot)$ is a concave utility function, UP yields the following necessary

and sufficient first-order conditions, that correspond to variables a_1, b_1, a_2 and b_2, respectively:

$$\beta R_1 R_2 (p_2 + R_2)^{-1}[(1 - a_2^* - b_2^*)u'(c_2^*) + \beta[p_2 + R_2(a_2^* + p_2 b_2^*)]u'(c_3^*)] \leqslant u'(c_1^*) \tag{2a}$$

$$\beta p_1 R_1[(1 - a_2^* - b_2^*)u'(c_2^*) + \beta R_2(a_2^* + p_2 b_2^*)u'(c_3^*)] \leqslant u'(c_1^*) \tag{2b}$$

$$\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) \tag{2c}$$

$$\beta p_2 R_2 u'(c_3^*) \leqslant u'(c_2^*) \tag{2d}$$

3.2 Optimal asset allocation

Through first-order conditions, we characterize possible optima in terms of each variable choice location with respect to its domain. Assuming that the marginal utility of consumption c tends to infinity as c tends to zero, an agent will always consume at least a fraction of the disposable wealth in each period, thus $a_t^* + b_t^* \in [0, 1)$.[21] Hence, we define possible *locus* for an optimal variable as: *i*) corner $[a_t^*, b_t^* = 0]$; or *ii*) interior $[a_t^*, b_t^* \in (0, 1)]$, and identify each solution by 0 and 1, respectively. Under this notation, for instance, optimal asset allocation $a_1^* b_1^* | a_2^* b_2^*$ such that only the risk-free asset is bought in $t = 1$ would be represented by $01|00$. We will use this notation throughout this paper.

Straight from first-order conditions yet, since $p_2 < 1$, it is easy to see that inequality (2c) implies inequality (2d), meaning that the risk-free bond is never optimally bought in the second period, that is: $b_2^* = 0$, regardless the parameters. Such optimal behavior comes simply from the excess return an annuity provides over the bond ($R_2/p_2 - R_2$, in $t = 2$). Because in the second period both assets differ only in terms of its return, if a security is to be bought it must be the one that has the greater rate of return, thus the annuity.

Then, with the value of three choice variables still to be defined, there are left 2^3 combinations of possible optimal *locus*. From these, it can be shown that portfolios $01|10$ and $11|10$ are not optimally chosen. Technically, allocation $a_1^* b_1^* | a_2^* b_2^* = 01|10$ $(11|10)$ would imply $p_1 \geqslant (=)\ 1$. Note that despite its greater rate of return, as benefits are fixed and the asset can not be sold, investment in annuities implies less flexibility in wealth management: once bought, an annuity transfers wealth to all subsequent periods. Therefore, since concavity of utility function induces a smooth optimal-path of consumption, some lack of wealth in period 2 could optimally justify the acquisition of the less profitable but more flexible asset in $t = 1$. Buying the risk-free bond is not compatible with following saving, though. An agent would optimally save in both periods only if there are enough means to be transferred to the third period. In this case, as no flexibility would be needed along the way, wealth is more efficiently saved in the form of annuities and, hence, only the annuity would be bought in period 1. Making it short: the difference between total return of transferring the the amount w of wealth from the first to the third period through strategies *bond in* $t = 1|$*annuity in* $t = 2$ and *annuity*|*annuity* depends on p_1:

$$\frac{R_2}{p_2}(R_1 w) = w\frac{R_1 R_2}{p_2} \lesseqgtr \frac{R_2}{p_2}\left(w\frac{R_1 R_2}{p_1(p_2 + R_2)}\right) + w\frac{R_1 R_2}{p_1(p_2 + R_2)} = w\frac{R_1 R_2}{p_1 p_2} \Leftrightarrow p_1 \lesseqgtr 1.$$

Since $p_1 < 1$, follows the non-optimality of $01|10$ and $11|10$.

[21] It is precisely for this reason that we are able to omit the multipliers and properly determine the inequality in each first-order condition of UP.

Interestingly, in response to some lack of wealth in the second period, as stated, holding money in both assets in period 1 combined with no savings in period 2, 11|00, and investing **only in the risk-free bond**, 01|00, are possible optimal allocations. Nevertheless, first order conditions imply $\phi > 0$, where $\phi = p_1(p_2 + R_2) - R_2$, as a necessary condition to the possible optimality of these portfolios. Restriction $\phi > 0$ is equivalent to $R_2 < p_1 p_2/(1 - p_1)$, which seems to represent a cap on R_2 above which the bond is not demanded. The real driver, though, is the fact that

$$\phi > 0 \Leftrightarrow \frac{1}{R_1 R_2}\phi = \frac{p_1(p_2 + R_2)}{R_1 R_2} - \frac{1}{R1} > 0. \tag{3}$$

That is: the price paid for a **life** unit benefit must be greater than that of a bond paying a **unique** unit payoff, or optimal demand for the bond would be zero.[22] In the absence of bequest motives, this should be trivial. Even though $\phi > 0$ is not a totally loose restriction, paying less for an annuity than for a bond seems quite unreasonable considering agents with not extremely low survival probabilities.[23] We assume, therefore, that ϕ is greater than zero.

Having discarded non-optimal behavior, we shall focus on the remaining possibilities. The goal is to analytically determine all optimal portfolio allocation strategies and show how they are related to endowments distribution along time. As a consequence, we also establish the optimal timing of annuitization. Hence, given β, p_t and r_t, figure 1 relate any triplet (e_1, e_2, e_3) to a correspondent optimal portfolio $a_1^* b_1^* | a_2^* b_2^* =$ 00|00, 00|10, 01|00, 10|00, 10|10, or 11|00. Optimal choices are shown within the *endowment space* E, and utility $u(\cdot)$ is assumed to be a CRRA function with parameter γ:

$$E = \frac{e1}{e_2} \times \frac{e3}{e_2}; \; u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

Before moving on, a technical note. From first-order conditions we completely and directly determine the regions within E that corresponds to each of the possible portfolio allocations just cited, except for the one related to $a_1^* b_1^* | a_2^* b_2^* =$ 10|00. The reason is that, unless e_2 is a function of e_3, we are not able to isolate a_1^* in first-order equation (2a) under 10|00:

$$\beta R_1 R_2 (p_2 + R_2)^{-1} \left[(c_2^*)^{-\gamma} + \beta p_2 (c_3^*)^{-\gamma} \right] = (c_1^*)^{-\gamma}$$

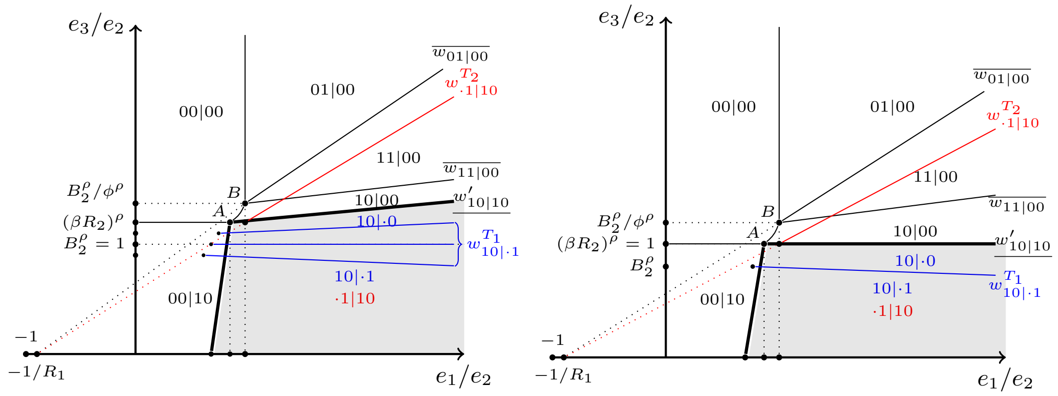
The subset of E where there is saving only through the annuity in the first period, nevertheless, has to be the complement of those that correspond to all other possible optimal portfolio allocation. The intuition is fairly straight forward. We are maximizing a strict concave function on a compact set, so the unique solution is always attained. Under these circumstances, the mentioned subsets of E must be disjoints, due to the uniqueness of the solution, and together they must fulfill the whole space, because there is always a solution. A formal proof derives directly from lemma 1, stated in Appendix A.

Interpretation of results is quite intuitive. Roughly speaking, if the respective endowments ratio is lower than marginal utility between the first and second periods, saving in $t = 1$ is null;

[22] Since the annuity price decreases as R_2 increases, the cap on R_2.

[23] As example, assume that interest-rates are fixed at 2% per year, take survival probabilities from the Period Life Table, 2007, of the US Social Security Administration, and consider that the probability of living another year is constant after 119 years, then the life annuity fair price for a unit benefit regarding a **106-year-old agent** is indeed lower than 1/1.02.

Figure 1: Optimal portfolio choice within endowment space E. Unless specified otherwise, all curves are derived from *foc* of UP. Except for those parallel to one of the axis, their inclinations are approximations, since parameters are left undetermined. In all *white* subsets of E, presented optimal allocations are also solution to RP. The *shaded area* exhibits main solutions to T1 and T2 problems; asset choice to RP equals max{T1,T2}. Graphs differ only on the value of βR_2, that is assumed to be greater than 1 on the left and equal to 1 on the right. The case in which $\beta R_2 < 1$ as well as functional form of every curve are shown in the appendix.



the same optimal behavior applies when e_3/e_2 is greater than the proper marginal utility, i.e.:

$$\frac{e_1}{e_2} < B_1^{-\rho} = \left[\frac{u'(c_1^*)}{u'(c_2^*)}\right]^{-\rho} \Rightarrow a_1^*, b_1^* \approx 0; \ \frac{e_3}{e_2} > (\beta R_2)^\rho = \left[\frac{u'(c_2^*)}{u'(c_3^*)}\right]^{-\rho} \Rightarrow a_2^* \approx 0,$$

where $B_t = \beta p_t R_t$, and ρ stands for γ^{-1}. Fixing e_2 and having its value as reference, we are just saying that if initial wealth e_1 is small, saving in period 1 is trivial; in the same way, if e_3 is **not** small, all disposable wealth is consumed in period 2. Optimal investment within the whole endowment space is directly attached to these ordinary observations.

In both graphs of figure 1, starting from the region close to the origin of the endowment space, where e_1 and e_3 are small, always relatively to e_2, it is optimal to save only in $t = 2$, thus 00|10. This is a trivial annuitization postponement, since wealth in $t = 1$ is humble. Moving clockwise, small e_1 and not small e_3 imply no savings at all, thus 00|00. Not small e_1 and not small e_3 has three different subregions: *i*) low e_1 and high e_3; *iii*) high e_1 and low e_3; and *ii*) the area between *i* and *iii*. Region *i* imply the mentioned interesting optimal behavior: buy the risk-free bond in $t = 1$ and nothing else, thus 01|00, in a "perpetual" annuitization postponement. As initial wealth increases, like in region *ii*, the excess return of the annuity starts to overcome the inflexibility it imposes to portfolio management, thus 11|00. Eventually, as in region *iii*, the annuity totally dominates the bond also in $t = 1$, thus 10|00. This is the full annuitization result of Yaari (1965) and Davidoff et al. (2005), but under incomplete annuity markets and with endowments in every period. Allocations 01|00 and 11|00 are quite interesting. They show that, depending on endowments, even fair annuities may be left out of a nontrivial optimal portfolio. However, being more straightforward and focusing on the graph on the right, where $\beta R_2 = 1$, whenever $e_2 = e_3$, wealth is saved, if so, only through annuities. Then, if the consumer's problem refers to a retiree, full annuitization is an inexorable result.

At last, still moving clockwise, we get to the shaded area where e_3 is small and e_1 is not, implying that it would be optimal, if possible, to buy annuities both in periods 1 and 2. The optimal asset allocation, then, has to be determined through problem max{T1,T2}. Not surprisingly, T1 is very similar to the unrestricted problem we have been detailing so far. The only difference is that just the risk-free bond is available in $t = 2$. For this reason, marginal utility relation βR_2 from UP changes to $B_2 = \beta p_2 R_2$. When $\beta R_2 > 1$, as in the left graph of figure 1, the solution to problem T1 is $10|{\cdot}1$ or $10|{\cdot}0$, depending if the pair $(e_1/e_2, e_3/e_2)$ is below or above curve $w^{\mathrm{T1}}_{10|\cdot 1}$. The upper, the middle, or the lower line is the actual reference, conditional on the value of B_2 being, respectively, greater, equal, or lower than 1. The unique $w^{\mathrm{T1}}_{10|\cdot 1}$ curve in the right graph comes from the fact that $\beta R_2 \leqslant 1$ implies $B_2 < 1$. With only two choices to be made, as the bond is a dominated asset in $t = 2$, problem T2 is much simpler to be solved. Solution to T2 is represented by $\cdot 1|10$, under the $w^{\mathrm{T2}}_{\cdot 1|10}$ line.[24]

There is no simple analytical answer to problem max{T1,T2}. Even numerically there is no easy solution since an explicit form for a_1^* as a function of the parameters, in portfolio $10|{\cdot}0$, depends on e_3 being itself a function of e_2. Therefore, to solve RP in the shaded area we must assign values to the parameters.

3.3 Parametrization

Considering that there is no other way to solve problem RP when annuities would be optimally bough in the first two periods, we must assign values to the parameters. Since we are attributing values, we will also analyze the sensitivity of the results to parameter variation. As a rule, we are taking the discount factor β to be equal to 1.02^{-1}, fixing interest-rates at 2 percent, and assuming that γ measures 2. Determining survival probabilities is not so straightforward. We detail the procedure with the help of one particular case, for which the results are shown below. We assume that each period in the model represents 10 "actual" years, so the first period would correspond to a "60/69" year-old consumer; the second period to a "70/79" year-old; and the last to a "80/89" year-old. We also assume that the survival probabilities will not change with time, i.e.: the probability of a, say, sixty year-old agent live one more period two years from *now* is the current probability associated to a 62 year-old. In the model there is no such thing as dying within a period, so that p_1, the probability of the "60/69" year-old agent being alive in $t = 2$, is the accumulated probability of an 70 year-old agent being alive at the age of 79. The same reasoning applies to p_2. Then, we correct the obtained probabilities, using a single factor, in order to fit the model life expectation to the empirical expectation of a 69 year-old individual. This proceeding is repeated for the case in which one model-period represents 15 years. That said, figures 2 and 3 exhibit the results for the "10" and "15" cases, respectively.

Because we have kept all parameters fixed but survival probabilities, only regions 10|00, 11|00, and 01|00 present significant change (left graphs of figures 2 and 3). As expected, because annuities' return grows when survivability decreases, annuitization increases as agents get older. This is particularly evidenced in the shift of region 11|00, that moves towards region 01|00, narrowing the area within the endowment space where buying only the risk-free bond is optimal,

[24] Although presented solutions ($10|{\cdot}1$,$10|{\cdot}0$ and $\cdot 1|10$) account for the vast majority of the subspace, they are not the only possible optimal portfolios solving problems T1 and T2 in the shaded area. Please refer to the appendix C for a fully reveling solution.

Figure 2: Optimal portfolio choice within endowment space E for specified parameters values when 1 period in the model corresponds to 10 years in a consumer's life. The graph on the left shows results sensitivity to survival probabilities in UP; the graph on the right presents the solution to RP. Solution to T1 is 10|·0 above line wT1, and 10|·1 bellow it; solution to T2 is ·1|10; see the appendix for details.

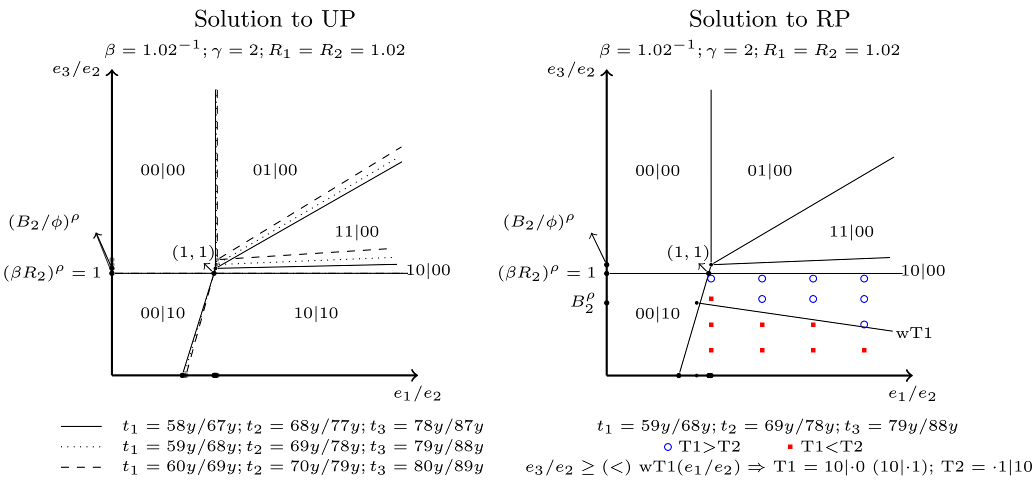
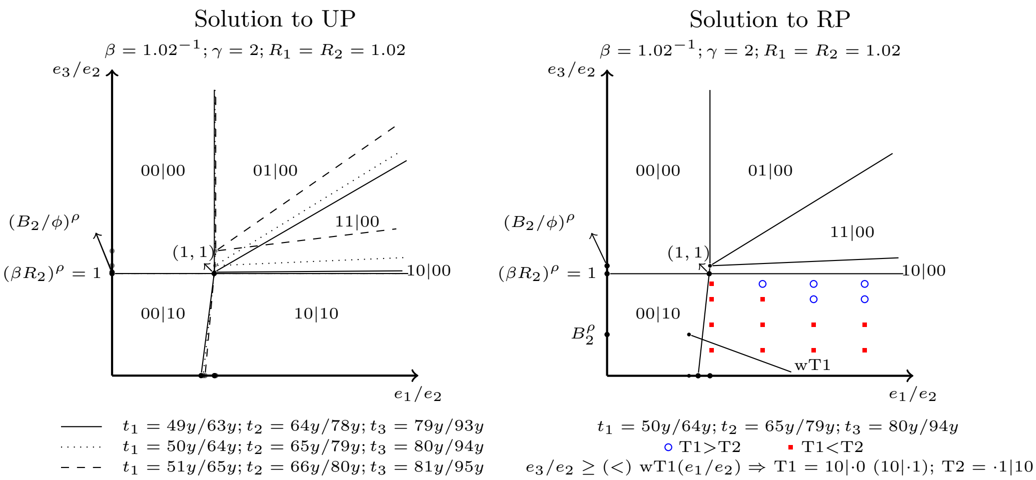


Figure 3: Optimal portfolio choice within endowment space E for specified parameters values when 1 period in the model corresponds to 15 years in a consumer's life. The graph on the left shows results sensitivity to survival probabilities in UP; the graph on the right presents the solution to RP. Solution to T1 is 10|·0 above line wT1, and 10|·1 bellow it; solution to T2 is ·1|10; see the appendix for details.



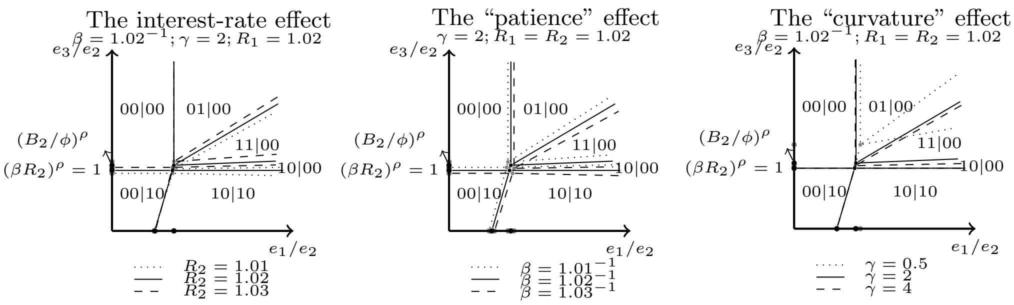
and widening the area related to full annuitization (10|00). Such movement is more accentuated in figure 3 due to the greater numbers of years encompassed in one model-period.

Optimal annuitization timing is shown in the right graphs of both figures. Within the region in which, if possible, it would be optimal to buy annuities more than once (roughly, the region delimited by the line $e_3 = e_2$), blue circles represent the area where the solution to problem T1 is greater to that on problem T2, i.e.: it would be better to annuitize in the first period. Red squares mark the opposite: it pays off to postpone annuitization to period 2. When e_3 is not

much lower than e_2, immediate annuitization is optimal, almost regardless the initial wealth. Having a fixed e_2 as reference, when e_3 decreases, the optimality of immediate annuitization depends on higher initial wealth. Delaying the purchasing of annuities usually pays off when middle age endowments are not much lower than initial wealth and, if they are, when there is another sharp decrease from the second to the third period. Thus, restricted to being lower than initial wealth, the higher the endowment in midlife, the more profitable the postponement of annuitization. Again, precisely because of the larger number of years, survival probability decreases faster in figure 3 making the highlighted effects are more pronounced.

Asset allocation sensitivity to interest-rate in period two R_2 (left graph), discount factor β (middle graph), and curvature γ (right graph) is shown in figure 4. We do not analyze changes in R_1 because it barely affects the results; it only slightly moves the lines that separates regions 00|10 and 10|10. Survival probabilities correspond to those in $t_1 = 59y/68y; t_2 = 69y/78y; t_3 = 79y/88y$.

Figure 4: Asset allocation sensitivity to selected parameters of solution to problem UP within endowment space E. Each period in the model corresponds to 10 years in a consumer's life.



In what regards R_2, having $R_2 = \beta^{-1}$ as reference, a higher rate dislocates upwards region 10|00, generally augmenting annuitization; with lower rates, the opposite happens. Indeed, propensity to saving increases when interests rates are higher than the discount factor; annuities are favored because, since we assume that $\phi = p_1(p_2 + R_2) - R_2 > 0$, the price derivative on R_2 is smaller (and negative, of course) for the contingent notes.[25] Sensitivity to discount factor β is similar to those regarding the interest rate in the second period, but the preference for saving in annuities is not so clear. As for the curvature parameter, we show asset allocation for γ equal to 0.5, 2 and 4. Commonly, the greater the curvature, the more sensitive an agent is to fluctuation on consumption. Since that with greater return comes less liquidity, the area of full annuitization is notably reduced when γ moves from a value bellow one, as half, to one above one; within values above one, however, when $R_2 = \beta^{-1}$, there is not much change in allocation.

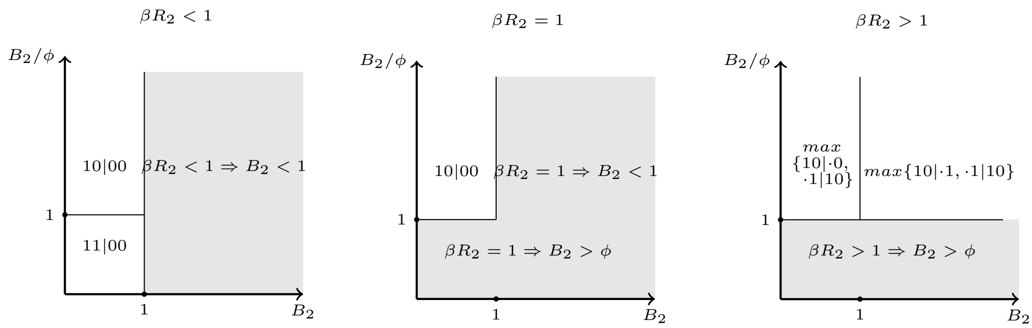
3.4 Optimal portfolio under zero-endowments

In order to illustrate the dependence of optimal asset allocation on the endowments, we solve problem (RP) supposing that only initial wealth differs from zero. Under this assumption, full conversion is frequently optimal, and there is always some level of annuitization. As expected,

[25]See equation (3).

such results are quite similar to those of Davidoff et al. (2005), when the authors assume, as we do, a CRRA utility functional form. The difference is that Davidoff et al. (2005) just consider the case where annuities are available only in the first period. Figure 5 exhibits optimal allocation conditional to parameters.

Figure 5: Solution to the restricted problem (RP) when only initial endowment differs from zero. Within each graph, filled (gray) areas are not in the domain of considered parameters, given the value of βR_2.



Optimal strategy is *buy the annuity immediately, and only the annuity* (10|00) whenever βR_2 equals 1. When βR_2 is lower than 1, best timing for annuitization is, again, always early in life; and the risk-free bond will be optimally bought only if $B_2 < \phi$. which implies

$$p_1 > \frac{\beta R_2 p_2 + R_2}{p_2 + R_2}$$

Assuming that βR_2 deviates from unit only marginally, p_1 must be nearly 1, or the risk-free bond would not be bought at all. Even if B_2 is indeed lower than ϕ, bond holdings would represent a tiny share of the portfolio, unless interests rate in $t = 2$ are way bellow $1/\beta$.[26]

There is no general analytical solution to the restricted problem if $\beta R_2 > 1$, since $max\{\text{T1,T2}\}$ is a non-trivial function of the parameters. The solution to T1 is $10|\cdot0$ or $10|\cdot1$, when $B_2 \leqslant 1$ or $B_2 > 1$, respectively. Solution to T2 is always $\cdot1|10$. Independently of the value of B_2, a sufficient condition for utility under T1 being greater than under T2 is

$$p_1 < \frac{1 + R_2}{p_2 + R_2} \frac{p_2(\beta R_2)^\rho + R_2}{(\beta R_2)^\rho + R_2}$$

When $B_2 > 1$, T2 dominates T1 if

$$p_1 > \frac{1 + R_2}{p_2 + R_2} \frac{p_2 B_2^\rho + R_2}{B_2^\rho + R_2}$$

Both sufficient conditions, when $B_2 > 1$, are obtained by fixing the optimal choice under Ti and comparing $c_3^*(\text{T}i)$ to a c_3 obtained using the optimal assets of the opposite problem in weights such that $c_1 = c_1^*(\text{T}i)$ and $c_2 = c_2^*(\text{T}i)$. Sufficient condition for T1 dominates T2 when $B_2 \leqslant 1$ is the same as in $B_2 > 1$ and it relies on the fact that under such parameters values utility from strategy $10|\cdot0$ is greater than that from $10|\cdot1$.

[26]For a detailed solution under the zero-endowments assumptions, see appendix G.

When agents are not entitled to endowments other than the initial one, then, annuities are even more attractive. Considering $\beta R_2 \leqslant 1$, optimal behavior precludes the postponement of annuitization and also savings in the risk-free bond, if $B_2 \geqslant \phi$. When interest rates are above β^{-1}, annuitization timing is determined by relative premium of conversion in the first or in the second period, which is captured by the survival probability p_1.

3.5 The optimality of constant benefits

We have been considering annuities with constant real benefits, regardless the optimality of this cash flow scheme. A framework where it is possible for an agent to choose the benefits distribution and where annuities are available in every period is equivalent to a complete annuity's market framework, so choosing the benefits distribution is not an issue. Therefore, we reaffirm the hypothesis that an annuity can be bought only once in life and relax the constant benefits restriction in order to verify if and when such payment plan is actually optimal. Since the benefits flow of an annuity bought in $t = 2$ is trivial in a three-period-model and to focus only on the flow issue, we ignore the optimal annuitization timing matter by supposing that annuities are available only in the first period. Agents, then, face a "flexible" T1 problem, where they may choose the fraction, α_t, of the total annuity benefit they want to receive in each period:

$$\begin{gathered}\max_{a_1,\alpha,b_1,b_2} \left[u(c_1) + \beta p_1 u(c_2) + \beta^2 p_1 p_2 u(c_3)\right] \\ s.t. : a_1, \alpha, b_1, b_2 \text{ and } a_1 + b_1 \in [0, 1]\end{gathered} \tag{FT1}$$

where

$$\begin{aligned}c_1 &= (1 - a_1 - b_1)w_1;\ w_1 = e_1 \\ c_2 &= (1 - b_2)w_2;\ w_2 = e_2 + R_1 b_1 w_1 + \alpha \frac{R_1}{p_1} a_1 w_1 \\ c_3 &= w_3 = e_3 + R_2 b_2 w_2 + (1 - \alpha) \frac{R_1 R_2}{p_1 p_2} a_1 w_1\end{aligned}$$

Necessary and sufficient first-order conditions are:

$$\beta R_1 \left[(1 - b_2^*)\alpha^* u'(c_2^*) + (1 - \alpha^*)\beta R_2 u'(c_3^*) + \beta p_2 R_2 b_2^* \alpha^* u'(c_3^*)\right] \leqslant u'(c_1^*) \tag{5a}$$

$$(1 - b_2^*)u'(c_2^*) - \beta R_2 u'(c_3^*) + \beta p_2 R_2 b_2^* u'(c_3^*) + \lambda_1 - \lambda_2 = 0 \tag{5b}$$

$$\beta p_1 R_1 [(1 - b_2^*)u'(c_2^*) + \beta R_2 p_2 b_2^*)u'(c_3^*)] \leqslant u'(c_1^*) \tag{5c}$$

$$\beta p_2 R_2 u'(c_3^*) \leqslant u'(c_2^*) \tag{5d}$$

Once again, through first-order conditions we characterize possible optima in terms of each variable choice location with respect to its domain. As in previous problems, we assume an utility function such that some fraction of the disposable wealth is always consumed. Thus, no optimal choice regarding investment in assets equals 1; α^*, however, can be and in fact will be 1 under some parameters conditions. Following prior notation, then, 0,1,+1 indicates, respectively, that an optimal choice is zero, belongs to (0,1), or equals 1. Our interest resides in solutions where the annuity is optimally bought and $\alpha^* \in (0, 1)$. From all four possible asset

combinations of this type, the only solution optimally sustainable is $a_1^*\alpha^*b_1^*|\cdot b_2^* = 110|\cdot 0$, that is: if there is annuities payment in every (subsequent) period (after the annuity purchase), the optimal portfolio has no risk-free asset. In fact, with the flexibility the α brings to annuities, the risk-free bond is never bought in $t = 1$ in any circumstance, because wealth is more efficiently transferred to period 2 through the former asset, meaning that $b_1^* = 0$. Optimal portfolio may contain annuities, bought in $t = 1$, and the risk-free asset, bought in $t = 2$, if the agent has a substantial need of wealth in $t = 3$. However, if this is the case and if the agent has spare initial wealth, it is always preferable to transfer means to $t = 3$ straight from $t = 1$ than having it passing through $t = 2$ because the rate of return on each investment is, respectively:

$$\frac{R_1R_2}{p_1p_2};\ \text{and } R_2\frac{R_1}{p_1}.$$

Therefore, when b_2^* is greater than zero, if the annuity is bought, α^* must equal zero.

As before, assuming $u(\cdot)$ to be a CRRA with parameter γ and letting $\rho = 1/\gamma$, when solving FT1 for $a_1^*\alpha^*b_1^*|\cdot b_2^* = 110|\cdot 0$ we obtain:

$$a_1^* = \frac{p_1\left[(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2]e_1 - (R_2e_2 + p_2e_3)\right]}{\left[p_1(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2] + R_1R_2\right]e_1};\ \text{and} \tag{6a}$$

$$\alpha^* = \frac{(\beta R_1)^\rho(R_1R_2e_1 + p_1p_2e_3) - [R_1R_2 + p_1p_2(\beta^2R_1R_2)^\rho]e_2}{R_1\left[(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2]e_1 - (R_2e_2 + p_2e_3)\right]} \tag{6b}$$

Stated in terms of the endowment in the second period, from equation (6a) we have that $0 < a_1^* < 1 \Leftrightarrow$

$$e_2 < \left[(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2]e_1 - p_2e_3\right]R_2^{-1} \tag{7}$$

Respecting restriction given by (7), from equation (6b), α^* is in (0,1) only, and only if, the denominator of (6b) is greater than its positive numerator. These conditions translate to $0 < \alpha^* < 1 \Leftrightarrow$

$$e_2 < \left[(\beta R_1)^\rho(R_1R_2e_1 + p_1p_2e_3)\right]\left[R_1R_2 + p_1p_2(\beta^2R_1R_2)^\rho\right]^{-1};\ \text{and} \tag{8a}$$

$$e_2 > \left[[p_1(\beta R_1)^\rho + R_1]e_3 - R_1(\beta^2R_1R_2)^\rho e_1\right]\left[p_1(\beta^2R_1R_2)^\rho\right]^{-1} \tag{8b}$$

The right-hand side of inequality (8a) is greater than its equivalent in (8b) if, and only if, $(\beta^2R_1R_2)^\rho e_1 > e_3$. Under this last restraint, restriction in (7) will be loose if (8a) is attended. Therefore, provided that the endowment in the second period is in the region defined by (8a) and (8b) and that $(\beta^2R_1R_2)^\rho e_1 > e_3$ the solution to (FT1) will be such that $a_1^*\alpha^*b_1^*|\cdot b_2^* = 110|\cdot 0$ and optimal values of a_1^* and α^* will be given by (6a) and (6b), respectively.

The question now is if such α^* and region of endowments distribution are compatible with $\overline{\alpha} = R_2(p_2+R_2)^{-1}$, the value that makes annuities payments equal. The answer is yes. Condition on endowments for α^* being equal to $R_2(p_2 + R_2)^{-1}$ is given by:

$$R_1R_2(\beta R_1)^\rho[1 - (\beta R_2)^\rho]e_1 - (R_1R_2)(e_2 - e_3) + p_1(p_2 + R_2)(\beta R_1)^\rho[(\beta R_2)^\rho e_2 - e_3] = 0 \tag{9}$$

Since $\alpha^* = R_2(p_2 + R_2)^{-1}$ is indeed in (0,1), optimality of such choice depends only on the fulfillment of restriction (7) or, equivalently, on $(\beta R_1R_2)^\rho e_1$ being greater than the third period

endowment, which will be attained if $(\beta R_2)^\rho e_2 > e_3$.

Summing up then, when $(\beta R_2)^\rho \neq 1$, if endowments are such that equation (9) is true and $(\beta R_2)^\rho e_2 > e_3$, or when $(\beta R_2)^\rho = 1$, if $(\beta R_1)^\rho e_1 > e_3$, constant benefits are optimal. This result may seem a bit vague, but together with the solution of problems UP and RP when $(\beta R_2)^\rho = 1$, it does almost all the job of proving the convenient proposition below.

Proposition 1 *When interest-rate in the second period equals the inverse of the discount factor or, more properly, when $(\beta R_2)^\rho = 1$, if endowments in periods two and three are equal, $e_2 = e_3 = e$, and if initial wealth, e_1, is such that $(\beta R_1)^\rho e_1 > e$, it is optimal to immediately buy life annuities with constant benefits, and only these assets, even if the annuity's market is complete and regardless the restriction of one-time annuitization.*

Proof: Having showed that - if $(\beta R_2)^\rho = 1$, $e_2 = e_3 = e$, and $(\beta R_1)^\rho e_1 > e$ - $10|00$ is the solution of UP and that constant benefits are optimal in (FT1), the only item still to be proved is that $10|00$ with constant benefits is a solution to a problem similar to UP under complete annuity's market and choice over benefits. This is done is appendix B. ■

Relying on the hypothesis that the discount factor β equals the interest rate, we will actually extend this result to the n-period case. Although full annuitization in constant-benefit life annuities is a powerful result, it is not a total surprise, since consumers under such conditions and facing a concave utility will seek a constant consumption path, which is exactly what the annuities provide with an excess return over the regular bond.

3.6 Bequest Motives

For simplicity, we keep the hypothesis that an agent who gets to the last period consumes all the disposable wealth.[27] Along the way, agents that save in the risk-free bond in period t and do not survive to $t+1$ still enjoy utility from this investment, which is assumed to be left as an inheritance. This does not mean that we are ruling out intentional bequest, just that no "donation" is made when the consumer is still alive. Subjected to the same constraints from the restricted problem (RP) and facing bequest motives the consumer's problem is:

$$\begin{aligned} \max_{a_1,a_2,b_1,b_2} & \; [\, u(c_1) + \beta[p_1 u(c_2) + (1-p_1)v(sb_1)] + \\ & + \beta^2[p_1 p_2 u(c_3) + p_1(1-p_2)v(sb_2)]\,] \\ s.t.: & \; a_1, a_2, b_1, b_2, a_1+b_1, a_2+b_2 \in [0,1] \text{ and } a_1 a_2 = 0 \end{aligned} \tag{RPB}$$

where $sb_t = R_t b_t w_t$, $v(\cdot)$ is an increasing function representing the utility an agent extracts from leaving a bequest, and $u(\cdot)$ is an increasing concave function.

Assuming that $v'(\cdot)$ goes to infinity as the bequest tends to zero, there is always saving in the risk-free bond, so both b_1^* and b_2^* are greater than zero. Following prior notation, then, possible optimal portfolio allocation are restricted to: $a_1^* b_1^* | a_2^* b_2^* = 01|01$, $01|11$, $11|01$ and $11|11$. However, if $v(\cdot)$ is a function such that the marginal gain from an extra unit of bequest is greater

[27] In the n-period case we show that this hypothesis makes only marginal difference when compared to the situation where agents also may leave bequests in the last period of the model.

than an extra unit of consumption, measured in $u(\cdot)$, when the level of consumption is greater than the level of bequest, the annuity is never bought in $t = 2$.

Assumption 1 *Utility function* $v(\cdot)$ *is such that* $v'(x) > u'(y)\ \forall\ y > x \geqslant sb_2^*$, *where* $sb_2^* = R_2 b_2^* w_2$

Proposition 2 *If* $v(\cdot)$ *is as in assumption 1, the annuity is never bought in* $t = 2$, *i.e.:* a_2^* *is always zero.*

Definition 1 *Denote any solution of problem RPB such that* $a_2^* > 0$ *by* $bp(a_1^* b_1^* | a_2^* b_2^*) = bp^*$. *Take* $\overline{b_2} = a_2^* + b_2^*$, $\overline{a_2} = 0$, *and let* $bp(a_1^* b_1^* | 0\overline{b_2}) = \overline{bp}$ *represent the total utility obtained under this alternative portfolio allocation.*

We will show that $\overline{\text{bp}}$ is always greater than bp^*, implying that any allocation with $a_2^* > 0$ can not be a solution to RPB. Note that $\overline{c_3} = e_3 + A_1 + R_2 \overline{b_2} w_2$; $c_3^* = \overline{c_3} + R_2 a_2^* w_2 (1/p_2 - 1)$; $\overline{c_2} = c_2^*$; and $\overline{c_1} = c_1^*$.

Proof:

$$\overline{\text{bp}} > \text{bp}^* \Leftrightarrow p_2 u(\overline{c_3^*}) + (1 - p_2) v(\overline{sb_2^*}) > p_2 u(c_3^*) + (1 - p_2) v(sb_2^*) \Leftrightarrow$$
$$\Leftrightarrow v(\overline{sb_2^*}) - v(sb_2^*) > p_2 [u(c_3^*) - u(\overline{c_3}) + v(\overline{sb_2^*}) - v(sb_2^*)]$$

Define

$$f(p_2) = p_2 [u(c_3^*) - u(\overline{c_3}) + v(\overline{sb_2^*}) - v(sb_2^*)]$$

and observe that in any interior critical point $f(p_2^*) = R_2 a_2^* w_2 u'(c_3^*)$ Then,

$$\frac{v(\overline{sb_2^*}) - v(sb_2^*)}{R_2 a_2^* w_2} \geqslant v'(\overline{sb_2^*}), \text{ if } v(\cdot) \text{ is concave in } [sb_2^*, \overline{sb_2}]; \text{ or}$$
$$\frac{v(\overline{sb_2}) - v(sb_2^*)}{R_2 a_2^* w_2} \geqslant v'(sb_2^*), \text{ if } v(\cdot) \text{ is convex in } [sb_2^*, \overline{sb_2}].$$

Since $sb_2^* < \overline{sb_2} < c_3^*$, using assumption 1 follows

$$\frac{v(\overline{sb_2}) - v(sb_2^*)}{R_2 a_2^* w_2} \geqslant v'(sb_2^i) > u'(c_3^*),\ \ sb_2^i \in \{sb_2^*, \overline{sb_2}\}, \Rightarrow \overline{\text{bp}} > \text{bp}^*$$

■

More directly, assuming for example that an agent values a bequest in the same way as his/her own consumption, i.e.: $v(\cdot) = u(\cdot)$, the annuity is never optimally bought in $t = 2$. This result is the reverse of that obtained without bequest motives and is also valid under a complete annuity's market in the sense of Davidoff et al. (2005). Remaining possible optimal allocations under assumption 1 are, then, $a_1^* b_1^* | a_2^* b_2^* = 01|01$ and $11|01$. Occurrence of each case

is determined by first-order conditions:

$$\beta\frac{R_1R_2}{(p_2+R_2)}[(1-a_2^*)u'(c_2^*)+\beta[p_2+R_2(a_2^*)]u'(c_3^*)]\leqslant u'(c_1^*) \tag{10a}$$

$$\beta p_1R_1[(1-a_2^*)u'(c_2^*)+\beta R_2a_2^*u'(c_3^*)]+\beta R_1(1-p_1)v'(sb_1^*)=u'(c_1^*) \tag{10b}$$

$$\beta R_2u'(c_3^*)\leqslant u'(c_2^*) \tag{10c}$$

$$\beta R_2(1-p_2)v'(sb_2^*)=u'(c_2^*)-\beta p_2R_2u'(c_3^*) \tag{10d}$$

Under assumption 1, i.e.: $a_2^*=0$, first-order conditions (10a), (10b) and (10c) imply that the annuity is optimally bought in the first period only, and only if, the marginal utility of consumption in $t=2$, if the agent survives, exceeds the marginal utility of the bequest, if the agent does not survive, i.e.: $u'(c_2^*)\geqslant v'(sb_1^*)$. Assuming for simplicity that $v(\cdot)=u(\cdot)$ and that $R_1=R_2=1/\beta$, follows that the annuity is never be bought if endowments are equal in every period. This is kind of a meaningless result, though, since with constant endowments and no reason to anticipate or postpone consumption other than the desire to bequeath, the agent will save only for this purpose. Under the same assumptions, also follows that the annuity is always bought if endowments are different from zero only in the first period:

$$a_1^*=0\xrightarrow{e_3=0}c_3^*=sb_2^*\xrightarrow{(10d)}u'(c_2^*)=u'(c_3^*)\rightarrow c_2^*=c_3^*\xrightarrow{(10a)(10b)}u'(c_2^*)\leqslant u'(sb_1^*)$$
$$u'(c_2^*)\leqslant u'(sb_1^*)\xrightarrow{e_2=0}(1-b_2^*)w_2=c_2^*\geqslant sb_1^*=w_2\rightarrow b_2^*\leqslant 0$$

A similar result is clear in Davidoff et al. (2005) with no restriction on the form of the utility functions: whenever feasible, consumers set aside the necessary to satiate the bequest motives and annuitize the remaining wealth.[28] Actually, when there are endowments in every period, particularly if income is constant after the initial period, because the marginal gain from a larger consumption would be greater than the marginal loss from a lower bequest, the annuity should always be bought whenever initial wealth were large enough. Making no assumption on endowments, however, even assuming that $v(\cdot)=u(\cdot)$ is a CRRA function it is not possible to completely solve problem (RPB) analytically.

4 Act II: N springs in a life

4.1 Introduction

The analytic foundation from the three-period model is a good reference regarding optimal asset allocation in a n-period environment primarily because the dynamics behind consumer's choices is usually maintained in larger horizons. This is of particular interest in an environment characterized by the presence of irreversible annuities, though, because explicit results can not be obtained in most of cases when $n>3$. Yet, under some particular assumptions, mainly the availability of credit markets, analytic results are reachable. Assuming that is possible to short-sell the assets - remarkably the annuity - up to the present value of future endowments, net

[28]According to (Davidoff et al., 2005), such result is also implicit in Yaari (1965).

from previous sales, this section presents brief representations in discrete time of the problems proposed by Yaari (1965). By doing so, important theoretical results regarding optimal asset allocation with annuities are reviewed, and distinguishable properties of this asset are made clear. Besides, we show that some important results does not depend on the completeness of the annuity market, an assumption both in Yaari (1965) and in Davidoff et al. (2005); notably: the optimality of (immediate) full annuitization, when bequest motives are not on the game; the optimality of partial immediate annuitization, when bequests are taken in consideration; and the constancy of the optimal consumption plan, in both cases.

Following Yaari (1965) in a "Fisher-type analysis", the problem FP below presents an elementary decision of asset allocation along time, under no uncertainty. An agent that lives T periods wants to find the plan $\{c_t\}_{t=1}^{\mathrm{T}}$ that maximizes the discounted utility from the consumption of such plan. To keep straight on the track, we assume that utility is represented by a strictly concave function such that: $u'(\cdot) > 0$; $u''(\cdot) < 0$, and $u'(c) \xrightarrow{c \to 0} \infty$. Then, because utility is strictly increasing, the budget restriction will be valid under equality. Thus, choosing b_t, the fraction of wealth invested in the available risk-free asset, with gross return of R_t per period, is equivalent to choosing c_t. Assuming that there is no credit and that β_t is the discount factor from time $t+1$ to time t, problem FP is given by:

$$\begin{aligned} \max_{\{b_t\}_{t=1}^{T}} \; & [u(c_1) + \beta_1 u(c_2) + \cdots + \beta_1\beta_2 \cdots \beta_{T-2}\beta_{T-1} u(c_T)] \\ s.t.: \; & b_t \in [0,1] \\ & w_{t>1} = R_{t-1} b_{t-1} w_{t-1} + e_t; \; w_1 = e_1 \\ & c_t = (1-b_t) w_t; \end{aligned} \tag{FP}$$

If solution is interior for every t, first-order condition on b_t yields:

$$u'(c_t^*) = \beta_t R_t u'(c_{t+1}^*) \tag{11}$$

Then, whenever $R_t = \beta_t^{-1}$, $u'(c_t^*) = u'(c_{t+1}^*)$, which implies the classical invariance of optimal consumption along time: $c_t^* = c_{t+1}^*$ for all $t < \mathrm{T}$, if $u'(\cdot)$, as assumed, is a strictly decreasing function.

When the length of live is uncertain, each term in the summation of the problem (FP) will be multiplied by its respective survival probability. All other specifications remain the same, so the only difference from equation (11) is that first-order condition now accounts for survivability:

$$u'(c_t^*) = \beta_t p_t R_t u'(c_{t+1}^*) \tag{12}$$

where p_t is the survival probability to period $t+1$, given that the agent is alive in t. Hence, if $R_t = \beta_t^{-1}$, a decreasing optimal consumption path:

$$u'(c_t^*) = \beta_t p_t R_t u'(c_{t+1}^*) \xrightarrow{p_t < 1} u'(c_t^*) < u'(c_{t+1}^*) \xrightarrow{u''(\cdot) < 0} c_t^* > c_{t+1}^*$$

As in Yaari (1965), however, the constancy of consumption is restored if fair annuities are

available and markets are complete in the sense of Davidoff et al. (2005). Completeness translates in practice by the availability in each period t of annuities paying off entirely in $t+1$, thus, a one-period annuity. In this way, the unique difference between both assets is that the agent has rights on annuities payoffs only if alive. Subjected to such conditions, the unrestricted (in what regards to multiple annutization) problem under complete annuities market hypothesis, problem $\mathrm{UP_c}$, is given by:

$$\begin{aligned} \max_{\{a_t,b_t\}_{t=1}^T} \; & [u(c_1) + \beta_1 p_1 u(c_2) + \cdots + \beta_1 \cdots \beta_{T-1} p_1 \cdots p_{T-1} u(c_T)] \\ s.t. : \; & a_t, b_1, \text{ and } a_t + b_t \in [0,1] \\ & w_{t>1} = R_{t-1}(a_{t-1}/p_{t-1} + b_{t-1})w_{t-1} + e_t; \; w_1 = e_1 \\ & c_t = (1 - a_t - b_t)w_t; \end{aligned} \tag{$\mathrm{UP_c}$}$$

Therefore, when bequest motives are not an issue, as in ($\mathrm{UP_c}$), since returns on annuities - R_t/p_t - are greater than those from bonds - R_t - optimal saving in these last assets are null: $b_t^* = 0$ for all t. That said, taken with respect to the percentage of disposable wealth invested in annuities, a_t, first-order condition in time t on problem $\mathrm{UP_c}$ is equal to (11):

$$u'(c_t^*) = \beta_t p_t \frac{R_t}{p_t} u'(c_{t+1}^*) \tag{13}$$

Trivially, if more consumption is better than less consumption, any annuities return R_t^a such that $R_t/p_t > R_t^a > R_t$ would imply the nullity of bond savings. Fairness, however, is necessary to restore the steadiness of consumption, as in Yaari (1965). Suppose that the price of a non-fair annuity were given by $(1+\delta)\frac{p_t}{R_t}$, $\delta > 0$, so that the return would be $\alpha \frac{R_t}{p_t}$, $\alpha = (1+\delta)^{-1}$. Then, under such assumptions, if $R_t^{-1} = \beta_t$, optimal consumption decreases along time:

$$u'(c_t^*) = \alpha u'(c_{t+1}^*) \xrightarrow{\alpha<1} c_t^* > c_{t+1}^* \tag{14}$$

Having empirical evidence as reference, completeness of the annuity markets is quite a strong assumption, though.[29] Then, as in the three-period model, we assume that only single premium life annuities with constant benefits (in real terms) are available. In this way, annuities and discount notes will also differ in what concerns to liquidity matters. Besides bequest motives then, the potential mismatch between intended consumptions plans and those that are feasible given the available annuities benefits, as Davidoff et al. (2005) affirm, diminishes the power of the actuarial bond. To address the question of optimal timing of annuitization, we will also assume that annuities may be bought only once in life. Under these assumptions, the problem approaches reality and, of course, loses analytical tractability. Even though, assuming that credit market is available, immediate full annutization is still an optimal result.

Definition 2 *Let UPcc be a unrestricted problem in what regards to multiple annutization, under complete annuities market hypothesis and with credit markets, i.e.: a problem similar to UP_c with credit. Let RP_{ic} be a restricted problem in what regards to multiple annutization, under incomplete annuities market hypothesis and with credit markets, i.e.: a problem similar to UP_{cc}*

[29] There many types of annuities in the market, but there is no one-period contingent asset available.

with a single buying of life annuities restriction.

Assumption 2 *There is a credit market supplied by risk neutral firms under perfect competition, and interest rates R_t are equal to the inverse of the discount factor β_t in every period.*

Proposition 3 *Under assumption 2, immediate full annuitization is a solution to problem such as UP_{cc} even when only life annuities are available for an unique purchase, i.e.: is a solution to RP_{ic}.*

Proof: A credit market under perfect competition of risk-neutral firms implies that an agent could trade the conditional on surviving endowment in time τ, e_τ, for an amount in any time t given by $\prod_{i=t}^{\tau-1} \frac{p_i}{R_i} e_\tau$. Assume that annuity markets are complete and that there is no restriction on multiple buying. So, $\prod_{i=t}^{\tau-1} \frac{p_i}{R_i} e_\tau$ in time t may be transfered to time τ, when it will be equal to e_τ. Since any value can be moved back and forward in time in equivalent rates, a problem with endowments e_t in every $t \in \{1, 2, ..., \mathrm{T}\}$ is identical to one where the agent has an initial wealth $w = \sum_{t=2}^{\mathrm{T}}(\prod_{i=1}^{t-1} \frac{p_i}{R_i} e_t) + e_1$ and no other nonfinancial income throughout life. Solution to such a problem is given by equation (13), meaning that optimal consumption is constant along life. An agent will not leave unconsumed wealth intentionally because utility is strictly increasing and will not buy the discount note, so:

$$w = \sum_{t=2}^{\mathrm{T}}(\prod_{i=1}^{t-1} \frac{p_i}{R_i} e_t) + e_1 = \sum_{t=2}^{\mathrm{T}}(\prod_{i=1}^{t-1} \frac{p_i}{R_i} c_t) + c_1 = c^*[1 + \sum_{t=2}^{\mathrm{T}}(\prod_{i=1}^{t-1} \frac{p_i}{R_i})]; \text{ then}$$

$$w \equiv \tilde{w} + c^* \Rightarrow \tilde{w} = c^* \sum_{t=2}^{\mathrm{T}}(\prod_{i=1}^{t-1} \frac{p_i}{R_i})$$

Finally, note that $\sum_{t=2}^{\mathrm{T}}(\prod_{i=1}^{t-1} \frac{p_i}{R_i})$ is exactly the fair price in $t = 1$ of an unit-benefit life annuity. An agent with initial wealth w will then consume c^* in the first period and annuitize $\tilde{w} = w - c^*$, which entitles him with a benefit of c^* throughout his life. Thus, even restrict to a single buying of life annuities, an agent is able to replicate the optimal consumption plan of a less restrictive environment through the acquisition of the annuity, and only the annuity, in $t = 1$.[30] As the optimal of a more restrict problem is never greater than that of a less restrict one, immediate full annuitization is a solution to problem $\mathrm{RP}_{\mathrm{ic}}$. ■

The single buying, the constancy and the irreversibility of benefits alone, although make the problem more close to what happens in the real world, can not explain the low demand for annuities. Of course, as shown in the three-period model, in the absence of a credit market - or, equivalently, with restriction to short-selling the assets -, (immediate) full annuitization will heavily depend on the endowments distribution. Demand for annuities does drop and the risk-free bond is not vanished from optimal portfolios, but the level of wealth conversion remains high under many circumstances. Even the presence of bequest motives, depending on its strength, will not be able to lower annuitization to observed levels.

Consider an <u>u</u>nrestricted, in what regards to multiple annutization, <u>p</u>roblem, with <u>b</u>equest motives, under <u>c</u>omplete annuities market hypothesis, and with <u>c</u>redit markets, where savings

[30]Certainly, the risk-free bond is not bought in any other period.

in the risk-free bond in period t is assumed to be left as inheritance if the agent does not survive to period $t+1$. Problem UPB_{cc} is given by:

$$\begin{aligned}
\max_{\{a_t,b_t\}_{t=1}^{\mathrm{T}}} \Bigg[& u(c_1) + \beta_1[p_1 u(c_2) + (1-p_1)v(sb_1)] + \\
& + \sum_{t=3}^{\mathrm{T}} \Bigg[\prod_{i=1}^{t-1} \beta_i [p_i u(c_t) + \prod_{j=1}^{t-2} p_j (1-p_{t-1}) v(sb_{t-1})] \Bigg] \Bigg] \\
s.t.:\ & a_t, b_t, a_t + b_t \in [0,1] \\
& w_1 = \sum_{t=2}^{\mathrm{T}} (\prod_{i=1}^{t-1} \frac{p_i}{R_i} e_t) + e_1; w_{t>1} = R_{t-1}(\frac{a_{t-1}}{p_{t-1}} + b_{t-1}) w_{t-1} \\
& c_t = (1 - a_t - b_t) w_t \text{ and } sb_t = R_t b_t w_t;
\end{aligned} \tag{UPB$_{\text{cc}}$}$$

where the equivalence between a problem with endowments in every period and one in which initial wealth equals the present value of such incomes (and $e_t = 0$ for all $t > 1$) have been used. Problem UPB_{cc} yields the following first-order conditions:

$$u'(c_t) \leqslant u'(c_{t+1});\ \text{and} \tag{a_t^*}$$

$$u'(c_t) - p_t u'(c_{t+1}) = (1-p_t) v'(sb_t) \tag{b_t^*}$$

To make our point more directly, assume that the agent values consumption and possible bequest in the same way and that the interest-rate, R_t, equals the inverse of the discount factor in each period, i.e.: $u(\cdot) = v(\cdot)$; and $R_t = \beta_t^{-1}$. Under such assumptions, equation (a_t^*) is always valid under equality, then: $c_{t+1}^* = c_t^* = sb_{t-1}^*$ for all t and optimal portfolio allocation is given by:

$$a_t^* = \frac{p_t(1 + R_{t+1} a_{t+1}^*)}{R_{t+1}(1+R_t) + p_t - R_{t+1} a_{t+1}^* (1 + R_t - p_t)}\ \forall\ t < \mathrm{T} - 1; \text{and} \tag{16a}$$

$$b_t^* = \frac{1 - a_t^*}{1 + R_t}\ \forall\ t < \mathrm{T} \tag{16b}$$

Because there is no intentional bequest in period T, the marginal gain of one unit of saving in the risk-free bond in $\mathrm{T}-1$ is always greater than the marginal gain correspondent to one unit of saving in the annuity, thus $a_{\mathrm{T}-1}^* = 0$.[31] Because the agent spares wealth to bequeath throughout life, however, optimal consumption is lower than that in problem UPcc and is given by c^* such that:

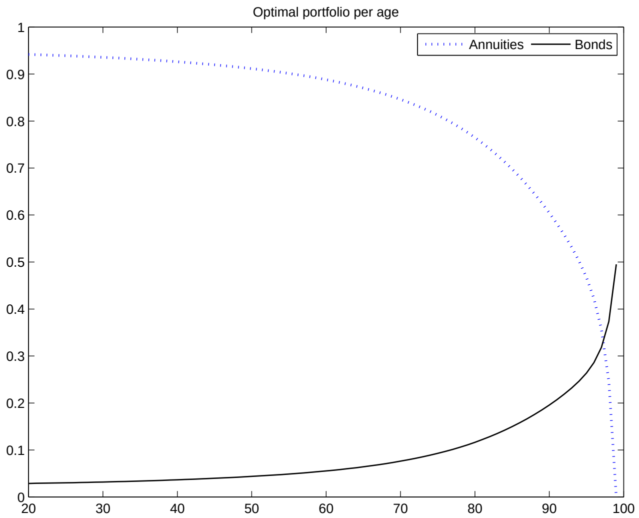
$$\sum_{t=2}^{\mathrm{T}} (\prod_{i=1}^{t-1} \frac{p_i}{R_i} e_t) + e_1 = c^* \Bigg[\sum_{t=2}^{\mathrm{T}-1} (\frac{1}{R_t} \prod_{i=1}^{t-1} \frac{p_i}{R_i}) + \frac{1}{R_1} + 1 \Bigg] \tag{17}$$

More explicitly, if the problem refers to a 20-year-old agent - subjected to: male survival probabilities as in the 2007 Period Life Table from the Social Security Statistical Tables; $R_t = 1.02 = B_t^{-1}$ for all t; and that T=100; i.e.: maximum possible age is 100-year-old -, optimal

[31] If there were intentional bequest in the last period, i.e.: if an extra term $v(sb_T)$ were added to UPBcc, then (16a) would be valid for every $t < \mathrm{T}$ and (16b) for all t.

portfolio would be largely composed by annuities, especially in dates corresponding to early stages of live. Moreover, savings in the contingent asset would considerably surpass that in the risk-free bond until late years. As figure 6 shows, despite the presence of bequest motives, optimal annuities conversion would be above 80 percent until the agent is almost 80-year-old.

Figure 6: Optimal portfolio choice in percentage of disposable wealth per age.



To satisfy optimal condition $c^*_{t+1} = c^*_t = sb^*_{t-1}$, the agent consumes c^*, saves exactly c^*/R_t in discount notes, and annuitizes all the remaining disposable wealth. As noted - and showed in their respective two-period models - by Davidoff et al. (2005) and Lockwood (2011), such asset allocation is precisely as implicit in Yaari (1965). This allocation and the large annuitization it establishes, however, is not due to the complete annuities market hypothesis. An agent would satisfy $c^*_{t+1} = c^*_t = sb^*_{t-1}$, with c^*_{t+1} as in equation (17), through a single and immediate purchase of an annuity that pays c^*/R_t each time, plus saving through the risk-free bond the referred sum of c^*/R_t in every period but in T. This is clearly shown in the right part of equation (17):

$$\underbrace{\sum_{t=2}^{\mathrm{T}-1}\left(\frac{1}{R_t}\prod_{i=1}^{t-1}\frac{p_i}{R_i}\right)c^*}_{i} + \underbrace{\frac{c^*}{R_1}}_{ii} + \underbrace{c^*}_{iii};$$

where i, ii, and iii are, respectively, the price in $t = 1$ of an annuity paying a constant benefit equal to c^*/R_t; the amount saved in risk-free assets; and optimal consumption per period.[32] Under the $R_t = R$ hypothesis, then, as in proposition 3, even if single buying constant-benefits annuities were available, the agent would be able to mimic the optimal consumption plan of a problem with complete annuity markets.

The presence of credit markets does help a lot on algebra, but it is not really a necessary assumption for proposition 3. In the absence of bequests, if $R_t = \beta_t^{-1}$, and when life endow-

[32]The annuity, in this case, should pay benefits only up to period T-1, violating the life lasting property of the considered annuities. However, the life lasting restriction would be respected if bequests to be left in period T and enjoyed in T+1 were considered. In both cases, minor changes that do not shadow the strength of a single-buying of life annuituies.

ments are **constant**, as if the agent is retired for example, proposition 3 is valid whenever initial wealth is greater than the life-endowment; otherwise, there is no savings at all. In the same way, still supposing that endowments are constant through life, when the agent cares about others, immediate annutization of all available resources above those spared and saved in the form of bonds to satisfy the bequest motives will take place whenever (initial) disposable wealth is large enough for that. Moreover, although bequests motives do play a major role in determining this threshold, thus the percentage of converted wealth, immediate full annuitization of resources above this boundary will happen regardless the functional form of bequests, as long as its marginal utility is limited. Thus, unless the way an agent values a potential bequest varies proportionally with possessions, a possible low rate of conversion attributed to this motives will be linked to a specific level of wealth. Then, under the constancy of endowments and the hypothesis of $R_t = \beta_t^{-1}$, the lack of credit has no significant impact on optimal consumer behavior, and life annuities with constant benefits, even with the single-buying restriction, remain very appealing.

The lack of credit markets together with endowments that are not constant, though, invalid proposition 3, with or without bequest motives, and make the timing of conversion a non-trivial topic, when single buying matters. To address this issue and avoid moral hazard issues regarding the selling of annuities, our focus will be on asset allocation in an environment where credit is not available. The drop of this market, however, has a huge negative impact on analytical tractability of the problems addressed in this paper when endowments are not constant through life; and so does the adoption of any uncertainty other than survivability, such as on endowments for instance. As usual, then, more generic assumptions lead to a computational approach. Before showing the numbers, though, it should be clear that life annuities are very appealing for an agent that seeks more and smooth consumption, and that demand can only be reduced by stressing the mismatch between desired and feasible consumption plans this actuarial bonds provide (Davidoff et al., 2005). Unfair prices, bequest motives and negative liquidity shocks - mainly health related - are the usual drivers that are explored in the attempt to find lower annuitization levels. Our numerical simulations, in line with Davidoff et al. (2005), show that the demand does not drop easily. Annuity demand only falls considerably when, associated to unfair prices and bequest motives, we assume that preferences are such that the agent gets more impatient towards future consumption as he/she gets older.

4.2 The n-period model

The maximum number of periods is now given by T, and we relax the price fairness assumption. Nothing else changes with respect to the three-period environment. The benchmark model still presumes no uncertainty besides survivability, there is no credit market, the only available type of actuarial notes is single premium (real) life annuities, which may be bought at most once. If an agent gets to period T, all disposable wealth is consumed, regardless bequest motives.[33] Besides annuities, agents may invest in the risk-free bond, with no restriction on

[33]As stated, this assumption is made just for simplicity; account for bequest in the last period produces no meaningful difference from this model.

multiple purchases. The consumer's problem is given by:

$$\max_{\{a_t,b_t\}_{t=1}^{\mathrm{T}}} \Bigg[u(c_1) + \beta_1[p_1 u(c_2) + (1-p_1)v(sb_1)] + \\ + \sum_{t=3}^{\mathrm{T}} \Big[\prod_{i=1}^{t-1} \beta_i [p_i u(c_t) + \prod_{j=1}^{t-2} p_j (1-p_{t-1}) v(sb_{t-1})]\Big]\Bigg] \\ s.t.:\ a_t, b_t, a_t + b_t \in [0,1] \text{ and } a_i a_j = 0\ \forall\ i \neq j \\ w_{t>1} = R_{t-1} b_{t-1} w_{t-1} + \mathcal{A}_t + e_t;\ w_1 = e_1 \\ c_t = \begin{cases} (1-a_t-b_t)w_t; & if\ \mathcal{A}_t = 0 \\ (1-b_t)w_t; & if\ \mathcal{A}_t > 0 \end{cases} \tag{$\mathrm{RP_n}$}$$

where a_t and b_t are the fraction of the disposable wealth w_t invested in annuities and risk-free bonds, respectively; R_t is the exogenously given gross return on non contingent bond savings; and $\mathcal{A}_t$ is the annuity benefit to be received in time t. The benefit equals zero, if the annuity has not been acquired before t; if it has been bought in optimal timing $\tau < t$, $\mathcal{A}_t$ is given by the investment divided by the annuity's time-τ unit cost:

$$\mathcal{A}_t = \sum_{n=1}^{t-1} A_n;\ \text{where } A_n = \frac{a_n w_n}{uc_n};\ \text{and } uc_n = (1+\delta_n) \sum_{j=n}^{T-1} (\prod_{i=n}^{j} \frac{p_i}{R_i})$$

Naturally, only A_τ will differ from zero, since the purchase of an annuity happens at most once in life, thus $\mathcal{A}_t = A_\tau$. Because assets can not be sold, $\mathcal{A}_t = \mathcal{A}_{t-1}$ whenever $\mathcal{A}_{t-1} > 0$; $\mathcal{A}_0$ equals zero. The parameter δ in the formula that gives the annuity premium measures the level of overprice (having actuarially fair prices as reference) in the market, so that $(1+\delta)$ equals the inverse of the *money's worth*. If $\delta = 0$, the the annuity is fairly priced. Despite differences over money's worth values among ages showed in section 2.3.1, in general, we will assume that the load is constant through live, i.e.: $\delta_t = \bar{\delta}$.

Stated recursively, problem $\mathrm{RP_n}$ is given by:

$$V_t^i(w_t^i, \mathcal{A}_t^i) = \max_{a_t,b_t} \left[u(c_t^i) + \beta_t[p_t^i V_{t+1}^i(w_{t+1}^i, \mathcal{A}_{t+1}^i) + (1-p_t^i)v(sb_t)]\right], \tag{18}$$

where the index i accounts for heterogeneity among individuals. In general, however, we assume that individuals differ only regarding their age, so that index t will do the job for index i, which will be unnecessary in most cases. Nevertheless, even considering that agents are homogeneous in all aspects but with respect to survivability, if interest-rates vary along time, in a deterministic or in a non-stationary way, indexes i and t will no longer be redundant. Despite that, c_T^* is always equal to w_T, thus $V_T = u(w_t)$, and, as usual, the problem is solved from period T backwards to period 1.

We assume that both utility on consumption and on bequests are constant relative risk aversion (CRRA) functions:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma};\ v(sb) = \eta u(sb)$$

The adopted functional form on consumption is quite standard; conversely, there is no easy

choice regarding bequeathed wealth. As commonly assumed in the literature, however, the bequest is valued by what it adds to some level, taken to be a measure of the luxury of the inheritance, and weighted by a second parameter that captures the strength of the motives. In De Nardi (2004), De Nardi et al. (2010), and Ameriks et al. (2011) for instance, $v(b)$ could be written as $\eta u(\omega + b)$, $u(\cdot)$ a CRRA function, where η and ω are constant parameters, generally used to estimation purposes. Depending on the value of η, of course, the utility of the bequest increases or decreases; because ω is greater than 0, this functional form exempt the agent of the necessity to bequeath, which seems to be a desirable property.

Another common functional form is simply $v(b) = \theta b$, as in Hurd and Smith (2002) and in Kopczuk and Lupton (2007). Again, the agent does not have to leave a bequest, because marginal utility does not goes to infinity when bequeathed wealth goes to zero. In fact, because marginal utility is constant in the linear form, the strength of bequest motives would be proportional to the consumption level, a property that looks reasonable. A disadvantage of both formulations is the necessity to assign values to the parameters and the dependency of the results to the assigned levels. Indeed, in the referred literature the value of the parameter related to the strength of the bequest is high, while the luxury related one is not. Therefore, we follow Fischer (1973) and Friedman and Warshawsky (1990) in our proposed formulation. Actually, trying to keep the model the simplest as possible, we are particularly interested in results when η is set to unit, thus when $v(b)$ is equal to $u(b)$.

4.3 The benchmark code

Using a standard value function approach, the problem is solved recursively based on the trivial optimal result: $c_T^* = w_T$, where T is such that the maximum possible age is equivalent to 100-year-old. Minimum age equals 20 years, and so: T = 81. That said, age and period will be treated indistinctly. Any possible problem specification that could invalid this conformity, such as non-constant interest rates, will be accommodated through index i. Fixed one age/period for a type-i individual (and dropping index i to easy notation), since endowments and interest rates, as survival probabilities, are exogenously given, two state variables are needed to fully characterize the problem in each period:

$$w_t \equiv e_t + \mathcal{A}_t + R_{t-1} b_{t-1} w_{t-1} = c_t + (a_t + b_t) w_t$$

Value functions are, then, found per level of disposable wealth, w_t, and the conditional on living level of annuities payments $\mathcal{A}_t$, to be received throughout live from time t (included). The choice of the benefit's level $\mathcal{A}$ seems natural because it specifies if the annuity has been bought or not, the magnitude of the purchase, and directly presents an endogenous component of the budget restriction that will be independent of current choices a_t and b_t, if $\mathcal{A} > 0$.[34] The use of disposable wealth w makes the solution more straightforward, as in $V_T(w_T, \cdot) = u(w_T)$, and is the only choice of state variable that, if continuous as it is, conciliates the grid choices of a_t and b_t to the available resources.

Solutions are obtained within a discrete grid of possible levels of benefits payments from

[34] As before, a consumer may invest in annuities only if this asset has not been bought before, i.e.: $\mathcal{A}_t = 0$; otherwise $a_t^* = 0$ and the benefit is already fixed.

annuities and, due to the *single buying* restriction, optimal annuitization timing will be particular sensitive to the size of the intervals determined by the points within this grid, overall in early stages of live. For that reason, robustness of results is highly dependent on the size of the vector containing possible levels of benefits, which will be in the order of thousands. Percentage saving in the risk-free bond, b_t, is also chosen from a grid, that starts in 1 percent and increases with incremental no greater than 50 basis points up to 100 percent, although, of course, $b_t = 1$ is never optimal because $u'(c) \xrightarrow{c \to 0} \infty$.

Hands-on, in line with Inkmann et al. (2011), we set a grid for possible levels of: disposable wealth (W), annuities benefits (A), and percentage saving in bonds (B). Given, $w \in W$ and $\mathcal{A} \in A$, the agent choses the feasible pair $(\mathcal{A}', b)$ - the level of income from annuities in next period and current percentage saving in the risk-free bond - that maximizes equation (18).[35] The percentage level of disposable wealth saved in the form of annuities is given by the identity $aw/Pr \equiv \mathcal{A}'$, where Pr is the current price of an unit-benefit life annuity starting payments in the following period. In the process, the only non-directly obtainable value is $V'(w', \mathcal{A}')$, implied by each feasible pair $(\mathcal{A}', b)$. Since the choice of benefits is restricted to values in grid A and since next period value functions are known for all $(w, \mathcal{A}) \in W \times A$, the result of $V(\tilde{w}, \mathcal{A})$ for $\tilde{w} \notin W$ is obtained through an interpolation referenced in $\{V(w \in W, \mathcal{A})\}$. We use cubic spline interpolation because it fits better the smooth concave form of function V. Nevertheless, we do not extrapolate the interpolation to values above the maximum wealth level in the grid (W).[36]

4.4 Data and parametrization

In the basic setup, interest-rates are constant at 2 percent; discount-factor β is always constant and equals the inverse of the gross interest-rate; utility curvature γ is set to 2; survival probabilities are those from the 2007 Period Life Table (male) from the Social Security Statistical Tables. Strictly, when modeling the optimal choices of an agent of a particular age, we would have to use cohort survival probabilities for a calendar year consistent with the chosen age and the date of analysis. Thus, if one is interested in choices of agents, say, from twenty to eighty years, 61 cohort tables would be needed; also, for example, the table of a 65-year-old female whose optimal choices were to be determined in 2011 should be different from that of a 65-year-old female analyzed in 2012. We disregard these structural differences between generations, although we are very aware of the relevance of the issue. Nevertheless, we perform a robustness check using probabilities of a cohort table of individuals born in 1940, thus 65-year-old adults in 2005. For the parameters we considered, results change marginally. Since in our model, maximum age is 100 years, we also perform simulations extending the maximum age to 120 years. Again, changes are marginal. The bottom line is: our results are wrong, but not too wrong, at least in what regards to survivability.

Endowments are based on Table 702 (Mean Earnings of Full-Time Year-Round Workers in Current Dollars by Educational Attainment, Sex, and Age: 2008) from section Income, Expenditures, Poverty and Wealth in the U.S. Census Bureau, Statistical Abstract of the United

[35] Only investment in the risk-free bond has to be chosen when $\mathcal{A} > 0$, but we focus on the more complex case $\mathcal{A} = 0$.

[36] The interpolation method, however, is not crucial to results, that are roughly the same if linear approximation is considered.

Table 3: Mean earnings of full-time year-round male workers per age and level of educational attainment.

Age	EA1 All workers	EA2 Less than 9th grade	EA3 9th to 12th grade (no diploma)	EA4 High school graduate (includes equivalency)	EA5 Some college, no degree	EA6 Associate degree	EA7 Bachelor's degree or more, total	EA8 Bachelor's degree
18 to 24 years old	28,246	19,896	21,305	26,218	27,591	31,992	44,888	44,656
25 to 34 years old	48,749	24,211	32,212	36,742	44,597	48,089	68,211	62,840
35 to 44 years old	65,839	27,366	34,998	47,057	53,937	57,183	97,334	86,705
45 to 54 years old	70,869	30,166	34,707	49,003	58,439	60,788	109,260	94,642
55 to 64 years old	72,773	34,106	45,244	47,568	56,486	58,959	104,983	92,615
65 years old and over	69,489	37,047	34,029	54,235	53,022	53,532	96,309	75,366

Source: Table 702. Mean Earnings of Full-Time Year-Round Workers in Current Dollars by Educational Attainment, Sex, and Age: 2008; Statistical Abstract of Current Population Survey

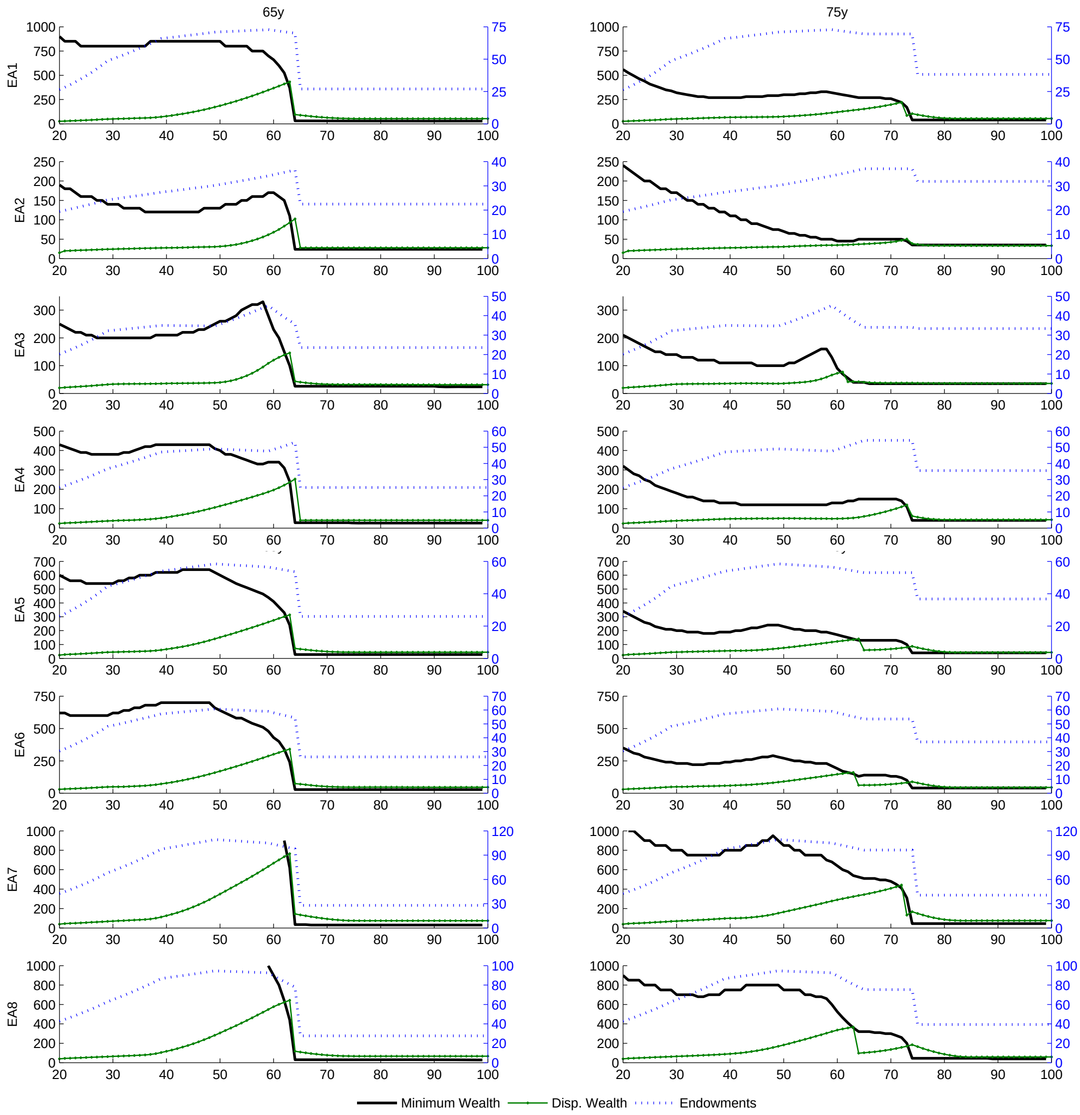
States: 2011, which is partially reproduced in table 3. We use 8 levels of educational attainment in Table 702: all workers (EA1); less than 9th grade (EA2); from 9th to 12th grade with no diploma (EA3); high school graduate (EA4); some college, no degree (EA5); associate degree (EA6); bachelor's degree or more (EA7); and bachelor's degree (EA8). Earnings are reported for age groups: 18 to 24 years; 25 to 34; 35 to 44; 45 to 54; 55 to 64; and above 65 years. To associate one endowment to each age from 20 to 64, we assumed that each reported level applies to the middle of the age group, for ages within two given earnings, endowment grows at a constant rate. For ages 65 and above, we consider two possibilities. Generally, we assume that consumers have just retired at 65 years, so endowments from age 65 to 100 are estimated retirement benefits provided by the Social Security through its *Online Calculator* based on life earnings as above. An alternative setup consider that consumers retire at 75 years; endowments at and above this age are also estimated by Social Security; endowments from 65 to 74 years are assumed to be those in Table 702 from the U.S. Census Bureau.

4.5 Results

We assume that agents receive in every period a deterministic income from work. Such income, as stated, are based on empirical earnings, are related to the level of education and usually grows with age until the date of retirement. A typical agent starts working at the age of 20 and leaves the firm the moment he/she turns 65. Alternatively, the agent works up to the age of 75. For each educational level, the initial wealth is the wage attributed to a 20-year-old employee. After retirement, individuals are entitled with Social Security benefits associated to the correspondent level of earnings during work life. In general, there is no uncertainty at all, except that linked to the length of life.

We first present the outcome regarding optimal annuitization timing. Bequests are not taken into consideration and annuities are fairly priced. Figure 7 shows results for the 8 available levels of education attainment, which as rule increase from the second to the eighth, while level 1 (EA1) refers to average income among all groups. Within each figure, the graphs on the left refer to agents that retire at the age of 65; for those on the right, retirement age is 75. In each graph we plot the wealth the agent has in the beginning of each period, which is composed by the income from work plus financial balances that result from previous optimal saving decisions, and the wealth level at which it becomes optimal to annuitize. When these lines first cross, the optimal timing is determined. Unless future endowments are constant, we do not have a result showing

Figure 7: Threshold wealth level (solid thick) above which is optimal to annuitize, and actual optimal level of wealth (cross marker) per age. Optimal annuitization timing is given when the curves intersect. Yearly earnings from work followed by yearly Social Security benefits are also plotted (dotted line). From top to bottom, earnings and benefits are related to educational attainment level from EA1 to EA8. Ages are in years, wealth (left axis) and endowments (right axis) scale is 1:1000. Left graphs are for 65-year-old retirees; right graphs refer to 75-year-old.



that, fixed one age, if it is optimal to annuitize when wealth level is $\bar{w}$, it is also optimal for all $w > \bar{w}$. Even numerical outcomes may not always exhibit this property. For instance, when the curvature parameter on the utility function is high ($\gamma = 10$), most likely because choice variables are not continuous and the agent is very sensitive even to small consumption fluctuations, it may be optimal to annuitize for wealth level w, but not for $\bar{w} > w$. In the case of the presented results ($\gamma = 2$), however, for wealth levels on the grid, the property holds. Endowments pattern for the respective educational level are also plotted.

Considering 65-year-old retirees, optimal age to buy annuities is, quite often, 63 years; when Social Security payments start at 75, this age moves up to 72 or 73 years, but, depending on

the educational attainment considered, it does not leave the early sixties neighborhood. The age an individual leaves work is relevant because retirement benefits are lower than the wage of an active worker. Therefore, keep working usually delays annuitization. At some point, highly dependent on the endowments pattern, annuitize and, afterwards, save wealth in the non contingent bond to balance current and future consumption become optimal. However, since annuitization if delayed could be done at lower prices, such allocation plan generally will be avoided (this is clear in figure 8). The age an individual claims Social Security is relevant because, up to a certain age, the later the claim takes place, the higher the payments are. Thus, the lower the difference between endowments of active and retired agents. Flatter endowments, for their turn, leads to earlier annuitization. We are assuming that the retiree leaves work and claim benefits concurrently. The combined contradictory effects of late retirement lead unstable optimal annuitization timing among 75-year-old retirees. Overall, then, optimal annuitization timing is closely related to the age of retirement, but the key driver is the endowment pattern: how it changes along live and not the level per se.

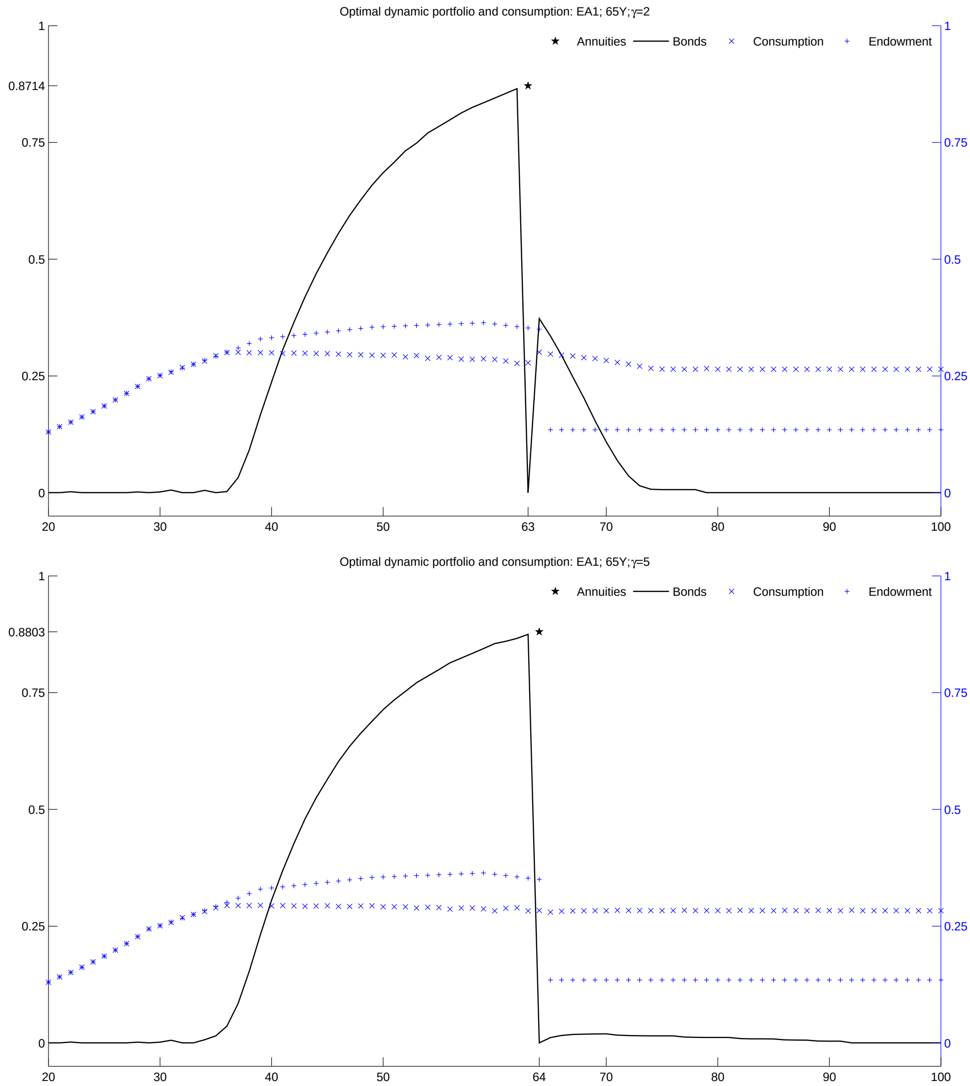
Figure 8 shows the dynamic optimal portfolio as well as the optimal consumption path, assuming that endowments are those related to EA1. The upper graph refers to choices when $\gamma = 2$; in the lower, this parameter is 5. Because endowments are not constant and credit markets are not available, proposition 3 does not hold, so that optimal consumption is not constant either. Since prices are actuarially fair, though, consumption plan is fairly flat and differences between results when gamma equals 2 or 5 are moderate. Even though, an agent whose preferences are such that $\gamma = 5$ is more averse to fluctuations and, thus, start saving earlier, slightly more, and annuitize later to guarantee a flatter consumption plan. On the other hand, when $\gamma = 2$, higher-early and lower-late consumption is preferred to a smoother plan. In both cases, optimal annuitization timing is close to or exactly in the last working year, usually marking the peak of the saving curve, just before the sharp reduction on endowments, and there is no saving in the risk-free bond when annuities are bought.

The graphs in figure 8 represent a general look of optimal dynamic choices, despite the educational attainment, age of retirement, and also the level of price unfairness, assuming that money's worth is independent of the consumer's age. Choices are very stable even considering a 25 percent load over the fair price. If the overcharge is above this level, assuming $\gamma = 2$, annuitization is reduced and postponed, while savings in the risk-free bond rise. In such case, the decline of optimal consumption over time becomes more clear. Under parameters as in the upper graph of figure 8 but with a 40 percent load, optimal age goes to 70 years, annuitization occurs after the peak on the saving curve, and risk-free bonds represent about one-third of the portfolio at the conversion date. Still, over half of the disposable wealth is used to buy annuities.

As a function of parameters δ (unfairness), η (bequest), and γ (utility curvature/risk aversion), table 4 presents the optimal rate of annuitization at the optimal timing and at the age of 65; in this case, the rate is linked to selected levels of disposable wealth. The wealth level of \$360,000 equals the actuarially-fair discounted present value of Social Security benefits and is of particular interest in the related literature. Simulations that account for certain medical expenses and for the insurer default probability, assuming that it applies, are also shown.[37] All

[37] Default probabilities are taken from Schuermann and Hanson (2004).

Figure 8: Optimal asset allocation and consumption per age when endowments are those related to educational attainment level EA1 followed by Social Security benefits starting at age 65. The left y-axis present the fraction of disposable wealth optimally saved in regular bonds and in annuities. The right y-axis shows endowment and consumption yearly levels (scale is 1:200,000). The upper graph refers to choices when $\gamma = 2$; in the lower, $\gamma = 5$. In all cases $R_t = 1.02 = \beta^{-1}$, and annuities are fairly priced.



results assume that endowments are as in EA1, which is the average income among all levels of educational attainment in the U.S. Statistical Abstract 2011, and that agents retire from work and simultaneously claim Social Security benefits in their 65th birthday.

Optimal timing is quite often in the early sixties and annuitization rates are high, not rarely above 80 percent of the disposable wealth. Considering only the effect of unfairness when $\gamma = 2$, it takes an overprice of 30 percent to annuitization levels drop below 70 percent.[38] With an extreme 40 percent load, there is no annuitization for wealth levels up to \$400,000; despite that, a consumer would optimally convert 55 percent of his/her wealth into annuities at the age of 70. Still assuming $\gamma = 2$, strong bequest motives alone, as when $\eta = 10$, do reduce annuitization,

[38] Brown et al. (2000) provides an estimate of 33.5 percent for the load of a real life annuity.

Table 4: Optimal rate of annuitization at the optimal timing and at the age of 65 for selected parameters. In all cases, $R_t = 1.02 = \beta^{-1}$; endowments are those related to educational attainment EA1; and social security benefits start at 65. Default probabilities are in basis points.

δ	η	γ	Out-of-pocket expense	DP	Optimal Annuitization Age	Optimal Annuitization Rate	Optimal annuitization rate per level of disposable wealth at 65 years					
							50k	100k	200k	360k	400k	1000k
-	-	2	-	-	63	0.871	0.43	0.68	0.81	0.86	0.87	0.91
0.1	-	2	-	-	63	0.871	0.33	0.64	0.77	0.84	0.85	0.89
0.15	-	2	-	-	63	0.871	0.22	0.56	0.72	0.81	0.82	0.88
0.2	-	2	-	-	62	0.862	-	0.46	0.68	0.77	0.78	
0.25	-	2	-	-	62	0.861	-	0.29	0.59	0.71	0.74	
0.3	-	2	-	-	65	0.668	-	0.11	0.46	0.64	0.66	
0.4	-	2	-	-	70	0.555	-	-	-	-	-	
-	-	5	-	-	64	0.880	0.43	0.68	0.81	0.86	0.87	0.91
0.2	-	5	-	-	63	0.885	0.26	0.65	0.76	0.84	0.83	
0.3	-	5	-	-	65	0.818	-	0.48	0.70	0.77	0.80	
-	1	2	-	-	62	0.830	-	0.39	0.63	0.74	0.75	0.83
-	2	2	-	-	61	0.814	-	0.28	0.57	0.69	0.71	0.79
-	10	2	-	-	61	0.706	-	-	0.29	0.50	0.53	0.66
0.15	1	2	-	-	61	0.788	-	0.05	0.43	0.61	0.62	0.74
0.2	1	2	-	-	62	0.694	-	-	0.33	0.54	0.56	
0.3	1	2	-	-	66	0.431	-	-	-	-	-	
0.2	1	5	-	-	63	0.787	-	0.20	0.52	0.67	0.69	
0.3	1	5	-	-	63	0.734	-	-	0.43	0.60	0.63	
-	-	2	50k/75Y	-	63	0.882	0.35	0.68	0.82	0.87	0.88	
-	-	2	100k/75Y	-	62	0.884	-	0.39	0.72	0.86	0.84	
-	-	2	50k/85Y	-	64	0.879	0.46	0.69	0.82	0.87	0.87	
-	-	2	100k/85Y	-	64	0.888	-	0.72	0.82	0.87	0.88	
-	-	2	50k/95Y	-	64	0.877	0.43	0.68	0.81	0.87	0.87	
-	-	2	100k/95Y	-	64	0.877	0.43	0.69	0.82	0.87	0.87	
-	-	2	-	AA- (0.25bp)	63	0.872	0.43	0.68	0.81	0.86	0.87	
-	-	2	-	A (0.56bp)	63	0.862	0.43	0.68	0.81	0.86	0.87	
-	-	2	-	BBB+ (3.6bp)	63	0.867	0.41	0.66	0.79	0.85	0.86	
-	-	5	-	AA- (0.25bp)	63	0.875	0.43	0.68	0.81	0.86	0.87	
-	-	5	-	A (0.56bp)	60	0.875	0.43	0.68	0.81	0.86	0.87	
-	-	5	-	BBB+ (3.6bp)	68	0.819	0.41	0.66	0.78	0.68	0.66	
0.25	1	2	-	BBB+ (3.6bp)	68	0.391	-	-	-	-	-	
0.25	1	5	-	BBB+ (3.6bp)	71	0.528	-	-	-	-	-	

but not by much. In association with unfairness, we only show results for simulations when utility on bequest is exactly the same as on consumption ($\eta = 1$). In complete markets, this would translate in a bequeathed wealth equal to the level of one-period consumption. Assuming, then, a 30 percent load and $\eta = 1$, optimal annuitization rate falls considerably, but remains high and above 40 percent. As in the case of figure 8, if $\gamma = 5$, optimal timing is usually slightly later, the level does not leave the 80 percent area with $\delta = 0.30$, and will not drop much below it even when bequest motives are considered. Regardless the level of the utility curvature/risk aversion, when prices are fair, certain out-of-pocket (medical) expenses and also the possibility of default on annuities payments do not change significantly the overall aspect of

optimal allocation. When we model the expenses to occur with probability one at some specific age after 65, the usual result is just that agents increase savings in the risk-free bond a few periods before the disbursement in anticipation to it.[39] When the out-of-pocket spending happens when the agent is 65-year-old, he/she anticipates annuitization and, as before, raises savings in the non-contingent after it to mitigate the impact of the expenses. Default probabilities are very low and Social Security benefits are certain. Thus, the impact of an insurer bankruptcy is modest. Following the Annuity Shopper publication of January 2012, most of insurance company ratings are in the AA- neighborhood. Despite that, assuming default probabilities related to the BBB+ rating, $\delta = 0.25$ and $\eta = 1$, optimal annuitization rate drops only to 0.39, if $\gamma = 2$, and to 0.52, if $\gamma = 5$. Interestingly, the effect of a higher gamma parameter is stronger regarding the desire for flatter consumption than on the wish to avoid the risk given by the default probabilities.[40]

Results on table 4, of course, refer to how agents should behave. In this "should be" world of presented simulations, optimal wealth at the age of annuitization is rarely below the \$400,000 level and, on average, is of about \$445,000. We are not addressing this matter, but the commonly low empirical saving's rate, with consequent reduction of consumption in late life, has also been puzzling economists for quite a while. In fact, Benartzi et al. (2011) affirm that, generally, "middle-class American households spend what they make". The main saving vehicles, when saving occur, continue the authors, are pension funds and home equity. In 2010, average account balance among 401(k) plans' participants in their 60's and that, at the time the data was collect, have been working for the same employer for at least 30 years, was just over \$200,000 (VanDerhei et al., 2011). Even for the simulations in which annuitization at 65 years is null for this level of wealth, as time passes, it would eventually become optimal to convert a sizable fraction of the disposable wealth into annuities. Assuming, as we are, that consumers do not hold company-sponsored pension plans, the presented annuitization rates are well above the inexpressive level observed in the immediate life annuities market. In this sense, table 4 consolidates and expand results already established in the literature. Although under some combinations of parameters the risk-free bond is also used to transfer wealth in time and the fraction of wealth converted into annuities drops considerably, annuitization rates are still too high.

Table 5: Optimal rate of annuitization at the optimal timing and at the age of 65 for selected parameters. In all cases, $R_t = 1.02$; $\beta_t = p_t$, endowments are those related to educational attainment EA1, and social security benefits start at 65.

δ	η	γ	Optimal Annuitization Age	Optimal Annuitization Rate	Optimal annuitization rate per level of disposable wealth at 65 years: 50k	100k	200k	360k	400k
-	-	2	60	0.790	-	0.31	0.58	0.69	0.71
0.15	-	2	58	0.732	-	-	0.19	0.43	0.46
0.2	-	2	58	0.514	-	-	-	0.31	0.35
0.25	-	2	62	0.081	-	-	-	0.16	0.22
0.3	-	2	-	0.000	-	-	-	-	0.05
0.15	1	2	58	0.363	-	-	-	0.21	0.17
0.2	1	2	60	0.035	-	-	-	0.09	0.04

[39] By how much and how many periods depend on the size of the expense.
[40] At least, for the considered level of default probabilities.

Despite unfair prices, moderate bequest motives, certain medical expenses, and even insurer's risk of default, the bottom line is that annuities are extremely attractive instruments. Assuming, however, that preferences are such that the optimal consumption plan would be decreasing even under fair prices and complete markets, the attractiveness of annuities reduces sharply. We model such preferences by assuming that the discount factor β decreases in time, that is: the older the agent, the more impatient he/she gets regarding future consumption. Based on evidence that consumption, or more properly, expenditures fall considerably after retirement, this is not a completely bad assumption.[41] Then, we assume that β_t equals the contemporary survival probability. Thus optimal consumption plan will be smooth, but not flat, and annuitization rates will dramatically decreases. A load of 0.25 alone will bring optimal annuitization rate down to 8 percent. A load of 0.20 together with moderate bequest motives ($\eta = 1$) reduces the rate to tiny 3.5 percent. Results are shown in table 5.

Figure 9: Optimal asset allocation and consumption per age when endowments are those related to educational attainment level EA1 followed by Social Security benefits starting at age 65. The left y-axis present the fraction of disposable wealth optimally saved in regular bonds and in annuities. The right y-axis shows endowment and consumption yearly levels (scale is 1:200,000). Utility on bequest equals utility on consumption; $\gamma = 2$; $R_t = 1.02$; and $\beta_t = p_t$.

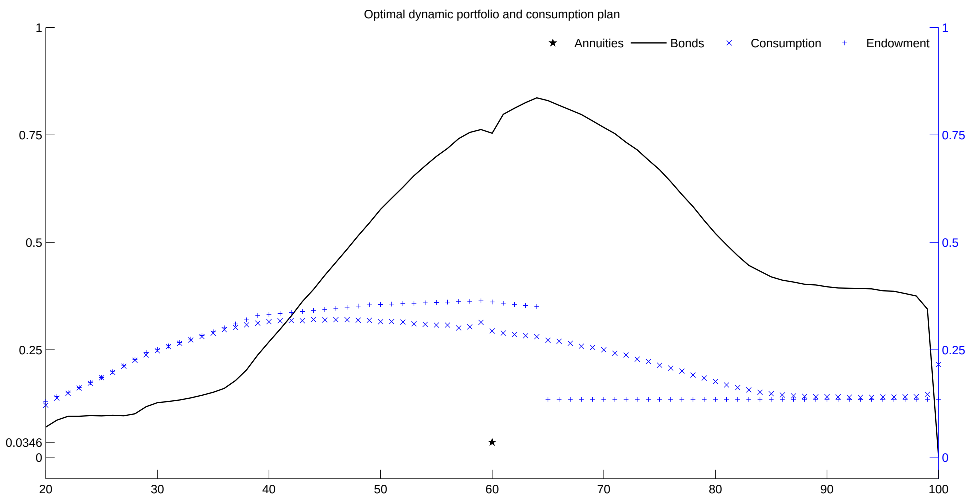


Figure 9 shows the optimal consumer's choices along life under the assumption that $\beta_t = p_t$. There is always savings in the risk-free bond because of the bequest motives. The saving rate increases with age/endowments until the agent retires from work, as in the standard case where $\beta_t = \beta$ in figure 8. However, because now preferences for higher earlier consumption are stronger, annuitization occurs sooner. Disposable wealth on optimal annuitization timing is about \$277,000. Saving in the risk-free bond as a form to balance consumption over time continues for a considerable period after annuitization, until early to mid 80's. After that, the risk-free bond is used only for the purpose of leaving bequeathable wealth. In late life, saving on the risk-free bond will be pretty stable around \$17,000. The sudden increase in consumption in the last period is just a consequence of the hypothesis of the absence of bequest motives when the individual knows he/she will no longer live.

[41] About the falling of consumption/expenditures, see Banks et al. (1998); Bernheim et al. (2001); Hurd and Rohwedder (2003); Thaler and Benartzi (2004).

These low annuitization levels, however, does not hold when gamma equals 5. The reason for humble rates in table 5 is that annuities benefits are constant and agents want a declining consumption plan. A higher gamma annuls the decreasing discount factor effect and implies a flatter optimal plan. Then, annuities are back in business, remarkably. Theoretically, even when $\gamma = 2$, higher levels of annuitization would probably be optimal if annuities benefits were decreasing in time, depending on the money's worth considered. Nevertheless, under preferences that indeed imply a downward slope of the optimal consumption path and considering current available annuities, the fraction of annuitized wealth, as shown, would be very low.

5 Conclusion

From many perspectives, annuities are outstandingly efficient assets. They offer higher returns, compared to non contingent bonds, enabling higher consumption. Also, for agents that dislike fluctuations, as usually assumed in Economics, the constancy of benefit payments will be really appreciated. These features will more than compensate, to a large extent, any lack of liquidity that might be associated to the indelible act of buying an annuity. In fact, as we have analytically shown, incompleteness of markets will not avoid the dominance of annuities over regular bonds. If prices are actuarially fair and non-financial endowments are constant through life, full annuitization is frequently optimal, even when the annuity may be bought only once and markets are incomplete.

The constancy of endowments and the fairness of prices, however, are not very realistic assumptions. For an active worker subject to non-constant salaries, as shown in the three-period model, asset allocation strongly depends on the endowment pattern. Since people eventually retire from work, though, constant endowments are not a major issue. As for annuity prices, we have provided time-series and cross-ages estimates of the *money's worth* of immediate annuities since 1991 through 2011. These results show that the *money's worth* of 0.85 usually assumed as a benchmark in the literature may be a conservative estimate of the actual market premium over fair prices. However, even assuming a load as high as 40 percent, annuities would still represent a large fraction of the optimal portfolio in numerical simulations. Thus, although prices are not fair, the effect of the mortality cross subsidy of annuities is so powerful that the annuity demand is strong even assuming extremely high loads.

Constant benefits associated to higher returns will indeed overcome many drawbacks related to the contingent notes, such as unfair prices, illiquidity or the desire to leave a bequest. Using empirical data on survival probabilities, endowments (for active workers) and Social Security benefits, and under many specifications of parameters, we have performed several simulations combining the cited drawbacks. In line with previous literature, we were not able to reach low levels of annuitization. We have also accounted for the risk of default by assuming that with some positive probability the insurer could permanently default on the promised benefit. Because we assume that retired agents receive Social Security benefits, the demand for contingent notes does not fall sharply even under the risk of the insurer going bankrupt.

The issue regarding optimal timing, when prices are fair and endowments are constant, is comprehended in the full annuitization result: it is optimal to buy life annuities, and only these assets, immediately. There is no gain in postponing annuitization. When endowments are not

constant, we were not able to completely elucidate this issue even in the three-period model with fair prices. Essentially, though, both in the three-period and in the n-period models, optimal timing is tightly attached to the endowments pattern. In our numerical simulations, because we suppose that agents retire from work at some point in life and because Social Security benefits are lower than the income of active workers, optimal annuitization timing will be closely related to the age of retirement.

All in, most of our results sustain the optimality of high annuity demand, accentuating the annuity puzzle. However, assuming preferences that imply a smooth but not constant optimal consumption plan, our simulations indicate that, under modest levels of unfairness in prices and/or bequest motives, life annuities will no longer be such a good deal. Assuming that the discount factor β gets lower as agents get older, as we do, a *money's worth* of 0.80 alone is capable of reducing the fraction of annuities in the optimal portfolio to 8.1 percent, compared with a figure greater than 86 percent in the standard case where β equals the inverse of the interest rate. In other words, we are saying that, if agents want to consume more today than later, the resulting decreasing optimal consumption path will induce tiny levels of annuitization.

Naturally, it also may be the case that we are not being able to correctly appraising the option associated to hold wealth in more liquid assets. Thus, the resilient high levels of annuitization under more standard preferences. Moreover, it is possible that we are not accurately measuring utility. As Aguiar and Hurst (2005) showed, approximate utility on consumption using only money expenditures may not be fully revealing. Further advances in this matter as well as in how preferences change through time, particularly at old ages, will almost certainly bring light and enhance the debate on the annuity puzzle.

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In all appendices we will use the following definitions: $w = e_1$; $\rho = 1/\gamma$; $B_t = \beta p_t R_t$; $\phi = p_1(p_2 + R_2) - R_2$; and $E = e_1/e_2 \times e_3/e_2$, $e_t \in \mathbb{R}_{++}$

A Lemma 1

Definition 3 *Fix two finite sets $N = \{1, 2, ..., n\}$ and $K = \{1, 2, ..., k\}$. Let $X = \prod_{i \in N} X_i \subset \mathbb{R}^n$ - where $X_i = [\underline{x_i}, \overline{x_i}] \subset \mathbb{R}_+$, $\underline{x_i} < \overline{x_i}$ - represent the domain of choice variable x. Define $P \subset \mathbb{R}^k$ as the parameter space; and S as the set of combinations of possible locus - left corner (-1), interior (0), or right corner $(+1)$ - for the location of a maximum candidate of each choice variable x_i, i.e.: $S = \{s : s \in \{-1, 0, +1\}^n\}$.*

Lemma 1 *In a maximization problem where a function $f|_{p \in P} : X \to \mathbb{R}$ always attains its unique maximum $x^*(p)$, combinations of possible locus for a potential maximum of each choice variable imply a partition of the parameter space P. That is: there exists a partition $\{P_s\}_{s \in S}$ such that $p \in P_s$ if and only if $x^*(p)$ is chosen as in s.*

Proof: For all $s \in S$, define P_s as follows: $\forall\ p \in P,\ p \in P_s$ if, and only if,

$$[\forall\ i \in N] : \ s_i = -1 \ \Rightarrow \ x_i^*(p) = \underline{x_i}, \ s_i = 0 \ \Rightarrow \ x_i^*(p) \in (\underline{x_i}, \overline{x_i}), \ s_i = 1 \ \Rightarrow \ x_i^*(p) = \overline{x_i}.$$

By the definition of P_s and using the fact that a solution always exist, it is guaranteed that $P = \cup_{s \in S} P_s$. Now assume by way of contradiction that, for some $\bar{p} \in P$, there exist $s, s' \in S$, with $s \neq s'$, such that $\bar{p} \in P_s \cap P_{s'}$. However, $s \neq s'$ implies $s_i \neq s'_i$ for some i, entailing $x_i^*(p \in P_s) \neq x_i^*(p \in P_{s'})$ and, therefore, $x^*(p \in P_s) \neq x^*(p \in P_{s'})$ for any $p \in P$, and in particular for $\bar{p} \in P$, which contradicts the uniqueness of the solution. ■

B Proof of proposition 1

There are at least two ways to proof proposition 1: straight by solving the consumer's problem under complete annuities market and with the possibility of choosing the benefits distribution, or by solving the previous problem without the choice over benefits distribution and comparing its solution with the one that solves problem RT1. We will follow the second approach and, so, the consumer problem - already using the optimal result that annuities completely dominate the risk-free bond - is given by:

$$\begin{aligned} &\max_{a_1, a_2} \left[u(c_1) + \beta p_1 u(c_2) + \beta^2 p_1 p_2 u(c_3) \right] \\ &\quad s.t. : \ a_1, a_2 \in [0, 1] \end{aligned} \tag{CMP}$$

where:

$$\begin{aligned} c_1 &= (1 - a_1) w_1; \ w_1 = e_1; \\ c_2 &= (1 - a_2) w_2; \ w_2 = e_2 + R_1 p_1^{-1} a_1 w_1; \\ c_3 &= w_3 = e_3 + R_2 p_2^{-1} a_2 w_2 \end{aligned}$$

We are particularly interested in a solution to CMP where it is optimal to buy annuities in periods 1 and 2. Such solution, assuming, as usual, $u(\cdot)$ to be a CRRA with paramenter γ and letting $\rho = 1/\gamma$, is given by:

$$a_1^* = \frac{p_1\left[(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2]e_1 - (R_2e_2 + p_2e_3)\right]}{\left[p_1(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2] + R_1R_2\right]e_1};\ \text{and} \tag{20a}$$

$$a_2^* = \frac{p_2\left[(\beta^2 R_1R_2)^\rho(R_1e_1 + p_1e_2) - [p_1(\beta R_1)^\rho + R_1]e_3\right]}{(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2](R_1e_1 + p_1e_2) - R_1p_2e_3} \tag{20b}$$

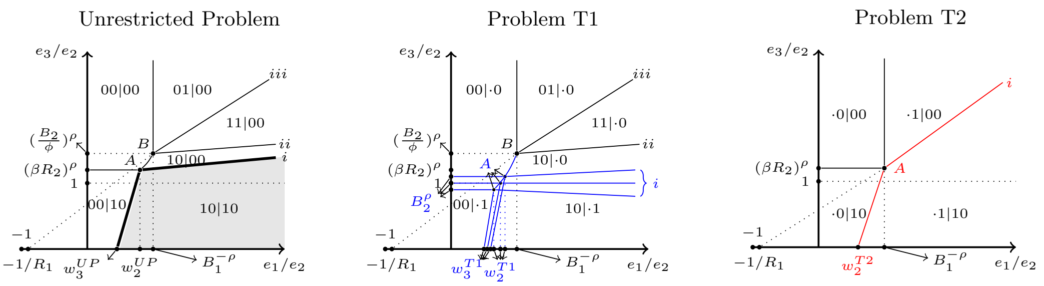
Proof: Within the subspace of parameters where $(\beta R_2)^\rho = 1$ and $e_2 = e_3 = e$, $(\beta R_1)^\rho e_1 > e$ is a necessary and sufficient condition for $a_1^*, a_2^* \in (0,1)$. Moreover, equation (20a) is equal to equation (6a), meaning that optimal investment in annuities in period 1 is the same in CMP and in FT1. Thus, clearly, optimal consumption in both problems in $t = 1$ will be the same as well. Finally, in CMP, first-order condition on a_2 together with $(\beta R_2)^\rho = 1$ imply $c_2^* = c_3^* = c^*(\text{CMP})$. This equality of consumption on second and third periods also happen in FT1, always assuming that parameters are as proposition 1 intend. Therefore, if $c^*(\text{CMP}) = c^*(\text{FT1})$, $a_1^*\alpha^*b_1^*|{\cdot}b_2^* = 110|{\cdot}0$, with a_1^* given by equation (6a) and $\alpha^* = R_2(p_2+R_2)^{-1}$ is also a solution to the consumer's problem under complete annuities market and choice over benefits distribution. In fact:

$$\begin{aligned} c^*(\text{FT1}) = \frac{R_1}{p_1}\alpha^* a_1^* e_1 + e = \frac{(\beta R_1)^\rho[R_1R_2e_1 + p_1(p_2 + R_2)e]}{(\beta R_1)^\rho p_1(p_2 + R_2) + R_1R_2} = \\ = (1 - a_2^*)(\frac{R_1}{p_1}a_1^* e_1 + e) = c^*(\text{CMP}) \end{aligned}$$

■

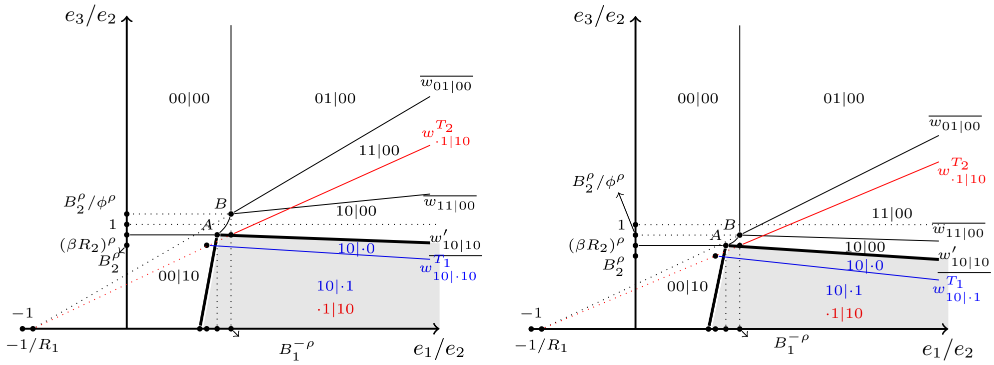
C Graphical solution to problem (RP)

Figure 10: Graphical representation of solutions to problems: unrestricted (left), T1 (center) and T2 (right).



- $B_1^{-\rho} > w_2^{UP} = p_1^{\rho} B_1^{-\rho} > w_3^{UP} = \frac{R_2}{p_2(\beta R_2)^{\rho}+R_2} w_2^{UP}$
- $w_2^{T1} = \frac{(p_2+R_2)^{\rho}}{(1+R_2)^{\rho}} p_1^{\rho} B_1^{-\rho} > w_3^{T1} = \frac{R_2}{B_2^{\rho}+R_2} w_2^{T1}$
- $w_2^{T2} = \frac{R_2}{p_2(\beta R_2)^{\rho}+R_2} B_1^{-\rho}$

Figure 11: Optimal portfolio choice when $\beta R_2 < 1$: $\phi > B_2$ (left) and $\phi < B_2$ (right)



D Unrestricted problem optimal *locus* solution

D.A Optimal asset allocation 00|00

First-order conditions and consumption path are as follow:

$$\beta R_1 R_2 (p_2 + R_2)^{-1} [u'(c_2^*) + \beta p_2 u'(c_3^*)] \leqslant u'(c_1^*) \tag{21a}$$

$$\beta p_1 R_1 u'(c_2^*) \leqslant u'(c_1^*) \tag{21b}$$

$$\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) \tag{21c}$$

$$c_1^* = w; \; c_2^* = e_2; \; c_3^* = e_3$$

Assuming $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $\phi > 0$:

$$(21a) \Rightarrow e_1 \leqslant \left[\beta R_1 R_2 (p_2 + R_2)^{-1} (e_2^{-\gamma} + \beta p_2 e_3^{-\gamma})\right]^{-\rho} = \overline{w_{a00|00}}$$
$$(21b) \Rightarrow e_1 \leqslant e_2 B_1^{-\rho} = \overline{w_{b00|00}}$$
$$(21c) \Rightarrow (\beta R_2)^{\rho} e_2 \leqslant e_3$$

The curve defined by $\overline{w_{a00|00}}$ is increasing in e_3;

$$\begin{aligned} p_2 + R_2 > p1(p_2 + R_2) \Rightarrow p_2 > \phi \Rightarrow B_2\phi^{-1} > \beta R_2 \\ e_3 = (\beta R_2)^{\rho} e_2 \Rightarrow \overline{w_{a00|00}} = e_2 (\beta R_1)^{-\rho} \\ e_3 = (B_2 \phi^{-1})^{\rho} e_2 \Rightarrow \overline{w_{a00|00}} = e_2 (B_1)^{-\rho} = \overline{w_{b00|00}} \\ e_3 > (\frac{B_2}{\phi})^{\rho} e_2 \Rightarrow \overline{w_{b00|00}} < \overline{w_{a00|00}} \end{aligned}$$

Hence, $\overline{w_{a00|00}}$ delimit the values of e_1 when $e_3 e_2^{-1} \in [(\beta R_2)^{\rho}, (B_2\phi^{-1})^{\rho}]$. For $e_3 e_2^{-1} > (B_2\phi^{-1})^{\rho}$ the restriction imposed by $\overline{w_{a00|00}}$ will be loose, and $e_1 \leqslant \overline{w_{b00|00}} = e_2 B_1^{-\rho}$ will be active.

- Summing up: assuming $\phi > 0$

$$a_1^* = 0, b_1^* = 0, a_2^* = 0, b_2^* = 0 \Leftrightarrow \begin{cases} (\beta R_2)^{\rho} e_2 \leqslant e_3 & \\ e_1 \leqslant \overline{w_{a00|00}}, & if\ e_3 \in [(\beta_2 R_2)^{\rho} e_2, (\frac{B_2}{\phi})^{\rho} e_2] \\ e_1 \leqslant \overline{w_{b00|00}}, & if\ e_3 \geqslant (\frac{B_2}{\phi})^{\rho} e_2 \end{cases}$$

D.B Optimal asset allocation 00|10

First-order conditions and consumption path are as follow:

$$\beta R_1 R_2 (p_2 + R_2)^{-1} [(1 - a_2^*) u'(c_2^*) + \beta (p_2 + R_2 a_2^*) u'(c_3^*)] \leqslant u'(c_1^*) \tag{24a}$$
$$\beta p_1 R_1 [(1 - a_2^*) u'(c_2^*) + \beta R_2 a_2^* u'(c_3^*)] \leqslant u'(c_1^*) \tag{24b}$$
$$\beta R_2 u'(c_3^*) = u'(c_2^*) \tag{24c}$$
$$c_1^* = e_1;\ \ c_2^* = (1 - a_2^*) e_2;\ \ c_3^* = e_3 + R_2 p_2^{-1} a_2^* e_2$$

$$(24c) \text{ in } (24a) \Rightarrow \beta^2 R_1 R_2 u'(c_3^*) \leqslant u'(c_1^*) \tag{25a}$$
$$(24c) \text{ in } (24b) \Rightarrow p_1 \beta^2 R_1 R_2 u'(c_3^*) \leqslant u'(c_1^*) \tag{25b}$$
$$(25a) \Rightarrow (25b)$$

Assuming $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$\begin{aligned}
(24c) &\Rightarrow (\beta R_2)^\rho = c_3^*/c_2^* \\
&\Rightarrow a_2^* = \frac{p_2[(\beta R_2)^\rho e_2 - e_3]}{p_2(\beta R_2)^\rho e_2 + R_2 e_2} \Rightarrow \qquad (a^*_{200|10}) \\
&\Rightarrow 0 < a_2^* < 1 \Leftrightarrow (\beta R_2)^\rho e_2 > e_3 \\
(25a) &\Rightarrow (\beta^2 R_1 R_2)^\rho \leqslant c_3^*/c_1^* \Rightarrow e_1 \leqslant \frac{1}{(\beta R_1)^\rho} \frac{R_2 e_2 + p_2 e_3}{p_2(\beta R_2)^\rho + R_2} = \overline{w_{00|10}}
\end{aligned}$$

The curve defined by $\overline{w_{00|10}}$ is increasing in e_3;

$$\begin{aligned}
e_3 = 0 &\Rightarrow \overline{w_{00|10}} = (\beta R_1)^{-\rho} R_2 e_2 [p_2(\beta R_2)^\rho + R_2]^{-1} \\
e_3 \uparrow (\beta R_2)^\rho e_2 &\Rightarrow \overline{w_{00|10}} = e_2(\beta R_1)^{-\rho}
\end{aligned}$$

- Summing up:

$$a_1^* = 0, b_1^* = 0, a_2^* = (a^*_{200|10}), b_2^* = 0 \Leftrightarrow \begin{cases} (\beta R_2)^\rho e_2 > e_3 \\ e_1 \leqslant \overline{w_{00|10}} \end{cases}$$

D.C Optimal asset allocation $01|00 : a_1^* = 0, b_1^* > 0 | a_2^* = 0, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\beta R_1 R_2 (p_2 + R_2)^{-1} [u'(c_2^*) + \beta p_2 u'(c_3^*)] \leqslant u'(c_1^*) \tag{28a}$$

$$\beta p_1 R_1 u'(c_2^*) = u'(c_1^*) \tag{28b}$$

$$\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) \tag{28c}$$

$$c_1^* = (1 - b_1^*) e_1; \;\; c_2^* = e_2 + R_1 b_1^* e_1; \;\; c_3^* = e_3$$

$$\begin{aligned}
(28b) \text{ in } (28a) &\Rightarrow \beta R_1 R_2 (p_2 + R_2)^{-1} [u'(c_2^*) + \beta p_2 u'(c_3^*)] \leqslant \beta p_1 R_1 u'(c_2^*) \\
&\Rightarrow (\beta p_2 R_2) u'(c_3^*) \leqslant (p_1(p_2 + R_2) - R_2) u'(c_2^*) \qquad (29a) \\
(28c) &\Rightarrow (\beta p_2 R_2) u'(c_3^*) \leqslant p_2 u'(c_2^*) \\
(29a) &\Rightarrow (28c)
\end{aligned}$$

Assuming $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $\phi > 0$:

$$\begin{aligned}
(28b) &\Rightarrow (\beta p_1 R_1)^\rho (1 - b_1^*) e_1 = e_2 + R_1 b_1^* e_1 \\
&\Rightarrow b_1^* = \frac{B_1^\rho e_1 - e_2}{(B_1^\rho + R_1) e_1} \Rightarrow \qquad (b^*_{101|00}) \\
&\Rightarrow 0 < b_1^* < 1 \Leftrightarrow e_1 > e_2 B_1^{-\rho} = \underline{w_{01|00}} \\
(29a) &\Rightarrow B_2^\rho (e_2 + R_1 b_1^* e_1) \leqslant \phi^\rho e_3 \Rightarrow e_1 \leqslant \frac{\phi^\rho (B_1^\rho + R_1) e_3 - (B_1 B_2)^\rho e_2}{R_1 (B_1 B_2)^\rho} = \overline{w_{01|00}}
\end{aligned}$$

$$\underline{w_{01|00}} < \overline{w_{01|00}} \Leftrightarrow \phi^\rho e_3 > B_2^\rho e_2$$

- Summing up:

$$a_1^* = 0, b_1^* = (b^*_{101|00}), a_2^* = 0, b_2^* = 0 \Leftrightarrow \begin{cases} B_2^\rho e_2 < \phi^\rho e_3 \\ \underline{w_{01|00}} < e_1 \leqslant \overline{w_{01|00}} \end{cases}$$

D.D Optimal asset allocation $10|00$

First-order conditions and consumption path are as follow:

$$\beta R_1 R_2 (p_2 + R_2)^{-1} [u'(c_2^*) + \beta p_2 u'(c_3^*)] = u'(c_1^*) \tag{31a}$$
$$\beta p_1 R_1 u'(c_2^*) \leqslant u'(c_1^*) \tag{31b}$$
$$\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) \tag{31c}$$
$$c_1^* = (1 - a_1^*) e_1; \; c_2^* = e_2 + A_1; \; c_3^* = e_3 + A_1; \; A_1 = R_1 R_2 [p_1 (p_2 + R_2)]^{-1} a_1^* e_1$$

Optimal choice is such that $a_1^* b_1^* | a_2^* b_2^* = 10|00$ in a subset of the endowment space E that is the complement (with respect to E) of all other cases. See lemma 1.

When $e_2 = e_3 = e$, however, optimal choice is

$$a_1^* = \frac{p_1 (p_2 + R_2) \left[[\beta R_1 R_2 (1 + \beta p_2)]^\rho e_1 - (p_2 + R_2)^\rho e \right]}{\left[p_1 (p_2 + R_2) [\beta R_1 R_2 (1 + \beta p_2)]^\rho + (p_2 + R_2)^\rho R_1 R_2 \right] e_1}$$

D.E Optimal asset allocation $01|10$

First-order conditions are as follow:

$$\beta R_1 R_2 (p_2 + R_2)^{-1} [(1 - a_2^*) u'(c_2^*) + \beta (p_2 + R_2 a_2^*) u'(c_3^*)] \leqslant u'(c_1^*) \tag{32a}$$
$$\beta p_1 R_1 [(1 - a_2^*) u'(c_2^*) + \beta R_2 a_2^* u'(c_3^*)] = u'(c_1^*) \tag{32b}$$
$$\beta R_2 u'(c_3^*) = u'(c_2^*) \tag{32c}$$

- Not possible

$$(32c) \text{ in } (32a) \Rightarrow \beta^2 R_1 R_2 u'(c_3^*) \leqslant u'(c_1^*) \tag{33a}$$

$$(32c) \text{ in } (32b) \Rightarrow p_1 \beta^2 R_1 R_2 u'(c_3^*) = u'(c_1^*) \tag{33b}$$

$$(33a) \text{ in } (33b) \Rightarrow p_1 \geqslant 1$$

D.F Optimal asset allocation 10|10

First-order conditions and consumption path are as follow:

$$\beta R_1 R_2 (p_2 + R_2)^{-1} [(1 - a_2^*) u'(c_2^*) + \beta (p_2 + R_2 a_2^*) u'(c_3^*)] = u'(c_1^*) \tag{34a}$$

$$\beta p_1 R_1 [(1 - a_2^*) u'(c_2) + \beta R_2 b_2 u'(c_3^*)] \leqslant u'(c_1^*) \tag{34b}$$

$$\beta R_2 u'(c_3^*) = u'(c_2^*) \tag{34c}$$

$$c_1 = (1 - a_1^*) e_1; \; c_2 = (1 - a_2^*)(e_2 + A_1); \; c_3 = e_3 + R_2 p_2^{-1} a_2^* (e_2 + A_1) + A_1;$$

$$A_1 = R_1 R_2 [p_1 (p_2 + R_2)]^{-1} a_1^* e_1; \; A_1' = A_1 / e_1$$

$$(34c) \text{ in } (34a) \Rightarrow \beta^2 R_1 R_2 u'(c_3^*) = u'(c_1^*) \tag{35a}$$

$$(34c) \text{ in } (34b) \Rightarrow p_1 \beta^2 R_1 R_2 u'(c_3^*) \leqslant u'(c_1^*) \tag{35b}$$

$$(35a) \Rightarrow (35b)$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$\begin{aligned} (35a) &\Rightarrow (\beta^2 R_1 R_2)^\rho (1 - a_1^*) e_1 = e_3 + R_2 p_2^{-1} a_2^* (e_2 + A_1) + A_1 \Rightarrow \\ &\Rightarrow a_1^* = \frac{(\beta^2 R_1 R_2)^\rho e_1 - R_2 p_2^{-1} a_2^* e_2 - e_3}{(\beta^2 R_1 R_2)^\rho e_1 + A_1' (1 + R_2 p_2^{-1} a_2^*)} \end{aligned} \tag{36a}$$

$$\begin{aligned} (34c) &\Rightarrow (\beta R_2)^\rho (1 - a_2^*)(e_2 + A_1) = e_3 + R_2 p_2^{-1} a_2^* (e_2 + A_1) + A_1 \Rightarrow \\ &\Rightarrow (\beta R_2)^\rho (1 - a_2^*)(e_2 + A_1) = (\beta^2 R_1 R_2)^\rho (1 - a_1^*) e_1 \Rightarrow \\ &\Rightarrow a_1^* = \frac{(\beta R_1)^\rho e_1 - (1 - a_2^*) e_2}{(\beta R_1)^\rho e_1 + (1 - a_2^*) A_1'} \end{aligned} \tag{36b}$$

Solving (iii) and (iv) for a_1^* and a_2^*, we obtain:

$$a_1^* = \frac{p_1\left[(\beta R_1)^\rho w[p_2(\beta R_2)^\rho + R_2] - (R_2 e_2 + p_2 e_3)\right]}{p_1(\beta R_1)^\rho w[R_2 + p_2(\beta R_2)^\rho] + R_1 R_2 w};$$

$$0 < a_1^* < 1 \Leftrightarrow w > \frac{R_2 e_2 + p_2 e_3}{(\beta R_1)^\rho [p_2(\beta R_2)^\rho + R_2]} = \underline{w_{10|10}}$$

$$a_2^* = \frac{p_2\left[R_1 R_2(e_2 - e_3) + (\beta R_1)^\rho(\phi + R_2)[(\beta R_2)^\rho e_2 - e_3] + (\beta R_1)^\rho[(\beta R_2)^\rho - 1] R_1 R_2 w\right]}{p_2 R_1 R_2(e_2 - e_3) + (\beta R_1)^\rho(\phi + R_2)[p_2(\beta R_2)^\rho + R_2](A' + e_2)};$$

$$0 < a_2^* < 1 \Leftrightarrow$$

$$w < \frac{R_1 R_2(e_2 - e_3) + (\beta R_1)^\rho(\phi + R_2)[(\beta R_2)^\rho e_2 - e_3]}{(\beta R_1)^\rho R_1 R_2[1 - (\beta R_2)^\rho]} = \overline{w_{10|10}}, \qquad if\ \beta R_2 < 1;$$

$$\underline{w_{10|10}} < \overline{w_{10|10}} \Leftrightarrow (\beta R_2)^\rho e_2 > e_3;$$

$$(\beta R_2)^\rho e_2 > e_3, \qquad if\ \beta R_2 = 1;$$

$$w > \frac{R_1 R_2(e_3 - e_2) + (\beta R_1)^\rho(\phi + R_2)[e_3 - (\beta R_2)^\rho e_2]}{(\beta R_1)^\rho R_1 R_2[(\beta R_2)^\rho - 1]} = \underline{w'101}, \qquad if\ \beta R_2 > 1.$$

$$\underline{w_{10|10}} \gtrless \underline{w'_{10|10}} \Leftrightarrow (\beta R_2)^\rho e_2 \gtrless e_3$$

Focusing on the case where $\beta R_2 < 1$, the reason for an upper bound on w comes from (a_2), that would imply $u'(c_3) > u'(c_2) \Leftrightarrow c_2 > c_3 \Leftrightarrow e_2 - (1 + \frac{R_2}{p_2})a_2^* e_2 > e_3 + (1 + \frac{R_2}{p_2})a_2^* A$, which may not be true even with a very small a_2^* if A is too high, i.e.: for a fixed a_1^*, if w is too high.

Summing up: $a_1^* > 0$, $b_1^* = 0$, and $a_2^* > 0 \Leftrightarrow$

- If $(\beta R_2)^\rho e_2 < e_3$

$$w\ is\ s.t. : \begin{cases} Not\ possible, & if\ \beta R_2 \leqslant 1 \\ w > \underline{w'_{10|10}}, & if\ \beta R_2 > 1 \end{cases}$$

- If $(\beta R_2)^\rho e_2 = e_3$

$$w\ is\ s.t. : \begin{cases} Not\ possible, & if\ \beta R_2 \leqslant 1 \\ w > \underline{w_{10|10}} = \underline{w'_{10|10}}, & if\ \beta R_2 > 1 \end{cases}$$

- If $(\beta R_2)^\rho e_2 > e_3$

$$w\ is\ s.t. : \begin{cases} \underline{w_{10|10}} < w < \overline{w_{10|10}}, & if\ \beta R_2 < 1 \\ w > \underline{w_{10|10}}, & if\ \beta R_2 \geqslant 1 \end{cases}$$

D.G Optimal asset allocation $11|00 : a_1^* > 0, b_1^* > 0 | a_2^* = 0, b_2^* = 0$

First-order conditions are as follow:

$$\beta \tfrac{R_1 R_2}{(p_2+R_2)}[u'(c_2^*) + \beta p_2 u'(c_3^*)] = u'(c_1^*) \quad (a_1)$$
$$\beta p_1 R_1 u'(c_2^*) = u'(c_1^*) \quad (b_1)$$
$$\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) \quad (a_2)$$

- $c_1^* = (1 - a_1^* - b_1^*)w;\ c_2^* = e_2 + R_1 b_1^* w + \tfrac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w;\ c_3^* = e_3 + \tfrac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w$

$$
\begin{array}{ll}
(b_1) \text{ in } (a_1) & \Rightarrow \beta \dfrac{R_1 R_2}{(p_2+R_2)}[u'(c_2) + \beta p_2 u'(c_3)] = \beta p_1 R_1 u'(c_2) \\
 & \Rightarrow \beta p_2 R_2 u'(c_3) = [p_1(p_2+R_2) - R_2]u'(c_2)\ (i) \Rightarrow R_2 < \dfrac{p_1 p_2}{1-p_1}; \\
(i) & \Rightarrow (a_2),\ \text{since } \tfrac{\phi}{p_2} < 1
\end{array}
$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and defining the annuity benefit as $A = \frac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w$; $A' = A/a_1^*$:

$$
\begin{array}{ll}
(i) & \Rightarrow B_2^\rho(e_2 + R_1 b_1^* w + A) = \phi^\rho(e_3 + A) \\
 & \Rightarrow a_1^* = \dfrac{B_2^\rho(e_2 + R_1 b_1^* w) - \phi^\rho e_3}{A'(\phi^\rho - B_2^\rho)}\ (ii) \\
(b_1) & \Rightarrow B_1^\rho(1 - a_1^* - b_1^*)w = e_2 + R_1 b_1^* w + A\ (iii)
\end{array}
$$

Solving (ii) and (iii) for a_1^* and b_1^*, we obtain:

$$a_1^* = \frac{(\phi + R_2)\,[(B_1 B_2)^\rho e_2 - \phi^\rho(B_1^\rho + R_1)e_3 + (B_1 B_2)^\rho R_1 w]}{[\phi^\rho(B_1^\rho + R_1)R_2 + \phi(B_1 B_2)^\rho]R_1 w};$$

$$b_1^* = \frac{\phi^\rho[R_1 R_2 + B_1^\rho(\phi + R_2)]e_3 - [\phi^\rho R_1 R_2 + (B_1 B_2)^\rho(\phi + R_2)]e_2 + B_1^\rho(\phi^\rho - B_2^\rho)R_1 R_2 w}{[\phi^\rho(B_1^\rho + R_1)R_2 + \phi(B_1 B_2)^\rho]R_1 w};$$

$$a_1^* + b_1^* = \frac{[\phi(B_1 B_2)^\rho + \phi^\rho B_1^\rho R_2]w - \phi^\rho R_2 e_2 - \phi^\rho \phi R_1 e_3}{[\phi(B_1 B_2)^\rho + \phi^\rho(B_1^\rho + R_1)R_2]w} < 1 \Rightarrow a_1^*, b_1^* < 1,\ if\ a_1^*, b_1^* > 0:$$

$$0 < a_1^* \Leftrightarrow w > \frac{\phi^\rho(B_1^\rho + R_1)e_3 - (B_1 B_2)^\rho e_2}{(B_1 B_2)^\rho R_1} = \underline{w_{11|00}}$$

$0 < b_1^* \Leftrightarrow$

$$w > \frac{[\phi^\rho R_1 R_2 + (B_1 B_2)^\rho(\phi + R_2)]e_2 - \phi^\rho[R_1 R_2 + B_1^\rho(\phi + R_2)]e_3}{B_1^\rho(\phi^\rho - B_2^\rho)R_1 R_2} = \underline{w'_{11|00}}, \quad if\ B_2 < \phi$$

$$\underline{w_{11|00}} \gtrless \underline{w'_{11|00}} \Leftrightarrow B_2^\rho e_2 \lessgtr \phi^\rho e_3$$

$$B_2^\rho e_2 < \phi^\rho e_3, \quad if\ B_2 = \phi$$

$$w < \frac{\phi^\rho[R_1 R_2 + B_1^\rho(\phi + R_2)]e_3 - [\phi^\rho R_1 R_2 + (B_1 B_2)^\rho(\phi + R_2)]e_2}{B_1^\rho(B_2^\rho - \phi^\rho)R_1 R_2} = \overline{w_{11|00}}, \quad if\ B_2 > \phi$$

$$\underline{w_{11|00}} < \overline{w_{11|00}} \Leftrightarrow B_2^\rho e_2 < \phi^\rho e_3$$

Summing up: Assuming $R_2 < \dfrac{p_1 p_2}{1-p_1}$, $a_1^* > 0$, $b_1^* > 0$, and $a_2^* = 0 \Leftrightarrow$

- If $B_2^\rho e_2 < \phi^\rho e_3$

$$w \; is \; s.t. \begin{cases} w > \underline{w_{11|00}}, & if \; B_2 < \phi \\ w > \underline{w_{11|00}} = \underline{w'_{11|00}}, & if \; B_2 = \phi \\ \underline{w_{11|00}} < w < \overline{w_{11|00}}, & if \; B_2 > \phi \end{cases}$$

- If $B_2^\rho e_2 = \phi^\rho e_3$

$$w \; is \; s.t. \begin{cases} w > \underline{w_{11|00}} = \underline{w'_{11|00}}, & if \; B_2 < \phi \\ Not \; possible, & if \; B_2 \geqslant \phi \end{cases}$$

- If $B_2^\rho e_2 > \phi^\rho e_3$

$$w \; is \; s.t. \begin{cases} w > \underline{w'_{11|00}}, & if \; B_2 < \phi \\ Not \; possible, & if \; B_2 \geqslant \phi \end{cases}$$

D.H Optimal asset allocation $11|10 : a_1^* > 0, b_1^* > 0 | a_2^* > 0, b_2^* = 0$

First-order conditions are as follow:

$$\begin{array}{lr} \beta \frac{R_1 R_2}{(p_2+R_2)}[(1-a_2^*)u'(c_2^*) + \beta(p_2 + R_2 a_2^*)u'(c_3^*)] = u'(c_1^*) & (a_1) \\ \beta p_1 R_1 [(1-a_2^*)u'(c_2^*) + \beta R_2 a_2^* u'(c_3^*)] = u'(c_1^*) & (b_1) \\ \beta R_2 u'(c_3^*) = u'(c_2^*) & (b_2) \\ \bullet \text{ Not possible} & \end{array}$$

$$\begin{array}{ll} (a_2) \text{ in } (a_1) & \Rightarrow \beta^2 R_1 R_2 u'(c_3^*) = u'(c_1^*) \; (i) \\ (b_2) \text{ in } (b_1) & \Rightarrow p_1 \beta^2 R_1 R_2 u'(c_3^*) = u'(c_1^*) \; (ii) \\ (i) \text{ and } (ii) & \Rightarrow p_1 = 1 \end{array}$$

E Problem T1 optimal *locus* resolution

E.A Optimal asset allocation $00|{\cdot}0 : a_1^* = 0, b_1^* = 0| \cdot, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\begin{array}{lr} \beta \frac{R_1 R_2}{(p_2+R_2)}[u'(c_2^*) + \beta p_2 u'(c_3^*)] \leqslant u'(c_1^*) & (a_1) \\ \beta p_1 R_1 u'(c_2^*) \leqslant u'(c_1^*) & (b_1) \\ \beta p_2 R_2 u'(c_3^*) \leqslant u'(c_2^*) & (b_2) \\ \bullet \; c_1^* = w; \; c_2^* = e_2; \; c_3^* = e_3 & \end{array}$$

This case is exactly like 00|00 in the unrestricted problem, except for the survival probability p_2 in first-order equation (b_2).

Summing up: Assuming $\phi > 0, a_1^* = 0$, $b_1^* = 0$, and $b_2^* = 0 \Leftrightarrow$

- $(\beta p_2 R_2)^\rho e_2 \leqslant e_3$; and

- $w\ is\ s.t.:\begin{cases} w \leqslant \dfrac{1}{[\beta\frac{R_1R_2}{(p_2+R_2)}(\frac{1}{e_2^{\gamma}}+\beta p_2\frac{1}{e_3^{\gamma}})]^{\rho}} = \overline{w_{00|\cdot 0}}, & if\ e_3 \in [B_2^{\rho}e_2, (\frac{B_2}{\phi})^{\rho}e_2] \\ w \leqslant \dfrac{e_2}{B_1^{\rho}} = \overline{w'_{00|\cdot 0}}, & if\ e_3 \geqslant (\frac{B_2}{\phi})^{\rho}e_2 \end{cases}$

- $\overline{w_{00|\cdot 0}} = \overline{w_{00|00}}$; $\overline{w'_{00|\cdot 0}} = \overline{w'_{00|00}}$; and $(\dfrac{1}{B_1^{\rho}}, 0) \in E = w_1^{t1} = w_1^{up}$.

E.B Optimal asset allocation $00|\cdot 1 : a_1^* = 0, b_1^* = 0| \cdot, b_2^* > 0$

First-order conditions and consumption path are as follow:

$$\begin{array}{lr} \beta\frac{R_1R_2}{(p_2+R_2)}[(1-b_2^*)u'(c_2^*) + \beta p_2(1+R_2b_2^*)u'(c_3^*)] \leqslant u'(c_1^*) & (a_1) \\ \beta p_1R_1[(1-b_2^*)u'(c_2^*) + \beta R_2b_2^*u'(c_3^*)] \leqslant u'(c_1^*) & (b_1) \\ \beta p_2R_2u'(c_3^*) = u'(c_2^*) & (b_2) \\ \bullet\ c_1^* = w;\ c_2^* = (1-b_2^*)e_2;\ c_3^* = e_3 + R_2b_2^*e_2 & \end{array}$$

$$\begin{array}{ll} (b_2) \text{ in } (a) & \Rightarrow \dfrac{(\beta R_1)(1+R_2)}{(p_2+R_2)}u'(c_2) \leqslant u'(c_1)\ (i) \\ (b_2) \text{ in } (b_1) & \Rightarrow (\beta p_1R_1)u'(c_2) \leqslant u'(c_1)\ (ii) \\ (i) \Rightarrow (ii) & \end{array}$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$\begin{array}{ll} (b_2) & \Rightarrow (\beta p_2R_2)^{\rho} = \frac{c_3^*}{c_2^*} \\ & \Rightarrow b_2^* = \dfrac{(\beta p_2R_2)^{\rho}e_2 - e_3}{(\beta p_2R_2)^{\rho}e_2 + R_2e_2}; 0 < b_2^* < 1 \Leftrightarrow (\beta p_2R_2)^{\rho}e_2 > e_3 \\ (i) & \Rightarrow \dfrac{(\beta R_1)^{\rho}(1+R_2)^{\rho}}{(p_2+R_2)^{\rho}} \leqslant \frac{c_2^*}{c_1^*} \overset{(b_2^*)}{\Rightarrow} w \leqslant \dfrac{1}{(\beta R_1)^{\rho}}\dfrac{(p_2+R_2)^{\rho}(R_2e_2+e_3)}{(B_2^{\gamma}+R_2)(1+R_2)^{\rho}} = \overline{w_{00|\cdot 1}} \end{array}$$

Note that: $\overline{w_{00|\cdot 1}}$ is increasing in e_3; $e_3 = 0 \Rightarrow \overline{w_{00|\cdot 1}} = \dfrac{1}{(\beta R_1)^{\rho}}\dfrac{(p_2+R_2)^{\rho}R_2e_2}{(B_2^{\gamma}+R_2)(1+R_2)^{\rho}}$; and that $e_3 \uparrow (\beta p_2R_2)^{\rho}e_2 \Rightarrow \overline{w_{00|\cdot 1}} = \dfrac{1}{(\beta R_1)^{\rho}}\dfrac{(p_2+R_2)^{\rho}e_2}{(1+R_2)^{\rho}}$.

Define point $(\dfrac{(p_2+R_2)^{\rho}}{(1+R_2)^{\rho}}\dfrac{1}{(\beta R_1)^{\rho}}, 0) \in E$ as w_2^{t1}; and $(\dfrac{R_2}{B_2^{\rho}+R_2}\dfrac{(p_2+R_2)^{\rho}}{(1+R_2)^{\rho}}\dfrac{1}{(\beta R_1)^{\rho}}, 0) \in E$ as w_3^{t1}.

Summing up: $a_1^* = 0$, $b_1^* = 0$, and $b_2^* > 0 \Leftrightarrow$

- $b^*_{200|\cdot 0} = \dfrac{(\beta p_2R_2)^{\rho}e_2 - e_3}{(\beta p_2R_2)^{\rho}e_2 + R_2e_2}$;

- $(\beta p_2R_2)^{\rho}e_2 > e_3$; and

- $w \leqslant \overline{w_{00|\cdot 1}} = \dfrac{1}{(\beta R_1)^{\rho}}\dfrac{(p_2+R_2)^{\rho}(R_2e_2+e_3)}{(B_2^{\rho}+R_2)(1+R_2)^{\rho}}$

E.C Optimal asset allocation $01|{\cdot}0 : a_1^* = 0, b_1^* > 0| \cdot, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\begin{aligned}
&\beta \tfrac{R_1 R_2}{(p_2+R_2)}[u'(c_2^*) + \beta p_2 u'(c_3^*)] \leqslant u'(c_1^*) && (a_1)\\
&\beta p_1 R_1 u'(c_2^*) = u'(c_1^*) && (b_1)\\
&\beta p_2 R_2 u'(c_3^*) \leqslant u'(c_2^*) && (b_2)\\
&\bullet\ c_1^* = (1-b_1^*)w;\ c_2^* = e_2 + R_1 b_1^* w;\ c_3^* = e_3
\end{aligned}$$

$$\begin{aligned}
&(b_1) \text{ in } (a_1) && \Rightarrow \beta \tfrac{R_1 R_2}{(p_2+R_2)}[u'(c_2^*) + \beta p_2 u'(c_3^*)] \leqslant \beta p_1 R_1 u'(c_2^*)\\
& && \Rightarrow (\beta p_2 R_2) u'(c_3^*) \leqslant (p_1(p_2+R_2) - R_2) u'(c_2^*)\ (i) \Rightarrow R_2 < \frac{p_1 p_2}{1-p_1}\\
&(b_2) && \Rightarrow (\beta p_2 R_2) u'(c_3^*) \leqslant u'(c_2^*)\\
&(i) && \stackrel{\phi<1}{\Rightarrow} (b_2)
\end{aligned}$$

Noting that (b_1) and (i) are the same as in the unrestricted problem, T1 and UP are equivalent regarding $01|{\cdot}0$ and $01|00$ portfolio allocations.

E.D Optimal asset allocation $10|{\cdot}0 : a_1^* > 0, b_1^* = 0| \cdot, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\begin{aligned}
&\beta \tfrac{R_1 R_2}{(p_2+R_2)}[u'(c_2^*) + \beta p_2 u'(c_3^*)] = u'(c_1^*) && (a_1)\\
&\beta p_1 R_1 u'(c_2^*) \leqslant u'(c_1^*) && (b_1)\\
&\beta p_2 R_2 u'(c_3^*) \leqslant u'(c_2^*) && (b_2)\\
&\bullet\ c_1^* = (1-a_1^*)w;\ c_2^* = e_2 + \tfrac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w;\ c_3^* = e_3 + \tfrac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w
\end{aligned}$$

Optimal choice is such that $a_1^* b_1^* | \cdot b_2^* = 01|{\cdot}0$ in a subset of the endowment space E that is the complement (with respect to E) of all other cases. See lemma 1.

E.E Optimal asset allocation $01|{\cdot}1 : a_1^* = 0, b_1^* > 0| \cdot, b_2^* > 0$

First-order conditions are as follow:

$$\begin{aligned}
&\beta \tfrac{R_1 R_2}{(p_2+R_2)}[(1-b_2^*) u'(c_2^*) + \beta p_2 (1 + R_2 b_2^*) u'(c_3^*)] \leqslant u'(c_1^*) && (a_1)\\
&\beta p_1 R_1 [(1-b_2^*) u'(c_2^*) + \beta p_2 R_2 b_2^* u'(c_3^*)] = u'(c_1^*) && (b_1)\\
&\beta p_2 R_2 u'(c_3^*) = u'(c_2^*) && (b_2)\\
&\bullet\ \text{Not possible}
\end{aligned}$$

$$\begin{aligned}
&(b_2) \text{ in } (a_1) && \Rightarrow \frac{\beta R_1 (1+R_2)}{(p_2+R_2)} u'(c_2^*) \leqslant u'(c_1^*)\ (i)\\
&(b_2) \text{ in } (b_1) && \Rightarrow \beta p_1 R_1 u'(c_2^*) = u'(c_1^*)\ (ii)\\
&(i) \text{ and } (ii) && \Rightarrow p_1 \geqslant \tfrac{(1+R_2)}{(p_2+R_2)} > 1
\end{aligned}$$

E.F Optimal asset allocation $10|{\cdot}1 : a_1^* > 0, b_1^* = 0| \cdot, b_2^* > 0$

First-order conditions and consumption path are as follow:

$$\begin{array}{ll}
\beta \frac{R_1 R_2}{(p_2+R_2)}[(1-b_2^*)u'(c_2^*) + \beta p_2(1+R_2 b_2^*)u'(c_3^*)] = u'(c_1^*) & (a_1)\\
\beta p_1 R_1[(1-b_2^*)u'(c_2) + \beta p_2 R_2 b_2^* u'(c_3^*)] \leqslant u'(c_1^*) & (b_1)\\
\beta p_2 R_2 u'(c_3^*) = u'(c_2^*) & (b_2)
\end{array}$$

- $c_1 = (1-a_1^*)w;$
- $c_2 = (1-b_2^*)(e_2 + \frac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w);\ and$
- $c_3 = e_3 + R_2 b_2^*(e_2 + \frac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w) + \frac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w$

$$\begin{array}{ll}
(b_2) \text{ in } (a_1) & \Rightarrow \beta R_1 \dfrac{(1+R_2)}{(p_2+R_2)} u'(c_2^*) = u'(c_1^*)\ (i)\\
(b_2) \text{ in } (b_1) & \Rightarrow \beta p_1 R_1 u'(c_2^*) \leqslant u'(c_1^*)\ (ii)\\
(i) & \Rightarrow (ii)
\end{array}$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, defining the annuity benefit as $A = \frac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w$, $A' = A/a_1^*$ and $\varphi = (\beta R_1)^\rho (1+R_2)^\rho$:

$$\begin{array}{ll}
(i) & \Rightarrow (\beta R_1)^\rho \dfrac{(1+R_2)^\rho}{(p_2+R_2)^\rho}(1-a_1^*)w = (1-b_2^*)(e_2+A) \Rightarrow\\
 & \Rightarrow b_2^* = \dfrac{e_2 + A + (\beta R_1)^\rho \frac{(1+R_2)^\rho}{(p_2+R_2)^\rho}(1-a_1^*)w}{e_2+A}\ (iii)\\
(b_2) & \Rightarrow (\beta p_2 R_2)^\rho (1-b_2^*)(e_2+A) = e_3 + R_2 b_2^*(e_2+A) + A\\
(i) \text{ and } (b_2) & \Rightarrow (\beta p_2 R_2)^\rho (\beta R_1)^\rho \frac{(1+R_2)^\rho}{(p_2+R_2)^\rho}(1-a_1^*)w = e_3 + R_2 b_2^*(e_2+A) + A\ (iv)
\end{array}$$

Solving (iii) and (iv) for a_1^* and b_2^*, we obtain:

$$a_1^* = \frac{p_1\left[(B_2^\rho + R_2)(\beta R_1)^\rho (1+R_2)^\rho w - (p_2+R_2)^\rho (R_2 e_2 + e_3)\right]}{\left[p_1(B_2^\rho + R_2)(\beta R_1)^\rho (1+R_2)^\rho + (1+R_2)(p_2+R_2)^{\rho-1} R_1 R_2\right] w};$$

$$0 < a_1^* < 1 \Leftrightarrow w > \frac{(p_2+R_2)^\rho (R_2 e_2 + e_3)}{(B_2^\gamma + R_2)(\beta R_1)^\rho (1+R_2)^\rho} = \underline{w_{10|\cdot 1}}$$

$$b_2^* = \frac{(p_2+R_2)^\rho R_1 R_2 (e_2 - e_3) + \varphi(\phi + R_2)[(\beta p_2 R_2)^\rho e_2 - e_3] + \varphi R_1 R_2[(\beta p_2 R_2)^\rho - 1]w}{(p_2+R_2)^\rho R_1 R_2 (e_2 - e_3) + \varphi(\phi + R_2)[(\beta p_2 R_2)^\rho + R_2]e_2 + \varphi R_1 R_2[(\beta p_2 R_2)^\rho + R_2]w};$$

$0 < b_2^* < 1 \Leftrightarrow$

$$w < \frac{(p_2+R_2)^\rho R_1 R_2 (e_2 - e_3) + \varphi(\phi + R_2)[B_2^\rho e_2 - e_3]}{\varphi R_1 R_2[1 - (\beta p_2 R_2)^\rho]} = \overline{w_{10|\cdot 1}}, \qquad if\ \beta p_2 R_2 < 1;$$

$\underline{w_{10|\cdot 1}} < \overline{w_{10|\cdot 1}} \Leftrightarrow (\beta p_2 R_2)^\rho e_2 > e_3;$

$(\beta p_2 R_2)^\rho e_2 > e_3, \qquad if\ \beta p_2 R_2 = 1;$

$$w > \frac{(p_2+R_2)^\rho R_1 R_2 (e_3 - e_2) + \varphi(\phi + R_2)[e_3 - (\beta p_2 R_2)^\rho e_2]}{\varphi R_1 R_2[(\beta p_2 R_2)^\rho - 1]} = \underline{w'_{10|\cdot 1}}, \qquad if\ \beta p_2 R_2 > 1.$$

$\underline{w_{10|\cdot 1}} \gtrless \underline{w'_{10|\cdot 1}} \Leftrightarrow (\beta p_2 R_2)^\rho e_2 \gtrless e_3$

Focusing on the case where $\beta p_2 R_2 < 1$, the reason for an upper bound on w comes from (b_2), that would imply $u'(c_3) > u'(c_2) \Leftrightarrow c_2 > c_3 \Leftrightarrow e_2 - (1+\frac{R_2}{p_2})b_2^* e_2 > e_3 + (1+\frac{R_2}{p_2})b_2^* A$, which may not be true even with a very small b_2^* if A is too high, i.e.: for a fixed a_1^*, if w is too high.

Summing up: $a_1^* > 0$, $b_1^* = 0$, and $b_2^* > 0 \Leftrightarrow$

- If $(\beta p_2 R_2)^\rho e_2 < e_3$

$$w \; is \; s.t. \begin{cases} Not \; possible, & if \; \beta p_2 R_2 \leqslant 1 \\ w > \underline{w'_{10|\cdot 1}}, & if \; \beta p_2 R_2 > 1 \end{cases}$$

- If $(\beta p_2 R_2)^\rho e_2 = e_3$

$$w \; is \; s.t. \begin{cases} Not \; possible, & if \; \beta p_2 R_2 \leqslant 1 \\ w > \underline{w_{10|\cdot 1}} = \underline{w'_{10|\cdot 1}}, & if \; \beta p_2 R_2 > 1 \end{cases}$$

- If $(\beta p_2 R_2)^\rho e_2 > e_3$

$$w \; is \; s.t. \begin{cases} \underline{w_{10|\cdot 1}} < w < \overline{w_{10|\cdot 1}}, & if \; \beta p_2 R_2 < 1 \\ w > \underline{w_{10|\cdot 1}}, & if \; \beta p_2 R_2 \geqslant 1 \end{cases}$$

E.G Optimal asset allocation $11|\cdot 0 : a_1^* > 0, b_1^* > 0| \; \cdot, b_2^* = 0$

First-order conditions are as follow:

$$\beta \tfrac{R_1 R_2}{(p_2+R_2)}[u'(c_2^*) + \beta p_2 u'(c_3^*)] = u'(c_1^*) \qquad (a_1)$$
$$\beta p_1 R_1 u'(c_2^*) = u'(c_1^*) \qquad (b_1)$$
$$\beta p_2 R_2 u'(c_3^*) \leqslant u'(c_2^*) \qquad (b_2)$$

- $c_1^* = (1 - a_1^* - b_1^*)w;\; c_2^* = e_2 + R_1 b_1^* w + \frac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w;\; c_3^* = e_3 + \frac{R_1 R_2}{p_1(p_2+R_2)} a_1^* w$

$$\begin{aligned} (b_1) \text{ in } (a_1) \quad & \Rightarrow \beta \frac{R_1 R_2}{(p_2+R_2)}[u'(c_2) + \beta p_2 u'(c_3)] = \beta p_1 R_1 u'(c_2) \\ & \Rightarrow \beta p_2 R_2 u'(c_3) = [p_1(p_2+R_2) - R_2]u'(c_2) \; (i) \Rightarrow R_2 < \frac{p_1 p_2}{1-p_1}; \\ (i) \quad & \Rightarrow (b_2), \text{ since } \phi < 1 \end{aligned}$$

Noting that (b_1) and (i) are the same as in the unrestricted problem, T1 and UP are equivalent regarding $11|\cdot 0$ and $11|00$ portfolio allocations.

E.H Optimal asset allocation $11|\cdot 1 : a_1^* > 0, b_1^* > 0| \; \cdot, b_2^* > 0$

First-order conditions are as follow:

$$\beta \tfrac{R_1 R_2}{(p_2+R_2)}[(1-b_2^*)u'(c_2^*) + \beta p_2(1 + R_2 b_2^*)u'(c_3^*)] = u'(c_1^*) \qquad (a_1)$$
$$\beta p_1 R_1[(1-b_2^*)u'(c_2^*) + \beta p_2 R_2 b_2^* u'(c_3^*)] = u'(c_1^*) \qquad (b_1)$$
$$\beta p_2 R_2 u'(c_3^*) = u'(c_2^*) \qquad (b_2)$$

- Not possible

$$\begin{aligned}
&(b_2) \text{ in } (a_1) && \Rightarrow \frac{\beta R_1(1+R_2)}{(p_2+R_2)}u'(c_2^*) \leqslant u'(c_1^*) \ (i)\\
&(b_2) \text{ in } (b_1) && \Rightarrow \beta p_1 R_1 u'(c_2^*) = u'(c_1^*) \ (ii)\\
&(i) \text{ and } (ii) && \Rightarrow p_1 = \tfrac{(1+R_2)}{(p_2+R_2)} > 1
\end{aligned}$$

F Problem T2 optimal *locus* resolution

F.A Optimal asset allocation $\cdot 0|00 : \cdot, b_1^* = 0 | a_2^* = 0, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\begin{aligned}
&\beta p_1 R_1 u'(c_2^*) \leqslant u'(c_1^*) && (b_1)\\
&\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) && (a_2)\\
&\bullet\ c_1^* = w;\ c_2^* = e_2;\ c_3^* = e_3
\end{aligned}$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$\begin{aligned}
&(b_1) && \Rightarrow w \leqslant \frac{e_2}{B_1^\rho} = \overline{w_{\cdot 0|00}};\\
&(a_2) && \Rightarrow (\beta R_2)^\rho e_2 \leqslant e_3
\end{aligned}$$

Summing up: $b_1^* = 0$ and $a_2^* = 0 \Leftrightarrow$

- $(\beta R_2)^\rho e_2 \leqslant e_3$; and
- $w \leqslant \dfrac{e_2}{B_1^\rho} = \overline{w_{\cdot 0|00}}$
- Define point $(\dfrac{1}{B_1^\rho}, 0) \in E$, as w_1^{t2}.

F.B Optimal asset allocation $\cdot 0|10 : \cdot, b_1^* = 0 | a_2^* > 0, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\begin{aligned}
&\beta p_1 R_1[(1-a_2^*)u'(c_2^*) + \beta R_2 a_2^* u'(c_3^*)] \leqslant u'(c_1^*) && (b_1)\\
&\beta R_2 u'(c_3^*) = u'(c_2^*) && (a_2)\\
&\bullet\ c_1^* = w;\ c_2^* = (1-a_2^*)e_2;\ c_3^* = e_3 + \tfrac{R_2}{p_2} a_2^* e_2
\end{aligned}$$

$$(a_2) \text{ in } (b_1) \quad \Rightarrow B_1 \beta R_2 u'(c_3^*) \leqslant u'(c_1^*) \ (i)$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$\begin{aligned}
&(a_2) && \Rightarrow (\beta R_2)^\rho = \tfrac{c_3^*}{c_2^*}\\
& && \Rightarrow a_2^* = \frac{p_2[(\beta R_2)^\rho e_2 - e_3]}{p_2(\beta R_2)^\rho e_2 + R_2 e_2}; 0 < a_2^* < 1 \Leftrightarrow (\beta R_2)^\rho e_2 > e_3\\
&(i) && \Rightarrow (B_1 \beta R_2)^\rho \leqslant \tfrac{c_3^*}{c_1^*} \overset{(a_2^*)}{\Rightarrow} w \leqslant \frac{1}{(\beta p_1 R_1)^\rho} \frac{R_2 e_2 + p_2 e_3}{p_2(\beta R_2)^\rho + R_2} = \overline{w_{\cdot 0|10}}
\end{aligned}$$

Note that: $\overline{w_{\cdot 0|10}}$ is increasing in e_3; $e_3 = 0 \Rightarrow \overline{w_{\cdot 0|10}} = \dfrac{1}{(B_1)^\rho}\dfrac{R_2 e_2}{p_2(\beta R_2)^\rho + R_2}$; and $e_3 \uparrow (\beta R_2)^\rho e_2 \Rightarrow \overline{w_{\cdot 0|10}} = \dfrac{e_2}{(B_1)^\rho}$. Define point $(\dfrac{R_2}{p_2(\beta R_2)^\rho + R_2}\dfrac{1}{(B_1)^\rho}, 0) \in E$ as w_2^{t2}.

Summing up: $b_1^* = 0$ and $a_2^* > 0 \Leftrightarrow$

- $a^*_{2\cdot 0|10} = \dfrac{p_2[(\beta R_2)^\rho e_2 - e_3]}{p_2(\beta R_2)^\rho e_2 + R_2 e_2}$;
- $(\beta R_2)^\rho e_2 > e_3$; and
- $w \leqslant \dfrac{1}{(\beta p_1 R_1)^\rho}\dfrac{R_2 e_2 + p_2 e_3}{p_2(\beta R_2)^\rho + R_2} = \overline{w_{\cdot 0|10}}$

F.C Optimal asset allocation $\cdot 1|00 : \cdot, b_1^* > 0 | a_2^* = 0, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\begin{aligned}
&\beta p_1 R_1 u'(c_2^*) = u'(c_1^*) && (b_1)\\
&\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) && (a_2)\\
&\bullet\ c_1^* = (1-b_1^*)w;\ c_2^* = e_2 + R_1 b_1^* w;\ c_3^* = e_3
\end{aligned}$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$\begin{aligned}
(b_1) \quad & \Rightarrow (\beta p_1 R_1)^\rho (1-b_1^*)w = e_2 + R_1 b_1^* w\\
& \Rightarrow b_1^* = \frac{B_1^\rho w - e_2}{(B_1^\rho + R_1)w}; 0 < b_1^* < 1 \Leftrightarrow w > \frac{e_2}{B_1^\rho} = \underline{w_{\cdot 1|00}}\\
(a_2) \quad & \Rightarrow (\beta R_2)^\rho \leqslant \tfrac{c_3^*}{c_2^*}\\
& \Rightarrow (\beta R_2)^\rho (e_2 + R_1 b_1^* w) \leqslant e_3\\
& \overset{(b_1^*)}{\Rightarrow} w \leqslant \frac{(B_1^\rho + R_1)e_3 - (B_1 \beta R_2)^\rho e_2}{(B_1\beta R_2)^\rho R_1} = \overline{w_{\cdot 1|00}}\\
\underline{w_{\cdot 1|00}} < \overline{w_{\cdot 1|00}} \quad & \Leftrightarrow e_3 > B_2^\rho e_2
\end{aligned}$$

Note that: $\overline{w_{\cdot 1|00}}$ is increasing in e_3; $e_3 = 0 \Rightarrow \overline{w_{\cdot 1|00}} = -\dfrac{e_2}{R_1}$; and $e_3 \downarrow (\beta R_2)^\rho e_2 \Rightarrow \overline{w_{\cdot 1|00}} = \dfrac{e_2}{(B_1)^\rho}$.

Summing up: $b_1^* > 0$ and $a_2^* = 0 \Leftrightarrow$

- $b^*_{1\cdot 1|00} = \dfrac{B_1^\rho w - e_2}{(B_1^\rho + R_1)w}$;
- $(\beta R_2)_2^\rho e_2 < e_3$; and
- $\dfrac{e_2}{B_1^\rho} < w \leqslant \dfrac{(B_1^\rho + R_1)e_3 - (B_1\beta R_2)^\rho e_2}{(B_1\beta R_2)^\rho R_1}$

F.D Optimal asset allocation $\cdot 1|10 : \cdot, b_1^* > 0 | a_2^* > 0, b_2^* = 0$

First-order conditions and consumption path are as follow:

$$\begin{aligned}
&\beta p_1 R_1[(1-a_2^*)u'(c_2^*) + \beta R_2 a_2^* u'(c_3^*)] = u'(c_1^*) && (b_1)\\
&\beta R_2 u'(c_3^*) = u'(c_2^*) && (a_2)\\
&\bullet\ c_1^* = (1-b_1^*)w;\ c_2^* = (1-a_2^*)(e_2 + R_1 b_1^* w);\ c_3^* = e_3 + \tfrac{R_2}{p_2} a_2^*(e_2 + R_1 b_1^* w)
\end{aligned}$$

$$(a_2) \text{ in } (b_1) \quad \Rightarrow B_1\beta R_2 u'(c_3^*) = u'(c_1^*)\ (i)$$

Using $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$:

$$\begin{aligned}
&(i) && \Rightarrow (B_1\beta R_2)^\rho(1-b_1^*)w = e_3 + \tfrac{R_2}{p_2}a_2^*(e_2 + R_1 b_1^* w) \\
&(a_2) && \Rightarrow (\beta R_2)^\rho(1-a_2^*)(e_2 + R_1 b_1^* w) = e_3 + \tfrac{R_2}{p_2}a_2^*(e_2 + R_1 b_1^* w) \\
&(i),(a_2) && \Rightarrow a_2^*(b_1^*) = \frac{e_2 + [(B_1^\rho + R_1)b_1^* - B_1^\rho]w}{e_2 + R_1 b_1^* w} \\
&(i) && \overset{(a_2^*)}{\Rightarrow} b_1^* = \frac{B_1^\rho[p_2(\beta R_2)^\rho + R_2]w - (R_2 e_2 + p_2 e_3)}{[B_1^\rho[p_2(\beta R_2)^\rho + R_2] + R_1 R_2]w} \\
&0 < b_1^* < 1 && \Leftrightarrow w > \frac{R_2 e_2 + p_2 e_3}{B_1^\rho[p_2(\beta R_2)^\rho + R_2]} = \underline{w'_{\cdot 1|10}} \\
&a_2^*(b_1^*) && \overset{(b_1^*)}{\Rightarrow} a_2^* = \frac{p_2[(B_1\beta R_2)^\rho e_2 - (B_1^\rho + R_1)e_3] + B_1^\rho p_2(\beta R_2)^\rho R_1 w}{B_1^\rho[p_2(\beta R_2)^\rho + R_2]e_2 - p_2 R_1 e_3 + B_1^\rho[p_2(\beta R_2)^\rho + R_2]R_1 w} \\
&0 < a_2^* < 1 && \Leftrightarrow w > \frac{(B_1^\rho + R_1)e_3 - (B_1\beta R_2)^\rho e_2}{(B_1\beta R_2)^\rho R_1} = \underline{w_{\cdot 1|10}} \\
& && \underline{w_{\cdot 1|10}} \gtrless \underline{w'_{\cdot 1|10}} \Leftrightarrow e_3 \gtrless (\beta R_2)^\rho e_2
\end{aligned}$$

Summing up: $b_1^* > 0$ and $a_2^* > 0 \Leftrightarrow$

- $w > \dfrac{R_2 e_2 + p_2 e_3}{B_1^\rho[p_2(\beta R_2)^\rho + R_2]}$, if $e_3 < (\beta R_2)^\rho e_2$;
- $w > \dfrac{(B_1^\rho + R_1)e_3 - (B_1\beta R_2)^\rho e_2}{(B_1\beta R_2)^\rho R_1}$; if $e_3 \geqslant (\beta R_2)^\rho e_2$

G Optimal asset allocation under zero-endowments

G.A Unrestricted problem under zero-endowments

Noticing that $c_2^* = c_3^* = c^*$, from subsection D.D, optimal choices are such that $a_1^* b_1^* | a_2^* b_2^* = 10|00$ if, and only if:

- $\beta R_2 u'(c^*) \leqslant u'(c^*) \Rightarrow \beta R_2 \leqslant 1$;
- $B_1 u'(c^*) \leqslant u'(c_1^*) \xrightarrow{(31a)} p_1(p_2 + R_2) - R_2 = \phi \leqslant B_2$.

Within this subspace of parameters,

- $a_1^* = \dfrac{p1(p_2 + R_2)^{(1-\rho)}[\beta R_1 R_2(1+\beta p_2)]^\rho}{p1(p_2 + R_2)^{(1-\rho)}[\beta R_1 R_2(1+\beta p_2)]^\rho + R_1 R_2}$.

Having first-order conditions as in subsection D.G and noticing that $c_2^* > c_3^*$, optimal choices are such that $a_1^* b_1^* | a_2^* b_2^* = 11|00$ if, and only if:

- $\beta R_2 u'(c_3^*) \leqslant u'(c_2^*) \Rightarrow \beta R_2 < 1$;
- $\phi > B_2$.

Within this subspace of parameters,

- $a_1^* = \dfrac{(\phi + R_2)(B_1B_2)^\rho}{\phi(B_1B_2)^\rho + \phi^\rho(B_1^\rho + R_1)R_2}$; and
- $b_1^* = \dfrac{B_1^\rho(\phi^\rho - B_2^\rho)}{\phi(B_1B_2)^\rho + \phi^\rho(B_1^\rho + R_1)R_2}$.

Optimal choices are such that $a_1^*b_1^*|a_2^*b_2^* = 10|10$ if, and only if:

- $\beta R_2 > 1$.

Within this subspace of parameters, optimal choices would be:

- $a_1^* = \dfrac{p1(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2]}{p1(\beta R_1)^\rho[p_2(\beta R_2)^\rho + R_2] + R_1R_2}$; and
- $a_2^* = \dfrac{(\beta R_2)^\rho - 1}{p_2(\beta R_2)^\rho + R_2}$.

G.B Problem T1 under zero-endowments

Restricting the analysis of T1 to parameters such that $\beta R_2 > 1$, there are only two admissible asset allocations. If $B_2 \leqslant 1$, optimal portfolio is $10|\cdot 0$ and a_1^* is equal to the solution of $10|00$ in UP with zero endowments. If $B_2 > 1$, solution is $10|\cdot 1$ and optimal choices are:

- $a_1^* = \dfrac{p1(\beta R_1)^\rho(B_2^\rho + R_2)}{p1(\beta R_1)^\rho(B_2^\rho + R_2) + (1 + R_2)^{1-\rho}(p_2 + R_2)^{\rho-1}R_1R_2}$; and
- $b_2^* = \dfrac{B_2^\rho - 1}{B_2^\rho + R_2}$.

G.3 Problem T2 under zero-endowments

The only optimal choice is $\cdot b_1^*|a_2^*0 = \cdot 1|10$, with:

- $b_1^* = \dfrac{B_1^\rho(p_2(\beta R_2)^\rho + R_2)}{B_1^\rho(p_2(\beta R_2)^\rho + R_2) + R_1R_2}$; and
- $a_2^* = \dfrac{p_2(\beta R_2)^\rho}{p_2(\beta R_2)^\rho + R_2}$.