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The implications of embodiment and Putty-Clay to economic development

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The Implications of Embodiment and Putty-Clay to Economic Development *

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Abstract

In this paper we construct and analyze a growth model with the following three ingredients. (i) Technological progress is embodied. (ii) The production function of a firm is such that the firm makes both technology upgrade as well as capital and labor decisions. (iii) The firm's production technology is putty-clay. We assume that there are disincentives to the accumulation of capital, resulting in a divergence between the social and the private cost of investment. We solve a single firm's problem in this environment. Then we determine general equilibrium prices of capital goods of different vintages. Using these prices we aggregate firms' decisions and construct the theoretical analogues of National Income statistics. This generates a relationship between disincentives and per capita incomes. We analyze this relationship and show the quantitative and qualitative roles of embodiment and putty-clay. We also show how the model is taken to data, quantified and used to determine to what extent income gaps across countries can be attributed to disincentives.

Key Words: Vintage Capital, Putty-Clay, Embodied Technological Change, Capital Theory, Distortions to Capital Accumulation.

JEL Classification: D92, E13, O40

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1 Introduction

It has long been suggested that distortions affect capital accumulation and can create significant per capita income differences across countries. This suggestion is based on growth theory, which emphasizes capital accumulation as the source of riches and, by implication, considers distortions to capital accumulation as one cause for poverty. While this general idea is hardly disputable, it brings up several, more detailed, questions. Among those questions are: To what extent is a model of economic growth into which distortions are incorporated quantitatively consistent with this view? Does the answer depend on whether technological progress is embodied or disembodied? Does it depend on whether the technology is putty-clay or putty-putty? What are the important parameters on which the answers to these questions hinge? Are there any other observables that can corroborate the theory? This paper develops a theory to address these questions and suggests ways of implementing the theory.

More specifically, we construct and analyze a model of technological progress and economic growth. The model contains the following three ingredients. A. Technological progress is embodied, i.e., new capital embodying the new technology has to be installed in order for technological progress to take effect. B. The production function of a firm is such that the firm makes both technology upgrade as well as capital and labor decisions - there is both a qualitative and a quantitative margin. C. Once a firm determines its capital-labor ratio for a new technology it cannot change it until its next technological upgrade, the so called putty-clay assumption. Analysis of this model and comparison to alternative models (that exclude one or more of these three ingredients) shows that per capita income is more sensitive to distortions in our model than in alternative models. The analysis also yields several implications that describe how certain observables are expected to behave over time and how they are expected to compare across countries. Some of these implications are as follows.

The first implication is that capital goods are eventually scrapped and replaced by new capital goods that reflect better technology. How long capital is held for is endogenous and depends on the price of capital, which reflects distortions. The more distorted the economy, the longer is capital held for, the more antiquated it is (on average), and the lower is its average quality. Hence distortions in our model affect not only the quantity of capital that a country accumulates but also the age structure of capital and hence its average quality.

Mirroring this, the second implication relates to the price profile of capital goods, i.e., the function that relates the vintage of a capital good to its price on second hand markets.¹ If one considers two different countries that have two different distortion levels, i.e., face different prices for new vintage capital, then their price profiles are different. The more distorted economy has a higher price profile and the price of capital in it declines more slowly (over a certain range). The implication therefore is that the price of capital on second hand markets is higher (for any given vintage) and that the price of capital goods declines more slowly as they age. This appears consistent with casual empiricism. Capital goods that have long been scrapped in rich (less distorted) economies and command therefore zero price are still in usage in poor economies and command a positive price.

The third implication of the theory is that wage rates are not equal to the marginal product of labor. Instead the wage rate equals the output of the most antiquated plant in the economy. As a consequence, workers working in the same type of establishment, say McDonald's, and operating the same equipment and structures earn much less in poor countries than in rich countries. Furthermore, wages decline faster with distortions in our model than in the neoclassical model (i.e., the model with disembodied technological progress and putty-putty technology).

The fourth implication is that investment at the level of the individual plant is lumpy, while investment at the level of the economy is smooth. This is consistent with empirical findings by Doms and Dunne (1998).

Our model is constructed with a view towards how it maybe quantified and applied to development issues. Motivated by this goal we specify a parametric family of aggregate production functions, the C.E.S. family. Then we derive an equilibrium and show how it relates to parameters of this family. This relationship allows us to recover the values of parameters of the aggregate production function from values of parameters that are estimable from data. Once we insert these estimated values (that we estimate elsewhere) into a production function, we simulate the model and show how per capita incomes depend on distortions over a reasonable range of parameter values (where "reasonable" means it is consistent with the Summers Heston data set). As it turns out per capita incomes are quite sensitive to distortions in our simulated model and are more so than in models that rely on a different

¹To facilitate cross country comparisons, we normalize this profile, i.e., divide the price of each vintage by the price of the new vintage.

production function. Hence one contribution of the paper is to highlight the role of the aggregate production function in explaining income gaps and propose an empirically workable framework in which this role can be studied.

Going beyond the particular model we analyze here we show the role of embodied versus disembodied technological progress, the role of putty-clay versus putty-putty technology, and the interplay between them. Each combination of these modeling choices generates a different model and different predictions (sometimes qualitative and sometimes quantitative) regarding how distortions affect per capita income. In this paper we work out the details of the embodied/putty-clay model and develop a methodology to show how it compares to other models. In a companion paper we apply this methodology to actual data, perform quantitative exercises, and assess the embodied/putty-clay model's predictions.

Although there is a fairly extensive literature that uses growth theory to assess to what extent distortions affect per capita incomes, the paper that pioneered the modeling approach we pursue here is Jovanovic and Rob (1997). The main difference is that we consider a more general theory in which the firm decides both the quantity and the quality of capital, which, as explained above, brings the model closer to data. In addition, we derive the general equilibrium price profile of capital goods which again brings us closer to data (because one is able to assess the relevance of the model by studying the behavior of capital good prices on second hand markets). Also, once we have a price profile we generate a theoretical counterpart to the concept of "capital at market prices," which is analogous to the way NIPA evaluates and aggregates capital stocks. We supply a more detailed and more comprehensive comparison to previous literature in the body of the paper.

The plan of the paper is as follows. The next subsection provides a verbal description of the model. Section 2 sets up the model and the maximization program of an individual firm. Section 3 provides the solution to the firm's program. Section 4 provides comparative statics properties of this solution. Section 5 treats general equilibrium aspects of the model, including the determination of wages, rental rates of capital and the price profile of capital goods. Section 6 uses this information to derive the theoretical counterparts of national income statistics and show their comparative statics properties. Section 7 provides a detailed discussion of how our model relates to alternative models of technological progress and to previous literature. An Appendix contains all technical details.

1.1 Verbal description of the Model and its Mechanics

The unit of analysis in the model is the individual firm. The production function of a firm depends on labor, capital and the vintage of this capital. A firm faces exogenous technological progress and has to decide when to adopt new technology. The effect of adopting a new technology is that it raises the efficiency (or productivity) of workers. When a new technology is adopted, however, a firm's pre-existing capital stock (which reflects the firm's old technology) is rendered useless. If the firm wants its workers to work with a positive quantity of capital, it must buy capital that reflects the new technology, new vintage capital. Therefore, a firm's decision is twofold: it chooses when to adopt a new technology and how much capital of the new vintage to install.

The economy is populated by a continuum of identical firms, which upgrade their technology in sequence. As a result of these sequential upgrading decisions the economy has, at any moment in time, a window of capital goods of different vintages. The width of this window corresponds to the holding period of capital and the height of the window corresponds to how much capital is being installed. In addition, at any moment in time one has a general equilibrium price profile of capital goods, i.e., a function relating the vintage of a capital good to its price. We determine how the window and prices of capital goods are affected by distortions and other parameters. Once we determine these effects, we determine the effect of distortions on various national income statistics. In particular, we determine the effect of distortions on per capita income, the investment-capital ratio, the investment-output ratio, and wage rates.

2 The Model

We consider an infinite horizon, one good economy. Time is continuous and indexed by $t \in [0, \infty)$.

Agents and goods. The economy is populated by a continuum of identical, infinitely lived individuals. Each individual buys output (in her capacity as a consumer) and sells labor (in her capacity as a worker). The population of individuals at date 0 is of measure 1 and it grows at the rate n . Thus population at date t is of measure

$$N(t) = e^{nt}.$$

There is a continuum of identical, infinitely lived firms. Each firm hires workers, buys capital, operates a constant returns to scale production function, and sells output. Individuals and firms take prices as given. Since production exhibits constant returns to scale and since firms are price takers, the equilibrium profit of each firm is zero and the size of firms is indeterminate. For convenience and without loss of generality we normalize the size of firms so that each firm employs one worker.

There is one output in the economy, which is used both for consumption and investment. The technical rate of transformation between consumption and investment is constant and normalized to one.

Technological progress, vintage capital and embodiment. As time progresses better technology becomes (exogenously) available. Technologies are indexed by their vintage so that vintage s technology is a technology that becomes available at date s . Technological progress is labor augmenting, i.e., it makes workers more productive. More specifically a worker using vintage s technology delivers $A(s)$ efficiency units of labor, where

$$A(s) = e^{gs}.$$

g is called the rate of technological progress.

Capital goods are also indexed by vintage and vintage s capital is capital that embodies vintage s technology. Capital of an earlier vintage cannot be used - either by itself or in combination with vintage s capital - to operate vintage s technology. Therefore if a firm is to use capital with a new technology, new capital goods that embody the new technology must be put in place.²

Production in a single firm at a given point in time. Consider a point in time, say t . At t a firm is characterized by its most recently adopted technology, say s ($s \leq t$),

²The firm has the option of adopting new technology, enjoying the increase in efficiency units of labor that it delivers, but not using any capital with the new technology. In this case the firm foregoes the benefit of combining labor with capital goods. Whether this option is profitable depends on properties of the production function and in particular on whether the marginal productivity of capital is finite or infinite at $K = 0$.

On the other hand, if the firm chooses to use capital with the new technology, it must use new vintage capital that is “customized” to the new technology. This is the sense in which technological progress is embodied.

and its vintage s capital input, K . We refer to such firm as an (s, K) firm. At $t = 0$, a firm starts from some initial condition (s_0, K_0) , with $K_0 \geq 0 \geq s_0$.

An (s, K) firm produces output according to a C.E.S. production function. Namely, the date t output flow of such firm is

$$Y(s, t) = e^{-\delta(t-s)} A(s) f(k), \tag{1}$$

where δ is a (physical) depreciation factor,³

$$k \equiv \frac{K}{A(s)}$$

is capital per efficiency unit of labor,

$$f(k) \equiv \left(1 - \alpha + \alpha k^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

is the C.E.S. production function and $0 \leq \alpha < 1$ is the distribution parameter.⁴

Upgrading. Consider a point in time, say s , at which a firm upgrades its technology. At that point the firm also chooses how much capital K of vintage s to install and how many workers to employ (which we normalize to one). Once the firm makes that determination it is committed to the vintage s technology and to the capital and labor inputs it had pre-selected - until its next technological upgrade. Therefore, a firm has full flexibility ex-ante and full rigidity ex-post. This is commonly referred to as the putty-clay formulation.⁵ Between upgrades the firm makes no economic decisions; it merely collects the output flow specified by (1), and pays wages.

The private cost of installing capital of a new vintage is p (units of the consumption good) per-unit capital. Thus, if a firm upgrades at date s and installs K units of vintage s capital, it pays pK . pK is paid up front (at s) and p is independent of s .

³Note that depreciation affects output, not capital. We do this for tractability and ease of exposition. In our companion, applications paper we consider also the case in which depreciation affects capital.

⁴In this section we only consider an individual firm. Later when we consider many firms we assume that the values of the above parameters (α , σ etc.) are the same for all firms. The only difference between firms is that they adopt new technologies at different points in time and hence install different quantities of capital.

⁵Although the model and analysis here is confined to the putty-clay case, we provide a full fledged comparison between models of putty-putty and putty-clay in Section 7.

When a firm buys new capital goods it scraps its old capital goods. Old capital goods cannot be converted back into output or into capital goods of a newer vintage (and, as mentioned earlier, the production function is such that it is not possible to combine old and new capital goods). We further assume that the scrap value of capital goods is zero. Later we introduce second hand markets explicitly and show that the market value of scrapped capital is in fact zero. In doing so we “endogenize” the assumption that the firm gets zero for capital it scraps.⁶

The firm’s maximization problem. The objective of a firm is to maximize the discounted value of revenues net of wages and capital installation costs. Discounting is with respect to the constant, instantaneous interest rate r , which the firm takes as given. We assume that $r > g$, which is necessary to ensure boundedness of the firm’s objective function.

The economic decision facing a firm at a given point in time, say t , is as follows. The firm operates with a certain combination (s, K) . Then it has to decide on its next upgrade date, say $s + T$, and the quantity of vintage $s + T$ capital K to install at $s + T$.

The tradeoff governing these decisions is as follows. The capital stock the firm has on its hands is already paid for so it “comes for free.” On the other hand, as time goes on the technological frontier keeps moving out so this capital becomes more and more obsolete. On top of that, the firm had pre-committed itself to employ one worker whose equilibrium wage keeps increasing (see below).⁷ Therefore, a point comes where operating an old technology is no longer profitable, i.e., it pays to upgrade to a new technology.

Considering output used for consumption as the numeraire, the dynamic programming formulation of the firm’s problem is as follows

$$V(s, K, t) = \max_{T, K'} \left\{ \int_t^{s+T} e^{-r(\tau-t)} [Y(s, \tau) - w(\tau)] d\tau + e^{-r(s+T-t)} [V(s + T, K', s + T) - pK'] \right\}, \quad (2)$$

⁶Another way of saying this is that the introduction of second hand markets has no effect on the firm’s upgrading decision. This is because the equilibrium prices in these markets are such that whatever is optimal without second hand markets is still optimal given the equilibrium prices in second hand markets. The reason for explicitly introducing second hand markets anyway is to establish a correspondence between the stock of capital as it is measured by NIPA and an analogous concept of the theory.

⁷Obviously, the cost of employing workers is lessened if labor is divisible and perfectly mobile, the putty-putty case. Nonetheless, even if labor is perfectly mobile, scrapping and upgrading still occur as long as $\sigma < 1$.

where $w(\tau)$ is the wage rate at date τ . The firm takes p, r , and $w(\cdot)$ as given. As a matter of terminology we refer to T as the **lifespan of capital** and to the time interval of length T between upgrades as a **replacement cycle**.

Distortions. As stated earlier, the social technical rate of transformation between consumption and investment is one. Nonetheless, a firm privately pays p per unit capital. The reason is that there are taxes (or other distortions) on the installment of capital, which equal $p - 1$. The analysis applies also when $p < 1$, i.e., when capital is subsidized. With appropriate changes, p can alternatively be interpreted as the *technical* rate at which consumption goods are transformed into investment goods. However, in discussing our results we confine attention to the case where $p - 1$ is interpreted as distortions.⁸

3 Solution to the Firm’s Maximization Problem

We derive now a solution to the firm’s problem assuming that the firm faces wages that grow at the rate g , $w(t) = w_0 e^{gt}$, where w_0 is determined below. The appendix shows, under this assumption, that a solution to the firm’s problem exists and that it is unique. Furthermore, it shows that the lifespan of capital is equal across upgrades (starting with the first upgrade and onwards and excluding the lifespan of the firm’s initial capital) and that the solution is “balanced,” i.e., that capital, output and profits all grow at the rate g . In this section we exploit these properties and characterize the solution to the firm’s problem by manipulating first order conditions. In doing so we assume an interior solution, i.e., $0 < K, T < \infty$. Our derivations here are heuristic and are spelled out because they clarify the mechanics of the model and because the results are used repeatedly in what follows. The formal analysis is fully worked out in the Appendix.

Assume the firm last upgraded at s and that its next upgrade is scheduled for $s + T$. Then differentiating the objective (2) with respect to K and using the envelope theorem we get the following FOC for K

$$p = D(r + \delta, T) f'(k), \tag{3}$$

⁸This interpretation is consistent with Jones (1994) paper. Recently, Hsieh and Klenow (2003) argue that income gaps are due to low productivity of the investment sector in poor countries (as opposed to distortions). Our analysis can also be interpreted from this point of view. In particular our analysis can be viewed as quantifying the extent to which income gaps come from the differential *efficiency* of the investment sector.

where

$$D(r + \delta, T) \equiv \int_{s+T}^{s+2T} e^{-(r+\delta)[t-(s+T)]} dt = \frac{1 - e^{-(r+\delta)T}}{r + \delta},$$

$$k = \frac{K(s+T)}{e^{g(s+T)}}$$

and $K(s+T)$ is the optimal quantity of capital installed at $s+T$.

This condition is the usual equality between the unit cost of capital p and the return it generates, i.e., the discounted value of marginal products over the lifespan of installed capital.

The FOC for T is obtained by differentiating (2) with respect to T and using the balancedness conditions $\frac{\dot{K}}{K} = \frac{\dot{V}}{V} = g$ (\dot{V} being the total derivative of $V(t, K, t)$ with respect to t). This gives

$$Y(s, s+T) - w(s+T) - (r-g)[V(s+T, K(s+T), s+T) - pK(s+T)] = 0. \quad (4)$$

This condition equates the cost of postponing a technological upgrade, which is the loss of higher output, to the benefit, which is that technology improves in the interim and that the cost of installing new capital goods is delayed.

We eliminate V from (4) as follows. Let π be the discounted profit - in terms $s+T$ dollars - within the time interval $t \in (s+T, s+T+T]$. Since the solution is balanced

$$V(s+T, K(s+T), s+T) - pK(s+T) = \frac{\pi}{1 - e^{-(r-g)T}}, \quad (5)$$

where

$$\pi = D(r + \delta, T)Y(s+T, s+T) - D(r - g, T)w(s+T) - pK(s+T).$$

Using balancedness again this simplifies to

$$\pi = e^{gT} [D(r + \delta, T)Y(s, s) - D(r - g, T)w(s) - pK(s)]. \quad (6)$$

Substituting (6) into (5) and the result into (4), we get

$$Y(s, s) - \frac{e^{(g+\delta)T}}{D(r - g, T)} [D(r + \delta, T)Y(s, s) - pK(s)] = 0. \quad (7)$$

Combining the FOC for T with the FOC for k , i.e., substituting (1) into (7), the FOC for T boils down to

$$[D(r + \delta, T) - e^{-(g+\delta)T} D(r - g, T)] f(k) = pk. \quad (8)$$

(3) and (8) are the FOC's to the firm's problem. There is an alternative and sometimes more convenient way of expressing them. Let

$$u \equiv e^{-gT}.$$

Then, transforming T into u , (3) and (8) are written as

$$\begin{aligned} \frac{1}{g} \frac{1-u^{a+b}}{a+b} f'(k) - p &= 0 \\ \frac{1}{g} \left(\frac{1-u^{a+b}}{a+b} - u^{1+b} \frac{1-u^{a-1}}{a-1} \right) f(k) - pk &= 0, \end{aligned} \quad (9)$$

where

$$a \equiv \frac{r}{g} \text{ and } b \equiv \frac{\delta}{g}.$$

A shorter, albeit less explicit, derivation of these conditions is as follows. We assume the firm has zero capital at $t = 0$ and impose the requirement that k and u are the same for all upgrades. Then the firm's objective function is written as

$$F(k, u) \equiv \frac{\frac{1}{g} \frac{1-u^{a+b}}{a+b} f(k) - pk}{1 - u^{a-1}}. \quad (10)$$

Differentiating $F(k, u)$ with respect to k and u , we get the same FOC's, (9). The Appendix shows that the second order conditions corresponding to the maximization of (10) are satisfied if and only if

$$h(u, a-1) - h(u, a+b) < \frac{1+b}{\sigma}, \quad (11)$$

where

$$h(u, a) \equiv \frac{au^a}{1 - u^a}. \quad (12)$$

The appendix also shows that (11) holds for any $0 \leq \sigma \leq 2$. Finally it shows that (9) admits one and only one (u, k) solution.

These derivations are valid only if the firm's objective is bounded and the solution is

interior. Since the firm operates with a constant returns to scale production function and is a price taker, these requirements are satisfied if and only if the maximized value of the firm's objective is zero. And this, in turn, is satisfied if and only if the discounted value within each replacement cycle, π , is zero. Manipulating equation (6), this requirement amounts to⁹

$$w(t) = f(k)e^{-(g+\delta)T}e^{gt}, t \geq 0, \quad (13)$$

where T and k take on the values that solve (9). This pins down the value of w_0 . The net result is stated as follows.

Proposition 1 *Assume $w(t)$ satisfies (13). Assume also that $0 \leq \sigma < 1$ and $p > P(\sigma)$ where*

$$P(\sigma) \equiv \frac{1}{r + \delta} \alpha^{\frac{\sigma}{\sigma-1}}, \quad (14)$$

or that $1 < \sigma < 2$ and $p > P(\sigma)$. Then:

(i) Starting with the first upgrade, the optimal lifespans of capital are uniquely determined, are equal across upgrades, are strictly positive and finite. The optimal lifespan T is found from the solution to (9).

(ii) Consider a date s at which the firm upgrades. Then the optimal amount of capital it installs $K(s)$ is given by

$$K(s) = A(s)k, \quad (15)$$

k being the solution to (9).

(iii) The lifespan of the capital that the firm has on its hands initially depends of its vintage and on its quantity. The younger is the vintage and/or the larger is the quantity of capital the longer is this lifespan.

In addition to the restrictions we have already discussed, two further restrictions underlie Proposition 1. The first restriction relates to the price of capital. If $\sigma < 1$ and $p \geq P(\sigma)$, capital is so expensive in relation to its marginal product that the firm never upgrades the technology. This can be verified from equation (3), which shows that if $p \geq P(\sigma)$ and if capital that is installed is held forever, the price of capital is higher than the discounted

⁹If the initial wage $w(0) = f(k)e^{-(g+\delta)T}$ were lower, the firm's objective would be unbounded. If $w(0)$ were higher the firm would never upgrade.

value of the marginal products that it generates. If that is the case we get a corner solution, i.e., the firm never upgrades its technology and its output converges to zero in the long run. For this reason, and as a matter of terminology, we refer to $P(\sigma)$ as a **poverty trap**. At the other extreme, If $\sigma > 1$ and $p \leq P(\sigma)$, capital is so inexpensive in relation to its marginal product that the firm installs infinite quantity of capital and the value of its objective is also infinite (this again can be verified from equation (3)). Altogether the restriction on the price of capital ensures that the solution is interior, not corner.

The second restriction is that the elasticity of substitution is between 0 and 2. When the elasticity of substitution is above 2 a qualitatively different solution may arise. In particular the firm may choose to upgrade at each instant but install zero capital. Under such upgrade policy the firm's output grows without having to install (and pay for) new capital. Hence this situation resembles what happens under *disembodied* technological progress, and is dealt with in a separate paper.

4 Comparative Statics of the Firm's Optimum

In this section we determine how T and K vary with p , i.e., how distortions affect the lifespan and the quantity of installed capital. We determine these effects by implicit differentiation of (9). We present the results in terms of elasticities

$$\frac{p}{k} \frac{dk}{dp} = - \frac{h(u, a-1)}{h(u, a-1) - h(u, a+b)} \frac{h(u, a-1)}{h(u, a+b)} \frac{1+b - [h(u, a-1) - h(u, a+b)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]} \quad (16)$$

and

$$\frac{p}{u} \frac{du}{dp} = - \frac{h(u, a-1)}{h(u, a+b)} \frac{\frac{1}{\sigma} - 1}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}, \quad (17)$$

where h is given by (12). Invoking the second order condition (11), the fact that h is positive, and that h is decreasing in a (which is shown in the Appendix), we have

$$\frac{dk}{dp} < 0 \quad (18)$$

for $0 \leq \sigma \leq 2$, i.e., the demand for capital is decreasing in its own price. We also have that

$$\frac{du}{dp} < 0$$

for $0 \leq \sigma < 1$ and that

$$\frac{du}{dp} > 0 \tag{19}$$

for $1 < \sigma \leq 2$. Thus the lifespan of capital may either increase or decrease (as p increases), depending on the elasticity of substitution σ .

This last result, (19), is somewhat surprising. A more natural conjecture is that the lifespan of capital is lengthened as p increases, so as to spread the cost of capital over a longer time and offset the increase in p . There is a second force at work however. When $\sigma > 1$, capital and labor are substitutes. Therefore, if p increases the firm uses less capital (as per (18)) and because capital and labor are substitutes it uses more labor. But the way to use more labor is to upgrade more frequently. Then labor works with technology that is closer (on average) to the frontier and therefore delivers more efficiency units. (By contrast, if $\sigma < 1$ capital and labor are complements. Therefore, if p increases the firm uses less capital and less labor, which it effects by upgrading less frequently)

5 General Equilibrium

To this point we analyzed the problem of a single, price taking firm. In this section we consider a continuum of such firms and a continuum of consumers. The competitive interaction between these agents determines equilibrium prices. It also determines the mechanics of consumption and investment over time and across firms. To determine these endogenous variables, we start out by analyzing the household sector.

5.1 The household sector

Consider the problem facing an individual in her capacity as a consumer. Population growth is incorporated into this problem by assuming that each individual is part of a family that grows at the rate n and that each individual maximizes the lifetime utility of her family subject to the family's lifetime budget constraint. The flow utility function of each family

member is $\frac{c^{1-\gamma}}{1-\gamma}$, where c is flow consumption and γ is an elasticity parameter. The rate of time preference is constant and denoted by ρ . The lifetime utility of a representative family from a per capita consumption stream $c(\cdot)$ is

$$\int_0^{\infty} e^{-\rho t} N(t) \frac{c(t)^{1-\gamma}}{1-\gamma} dt.$$

The family has some initial wealth, call it ω , which comes from owning equity shares in firms. On top of that wealth, each member of the family receives a stream of wages $w(\cdot)$. The family's lifetime budget constraint is

$$\int_0^{\infty} e^{-rt} N(t) c(t) dt = \omega + \int_0^{\infty} e^{-rt} N(t) [w(t) + x(t)] dt, \quad (20)$$

where $x(t)$ are the date t proceeds from the taxation of capital.

To ensure that a (finite) solution to the household problem exists we assume that

$$\rho > (1 - \gamma)g + n. \quad (21)$$

This assumption together with equation (28) ensures that

$$r > n + g,$$

so that each family's wealth is finite and so that each firm's objective is bounded. Under these assumptions it is well-known that the solution to the family's problem is

$$c(t) = c_0 e^{\frac{r-\rho}{\gamma}t}, \quad (22)$$

where c_0 is such that the family's budget constraint is satisfied.

5.2 The relationship between rental rates and the prices of capital goods

Consider the capital stock held by some firm and assume it is of vintage s . Then this capital delivers a flow of rental rates which equals the output flow (generated by this capital and the one worker that operates it) minus wages. The discounted value of these rental rates is the shadow price of this capital in the second hand market¹⁰ or, equivalently, the shadow equity value of the firm holding this capital stock.¹¹ Let $R(s, \tau)$ be the date- τ rental rate of one unit of vintage- s capital (for $s \leq \tau \leq s + T$) and let r be the instantaneous interest rate. Then the date- t shadow price of one unit of vintage- s capital is

$$p(s, t) = \int_t^{s+T} R(s, \tau) e^{-r(\tau-t)} d\tau. \quad (23)$$

Rental rates adjust in equilibrium so that

$$p(s, s) = p \text{ for all } s.$$

Note that we define the rental rate of capital as residual earnings, i.e., as output minus wages. If wages were equal to the marginal productivity of labor the rental rate would equal the marginal productivity of capital (because the production function exhibits constant returns to scale). However, as we show later, the equilibrium in our model is such that wages and rental rates are *not* equal to the corresponding marginal productivity.

5.3 The staggering of firms in the economy

To focus on a balanced growth path we assume that firms are initially (at $t = 0$) staggered across vintages, so their upgrading decisions come in sequence. Let $M(s, 0)$ be the density

¹⁰Since all firms are identical there are no gains from trading capital across firms. Consequently, “shadow price” is understood as the price that makes the net demand for capital equal to zero.

¹¹The price of a share is understood as its *shadow* price since in our economy we don’t need a separate equity market to decentralize feasible allocations.

of vintage s firms at date 0 and assume

$$M(s, 0) = \begin{cases} me^{ns} & -T \leq s \leq 0 \\ 0 & s < -T \text{ or } s > 0 \end{cases}, \quad (24)$$

for some m . Since the labor force at date 0 is of measure 1 and since it is fully employed, m must be such that

$$\int_{-T}^0 me^{ns} ds = 1.$$

Solving for m we get

$$m = \frac{n}{1 - e^{-nT}}. \quad (25)$$

The quantity of capital held by a vintage s firm, for $-T \leq s \leq 0$, is $K(s) = ke^{gs}$, where k is the solution of (9). This pins down the economy's initial conditions.

Given the initial density (24), given that firms upgrade in sequence every T periods, and given the zero profit condition (which, together with the one firm/one worker convention, means that new firms come into existence to “create jobs” for newborns), (24) extends to all $t \geq 0$, i.e.,

$$M(s, t) = \begin{cases} me^{ns} & t - T \leq s \leq t \\ 0 & s < t - T \text{ or } s > t \end{cases}, \quad (26)$$

where $M(s, t)$ is the density of firms with vintage s technology at time t . To see this assume that $M(s, \tau)$ satisfies (26) for all $\tau < t$. Then at t a density $me^{n(t-T)}$ of existing firms upgrade their technology and a density ne^{nt} of firms come into existence. Both groups of firms adopt vintage t technology. This implies that the density of firms with vintage t technology at date t is

$$M(t, t) = me^{n(t-T)} + ne^{nt} = (me^{-nT} + n)e^{nt} = me^{nt},$$

where the last equality follows from (25). In consequence, if $M(s, \tau)$ satisfies (26) for $\tau < t$ it also satisfies it for $\tau = t$.

5.4 Equilibrium prices

We find now equilibrium wage rates, interest rates, rental rates and shadow prices of capital. To abbreviate we delete the adjective “shadow” from this point onwards.

Labor market. Given the solution to a representative firm's maximization problem

(see Section 3) the equilibrium wage rate at t must be such that firms make zero profits. According to (13) this happens if and only if

$$w(t) = f(k)e^{-(g+\delta)T}e^{gt}. \quad (27)$$

Therefore, rather than equaling the marginal productivity of labor, the equilibrium wage rate in our model equals the output, or the **value added**, of a firm with vintage $t - T$ capital. Thus, the wage rate in our model reflects the quality of the marginal (most antiquated) capital good. This suggests that wages are more sensitive to distortions in our model than in a model in which the wage rate reflects the quality of the *average* capital good in the economy (which is the case in a model with putty-putty technology and with wage equal to marginal productivity). We show later that this is indeed the case.

Although the wage rate in our model is not equal to the marginal productivity of labor, one can interpret the wage rate as equaling the social marginal contribution of an extra worker. If one such worker is added to the economy, while the economy's capital stock is held constant, this worker would be assigned to the marginal plant (which is about to be scrapped) and would increase the economy's output by whatever output this plant produces.

Interest rates. Since output net of investment grows at the rate g and since consumption grows at the rate $\frac{r-\rho}{\gamma}$, see equation (22), market clearing dictates $g = \frac{r-\rho}{\gamma}$. Thus,

$$r = \gamma g + \rho. \quad (28)$$

Dividends/Rental rates of capital. The rental rate of vintage- s capital at date t is computed as follows. (27) shows that the wage rate is the value added at the marginal plant. Consequently, the rental rate of capital of an infra-marginal plant is its value added over and above the value added of the marginal plant

$$\begin{aligned} R(s, t)K(s) &= e^{-\delta(t-s)}A(s)f(k) - w(t) \\ &= e^{-\delta(t-s)}\frac{f(k)}{k}[1 - e^{-(g+\delta)(s-(t-T))}]K(s), \end{aligned} \quad (29)$$

after we substitute in from (27). We refer sometimes to these rental rates as **dividends**.

Since, as noted above, the wage rate is not equal to the marginal productivity of labor, the rental rate is not equal (at each and every point in time) to the marginal productivity of

capital. Nonetheless, using (3) one can show that the discounted value of rental rates over a replacement cycle equals to the discounted value of the marginal productivities of capital over the same cycle. Hence capital is over-compensated early in the cycle (dividend exceeds the marginal productivity) and under-compensated late in the cycle.

Price profile. Substituting (29) into (23) we get

$$p(s, t) = pe^{-\delta(t-s)} \frac{D(r + \delta, T - (t - s))}{D(r + \delta, T)} \frac{1 - H(T - (t - s))}{1 - H(T)}, \quad (30)$$

where

$$H(T) \equiv \frac{D(r - g, T)}{D(r + \delta, T)} e^{-(g+\delta)T}. \quad (31)$$

Consider date t and a vintage s , $t - T < s < t$. Then (29) and (30) show that dividends and capital prices are independent of t . Rather, they depend on the *age* of capital only, $t - s$. Consequently, if calendar date is implicitly understood, one can write (which we do) $p(t - s)$, instead of $p(s, t)$ and $R(t - s)$, instead of $R(s, t)$. If $t - s > T$ vintage- s capital had already been scrapped so $p(t - s) = R(t - s) = 0$.

Given firm prices, as given by (30), ω is determined¹² as the equity value of all firms in the economy at $t = 0$

$$\omega = \int_{-T}^0 p(-s)K(s)ds.$$

6 Aggregation

Given the above equilibrium, we construct now the theoretical counterparts of national income statistics and perform comparative statics exercises with respect to them. The motivation for this is to establish a correspondence between observables and their theoretical analogues. This allows us to calibrate the model, perform quantitative exercises, and compare it (qualitatively and quantitatively) to alternative growth models.

At a point in time, say t , the aggregation procedure is to add across vintage- s firms, where $s \in [t - T, t]$. For example, given the convention that each firm operates with one

¹²Once ω is pinned down z_0 is determined, using (20), (22), (27), and (35), so that the consumer exhausts her budget.

worker, we confirm that the economy's labor force is fully employed

$$\text{Total measure of firms at } t = \int_{t-T}^t M(s, t) ds = m \int_{t-T}^T e^{ns} ds = e^{nt} = \text{Labor Force},$$

where (25) is invoked to prove the last equality.

6.1 National income statistics

In this subsection we derive the theoretical counterparts of national income statistics. This establishes a mapping between underlying parameters (e.g., α) and endogenous variables (e.g., y). This mapping forms the base to a calibration exercise, i.e., one can invert this mapping and use the values of observed variables to infer the values of unobserved ones.

Per-capita output. At t firms in the economy operate with vintage- s technology, where $s \in [t - T, t]$. Vintage- s firms employ $M(s, t)$ workers and produce a flow of output $Y(s, t)$ per worker. Thus the aggregate per-capita output flow at t is

$$\begin{aligned} y(t) &= \frac{1}{N(t)} \int_{t-T}^t M(s, t) Y(s, t) ds \\ &= e^{-nt} \int_{t-T}^t m e^{-(n+\delta)(t-s)} f(k) e^{gs} ds \\ &= y e^{gt}, \end{aligned} \tag{32}$$

where

$$y \equiv m f(k) D(g + n + \delta, T). \tag{33}$$

Investment. At t a density $m e^{nt}$ of firms invest in new capital. The quantity invested by each such firm is $K(t)$, which is given by (15). This implies that the economy wide per-capita investment at t is

$$i(t) = m k e^{gt}. \tag{34}$$

Given (34), tax proceeds are

$$x(t) = (p - 1) i(t) = (p - 1) m k e^{gt}. \tag{35}$$

Investment-output ratio. Let

$$i \equiv \frac{i(t)}{y(t)} = \frac{k}{f(k)} \frac{1}{D(g+n+\delta, T)}, \quad (36)$$

where the equality follows after we substitute in from (32), (33) and (34).

An alternative to (36) that we use later is

$$i = \left(\frac{\alpha}{p}\right)^\sigma \frac{[D(r+\delta, T)]^\sigma}{D(g+n+\delta, T)}. \quad (37)$$

This last equality follows from (36) after we substitute (65) and the first equation in (9) into it.

Factor-shares of income. Capital-income is the sum of rental rates on capital employed by all firms. Since there is a density me^{ns} of vintage- s firms at time t , where $s \in [t-T, t]$, capital-income at t is

$$me^{nt} \int_{t-T}^t e^{-n(t-s)} R(t-s) K(s) ds.$$

After we substitute in from (15) and (29), we get

$$\text{Capital Income} = e^{(g+n)t} mk \frac{f(k)}{k} \left[\frac{1 - e^{-(g+n+\delta)T}}{g+n+\delta} - e^{-(g+\delta)T} \frac{1 - e^{-nT}}{n} \right].$$

And dividing this by total output it follows that

$$\alpha_K = \frac{D(g+n+\delta, T) - e^{-(g+\delta)T} D(n, T)}{D(g+n+\delta, T)}, \quad (38)$$

where α_K is the capital-share of income.

By working out a similar computation, the labor-share of income is¹³

$$\alpha_L = \frac{e^{-(g+\delta)T} D(n, T)}{D(g+n+\delta, T)} = 1 - \alpha_K. \quad (39)$$

¹³Although it is “natural” that labor and capital shares add up to one this hinges on computing these shares at market prices. If we compute them at factor costs they add up to less than one. The difference is the “distortion-share of income.”

The value of the economy's capital stock at market prices. Given the market prices of capital and the (optimal) quantity of capital installed by each firm, the market value of the economy's capital stock at t is

$$\begin{aligned} k_M(t) &= \int_{t-T}^t p(t-s) K(s) M(s, t) ds \\ &= mk \int_0^T e^{-(g+n)\tau} p(\tau) d\tau. \end{aligned} \tag{40}$$

After substituting (23) into (40) and changing the order of integration we obtain

$$k_M(t) = \frac{e^{(g+n)t}}{r - (g+n)} mk \left[\int_0^T R(\tau) e^{-(g+n)\tau} d\tau - p \right]. \tag{41}$$

Substituting (29) into (41), performing the integration and substituting the second equation in (9) we get

$$k_M(t) = \frac{i_M(t)}{r - (g+n)} \left[\frac{D(g+\delta+n, T) - e^{-(g+\delta)T} D(n, T)}{D(r+\delta, T) - e^{-(g+\delta)T} D(r-g, T)} - 1 \right], \tag{42}$$

where

$$i_M(t) \equiv pmke^{gt},$$

i.e., $i_M(t) \equiv pi(t)$ is the market value of the date- t investment.

$k_M(t)$ correspond to an empirical concept of capital at market prices, which accounts for heterogeneity of capital goods. In particular it mirrors the NIPA practice of classifying capital goods into categories according to their vintage, determining the market value of the capital stock in each category by multiplying the quantity of capital by its market price, and adding up over all categories;¹⁴ see Fraumeni (1997) and Katz and Herman (1997).

Three kinds of depreciation. Capital in our model undergoes three kinds of depre-

¹⁴This link between a theoretically constructed concept of capital at market prices and its empirical counterpart is not found in any other paper that we are aware of.

ciation: physical, which is due to wear and tear and denoted by δ , obsolescence, which is due to the appearance of new and improved capital goods and denoted by g , and economic, which is due to scrapping. The first two depreciation rates are exogenously specified, model primitives. The third, economic depreciation, is endogenously determined by the lifespan of capital and its price. To define it let's first introduce

$$\frac{1}{\delta_{\text{EF}}} \equiv \frac{k_{\text{M}}(t)}{i_{\text{M}}(t)} = \frac{1}{r - (g + n)} \left[\frac{D(g + \delta + n, T) - e^{-(g+\delta)T} D(n, T)}{D(r + \delta, T) - e^{-(g+\delta)T} D(r - g, T)} - 1 \right] \quad (43)$$

and call δ_{EF} the effective depreciation rate. The analogue of δ_{EF} in the neoclassical model, namely, the model with disembodied technological progress and putty-putty technology (and hence where there is no such thing as scrapping) equals $g + n + \delta$. In fact, as we show in the Appendix,

$$\lim_{T \rightarrow \infty} \delta_{\text{EF}} = g + n + \delta.$$

Given this and given equation (49), δ_{EF} exceeds $g + n + \delta$. Accordingly, we define economic depreciation as the residual

$$\delta_e \equiv \delta_{\text{EF}} - (g + n + \delta).$$

In the neoclassical model capital is never scrapped. Therefore there is no distinction between physical and economic depreciation, so the calibrated value of physical depreciation reflect both depreciation rates. Here, because capital is eventually scrapped, we distinguish between physical and economic depreciation.

6.2 Comparative statics of national income statistics

In this subsection we show how distortions are manifested or, more specifically, how various observables vary with p and how this compares with alternative growth models.¹⁵ In doing so we suggest how our model can be assessed empirically and compared to alternative models.¹⁶

Comparative statics of per-capita output. Equation (32) shows that p has only a *level* effect on per-capita output, not a growth effect. To examine how p affects output, let

¹⁵The difference between what we do here and in Section 4 is that the comparative statics results here pertain to economy-wide variables (i.e., national income statistics), whereas the comparative statics results of Section 4 pertain to an individual firm's variables.

¹⁶The results we derive here analytically are illustrated quantitatively in our applications paper, using the U.S. calibrated values of parameters.

us assemble the causality links $p \rightarrow k$, $p \rightarrow T$, and $(k, T) \rightarrow y$ into one equation

$$y(p) \equiv \frac{1 - [u(p)]^{1+b+c}}{1 + b + c} \frac{c}{1 - [u(p)]^c} f(k(p)), \quad (44)$$

where $u(p)$ and $k(p)$ is the solution to (9) and

$$c \equiv \frac{n}{g}.$$

Equation (44) shows that distortions are manifested through a quantitative and a qualitative channel. The third term on the RHS of (44) is the traditional quantitative channel. Distortions stifle capital accumulation and thereby reduce output. This is reflected in $k(p)$. The first two terms represent the qualitative channel. Distortions lengthen the lifespan of capital, causing the inventory of capital goods in the economy to be of higher average age and lower average quality. This second effect is reflected in $u(p)$.

Equation (44) also shows that development data is interpreted differently depending on which model one considers as the data generating model. If one looks at data on per capita income and prices of capital goods (across countries) and considers it as being generated by the neoclassical model when it is, in fact, generated by our model the first two terms would appear as total factor productivity. In other words, the first two terms would have the effect of creating (“spurious”) variation which appears unexplained by variables that are included in the model (p). On the other hand, from the perspective of our model this variation *is* explained. In this sense our model accounts for a bigger fraction of per capita income across countries as coming from distortions than the neoclassical model. This contrast between different data generating models can be assessed by executing a quantitative exercise analogous to the one in Hall and Jones (1999).¹⁷

Equation (44) provides a related contrast between our model and the neoclassical model. Given Proposition 1, $y(p)$ vanishes at $p = P(\sigma)$, i.e., an economy becomes poverty trapped when distortions are sufficiently large. We can do the same exercise that led to equation (44) for the neoclassical model (which we do elsewhere), derive an analogous equation for $y(p)$ and study it for a commensurate value of σ . When we do that, an economy becomes poverty trapped at a *higher* value of p and the $y(p)$ curve spans a smaller range of incomes

¹⁷We execute this exercise in our applications paper.

as p varies by comparison with the $y(p)$ curve that our model generates. In this sense our model accounts again for a greater fraction of income variation across countries as coming from differential distortions.

Equation (44) expresses the level of income as a function of p . Sometimes it is more convenient to work with the elasticity of income with respect to p . To obtain an expression for that we substitute (16) and (17) into equation (44) and differentiate it. The result is

$$\begin{aligned} \frac{p \, dy}{y \, dp} &= -\frac{h(u, a-1) - h(u, a+b)}{h(u, a+b)} \\ &\times \frac{1+b - [h(u, a-1) - h(u, a+b)] - \frac{\sigma-1}{\sigma} [h(u, c) - h(u, 1+b+c)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}. \end{aligned} \quad (45)$$

As shown in the Appendix, h is positive and decreasing in a . Hence this elasticity is negative whenever the second order condition (11) is satisfied.

Comparative statics of the wage. Following analogous steps, we find the effect of distortions on the wage rate by differentiating equation (27) with respect to p

$$\frac{p \, dw}{w \, dp} = 1 - \frac{h(u, a-1)}{h(u, a+b)}, \quad (46)$$

where for brevity, w stands for w_0 . Since h decreases in a this elasticity is negative.

The Appendix shows that

$$\frac{p \, dw}{w \, dp} < -\frac{\alpha_K}{\alpha_L}.$$

The Appendix also derives $\frac{p \, dw}{w \, dp}$ for the neoclassical model and shows that it equals $-\frac{\alpha_K}{\alpha_L}$. Thus wages are affected more by distortions ($\frac{p \, dw}{w \, dp}$ is bigger in absolute value) in our model than in the neoclassical model.

Comparative statics of the price profile and economic depreciation. When $\sigma < 1$, an increase in p shifts the price profile upwards. This holds even after we divide by p , i.e., consider the **normalized price profile**, $\frac{p^{(t-s)}}{p}$. To show this, it is convenient to work with the transformed variable $v = e^{-g(t-s)}$. Then the effect of p on $\frac{p(v)}{p}$ is found by writing

(using (30)) the price profile as

$$\frac{p(v)}{p} = v^b \frac{1 - \left(\frac{u}{v}\right)^{a+b}}{a+b} - \left(\frac{u}{v}\right)^{1+b} \frac{1 - \left(\frac{u}{v}\right)^{a-1}}{a-1}, \quad (47)$$

where $v \equiv e^{-g(t-s)}$ and u is the optimal u . After some calculations, we get

$$\frac{u}{p(v)/p} \frac{dp(v)/p}{du} = (1+b) \frac{\frac{h(u,a+b)}{h(u,a-1)} - \frac{h\left(\frac{u}{v},a+b\right)}{h\left(\frac{u}{v},a-1\right)}}{\frac{h(u,a-1) - h(u,a+b)}{h(u,a+b)} \frac{h\left(\frac{u}{v},a-1\right) - h\left(\frac{u}{v},a+b\right)}{h\left(\frac{u}{v},a+b\right)}}. \quad (48)$$

The Appendix shows that $\frac{h(u,a+b)}{h(u,a-1)}$ is increasing in u . Hence the RHS of (48) is negative, which means that as p increases the normalized price profile rotates upwards.

This feature of the model can be contrasted with data. What the model says is that if we consider a specific capital good, say trucks or tractors (of a certain make and specification), they would be used for a longer time and their normalized price would decline more slowly in poor countries. Indeed casual empiricism suggests that trucks are used for a longer time in poor countries and that they command a positive market price when the same vintage truck has already been retired in richer countries.

>From (40) we also have that

$$\frac{d}{dp} \frac{1}{\delta_{EF}} = \frac{d}{dp} \int_0^T e^{-(g+n)\tau} \frac{p(\tau)}{p} d\tau > 0, \quad (49)$$

which is due to (19) and (48).

Comparative statics of the investment-output ratio. It follows from (37) that

$$\beta \equiv \frac{p}{i} \frac{di}{dp} = -\sigma + [h(u, 1+b+c) - \sigma h(u, a+b)] \frac{p}{u} \frac{du}{dp}. \quad (50)$$

Equation (50) provides a link between the estimable, macro parameter β and the unobserved, micro parameter σ . Indeed β can be estimated from cross country data on investment-output ratios and capital prices; see Pessoa *et al.* (2003). Then the RHS of (50) can be used to infer a value for σ .

If $\sigma < 1$ we have that

$$\frac{p}{i} \frac{di}{dp} < -\sigma.$$

This last inequality follows because $\frac{du}{dp} < 0$, h is decreasing in a and $1 + c < a$.

On the other hand, the same elasticity, β , is equal to σ in the neoclassical model. As we indicate in the next section this disparity between the estimable parameter β and the unobserved, micro parameter σ has important quantitative implications concerning the extent to which distortions affect per capita income.

Comparative statics of the wage-income covariance. It follows from (45) and (46) that

$$\eta \equiv \frac{w}{y} \frac{dy}{dw} = \sigma + \frac{1 - \sigma h(u, c) - h(u, 1 + b + c) - \sigma [h(u, a - 1) - h(u, a + b)]}{\frac{1+b}{\sigma} - [h(u, a - 1) - h(u, a + b)]}. \quad (51)$$

Equation (51), much like equation (50), links the estimable parameter η and the unobserved, micro parameter σ . Thus one can alternatively use (51) to infer an empirically relevant value for σ .

>From (51) and using properties of the h function we have that

$$\frac{w}{y} \frac{dy}{dw} > \sigma \text{ for } \sigma < 1.$$

On the other hand, in the neoclassical model the same elasticity, η , equals σ . Therefore wages vary more with incomes in our model than in the neoclassical model.

7 Empirical Implications of the Theory and Comparison to Related Growth Theories under Exogenous Technological Progress

A variety of growth models have been used to interpret income differences across countries and evaluate whether growth theory is quantitatively consistent with the view that income differences are due to investment distortions. In this section we discuss empirical implications of our theory and how they differ - either qualitatively or quantitatively - from other theories. We organize the list of empirical implications by contrasting them with three categories of

alternative theories.

7.1 Disembodied technological progress with putty-putty technology

1. The workhorse of growth and development is the neoclassical model. Lucas (1988), Mankiw (1995), Restruccia and Urritia (2001) and Chari *et al.* (1997) show how this model is applied to interpret income differences across countries.¹⁸ Technological progress in the neoclassical model is disembodied and labor is continuously mobile across technologies. Given these specifications, capital is a homogenous good (“jelly”) which is never scrapped - no matter how old it is. In our model capital goods are heterogenous, each capital good is eventually scrapped, the age at which it is scrapped depends on the distortion parameter p , and the price of capital in second hand markets depends on p as well. Therefore one can assess how differential distortions are reflected across countries by studying the service lifes of capital and the price of capital on second hand markets. In other words, one can assess whether the predictions expressed by equation (30) are borne out by data. In particular, if one considers two countries that face different p 's then the country with the higher p should have longer service lifes of capital and the inventory of capital goods in it should have an older average age. Furthermore the price profile of capital will be different in the two countries. If $\sigma < 1$, the empirically relevant case, the high p country will have a higher and flatter normalized price profile and capital will be retired later.¹⁹ Therefore to assess the validity of our theory one can simply look at the age and pricing structure of capital across countries. Alternatively, one can examine one country and assess how a change in governmental policy that affects p is translated into changes in the service lifes of capital (or its price on second hand markets). A recent paper that takes this approach is Kasahara (2002).

¹⁸An approach which uses much the same ideas is found in Greenwood *et al.* (1997); see also the survey paper by Jovanovic and Greenwood (2001). Although the Greenwood *et al.* (1997) paper considers investment-specific technological change (i.e., capital whose cost of production decreases over time), which shares some similarities with our embodied technological change approach, there is no such thing as vintage-capital in the Greenwood *et al.* model (1997). Instead, capital is envisioned as a homogenous good that enters into an aggregate production function. Thus, none of the issues relating to the lifespan of capital and the pricing of capital goods of different vintages arise in that paper. Furthermore, Greenwood *et al.* (1997) consider a single economy, the U.S., so they don't perform the comparative statics results with respect to p , which is one of the main focuses of our paper.

¹⁹The price profile in the neoclassical model is exponential, $e^{-(\delta+g)t}$, and is independent of p .

2. A second implication is that distortions have a greater impact on per capita income in our theory than in the neoclassical theory. In either theory the impact of distortions depends on the micro parameter σ . The lower is σ the lower is the marginal productivity of capital and the stronger is the impact of distortions. In particular, the $y(p)$ curve spans a larger range of incomes as p varies, and the lowest price at which an economy becomes poverty trapped is lower the lower is σ , see equation (14) and discussion following Proposition 1. Therefore, whether distortions account for income differences for a reasonable range of p 's depends on the σ that is consistent with this theory and with observed data. The observed data that allows one to recover σ is the elasticity β of the investment-output ratio with respect to p (or alternatively the elasticity η). Once this elasticity is known the value of σ that is consistent with it depends on the theoretical mapping from β to σ , see equation (50). The mapping that our theory produces is such that it assigns a lower value of σ for every β than the neoclassical theory. In this sense distortions play a more important role in our theory. The reason for this again is that distortions affect not only the quantity of installed capital but also its quality. Countries with a higher value of p operate with less capital, *and* with capital of lower average quality.

3. A third implication is that the relationship between the capital-output ratio and the distortion parameter p is u-shaped (for any value of σ , $0 \leq \sigma < 1$). This contrasts with the neoclassical theory where this relationship is downward sloping. This gives a simple criterion, in terms of observables, to distinguish between the theories. The reason for this u-shapedness is that capital is held for a longer time and that its price declines more slowly, which results in a higher market valuation of the capital stock.

7.2 Embodied technological progress with putty-putty technology

Another class of models that are potentially applicable to the development issues discussed above are the putty-putty models with embodied technological progress. The first model of this genre is due to Solow (1959) and its extensions are studied by Levhari and Sheshinski (1969, 1972).²⁰ In this class of putty-putty models, as in our model, technological progress is embodied in capital goods. The main difference is that labor is assumed mobile across vintages in the putty-putty model, whereas we assume that it is not; instead we assume that

²⁰Fisher (1965) shows the set of circumstances under which the aggregation properties used by Solow (1959) are valid with a generally specified production function.

once a plant is designed it retains its capital-labor ratio until it is scrapped. Based on the analysis here and on calibration of the putty-putty model, we find several empirical features that differentiate the embodied, putty-putty model from our model.

One difference is that, in the putty-putty model, scrapping either does not occur (when $\sigma \geq 1$) or that it occurs (when $\sigma < 1$) for unrealistically large values of T . Relatedly, the labor-capital ratio tends to zero as a plant ages in the putty-putty model, whereas in our model it remains bounded away from zero. Finally, wages equal the value of the marginal productivity of labor in the putty-putty model, while in our model they equal the output of the most antiquated plant in the economy (see equation (27)). This results from the assumption that labor is immobile and hence a theory in which the wage rate equals the marginal productivity of labor is not applicable. As a result of this, our model accounts for the fact that workers employed in similar establishments (same age plant and equipment, producing the same output) earn lower salaries in poor (highly distorted) countries than in richer countries. This prediction can be assessed by looking at the salary structure of multinational enterprises (McDonald, WalMart, etc.) that operate similar establishments in different countries. Assuming that one can control for variation in total factor productivity, one can also assess this prediction by looking at the empirical counterpart of (46) and determine how fast wages decline as the price of capital varies across countries.

7.3 Embodied technological progress with putty-clay technology

Our formulation is in the spirit of a class of models that were studied in the 1960's and include Johansen (1959), Phelps (1963), Solow *et al.* (1966), Sheshinski (1967), Bliss (1968) and Bardhan (1969). Arrow's (1962) learning by doing model also employs this assumption. Compared with this literature our model incorporates an extra ingredient, the distortion parameter p , and shows the role that this parameter plays and, in particular, its role in explaining income differences across countries. A further extension is that we derive the price profile of capital goods and various national income statistics, which allows us to calibrate and apply the model to development data.

A recent model along the same lines is Jovanovic and Rob (1997); see also Parente (2000) and Mateus-Planas (2002). The main difference from Jovanovic and Rob (1997) is that we specify a more general production function, namely we consider the family of C.E.S. production functions (instead of the Leontieff production function which is a particular

member of this family). This, together with the relationship that links estimable parameters and parameters of this production function (equation (50)), allow us to pin down a production function based on data. It also allows us to consider both the quantitative margin, i.e., how much capital the firm installs as well as the qualitative margin, i.e., how frequently the firm upgrades its technology and hence with what quality capital it operates. In addition to this, we analyze more fully general equilibrium features of the model, including the determination of wages and prices of capital of different vintages. This allows us to generate a theoretical counterpart to the concept of “capital at market prices” (see equation (42)), which is in accordance with NIPA practice. On all these counts, the C.E.S. approach seems to be richer and hence better suited to be taken to data and used to evaluate development facts. We do this in our companion, applications paper, and show that this generalized model delivers indeed better results.

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A Appendix

The appendix is divided into seven parts. In A.1 we show that the optimal lifespan of capital is no shorter than some critical age $\underline{\tau} > 0$ and thus that the firm's problem can be reduced to a program in which technology is upgraded at discretely spaced points in time. In A.2 we prove that a solution to this program exists. In A.3 we characterize this solution. In A.4 we analyze the first and second order conditions to the firm's problem and derive comparative statics properties of the solution. In A.5 we gather several results that are used in the course of this analysis. In A.6 we derive comparative static properties of national income statistics. In A.7 we prove properties of the price profile.

A.1 Reducing the problem to a sequence program

To this point we made no assumptions on the sets of upgrade dates that the firm is able to choose from. If any such set were feasible, it would not be possible to define the value of the firm's objective for some sets of upgrade dates. This is due to the fact that the firm's objective, i.e., its discounted value, is a certain integral (see below) and this integral is undefined if the set of upgrade dates is not measurable. To overcome this technical difficulty we restrict the set of upgrade dates to be such that the lifespan of capital is positive, i.e., the firm has to wait a positive amount of time between upgrades. Then we define the value of the firm's objective to be the supremum over all such plans when the lifespan of capital is allowed to be arbitrarily short. The message of this section is that the firm does not exercise the option to arbitrarily shorten the lifespan of capital. To the contrary, there is a positive length of time so that the firm holds capital for at least this length of time. This allows us to restrict attention to discrete upgrade plans with a countable number of upgrades and this, in turn, allows us to exploit the usual methods of dynamic optimization in discrete time. The first step in this approach is to define the value of the firm's objective (discounted profits) under this restriction.

Let $t' > 0$ and let

$$u_{t'}(s_0, K_0) = e^{\delta s_0} D(r + \delta, t') \left[(1 - \alpha) A(s_0)^{\frac{\sigma-1}{\sigma}} + \alpha K_0^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

be the increment to the firm's objective in the time interval $[0, t')$ when the firm holds on to its initial capital stock for $t' > 0$ units of time, and when (s_0, K_0) is the firm's initial

condition. Similarly let $t' > t \geq 0$ and let

$$v_{t,t'}(K) = D(r + \delta, t' - t) \left[(1 - \alpha)A(t)^{\frac{\sigma-1}{\sigma}} + \alpha K^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - pK$$

be the increment to the firm's objective in the time interval $[t, t')$ when capital is upgraded at dates t and t' , with K units of capital being installed at t .

Let $Z = (t_i, K_i)_{i=1}^{\infty}$, where t_i is the i^{th} upgrade time and K_i is the amount of capital installed at time t_i , be an upgrade plan. The set of feasible upgrade plans is such that $t_1 \geq 0$, $t_{i+1} > t_i$ and $K_i \geq 0$. Let J be the value of such plan

$$J(Z) = u_{t_1}(s_0, K_0) + \sum_{i=1}^{\infty} e^{-rt_i} v_{t_i, t_{i+1}}(K_i).$$

If $t_1 = 0$, the first term of $J(Z)$ is zero. Let $\varepsilon > 0$ and consider the problem of maximizing J over upgrade plans for which $t_{i+1} - t_i \geq \varepsilon$. Let the supremum of J over these plans be J_{ε}^* . Then the firm's objective is

$$J^* = \sup_{\varepsilon > 0} J_{\varepsilon}^*.$$

To analyze this objective we start out by considering the quantity-of-capital choice problem at the beginning of one replacement cycle.

$$\max_{K \geq 0} v_{t,\tau}(K), \tag{52}$$

where τ is understood now as the lifespan of capital of vintage t , i.e., $\tau = t' - t$.

We know that $v_{t,\tau}(K)$ is strictly concave in K , so the above programming problem has one solution at most. We analyze this problem by studying $0 \leq \sigma < 1$ and $1 < \sigma < 2$ separately.

(A) We start with the case where $0 \leq \sigma < 1$. Given that σ is in this range

$$Q_t(K) \equiv \left[(1 - \alpha)A(t)^{\frac{\sigma-1}{\sigma}} + \alpha K^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

is such that $\lim_{K \rightarrow \infty} Q'_t(K) = 0$, and thus $\lim_{K \rightarrow \infty} v_{t,\tau}(K) = -\infty$ (given any (t, τ) pair). Therefore (52) has a unique optimal solution. Since the constraint equation is linear, this solution is characterized by the Kuhn-Tucker conditions. We have the following lemma.

Lemma 1 (i) Suppose $\tau > 0$ is such that

$$D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} \leq p.$$

Then the unique optimal solution to (52) is $K^* = 0$. The maximized value $v_{t,\tau}^* \equiv v_{t,\tau}(0)$ is zero as well.

(ii) Suppose that $\tau > 0$ is such that

$$D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} > p.$$

Then the unique optimal solution K^* to (52) is positive and given by

$$K^* = A(t) \left\{ \frac{1}{1-\alpha} \left[\frac{p}{D(r + \delta, \tau)\alpha} \right]^{\sigma-1} - \frac{\alpha}{1-\alpha} \right\}^{\frac{\sigma}{1-\sigma}}. \quad (53)$$

This K^* is increasing, finite, and continuous in τ . The maximized value $v_{t,\tau}^* \equiv v_{t,\tau}(K^*)$ is positive, continuous and increasing in t and τ and given by

$$v_{t,\tau}^* = pA(t) \left(\frac{1-\alpha}{\alpha} \right)^{\frac{\sigma}{\sigma-1}} \left\{ [D(r + \delta, \tau)]^{1-\sigma} \frac{p^{\sigma-1}}{\alpha^\sigma} - 1 \right\}^{\frac{1}{1-\sigma}}. \quad (54)$$

Proof: We know that (52) has a unique optimal solution that is characterized by the Kuhn-Tucker conditions

$$\begin{aligned} v'_{t,\tau}(K) + \lambda &= 0 \\ \lambda K &= 0 \end{aligned} \quad (55)$$

with λ non-negative. One can show that

$$v'_{t,\tau}(0) = D(r + \delta, \tau) \lim_{K \rightarrow 0} Q'_t(K) - p = D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} - p.$$

(i) Hence $D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} \leq p$ implies that $v'_{t,\tau}(0) \leq 0$, meaning that $\lambda = -v'_{t,\tau}(0) \geq 0$ and $K^* = 0$ is the (unique) solution to (55).

(ii) Now suppose that $\tau > 0$ is such that

$$D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} > p.$$

In this case we have that $v'_{t,\tau}(0) > 0$, and so $K = 0$ cannot be part of a solution pair to (55). Instead the optimal solution K^* to (52) is given by the unique (unconstrained) solution to the first order condition $v'_{t,\tau}(K) = 0$. Tedious calculations show that K^* is given by equation (53). Plugging K^* into $v_{t,\tau}(K)$ we obtain, after some more calculations, that $v^*_{t,\tau}$ is given by (54). Since $D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} > p$ by assumption and $\sigma \in [0, 1)$, we have that

$$[D(r + \delta, \tau)]^{1-\sigma} > p^{1-\sigma}\alpha^\sigma \Rightarrow [D(r + \delta, \tau)]^{1-\sigma}\frac{p^{\sigma-1}}{\alpha^\sigma} > 1,$$

and so $K^*, v^*_{t,\tau} > 0$, as was claimed. That K^* is increasing and continuous in τ is verified by inspecting (53) and that $v^*_{t,\tau}$ is increasing in t and τ is verified by inspecting (54).

Q.E.D

Lemma 2 Suppose $\sigma \in [0, 1)$ and $p < P(\sigma)$, where

$$P(\sigma) = \frac{\alpha^{\frac{\sigma}{\sigma-1}}}{r + \delta}. \quad (56)$$

Then there exists a $\underline{\tau} > 0$ with the following property. The only capital upgrade plans that can be optimal are the ones where $t_{i+1} - t_i > \underline{\tau}$ and $K_i > 0$.

Proof: Consider the equation

$$D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} = p. \quad (57)$$

Since $D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}}$, considered as a function of τ alone, is strictly increasing, takes the value 0 at $\tau = 0$, and

$$\lim_{\tau \rightarrow \infty} D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} = \frac{\alpha^{\frac{\sigma}{\sigma-1}}}{r + \delta} = P(\sigma) > p,$$

we know that (57) has a positive solution $\underline{\tau}$. Furthermore, because of the monotonicity, $\tau \leq \underline{\tau}$ implies that $D(r + \delta, \tau)\alpha^{\frac{\sigma}{\sigma-1}} \leq p$.

Consider now an upgrade plan $Z = (t_i, K_i)_{i=1}^{\infty}$ and assume $t_{j+1} - t_j \leq \underline{\tau}$ for some j or $K_j = 0$ (or both). Then, by Lemma 1, $v_{t_j, t_{j+1}}^* = 0$. If the continuation profit from t_{j+1} onwards is zero as well, the firm can increase its profit by upgrading for the last time at t_j and installing a positive quantity of capital. This follows from Lemma 1. If the continuation profit from t_{j+1} onwards is not zero then, again by Lemma 1, we know that there must be an upgrade date $t_m \geq t_{j+1}$ for which $t_{m+1} - t_m > \underline{\tau}$ and $K_m > 0$. Consider the first such m and construct a new upgrade plan $Z' = (t'_i, K'_i)_{i=1}^{\infty}$ as follows. $t'_i = t_i$ and $K'_i = K_i$ for $i = 1, \dots, j-1$, $t'_j = t_j$, $K'_j = K_m e^{-g(t_m - t_j)}$, $t'_{j+n} = t_{m+n}$, $K'_{j+n} = K_{m+n}$ for $n = 1, 2, \dots$. Since $r > g$ the value of this plan is higher than the value of the original plan.

Q.E.D

Lemma 2 shows that if the firm installs a positive amount of capital it holds on to it for an amount of time, which is no shorter than some positive lower bound, $\underline{\tau}$. This has been shown for the case $0 \leq \sigma < 1$. The next set of results extends this to $1 < \sigma < 2$.

(B) The case where $1 < \sigma < 2$.

As in the case where $\sigma \in [0, 1)$, we want to solve the programming problem

$$\max_{K \geq 0} v_{t, \tau}(K). \quad (58)$$

We still have that the above problem is a concave programming problem, so that if it has an optimal solution, it must be unique. However, since now $\sigma \in (1, 2)$, we have that $\lim_{K \rightarrow \infty} Q'_t(K) = \alpha^{\frac{\sigma}{\sigma-1}}$, so that if p is too small, problem (58) will not have an optimal solution. The firm, in this case, will want to install an infinite amount of capital. A sufficient condition for (58) to have a (unique) finite optimal solution, no matter the value of τ , is that

$$p > \frac{\alpha^{\frac{\sigma}{\sigma-1}}}{r + \delta} = P(\sigma).$$

From now on, this will be our underlying assumption.

One can show that when $\sigma \in (1, 2)$, $\lim_{K \rightarrow 0} Q'_t(K) = +\infty$, which means that the firm will always install a positive amount K^* of capital no matter how small τ is. This follows from the proof of Lemma 4, and is in sharp contrast with what happens when $\sigma \in (0, 1)$.

Remember that we defined $v_{t, \tau}^* \equiv v_{t, \tau}(K^*)$. Since the algebra involved in finding the unique solution to the first order condition $Q'_t(K) = p$ is independent of σ , we have that also

in this case

$$v_{t,\tau}^* = pA(t) \left(\frac{1-\alpha}{\alpha} \right)^{\frac{\sigma}{\sigma-1}} \left\{ [D(r+\delta, \tau)]^{1-\sigma} \frac{p^{\sigma-1}}{\alpha^\sigma} - 1 \right\}^{\frac{1}{1-\sigma}}.$$

The next 3 lemmas will be necessary in the proof of the main result of this section.

Lemma 3 *Let $h : [0, 1] \rightarrow \mathbf{R}$ be the function given by*

$$h(z) = \sigma q(\sigma)(a+b)z(1-z)^{\sigma-2} - (1+b)[1 - q(\sigma)(1-z)^{\sigma-1}], \quad (59)$$

where $a \geq 1$, $b \geq 0$ and $q(\sigma) \in (0, 1)$. Then there exists a unique $\bar{u} \in (0, 1)$ such that $h(\bar{u}^{a+b}) = 0$. Moreover, $h(v^{a+b}) \geq 0$ if, and only if, $v \geq \bar{u}$.

Proof: If we let $\xi = 1 - \bar{u}^{a+b}$ and $\eta = \frac{1+b}{a+b}$, then $h(\bar{u}^{a+b}) = 0$ if, and only if,

$$1 - \xi \left(1 - \frac{\eta}{\sigma} \right) = \frac{\eta}{\sigma} \frac{\xi^{2-\sigma}}{q(\sigma)}.$$

Denote by $f(\xi)$ and $g(\xi)$ the right-hand side and the left-hand side, respectively, of the above equation. Observe that f is strictly decreasing and, since $\sigma < 2$, that g is strictly increasing. Moreover, $f(0) = 1$, $f(1) = \frac{\eta}{\sigma}$, $g(0) = 0$ and $g(1) = \frac{\eta}{\sigma q(\sigma)}$. Since $q(\sigma) < 1$ by hypothesis, $f(1) < g(1)$. Hence $f(\xi) - g(\xi) = 0$ has a unique solution $\bar{\xi} \in (0, 1)$. We also know that $g(\xi) > f(\xi)$ if, and only if, $\xi > \bar{\xi}$. Now, since $h(\xi)$ is proportional to $f(\xi) - g(\xi)$, $h(\xi) \geq 0$ if, and only if, $\xi \leq \bar{\xi}$. Therefore $h(v^{a+b}) \geq 0$ if, and only if, $v \geq \bar{u}$, where $\bar{u}^{a+b} = 1 - \bar{\xi}$.

Q.E.D

Lemma 4 *There exists a $\underline{\tau}$ such that if $\tau \leq \underline{\tau}$, then*

$$v_{0,\tau} \geq v_{0,\tau_1} + e^{-(r-g)\tau_1} v_{\tau_1,\tau}$$

for all $0 \leq \tau_1 < \tau$.

Proof: Note first that $v_{0,\tau} \geq v_{0,\tau_1} + e^{-(r-g)\tau_1}v_{\tau_1,\tau}$ is equivalent to

$$\frac{1 - e^{-(r+\delta)\tau}}{\left\{1 - \left[\frac{P(\sigma)}{p} (1 - e^{-(r+\delta)\tau})\right]^{\sigma-1}\right\}^{\frac{1}{\sigma-1}}} \geq \frac{1 - e^{-(r+\delta)\tau_1}}{\left\{1 - \left[\frac{P(\sigma)}{p} (1 - e^{-(r+\delta)\tau_1})\right]^{\sigma-1}\right\}^{\frac{1}{\sigma-1}}} + e^{-(r-g)\tau_1} \frac{1 - e^{-(r+\delta)(\tau-\tau_1)}}{\left\{1 - \left[\frac{P(\sigma)}{p} (1 - e^{-(r+\delta)(\tau-\tau_1)})\right]^{\sigma-1}\right\}^{\frac{1}{\sigma-1}}}.$$

where $P(\sigma)$ is given by (56). If we let v, u, \bar{u} and $q(\sigma)$ be given by

$$v = e^{-g\tau_1}, \quad u = e^{-g\tau}, \quad \bar{u} = e^{-g\tau} \text{ and } q(\sigma) = \left[\frac{P(\sigma)}{p}\right]^{\sigma-1},$$

then the above inequality can be rewritten as

$$\frac{1 - u^{a+b}}{\left\{1 - q(\sigma) [1 - u^{a+b}]^{\sigma-1}\right\}^{\frac{1}{\sigma-1}}} \geq G(v, u),$$

where the pair (v, u) is such that $\bar{u} \leq u \leq v \leq 1$ and $G(v, u)$ is the function given by

$$G(v, u) = \frac{1 - v^{a+b}}{\left\{1 - q(\sigma) [1 - v^{a+b}]^{\sigma-1}\right\}^{\frac{1}{\sigma-1}}} + v^{a-1} \frac{1 - \left(\frac{u}{v}\right)^{a+b}}{\left\{1 - q(\sigma) \left[1 - \left(\frac{u}{v}\right)^{a+b}\right]^{\sigma-1}\right\}^{\frac{1}{\sigma-1}}},$$

with $a = \frac{r}{g}$ and $b = \frac{\delta}{g}$. Now fix $u \geq \bar{u}$ and consider $G(v, u)$ as a function of v alone, with $v \in [u, 1]$. Note that

$$G(u, u) = G(1, u) = \frac{1 - u^{a+b}}{\left\{1 - q(\sigma) [1 - u^{a+b}]^{\sigma-1}\right\}^{\frac{1}{\sigma-1}}},$$

and so, since G is smooth, it has at least one critical point in the open interval $(u, 1)$. After some algebra, one can show that

$$\frac{\partial^2 G}{\partial v^2} = \frac{a-2}{v} \frac{\partial G}{\partial v} +$$

$$\frac{1}{v} \left\{ \frac{(a+b)v^{a+b}}{[1-q(\sigma)(1-v^{a+b})^{\sigma-1}]^{\frac{2\sigma-1}{\sigma-1}}} h(v^{a+b}) + v^{a-1} \frac{(a+b)\left(\frac{u}{v}\right)^{a+b}}{[1-q(\sigma)(1-\left(\frac{u}{v}\right)^{a+b})^{\sigma-1}]^{\frac{2\sigma-1}{\sigma-1}}} h\left(\left[\frac{u}{v}\right]^{a+b}\right) \right\},$$

where h is the function defined in (59).

Let us define \bar{u} implicitly by $h(\bar{u}^{a+b}) = 0$. In Lemma 3 we saw that \bar{u} exists, is unique, and is such that $h(v^{a+b}) \geq 0$ if, and only if, $v \in [0, 1]$ is such that $v \geq \bar{u}$. Therefore any critical point $v \in (u, 1)$ of $G(v, u)$, with $u \geq \bar{u}$ fixed, is such that $\frac{\partial^2 G}{\partial v^2} > 0$. This is so because $v > u \geq \bar{u}$, so that $h(v^{a+b}) > 0$, and $\frac{u}{v} > u \geq \bar{u}$, so that $h\left(\left[\frac{u}{v}\right]^{a+b}\right) > 0$. Hence we must have $G(v, u) \leq G(u, u) = G(1, u)$ for all $v \in (u, 1)$, otherwise G will have a local maximum in $(u, 1)$, which we have just ruled out.

Q.E.D

Lemma 5 *Let $G(v, u)$ be the function introduced in Lemma 4. There exists a $\delta > 0$ such that for all $u \in (0, \bar{u})$, where \bar{u} was defined in the same lemma, the function $G_u(v) := G(v, u)$, with $v \in [u, 1]$, has no local maximum in the intervals $[1 - \delta, 1)$ and $(u, u + \delta]$.*

Proof: Let us show first that there exists a $\delta_1 > 0$ such that for all $u \in (\bar{u}, 1)$ the function $G_u(v)$ has no local maximum in $[1 - \delta_1, 1)$. For this let $\delta_0 \in (0, 1)$ be such that $u + \delta_0 < 1$ for all $u \in (0, \bar{u})$. Then $v \in [u + \delta_0, 1)$ implies that there is an $\eta \in (0, 1)$ with the property that $\frac{u}{v} < \eta$ for all $u \in (0, \bar{u})$. Since the function h defined in Lemma 3 is strictly increasing, and $a + b \geq 1$, this means that

$$\left| h\left(\left[\frac{u}{v}\right]^{a+b}\right) \right| \leq \max\{h(\eta^{a+b}), (1+b)[1-q(\sigma)]\} = M_1 < \infty.$$

Now observe that the function $f : [0, 1] \rightarrow \mathbf{R}$ given by

$$f(\xi; \alpha) = \frac{\xi^\alpha}{\{1 - q(\sigma)[1 - \xi^\alpha]^{\sigma-1}\}^{\frac{2\sigma-1}{\sigma-1}}},$$

where $\alpha > 0$, is non-negative and bounded above. If we let $M_2 = \sup\{f(\xi; a+b) : \xi \in [0, 1]\}$, then, for all $u \in (0, \bar{u})$ and all $v \in [u + \delta_0, 1)$, we have that

$$\left| v^{a-1} \frac{\left(\frac{u}{v}\right)^{a+b}}{\{1 - q(\sigma)[1 - \left(\frac{u}{v}\right)^{a+b}]^{\sigma-1}\}^{\frac{2\sigma-1}{\sigma-1}}} h\left(\left[\frac{u}{v}\right]^{a+b}\right) \right| < M_1 M_2 = M,$$

since $a \geq 1$. To finish note that $\lim_{v \rightarrow 1} h(v^{a+b}) = +\infty$. By taking δ_0 such that $u + \delta_0$ is sufficiently close to one, we can then have that $f(v^{a+b})h(v^{a+b}) > 2M$. Therefore, $v \geq 1 - \delta_1$, where $\delta_1 = 1 - (u + \delta_0)$ by definition, implies that

$$\frac{\partial^2 G}{\partial v^2} > \frac{a-2}{v} \frac{\partial G}{\partial v} + M,$$

and so if any critical point v^* of G lies in $[1 - \delta_1, 1)$, it cannot be a local maximum.

A similar argument shows that there exists $\delta_2 > 0$ such that if $v \in (u, u + \delta_2]$, then v cannot be a local maximum of $G_u(v)$ no matter what is the value of u in $(0, \bar{u})$. By taking $\delta = \min\{\delta_1, \delta_2\}$ we have the desired result.

Q.E.D

Lemmas 3-5 show that the lifespan of capital is bounded away from zero in case $1 < \sigma < 2$ and $p > P(\sigma)$. The next set of results apply to $0 \leq \sigma < 1$ and $p < P(\sigma)$ as well as to $1 < \sigma < 2$ and $p > P(\sigma)$

Lemma 6 *No plan with a last upgrade time is optimal.*

Proof: Consider a capital upgrade plan with a last upgrade. Let t be the date of this last upgrade and denote by $u_{0,t}$ the value of the plan in the interval $[0, t)$. The value of this plan is then given by

$$\begin{aligned} v &= u_{0,t} + e^{-rt} \left\{ \frac{1}{r + \delta} \left[(1 - \alpha)A(t)^{\frac{\sigma-1}{\sigma}} + \alpha K^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - pK \right\} \\ &= u_{0,t} + e^{-rt} v_{t,\infty}^* \end{aligned}$$

where K is the optimal amount of capital installed at t . Consider now the capital upgrade plan that is obtained from the one above by adding one more upgrade at time $t' > t$. The value of this new plan is

$$\begin{aligned} v' &= u_{0,t} + e^{-rt} \left\{ D(r + \delta, t' - t) \left[(1 - \alpha)A(t)^{\frac{\sigma-1}{\sigma}} + \alpha K^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - pK \right\} \\ &+ e^{-rt'} \underbrace{\left\{ \frac{1}{r + \delta} \left[(1 - \alpha)A(t')^{\frac{\sigma-1}{\sigma}} + \alpha K'^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - pK' \right\}}_{v'_{t',\infty}(K')}, \end{aligned}$$

where K' is the amount of capital installed at t' . Since $v_{t',\infty}^*$ is increasing in t' (see Lemma 1), we can take $v_{t',\infty}(K') = v_{t',\infty}^* > 0$ by appropriately choosing K' . Then

$$\begin{aligned} v' - v &= e^{-rt} \left[\frac{-e^{-(r+\delta)(t'-t)} Q_t(K)}{r + \delta} \right] + e^{-rt'} v_{t',\infty}(K') \\ &= -\frac{e^{\delta t} e^{-(r+\delta)t'} Q_t(K)}{r + \delta} + e^{-rt'} v_{t',\infty}(K') \\ &= e^{-rt'} \left\{ v_{t',\infty}(K') - \frac{e^{\delta t} e^{-\delta t'} Q_t(K)}{r + \delta} \right\}. \end{aligned}$$

Since $v_{t',\infty}(K')$ is a positive constant ($= v_{t',\infty}^*$) and since $e^{-\delta(t'-t)}$ converges to zero as $t' \rightarrow \infty$, by choosing t' sufficiently large we can make the term inside the braces strictly positive. But this contradicts the optimality of $v_{t,\infty}^*$. So no capital upgrade plan with a last upgrade time is optimal.

Q.E.D

Since for any given amount K of capital, $\lim_{t \rightarrow \infty} e^{-rt} Q_t(K) = \lim_{t \rightarrow \infty} e^{-\delta t} Q_t(K) = 0$, Lemma 6 implies the following corollary.

Corollary 1 *There exists a $\bar{\tau} > \underline{\tau}$ such that no capital upgrade plan with lifespan of capital $\tau > \bar{\tau}$ can be optimal.*

This corollary holds also for the firm's initial upgrade and is independent of how much capital the firm has to begin with, K_0 . We are now ready to state and prove the main result of this section.

Theorem 1 *Suppose $\sigma \in [0, 1)$ and $p < P(\sigma)$ or $\sigma \in (1, 2)$ and $p > P(\sigma)$, where $P(\sigma)$ is given by (56). Let $\mathfrak{T}(Z)$ be the set of upgrade times of Z where a positive amount of capital is installed. Then the only capital upgrade plans Z that can be optimal are the ones satisfying the following properties.*

- (i) *The set $\mathfrak{T}(Z)$ is infinite, has a first element, and for any $t, t' \in \mathfrak{T}(Z)$, $\bar{\tau} \geq |t - t'| \geq \underline{\tau}$, where $\underline{\tau}$ and $\bar{\tau}$ are given by Lemma 2, Lemma 4 and Corollary 1, respectively.*
- (ii) *Suppose $t, t' \in \mathfrak{T}(Z)$, with $t < t'$, be two consecutive elements of $\mathfrak{T}(Z)$. Then there is no upgrade time between t and t' .*

Proof: (i) From Lemma 6 we know that no plan with a finite number of upgrades can be optimal. Moreover, installing zero capital yields zero flow payoff. Therefore, $\mathfrak{T}(Z)$ must be infinite. Suppose now that $\mathfrak{T}(Z)$ has no first element. Since $\mathfrak{T}(Z) \subset \mathbf{R}_+$, there must be a sequence in $\mathfrak{T}(Z)$ that converges from above to some non-negative t . This implies that the interval $I = [t, t + \tau)$, with some $\tau \leq \underline{\tau}$, has two upgrade times where a positive amount of capital is installed. Then, according to Lemmas 2 and 4, Z cannot be optimal. Hence $\mathfrak{T}(Z)$ has a first element. Also from Lemmas 2 and 4 and Corollary 1, we know that for any $t, t' \in \mathfrak{T}(Z)$, we must have $\bar{\tau} \geq |t - t'| \geq \underline{\tau}$.

(ii) Let $t, t' \in \mathfrak{T}(Z)$, with $t < t'$, be two consecutive elements of $\mathfrak{T}(Z)$. Suppose that there is an upgrade time $t'' \in (t, t')$. Since t'' is not an element of $\mathfrak{T}(Z)$, zero capital is installed at t'' . By removing all such upgrade times, we strictly increase the payoff of the firm, and so Z cannot be optimal.

Q.E.D

Theorem 1 simplifies the task of determining whether there is an optimal capital upgrade plan, and establishing what it looks like. We can restrict our attention to capital upgrade plans such that:

- (i) The number of upgrade times where a positive amount of capital is installed is infinite.
- (i) There is a first time t_1 where a positive amount of capital is installed.
- (iii) After t_1 , only upgrades with a positive amount of capital installed are possible, and they must be discretely spaced.

For such capital upgrade plans, what happens before t_1 is irrelevant since, by the above, the firm installs zero capital at such times. Therefore we can assume, without loss of generality, that there are no upgrade times before t_1 .

A.2 Existence of an optimal plan

We have seen that any optimal plan, if it exists, must never have a last upgrade time, and it also cannot have more than 2 upgrade times within any interval of length $\tau \leq \underline{\tau}$, where $\underline{\tau}$ is some positive, finite real number. This means that we can restrict our attention to capital upgrade plans that have a countable number of discretely spaced upgrade times. In other words, the relevant capital upgrade plans are given by infinite double sequences $(t_i, K_i)_{i=1}^\infty$, where t_i is the i^{th} upgrade time, with $t_{i+1} - t_i \geq \underline{\tau}$, and K_i is the amount of capital installed at time t_i . Denote such a plan by Z . The value associated with Z is

$$J(Z) = e^{\delta s_0} D(r + \delta, t_1) \left[(1 - \alpha) A(s_0)^{\frac{\sigma-1}{\sigma}} + \alpha K_0^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \sum_{i=1}^{\infty} e^{-rt_i} \left\{ D(r + \delta, t_{i+1} - t_i) \left[(1 - \alpha) A(t_i)^{\frac{\sigma-1}{\sigma}} + \alpha K_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - pK_i \right\},$$

where (s_0, K_0) is the firm's initial condition.

Instead of the upgrade times, t_i , we let the elapsed times, τ_i , between the i^{th} and $(i+1)^{\text{th}}$ upgrade times be the choice variables. This means that $\tau_i = t_{i+1} - t_i$ and $t_i = \sum_{j=0}^{i-1} \tau_j$, with $\tau_0 = t_1$, so that

$$J(Z) = e^{\delta s_0} D(r + \delta, \tau_0) \left[(1 - \alpha) A(s_0)^{\frac{\sigma-1}{\sigma}} + \alpha K_0^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \sum_{i=1}^{\infty} e^{-r \sum_{j=0}^{i-1} \tau_j} \left\{ D(r + \delta, \tau_i) \left[(1 - \alpha) A\left(\sum_{j=0}^{i-1} \tau_j\right)^{\frac{\sigma-1}{\sigma}} + \alpha K_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - pK_i \right\},$$

and now $Z = \{\tau_0, (\tau_i, K_i)_{i=1}^\infty\} \in [0, \bar{\tau}] \times [\underline{\tau}, \bar{\tau}]^\infty \times [0, +\infty)^\infty$. Dividing and multiplying the i^{th} term by $e^{-g \sum_{j=0}^{i-1} \tau_j}$ we get

$$J(Z) = e^{\delta s_0} D(r + \delta, \tau_0) \left[(1 - \alpha) A(s_0)^{\frac{\sigma-1}{\sigma}} + \alpha K_0^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \sum_{i=1}^{\infty} e^{-(r-g) \sum_{j=0}^{i-1} \tau_j} \left\{ e^{-g \sum_{j=0}^{i-1} \tau_j} \left[D(r + \delta, \tau_i) \left[(1 - \alpha) A\left(\sum_{j=0}^{i-1} \tau_j\right)^{\frac{\sigma-1}{\sigma}} + \alpha K_i^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - pK_i \right] \right\}$$

Finally defining k_i to be $k_i = K_i e^{-g \sum_{j=0}^{i-1} \tau_j}$ we rewrite $J(Z)$ as

$$J(Z) = e^{\delta s_0} D(r + \delta, \tau_0) \left[(1 - \alpha) A(s_0)^{\frac{\sigma-1}{\sigma}} + \alpha K_0^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} + \sum_{i=1}^{\infty} e^{-(r-g) \sum_{j=0}^{i-1} \tau_j} \left\{ D(r + \delta, \tau_i) [1 - \alpha + \alpha k_i]^{\frac{\sigma}{\sigma-1}} - p_i k_i \right\}, \quad (60)$$

where again $Z \in [0, \bar{\tau}] \times [\underline{\tau}, \bar{\tau}]^\infty \times [0, +\infty)^\infty$.

By the corollary to Lemma 6 we know that $\tau_i \leq \bar{\tau}$ for $i \geq 1$. By Lemma 1 we know that the optimal amount k_i to be installed in the interval $[t_i, t_{i+1})$ is written as $\kappa(\tau_i)$, where $\kappa(\cdot)$ depends only and continuously on τ_i . Since, for all i , τ_i is restricted to $[\underline{\tau}, \bar{\tau}]$ and since $\kappa(\cdot)$ is increasing, we know that k_i must be restricted, for all i , to the set $[0, \bar{k}]$, where $\bar{k} \equiv \kappa(\bar{\tau})$. These arguments allow us to restrict Z to the set $\mathcal{S} = [0, \bar{\tau}] \times [\underline{\tau}, \bar{\tau}]^\infty \times [0, \bar{k}]^\infty$. Endow \mathcal{S} , which is an infinite product of compact sets, with the product topology. Then, by Tychonoff's theorem, \mathcal{S} is compact. If we show that J is continuous over \mathcal{S} we will have that there exists an optimal upgrade plan.

Since \mathcal{S} , endowed with the product topology, has a countable base, it is metrizable. A metric on \mathcal{S} that is compatible with the product topology is given by

$$d(Z, Z') = \sum_{i=0}^{\infty} 2^{-i} \|(\tau_i, k_i) - (\tau'_i, k'_i)\|,$$

where $\|\cdot\|$ is the Euclidian distance in \mathbf{R}^2 , $Z = \{\tau_0, (\tau_i, k_i)_{i=1}^\infty\}$ and $Z' = \{\tau'_0, (\tau'_i, k'_i)_{i=1}^\infty\}$ are two arbitrary elements of \mathcal{S} and $k_0 = k'_0 = 0$. Let $\varepsilon > 0$ be any positive number and consider the continuous functions L_i given by

$$\begin{aligned} L_0(\tau, k) &= e^{\delta s_0} D(r + \delta, \tau) [1 - \alpha + \alpha (K_0 e^{g s_0})^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} \\ L_i(\tau, k) &= D(r + \delta, \tau) [1 - \alpha + \alpha k^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} - p k, \quad i = 1, 2, \dots \end{aligned}$$

If we let $\bar{L} = \sup\{L_i(\tau, k) : \tau \in [\underline{\tau}, \bar{\tau}], k \in [0, \bar{k}], i = 0, 1, \dots\}$, we know that $\bar{L} < \infty$. Let $j \geq 2$ be such that

$$\sum_{i=j}^{\infty} 2\bar{L} e^{-(r-g)(\bar{\tau}-\underline{\tau})i} = \frac{2\bar{L} e^{-(r-g)(\bar{\tau}-\underline{\tau})j}}{1 - e^{-(r-g)(\bar{\tau}-\underline{\tau})}} < \frac{\varepsilon}{2}.$$

Because each L_i is continuous, we know that there exists a $\delta > 0$ so that if $\max\{|\tau - \tau'|, |k -$

$k'_i\} \leq \delta$, then $|L_i(\tau, k) - L_i(\tau', k')| \leq \frac{\varepsilon}{2^j}$, $i = 0, \dots, j-1$. Let $\delta' = \frac{\delta}{2^j}$ and suppose that $Z, Z' \in \mathcal{S}$ are such that $d(Z, Z') < \delta'$. Then, since $\max\{|\tau_i - \tau'_i|, |k_i - k'_i|\} \leq \|(\tau_i, k_i) - (\tau'_i, k'_i)\|$, we know that $\max\{|k_i - k'_i|, |\tau_i - \tau'_i|\} \leq 2^i \delta' \leq \delta$ for all $i \geq 0$. Therefore $d(Z, Z') < \delta'$ implies that

$$\begin{aligned} |J(Z) - J(Z')| &\leq \sum_{i=0}^{j-1} |L_i(\tau_i, k_i) - L_i(\tau'_i, k'_i)| e^{-(r-g)(\bar{\tau}-\underline{\tau})i} \\ &+ \sum_{i=j}^{\infty} 2\bar{L} e^{-(r-g)(\bar{\tau}-\underline{\tau})i} < \varepsilon, \end{aligned}$$

showing that J is indeed a continuous function from \mathcal{S} into \mathbf{R} .

Summarizing we have the following result.

Theorem 2 *Assume $\sigma \in [0, 1)$ and $p < P(\sigma)$ or $\sigma \in (1, 2)$ and $p > P(\sigma)$. Then there exists an optimal capital upgrade plan. It will have an infinite number of upgrades that are discretely spaced, with a minimum and a maximum time between any two consecutive upgrades.*

A.3 Characterization of the optimal plan

To simplify the exposition we assume first that the firm's initial condition is such that $(s_0, K_0) = (s_0, 0)$ with $s_0 \leq 0$. In this case the firm installs a positive amount of capital at $t = 0$, call it k_0 , which is shown in the next lemma.

Lemma 7 *Assume $K_0 = 0$. Then the first upgrade date is $t_1 = 0$ and the firm installs a positive quantity of capital at that time.*

Proof. Assume first $0 \leq \sigma < 1$ and consider an arbitrary upgrade plan $(t_i, K_i)_{i=1}^{\infty}$ with $t_1 > 0$. Then the firm collects zero during $[0, t_1)$. Consider the plan $(t_{i-1}, K_i e^{-gt_1})_{i=1}^{\infty}$, where $t_0 = 0$. Then, since $r > g$, this plan gives the firm a higher value.

Second assume $1 < \sigma < 2$ and consider an arbitrary upgrade plan $(t_i, K_i)_{i=1}^{\infty}$ with $t_1 > 0$. Then, since $\lim_{K \rightarrow 0} Q'_t(K) = +\infty$, one can choose a small enough k_0 so that the value of the plan with upgrade at $t_1 = 0$ and with k_0 capital installed at that time is higher than the value of the original plan. Theorem 1 implies then that $k_0 > 0$. ■

At the end of this subsection we discuss what changes if K_0 is arbitrarily specified.

Based on the analysis in Section A.2, the maximization program of the firm is written as

$$\begin{aligned} \max \quad & J(Z) \\ \text{s.t.} \quad & Z \in \mathcal{S} \end{aligned}$$

where $J(Z)$ is given by (60) and $\mathcal{S} = [\underline{z}, \bar{z}]^{\infty} \times [0, \bar{k}]^{\infty}$. If we separate the first upgrade decision from the remaining upgrade decisions we have that

$$J(Z) = D(r + \delta, \tau_1) [1 - \alpha + \alpha k_1^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}} - pk_1 + e^{-(r-g)\tau_1} J(Z \setminus (\tau_1, k_1)),$$

where now, as a matter of notation, $Z = (\tau_1, k_1, \tau_2, k_2, \dots)$ and $Z \setminus (\tau_1, k_1) = (\tau_2, k_2, \tau_3, k_3, \dots)$. This allows us to write the firm's problem as

$$\begin{aligned} \max \quad & D(r + \delta, \tau_1) f(k_1) - pk_1 + e^{-(r-g)\tau_1} J(Z \setminus (\tau_1, k_1)) \\ \text{s.t.} \quad & Z = (\tau_1, k_1, \tau_2, k_2, \dots) \in \mathcal{S}, \end{aligned}$$

where the function f is

$$f(k) = [1 - \alpha + \alpha k^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}.$$

The following simple lemma will be central.

Lemma 8 *Let f , g and h be any three finite functions, with h strictly positive. Then (x^*, y^*) maximizes $f(x) + h(x)g(y)$ if, and only if, y^* maximizes g and x^* maximizes $f(x) + h(x)g(y^*)$.*

Proof: Suppose (x^*, y^*) maximizes $f(x) + h(x)g(y)$. Then for every y we have that

$$f(x^*) + h(x^*)g(y^*) \geq f(x^*) + h(x^*)g(y),$$

which implies that $g(y^*) \geq g(y)$ for all y by the positivity of h . Now observe that for all x ,

$$f(x^*) + h(x^*)g(y^*) \geq f(x) + h(x)g(y^*),$$

thus establishing the necessity.

For the sufficiency note that

$$f(x^*) + h(x^*)g(y^*) \geq f(x) + h(x)g(y^*) \geq f(x) + h(x)g(y),$$

for all x, y , where the last inequality follows because h is positive.

Q.E.D

Therefore, $(\tau_1, k_1, \tau_2, k_2, \dots)$ is an optimal capital upgrade plan if, and only if, it has the following 2 properties.

(i) $Z' = (\tau_2, k_2, \dots)$ maximizes J , that is, Z' is itself an optimal capital upgrade plan for the firm.

(ii) The pair (τ_1, k_1) maximizes $D(r + \delta, \tau)f(k) - pk + e^{-(r-g)\tau}J^*$, where J^* is the value to the firm of any optimal capital upgrade plan.

From this observation we can prove the following result, telling us that among the optimal capital upgrade plans we can find ones in which the action choice of the firm is the same in every period.

Lemma 9 *Let J^* be the value of an optimal upgrade plan. Then the plan $(\tau_i, k_i)_{i=1}^{\infty}$, where $\tau_i = \tau$ and $k_i = k$, with the pair (τ, k) being an optimal solution to*

$$\max \{D(r + \delta, \tau)f(k) - pk + e^{-(r-g)\tau}J^*\},$$

is an optimal capital upgrade plan.

Proof: This follows from repeatedly applying the previous lemma plus the observation that follows it.

Q.E.D

Consequently, if we restrict ourselves to time invariant capital upgrade plans and look for one that is optimal in this subclass, then we will find one that is optimal among *all* possible capital upgrade plans. Moreover, if we can determine that there is a unique optimal time invariant plan, then this plan will be the unique optimal plan among the set of all possible capital upgrade plans. To see why this is the case, suppose, to the contrary, that there is a time varying plan that is optimal, but there is a unique optimal time invariant plan. Denote the latter by (τ, k) , and let t be the first date at which the time varying plan specifies a behavior different from (τ, k) . By hypothesis, we know that such a date exists. Let (τ', k') be what this time varying plan specifies at t . But we know, from the reasoning made above, that (τ', k') maximizes

$$D(r + \delta, \tau)f(k) - pk + e^{-(r-g)\tau} J^*,$$

and so the plan that specifies (τ', k') at every upgrade date is optimal according to the previous lemma. This, however, contradicts our assumption that there is a *unique* optimal time invariant capital upgrade plan.

With what was said in the previous paragraph in mind, let us determine whether an optimal time invariant capital upgrade plan exists and is unique. To this end let's write a restricted objective function, i.e., let's impose the requirement that the capital upgrade plan is time invariant on the objective function and that the initial condition is such that $K_0 = 0$. Then the objective function is

$$\sum_{i=0}^{\infty} [D(r + \delta, \tau)f(k) - pk] e^{-(r-g)i\tau} = \frac{D(r + \delta, \tau)f(k) - pk}{1 - e^{-(r-g)\tau}}.$$

Alternatively, using the transformation

$$u = e^{-g\tau},$$

the restricted objective function is written as

$$F(k, u) = \frac{\frac{1}{g} \frac{1-u^{a+b}}{a+b} f(k) - pk}{1 - u^{a-1}}. \quad (61)$$

It remains to show that (61) admits a unique maximum, which we do in Section A.4. Modulo that we have the following result.

Theorem 3 *Let the firm's initial condition (s_0, K_0) be such that $K_0 = 0 \geq s_0$. The problem of the firm has a unique optimal solution, and this optimal solution has the feature that all upgrades are done at equally spaced intervals with spacing τ and the amount of capital installed at each upgrade is ke^{gt} , where k is a constant and t is the upgrade time. The optimal (k, τ) is the unique maximizer of the restricted objective function (61).*

If (s_0, K_0) is arbitrarily specified, the analysis proceeds as follows. Let τ_0 be the first time the firm upgrades. We know that such time exists, otherwise the firm's continuation payoff converges zero, and she can do better by replacing her initial capital at some late enough date. From τ_0 on, the firm's problem is exactly as above. Let $(\tau_i, k_i)_{i=1}^\infty = (\tau, k)_{i=1}^\infty$ be the unique capital upgrade plan from τ_0 on, where τ and k are as in Theorem 3. Let J^* be the value of $(\tau_i, k_i)_{i=1}^\infty$. Then the value to the firm of upgrading her capital at τ_0 is

$$U(\tau_0) = D(r + \delta, \tau_0)Q_0 + e^{-r\tau_0}J^*,$$

where $Q_0 = e^{\delta s_0}Q_s(K_0)$, K_0 being the initial amount of capital the firm is endowed with and s_0 being the vintage of this capital. The problem of the firm is then to maximize $U(\tau_0)$. Differentiating U we get

$$U'(\tau_0) = \frac{e^{-(r+\delta)\tau_0}}{r + \delta}Q_0 - re^{-r\tau_0}J^*. \quad (62)$$

Multiplying the RHS of (62) by $e^{r\tau_0}$, we see that $U'(\tau_0) = 0$ has at most one positive solution and that U is single peaked. If it has a positive solution τ_0^* , then τ_0^* is the unique solution to the problem of maximizing $U(\tau_0)$ and, by (62), τ_0^* is increasing in K_0 . If (62) does not have a positive solution, then $\tau_0^* = 0$ is the unique solution to this problem. In any way, the choice of τ_0 is unique. If $s_0 + T \leq 0$ and $K_0 \leq K(s_0)$, $\tau_0 = 0$; if $s_0 + T \leq 0$ and $K_0 > K(s_0)$, $\tau_0 > 0$ or $\tau_0 < 0$ - depending on how large is K_0 . The larger is K_0 the larger is τ_0 . Similarly, if $s_0 + T > 0$ and $K_0 > K(s_0)$, $\tau_0 > 0$ and τ_0 is larger the larger is K_0 ; if $s_0 + T > 0$ and $K_0 \leq K(s_0)$, $\tau_0 \geq 0$ and τ_0 is larger the larger is K_0 .

In summary, we have the following theorem.

Theorem 4 *Let the firm's initial condition (s_0, K_0) be arbitrarily specified, $K_0 \geq 0 \geq s_0$. The problem of the firm in this case also has a unique solution. It is given by $\{\tau_0, (\tau, k)_{i=1}^{\infty}\}$, where τ, k are as in Theorem 3. The choice of the first period τ_0 of upgrade is either $\tau_0 = 0$ or the unique interior solution to $U'(\tau_0) = 0$, if it exists, where $U'(\tau_0)$ is given by equation (62). This initial τ_0 is larger the larger is the initial s_0 and the larger is K_0 .*

A.4 Analysis of the FOCs and the SOC's to problem (61) and comparative statics results.

A.4.1 Existence

To refresh our memory let's re-write the restricted objective function

$$F(k, u) = \frac{\frac{1}{g} \frac{1-u^{a+b}}{a+b} f(k) - pk}{1 - u^{a-1}}.$$

The FOC's of this problem are

$$F_k = \frac{\frac{1}{g} \frac{1-u^{a+b}}{a+b} f'(k) - p}{1 - u^{a-1}} \quad (63)$$

and

$$F_u = \frac{(a-1)u^{a-2} \left\{ \frac{1}{g} \left[\frac{1-u^{a+b}}{a+b} - u^{1+b} \frac{1-u^{a-1}}{a-1} \right] f(k) - pk \right\}}{(1 - u^{a-1})^2}. \quad (64)$$

Note that a (k, u) pair solves (63) and (64) if and only if it solves (9).

We employ the result that a C.E.S. production function satisfies

$$f'(k) = \alpha \left(\frac{f(k)}{k} \right)^{\frac{1}{\sigma}}. \quad (65)$$

Substituting (63) into (64) and using (65), we need to show that for some $u \in (0, 1)$

$$u^{1+b} \frac{1 - u^{a-1}}{a-1} - \frac{1 - u^{a+b}}{a+b} + \frac{\alpha^\sigma}{(pg)^{\sigma-1}} \left(\frac{1 - u^{a+b}}{a+b} \right)^\sigma = 0. \quad (66)$$

Assuming that $0 \leq \sigma < 1$ and $p < P(\sigma)$ we show in Section A.5 (Property 1) that the LHS of (66) is negative at $u = 0$ and positive in a left neighborhood of $u = 1$. Since the LHS is continuous this implies that a solution to (66) (and hence to equations (63) and (64)) exists.

A.4.2 Uniqueness

We prove uniqueness and maximality of a critical point by showing that the Hessian of F is negative definite at every critical point. Calculating the cross partials at a critical point we

get

$$\begin{aligned}
F_{kk} &= \frac{\frac{1}{g} \frac{1-u^{a+b}}{a+b} f''(k)}{1-u^{a-1}}, \\
F_{ku} = F_{uk} &= \frac{\frac{-1}{g} u^{a+b-1} f'(k)}{1-u^{a-1}}, \\
F_{uu} &= \frac{\frac{-f(k)}{g} (1+b) u^{a+b-2}}{1-u^{a-1}},
\end{aligned}$$

where F_{ku} incorporates the condition that $F_k = 0$ and F_{uu} incorporates the condition that $F_u = 0$. Factoring $\frac{1}{g(1-u^{a-1})}$ out of the Hessian matrix and incorporating the same conditions into it we get

$$\frac{1}{g(1-u^{a-1})} \begin{bmatrix} \frac{1-u^{a+b}}{a+b} f''(k) & -u^{a+b-1} f'(k) \\ -u^{a+b-1} f'(k) & -f(k)(1+b)u^{a+b-2} \end{bmatrix}.$$

Since f'' , $-f < 0$, this Hessian is negative definite at a critical point if its determinant Δ is positive

$$\Delta \equiv \frac{1}{g^2(1-u^{a-1})^2} \left\{ -\frac{1-u^{a+b}}{a+b} f''(k) f(k) (1+b) u^{a+b-2} - u^{2a+2b-2} [f'(k)]^2 \right\} > 0.$$

To show that $\Delta > 0$, we do the following calculations

$$\begin{aligned}
\Delta &= \frac{u^{a+b-2}}{g^2(1-u^{a-1})^2} \left\{ -\frac{1-u^{a+b}}{a+b} f''(k) f(k) (1+b) - u^{a+b} [f'(k)]^2 \right\} \\
&= \frac{u^{a+b-2}}{g^2(1-u^{a-1})^2} [f'(k)]^2 \left\{ \frac{1-u^{a+b}}{a+b} \frac{1+b}{\sigma} \frac{1-\alpha(k)}{\alpha(k)} - u^{a+b} \right\},
\end{aligned}$$

where

$$\alpha(k) = k \frac{f'(k)}{f(k)}, \tag{67}$$

and where we use the definition of σ

$$\sigma \equiv -\frac{f'(k) [f(k) - k f'(k)]}{k f(k) f''(k)} \tag{68}$$

to eliminate $f''(k)$. Continuing with the calculations

$$\Delta = \frac{u^{a+b-2}}{g^2(1-u^{a-1})^2} g^2 p^2 \left[\frac{a+b}{1-u^{a+b}} \right]^2 \left\{ \frac{1-u^{a+b}}{a+b} \frac{1+b}{\sigma} \frac{h(u, a+b)}{h(u, a-1) - h(u, a+b)} - u^{a+b} \right\},$$

where we use (63) to eliminate $f'(k)$ and we also use

$$\alpha(k) = 1 - \frac{h(u, a+b)}{h(u, a-1)}, \quad (69)$$

which follows when (63) and (64) are combined. Continuing

$$\begin{aligned} \Delta &= \frac{u^{a+b-2}}{(1-u^{a-1})^2} p^2 \left[\frac{a+b}{1-u^{a+b}} \right]^2 \left\{ \frac{1+b}{\sigma} \frac{u^{a+b}}{h(u, a-1) - h(u, a+b)} - u^{a+b} \right\} \\ &= \frac{p^2}{u^2(1-u^{a-1})^2} [h(u, a+b)]^2 \left\{ \frac{1+b}{\sigma} \frac{1}{h(u, a-1) - h(u, a+b)} - 1 \right\} \\ &= \frac{p^2}{u^2(1-u^{a-1})^2} \frac{[h(u, a+b)]^2}{h(u, a-1) - h(u, a+b)} \left\{ \frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)] \right\}. \end{aligned} \quad (70)$$

Since h is decreasing in a (see Property 2 in Section A.5), $\Delta > 0$ if and only if

$$\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)] > 0.$$

In Section A.5 we show (Property 2) that this is true whenever $0 \leq \sigma \leq 2$.

A.4.3 Comparative statics of the firm's problem

The procedure for finding comparative statics properties is to implicitly differentiate equations (63) and (64). When we do that we get

$$\frac{1}{g(1-u^{a-1})} \begin{bmatrix} \frac{1-u^{a+b}}{a+b} f''(k) & -u^{a+b-1} f'(k) \\ -u^{a+b-1} f'(k) & -f(k)(1+b)u^{a+b-2} \end{bmatrix} \begin{bmatrix} dk \\ du \end{bmatrix} = \begin{bmatrix} \frac{1}{1-u^{a-1}} \\ \frac{(a-1)u^{a-2}}{(1-u^{a-1})^2} k \end{bmatrix} dp.$$

To find $\frac{du}{dp}$ we compute Δ_p^u , defined as

$$\Delta_p^u = \begin{vmatrix} \frac{\frac{1-u^{a+b}}{a+b} f''(k)}{g(1-u^{a-1})} & \frac{1}{1-u^{a-1}} \\ -\frac{u^{a+b-1} f'(k)}{g(1-u^{a-1})} & \frac{(a-1)u^{a-2}}{(1-u^{a-1})^2} k \end{vmatrix}.$$

Invoking the definition of σ , (68), we re-write this as

$$\begin{aligned} \Delta_p^u &= \begin{vmatrix} -\frac{\frac{1-u^{a+b}}{a+b} f'(k)[f(k)-kf'(k)]}{g(1-u^{a-1})} & \frac{1}{1-u^{a-1}} \\ -\frac{u^{a+b-1} f'(k)}{g(1-u^{a-1})} & \frac{(a-1)u^{a-2}}{(1-u^{a-1})^2} k \end{vmatrix} \\ &= \frac{1}{g(1-u^{a-1})^2} \begin{vmatrix} -\frac{1-u^{a+b}}{a+b} \frac{f'(k)(1-\alpha(k))}{k\sigma} & 1 \\ -u^{a+b-1} f'(k) & u^{-1} h(u, a-1) k \end{vmatrix}. \end{aligned}$$

Substituting (63) into the above we get

$$\begin{aligned} \Delta_p^u &= \frac{1}{g(1-u^{a-1})^2} \begin{vmatrix} -\frac{1-\alpha(k)}{k\sigma} gp & 1 \\ -u^{-1} h(u, a+b) gp & u^{-1} h(u, a-1) k \end{vmatrix} \\ &= \frac{p}{u} \frac{1}{(1-u^{a-1})^2} \begin{vmatrix} -\frac{1-\alpha(k)}{k\sigma} & 1 \\ -h(u, a+b) & h(u, a-1) k \end{vmatrix} \\ &= \frac{p}{u} \frac{1}{(1-u^{a-1})^2} \left\{ h(u, a+b) - \frac{1-\alpha(k)}{\sigma} h(u, a-1) \right\}. \end{aligned}$$

Next, using (69), it follows that

$$\Delta_p^u = -\frac{p}{u} \frac{h(u, a+b)}{(1-u^{a-1})^2} \frac{1-\sigma}{\sigma}.$$

Consequently, substituting in from (70), we get

$$\begin{aligned} \frac{p}{u} \frac{du}{dp} &= \frac{p\Delta_p^u}{u\Delta} \\ &= -\frac{\frac{p^2}{u^2} \frac{h(u, a+b)}{(1-u^{a-1})^2} \frac{1-\sigma}{\sigma}}{\frac{p^2}{u^2(1-u^{a-1})^2} \frac{[h(u, a+b)]^2}{h(u, a-1)-h(u, a+b)} \frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]} \frac{1}{[h(u, a-1) - h(u, a+b)]} \\ &= -\frac{h(u, a-1) - h(u, a+b)}{h(u, a+b)} \frac{\frac{1-\sigma}{\sigma}}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}, \end{aligned}$$

which is equation (17) in the text.

As we show in Section A.5 (Property 2 and corollary to it), h is positive, decreasing in a and $\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)] > 0$ for $0 \leq \sigma \leq 2$. Consequently, if $0 \leq \sigma < 1$, $\frac{pdu}{udp} < 0$ while if $1 < \sigma \leq 2$, $\frac{pdu}{udp} > 0$.

To find $\frac{dk}{dp}$ we compute Δ_p^k , defined as

$$\begin{aligned} \Delta_p^k &= \begin{vmatrix} \frac{1}{1-u^{a-1}} & \frac{-u^{a+b-1}f'(k)}{g(1-u^{a-1})} \\ \frac{(a-1)u^{a-2}}{(1-u^{a-1})^2}k & \frac{-f(k)(1+b)u^{a+b-2}}{g(1-u^{a-1})} \end{vmatrix} \\ &= -\frac{1}{g(1-u^{a-1})} \frac{u^{a+b-2}}{1-u^{a-1}} \begin{vmatrix} 1 & f'(k) \\ h(u, a-1)k & f(k)(1+b) \end{vmatrix}. \end{aligned}$$

Substituting from (63) and (64) we get

$$\begin{aligned} \Delta_p^k &= -\frac{1}{g(1-u^{a-1})} \frac{u^{a+b-2}}{1-u^{a-1}} \begin{vmatrix} 1 & gp \frac{a+b}{1-u^{a=b}} \\ h(u, a-1)k & gpk \frac{1+b}{\frac{1-u^{a+b}}{a+b} - u^{1+b} \frac{1-u^{a-1}}{a-1}} \end{vmatrix} \\ &= -\frac{1}{g(1-u^{a-1})} \frac{u^{-2}}{1-u^{a-1}} \begin{vmatrix} 1 & gph(u, a+b) \\ h(u, a-1)k & gpk \frac{(1+b)h(u, a-1)h(u, a+b)}{h(u, a-1) - h(u, a+b)} \end{vmatrix} \\ &= -\frac{h(u, a-1)h(u, a+b)}{gu^2(1-u^{a-1})} \frac{gpk}{1-u^{a-1}} \begin{vmatrix} 1 & 1 \\ 1 & \frac{(1+b)}{h(u, a-1) - h(u, a+b)} \end{vmatrix}. \end{aligned}$$

Computing the determinant we obtain

$$\begin{aligned} \Delta_p^k &= -\frac{h(u, a-1)h(u, a+b)}{u^2(1-u^{a-1})} \frac{pk}{1-u^{a-1}} \left\{ \frac{(1+b)}{h(u, a-1) - h(u, a+b)} - 1 \right\} \\ &= -\frac{h(u, a-1)h(u, a+b)}{h(u, a-1) - h(u, a+b)} \frac{pk}{u^2(1-u^{a-1})^2} \{1+b - [h(u, a-1) - h(u, a+b)]\}. \end{aligned}$$

Consequently, substituting in from (70), we get

$$\begin{aligned}
\frac{p \, dk}{k \, dp} &= \frac{p \Delta_p^k}{k \Delta} \\
&= -\frac{\frac{h(u, a-1)h(u, a+b)}{h(u, a-1)-h(u, a+b)} \frac{p^2 k}{u^2(1-u^{a-1})^2} \frac{1+b - [h(u, a-1) - h(u, a+b)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}}{k \frac{p^2}{u^2(1-u^{a-1})^2} \frac{[h(u, a+b)]^2}{h(u, a-1)-h(u, a+b)}} \\
&= -\frac{h(u, a-1) \frac{1+b - [h(u, a-1) - h(u, a+b)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}}{h(u, a+b) \frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]},
\end{aligned}$$

which is equation (16) in the text.

As we show in Section A.5 (Property 2), h is positive and $\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)] > 0$ for $0 \leq \sigma \leq 2$. Consequently, if $0 \leq \sigma \leq 2$, $\frac{p \, dk}{k \, dp} < 0$.

A.5 Some properties of h

To complete the proof of existence in Section A.4 we show now the following result.

Property 1: Assume $p < P(\sigma)$ and $0 \leq \sigma < 1$. Then

$$(i) \quad \lim_{u \rightarrow 0} \left[u^{1+b} \frac{1 - u^{a-1}}{a-1} - \frac{1 - u^{a+b}}{a+b} + \frac{\alpha^\sigma}{(pg)^{\sigma-1}} \left(\frac{1 - u^{a+b}}{a+b} \right)^\sigma \right] < 0.$$

(ii) There exists a \bar{u} , $0 < \bar{u} < 1$ so that

$$(ii) \quad u^{1+b} \frac{1 - u^{a-1}}{a-1} - \frac{1 - u^{a+b}}{a+b} + \frac{\alpha^\sigma}{(pg)^{\sigma-1}} \left(\frac{1 - u^{a+b}}{a+b} \right)^\sigma > 0, u \in [\bar{u}, 1].$$

Proof: To show (i), set $u = 0$, which leaves us with

$$\lim_{u \rightarrow 0} -\frac{1}{a+b} \left[1 - \left(\frac{P(\sigma)}{p} \right)^{\sigma-1} \right].$$

This is negative because we are assuming $p < P(\sigma)$ and $\sigma - 1 < 0$.

To show (ii) we transform

$$u^{1+b} \frac{1 - u^{a-1}}{a-1} - \frac{1 - u^{a+b}}{a+b} + \frac{\alpha^\sigma}{(pg)^{\sigma-1}} \left(\frac{1 - u^{a+b}}{a+b} \right)^\sigma$$

into an equivalent expression. First we divide and multiply by u^{a+b} which gives

$$u^{a+b} \left\{ \left[\frac{1}{h(u, a-1)} - \frac{1}{h(u, a+b)} \right] + \frac{\alpha^\sigma}{p^{\sigma-1}} \left(\frac{1 - u^{a+b}}{g(a+b)} \right)^{\sigma-1} \left(\frac{1 - u^{a+b}}{u^{a+b}(a+b)} \right) \right\}.$$

Then we reorganize the above to obtain

$$\frac{u^{a+b}}{h(u, a-1)h(u, a+b)} \left\{ \left[\frac{P(\sigma)}{p} (1 - u^{a+b}) \right]^{\sigma-1} h(u, a-1) - [h(u, a-1) - h(u, a+b)] \right\}. \quad (71)$$

Now it suffices to show that

$$(a) \quad \lim_{u \rightarrow 1} \left[\frac{P(\sigma)}{p} (1 - u^{a+b}) \right]^{\sigma-1} h(u, a-1) = \infty$$

and

$$(b) \quad \lim_{u \rightarrow 1} [h(u, a-1) - h(u, a+b)] = \frac{1+b}{2}.$$

(a) follows from the definition of h

$$\lim_{u \rightarrow 1} (1 - u^{a+b})^{\sigma-1} h(u, a-1) = \lim_{u \rightarrow 1} (1 - u^{a+b})^{\sigma-1} \frac{(a-1)u^{a-1}}{1 - u^{a-1}} = \infty \text{ if } 0 \leq \sigma < 1.$$

(b) follows because

$$\begin{aligned} \lim_{u \rightarrow 1} [h(u, a-1) - h(u, a+b)] &= \lim_{u \rightarrow 1} \left[\frac{(a-1)u^{a-1}}{1 - u^{a-1}} - \frac{(a+b)u^{a+b}}{1 - u^{a+b}} \right] \\ &= (a+b) - (a-1) + \lim_{u \rightarrow 1} \left[\frac{a-1}{1 - u^{a-1}} - \frac{a+b}{1 - u^{a+b}} \right]. \end{aligned} \quad (72)$$

It remains to evaluate

$$\begin{aligned} &\lim_{u \rightarrow 1} \left[\frac{a-1}{1 - u^{a-1}} - \frac{a+b}{1 - u^{a+b}} \right] \\ &= \lim_{u \rightarrow 1} \left[\frac{(a-1)(1 - u^{a+b}) - (a+b)(1 - u^{a-1})}{(1 - u^{a-1})(1 - u^{a+b})} \right]. \end{aligned}$$

Since the numerator and denominator of the above tend to zero as $u \rightarrow 1$, we apply L'Hôpital's rule to evaluate the limit of the ratio

$$\begin{aligned} &= - \lim_{u \rightarrow 1} \left[\frac{-(a-1)(a+b)u^{a+b-1} + (a+b)(a-1)u^{a-2}}{(a-1)u^{a-2}(1 - u^{a+b}) + (a+b)u^{a+b-1}(1 - u^{a-1})} \right] \\ &= \lim_{u \rightarrow 1} \left[\frac{(a-1)(a+b)(u^{1+b} - 1)}{(a-1)(1 - u^{a+b}) + (a+b)u^{1+b}(1 - u^{a-1})} \right], \end{aligned}$$

where the last equality follows after we divide numerator and denominator by u^{a-2} . Continuing with the calculations we apply L'Hôpital's rule once again

$$\begin{aligned} &= - \lim_{u \rightarrow 1} \left[\frac{(a-1)(a+b)(1+b)u^b}{(a-1)(a+b)u^{a+b-1} + (a-1)(a+b)u^{1+b}u^{a-2} - (a+b)(1+b)u^b(1 - u^{a-1})} \right] \\ &= - \frac{(a-1)(a+b)(1+b)}{(a-1)(a+b) + (a-1)(a+b)} = - \frac{1+b}{2}. \end{aligned}$$

Combining the last equality with (72) gives the desired result.

The next result is central for the analysis of the SOC and the comparative statics results.

Property 2: Let $a > 0$ and let $0 < u < 1$. Then

$$-\frac{1}{2} < \frac{\partial}{\partial a} h(u, a) < 0. \quad (73)$$

Proof: We have to show that

$$-\frac{1}{2} < \frac{\partial}{\partial a} \left[\frac{au^a}{1-u^a} \right] = \frac{u^a(1-u^a + \ln u^a)}{(1-u^a)^2} < 0.$$

Let us call

$$h_1(x) \equiv \frac{x(1-x + \ln x)}{(1-x)^2}.$$

Note that

$$h_1(0) = 0, \quad h_1(1) = -\frac{1}{2},$$

where the last equality follows by application of L'Hôpital's rule. Thus (73) follows if we can show that

$$h_1'(x) = \frac{2 - 2x + (1+x) \ln x}{(1-x)^3} < 0 \text{ for } 0 < x < 1,$$

or equivalently if

$$h_2(x) \equiv 2 - 2x + (1+x) \ln x < 0.$$

Given that $h_2(0) = -\infty$ and that $h_2(1) = 0$ it suffices to show that $h_2'(x) > 0$. But,

$$\begin{aligned} h_2'(x) &= -1 + \ln x + \frac{1}{x} \\ &= -1 - \ln \frac{1}{x} + \frac{1}{x} \\ &> -1 - \left(\frac{1}{x} - 1 \right) + \frac{1}{x} = 0, \end{aligned}$$

where the inequality follows from the the concavity of \ln and from its first order Taylor expansion around $x = 1$.

The next result completes the proof of uniqueness in Section A.4.2 and is used in various comparative statics exercises.

Corollary from Property 2: (73) implies

$$0 < h(u, x) - h(u, x + b + 1) < \frac{1 + b}{2}$$

for any $x \geq 0$ and any $0 \leq \sigma \leq 2$. This implies

$$h(u, c) - h(u, 1 + b + c) < \frac{1 + b}{\sigma}$$

and

$$h(u, a - 1) - h(u, a + b) < \frac{1 + b}{\sigma}$$

for any $0 \leq \sigma \leq 2$.

The next result computes the effective depreciation rate when $T \rightarrow \infty$ or equivalently when $u \rightarrow 0$.

Property 3: $\lim_{u \rightarrow 0} \delta_{\text{EF}} = g + \delta + n$.

To refresh our memory let us re-write equation (43)

$$\frac{1}{\delta_{\text{EF}}} \equiv \frac{k_{\text{M}}(t)}{i_{\text{M}}(t)} = \frac{1}{r - (g + n)} \left[\frac{D(g + \delta + n, T) - e^{-(g+\delta)T} D(n, T)}{D(r + \delta, T) - e^{-(g+\delta)T} D(r - g, T)} - 1 \right].$$

Using the transformation $u = e^{-gT}$, the above is equivalent to

$$\frac{1}{\delta_{\text{EF}}} \equiv \frac{k_{\text{M}}(t)}{i_{\text{M}}(t)} = \frac{1}{g} \frac{1}{a - (1 + c)} \left[\frac{\frac{1 - u^{1+b+c}}{1+b+c} - u^{1+b} \frac{1 - u^c}{c}}{\frac{1 - u^{a+b}}{a+b} - u^{1+b} \frac{1 - u^{a-1}}{a-1}} - 1 \right].$$

>From this we get

$$\begin{aligned} g^{-1} \delta_{\text{EF}} &= [a - (1 + c)] \frac{\frac{1 - u^{a+b}}{a+b} - u^{1+b} \frac{1 - u^{a-1}}{a-1}}{\frac{1 - u^{1+b+c}}{1+b+c} - u^{1+b} \frac{1 - u^c}{c} - \left[\frac{1 - u^{a+b}}{a+b} - u^{1+b} \frac{1 - u^{a-1}}{a-1} \right]} \\ &= [a - (1 + c)] \frac{\frac{1 - u^{a+b}}{a+b} - u^{1+b} \frac{1 - u^{a-1}}{a-1}}{\frac{1 - u^{1+b+c}}{1+b+c} - \frac{1 - u^{a+b}}{a+b} - u^{1+b} \left[\frac{1 - u^c}{c} - \frac{1 - u^{a-1}}{a-1} \right]}. \end{aligned}$$

Taking the limit as $u \rightarrow 0$, it follows that

$$g^{-1} \lim_{u \rightarrow 0} \delta_{\text{EF}} = [a - (1 + c)] \frac{\frac{1}{a+b}}{\frac{1}{1+b+c} - \frac{1}{a+b}} = [a - (1 + c)] \frac{1 + b + c}{a + b - (1 + b + c)},$$

which implies

$$\lim_{u \rightarrow 0} \delta_{\text{EF}} = g(1 + b + c) = g + \delta + n.$$

The next result is used to establish the comparative statics properties of Section 6.2.

Property 4: Assume $a - 1 > c$. Then

(i)

$$\frac{h(u, a - 1) - h(u, a + b)}{h(u, a + b)} > \frac{h(u, c) - h(u, 1 + b + c)}{h(u, 1 + b + c)}$$

and (ii)

$$h(u, c) - h(u, 1 + b + c) > h(u, a - 1) - h(u, a + b).$$

Assume $0 < v < 1$. Then

(iii)

$$\frac{h(u, a + b)}{h(u, a - 1)} - \frac{h\left(\frac{u}{v}, a + b\right)}{h\left(\frac{u}{v}, a - 1\right)} < 0.$$

Proof: Since $a - 1 > c$, (i) holds if $\frac{h(u, x)}{h(u, 1 + b + x)}$ is increasing in x . So let's compute the partial derivative of $\frac{h(u, x)}{h(u, 1 + b + x)}$ with respect to x .

$$\begin{aligned} \frac{\partial}{\partial x} \left[\frac{h(u, x)}{h(u, 1 + b + x)} \right] &= \frac{h(u, x)}{h(u, 1 + b + x)} \\ &\times \left[\frac{1}{x} \frac{1 - u^x + \ln u^x}{1 - u^x} - \frac{1}{1 + b + x} \frac{1 - u^{1+b+x} + \ln u^{1+b+x}}{1 - u^{1+b+x}} \right] \\ &= \ln u \frac{h(u, x)}{h(u, 1 + b + x)} \\ &\times \left[\frac{1 - u^x + \ln u^x}{(1 - u^x) \ln u^x} - \frac{1 - u^{1+b+x} + \ln u^{1+b+x}}{(1 - u^{1+b+x}) \ln u^{1+b+x}} \right]. \end{aligned}$$

This last term is positive because

$$\frac{d}{dv} \left(\frac{1 - v + \ln v}{(1 - v) \ln v} \right) < 0$$

and because

$$1 > u^x > u^{1+b+x} > 0 \text{ for } u \in (0, 1) \text{ and } b, x > 0.$$

(ii) Likewise (ii) holds if $h(u, x) - h(u, x + 1 + b)$ is decreasing in x . So let's compute the partial derivative of $h(u, x) - h(u, x + 1 + b)$ with respect to x .

$$\frac{\partial}{\partial x} [h(u, x) - h(u, x + 1 + b)] = \frac{u^x (1 - u^x + \ln u^x)}{(1 - u^x)^2} - \frac{u^{1+b+x} (1 - u^{1+b+x} + \ln u^{1+b+x})}{(1 - u^{1+b+x})^2}.$$

This last term is negative by the proof of Property 2 (see the proof that $h'_1 < 0$).

(iii) Since $v < 1$, $\frac{u}{v} > u$ so it suffices to show that $\frac{h(u, a+b)}{h(u, a-1)}$ is increasing in u . To this end we compute

$$\begin{aligned} u \frac{h(u, a-1)}{h(u, a+b)} \frac{\partial}{\partial u} \left[\frac{h(u, a+b)}{h(u, a-1)} \right] &= \frac{u}{h(u, a+b)} \frac{\partial h(u, a+b)}{\partial u} - \frac{u}{h(u, a-1)} \frac{\partial h(u, a-1)}{\partial u} \\ &= \frac{a+b}{1-u^{a+b}} - \frac{a-1}{1-u^{a-1}}. \end{aligned}$$

This last term is positive because $\frac{d}{dx} \left[\frac{x}{1-u^x} \right] > 0$. The result follows now because $u \frac{h(u, a-1)}{h(u, a+b)}$ is positive.

A.6 Comparative statics results of national income statistics with respect to p

A.6.1 per capita income

>From (44) in the text, per capita income is

$$y(p) \equiv \frac{1 - [u(p)]^{1+b+c}}{1+b+c} \frac{c}{1 - [u(p)]^c} f(k(p)).$$

where $u(p)$ and $k(p)$ is the solution to (9) and $c \equiv \frac{n}{g}$.

Differentiating the logarithm of the above expression and multiplying by p we get

$$\frac{p}{y} \frac{dy(p)}{dp} = [h(u, c) - h(u, 1+b+c)] \frac{p}{u} \frac{du}{dp} + \alpha(k) \frac{p}{k} \frac{dk}{dp}.$$

Substituting (69), (16) and (17) into the above expression we obtain

$$\begin{aligned} \frac{p}{y} \frac{dy(p)}{dp} &= -[h(u, c) - h(u, 1+b+c)] \\ &\times \frac{h(u, a-1) - h(u, a+b)}{h(u, a+b)} \frac{\frac{1-\sigma}{\sigma}}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]} \\ &- \frac{h(u, a-1) - h(u, a+b)}{h(u, a-1)} \frac{h(u, a-1)}{h(u, a+b)} \frac{1+b - [h(u, a-1) - h(u, a+b)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}. \end{aligned}$$

Or, after some manipulations,

$$\begin{aligned} \frac{p}{y} \frac{dy(p)}{dp} &= -\frac{h(u, a-1) - h(u, a+b)}{h(u, a+b)} \\ &\times \frac{\frac{1+b}{2} - [h(u, a-1) - h(u, a+b)] + \frac{1+b}{2} - \frac{\sigma-1}{\sigma} [h(u, c) - h(u, 1+b+c)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}, \end{aligned}$$

which simplifies to equation (45) in the text. It follows from the corollary to Property 2 in Section A.5 that the above is negative as long as $0 \leq \sigma \leq 2$.

A.6.2 Comparative Statics of the wage

>From (13) and the transformation $u = e^{-gT}$, we know that

$$w_0 = u^{1+b} f(k).$$

Consequently

$$\frac{p}{w} \frac{dw}{dp} = (1+b) \frac{p}{u} \frac{du}{dp} + \alpha(k) \frac{p}{k} \frac{dk}{dp}.$$

Substituting (69), (16) and (17) it follows that

$$\begin{aligned} \frac{p}{w} \frac{dw}{dp} &= -(1+b) \frac{h(u, a-1) - h(u, a+b)}{h(u, a+b)} \frac{\frac{1-\sigma}{\sigma}}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]} \\ &\quad - \frac{h(u, a-1) - h(u, a+b)}{h(u, a-1)} \frac{h(u, a-1)}{h(u, a+b)} \frac{1+b - [h(u, a-1) - h(u, a+b)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}, \end{aligned}$$

or, after some simplifications, that

$$\begin{aligned} \frac{p}{w} \frac{dw}{dp} &= - \frac{h(u, a-1) - h(u, a+b)}{h(u, a+b)} \frac{1+b - [h(u, a-1) - h(u, a+b)] - \frac{\sigma-1}{\sigma} (1+b)}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]} \\ &= - \frac{h(u, a-1) - h(u, a+b)}{h(u, a+b)}, \end{aligned}$$

which is equation (46) in the text. Assuming that $a-1 > c$, Property 4 (i) in Section A.5 tells us that

$$\begin{aligned} \frac{p}{w} \frac{dw}{dp} &= - \frac{h(u, a-1) - h(u, a+b)}{h(u, a-1)} \frac{h(u, a-1)}{h(u, a+b)} \\ &< - \frac{h(u, c) - h(u, 1+b+c)}{h(u, c)} \frac{h(u, c)}{h(u, 1+b+c)} \\ &= - \frac{\alpha_K}{\alpha_L}, \end{aligned} \tag{74}$$

where the last equality follows from equations (38) and (39) and the transformation $u = e^{-gT}$.

On the other hand, for the neoclassical model we get

$$w = f(k) - kf'(k) \implies \frac{p}{w} \frac{dw}{dp} = -\frac{k^2 f''(k)}{f(k) - kf'(k)} \frac{p}{k} \frac{dk}{dp}.$$

Further, for the neoclassical model,

$$f'(k) = p(r + \delta) \implies \frac{p}{k} \frac{dk}{dp} = \frac{1}{k} \frac{f'(k)}{f''(k)}.$$

Consequently, in the neoclassical model,

$$\frac{p}{w} \frac{dw}{dp} = -\frac{kf'(k)}{f(k) - kf'(k)} = -\frac{\frac{kf'(k)}{f(k)}}{\frac{f(k) - kf'(k)}{f(k)}} = -\frac{\alpha_K}{\alpha_L}.$$

Therefore, $\frac{p}{w} \frac{dw}{dp}$ is smaller (bigger in absolute value) in our model than in the neoclassical model.

A.6.3 Comparative Statics of the price profile

Re-writing (47) in the text

$$\frac{p(v)}{p} = v^b \frac{\frac{1 - (\frac{u}{v})^{a+b}}{a+b} - (\frac{u}{v})^{1+b} \frac{1 - (\frac{u}{v})^{a-1}}{a-1}}{\frac{1 - u^{a+b}}{a+b} - u^{1+b} \frac{1 - u^{a-1}}{a-1}},$$

where $v \equiv e^{-g(t-s)}$ and u is the optimal u . Differentiating, we get

$$\begin{aligned} \frac{d \frac{p(v)}{p}}{du} &= -\frac{p(v)}{p} \frac{(1+b) \left(\frac{u}{v}\right)^{1+b} \frac{1 - (\frac{u}{v})^{a-1}}{a-1}}{\frac{1 - (\frac{u}{v})^{a+b}}{a+b} - \left(\frac{u}{v}\right)^{1+b} \frac{1 - (\frac{u}{v})^{a-1}}{a-1}} \frac{1}{u} \\ &+ \frac{p(v)}{p} \frac{(1+b) u^{1+b} \frac{1 - u^{a-1}}{a-1}}{\frac{1 - u^{a+b}}{a+b} - u^{1+b} \frac{1 - u^{a-1}}{a-1}} \frac{1}{u}. \end{aligned}$$

Or, equivalently

$$\frac{u}{\frac{p(v)}{p}} \frac{d\frac{p(v)}{p}}{du} = (1+b) \left\{ \frac{u^{1+b} \frac{1-u^{a-1}}{a-1}}{\frac{1-u^{a+b}}{a+b} - u^{1+b} \frac{1-u^{a-1}}{a-1}} - \frac{\left(\frac{u}{v}\right)^{1+b} \frac{1-\left(\frac{u}{v}\right)^{a-1}}{a-1}}{\frac{1-\left(\frac{u}{v}\right)^{a+b}}{a+b} - \left(\frac{u}{v}\right)^{1+b} \frac{1-\left(\frac{u}{v}\right)^{a-1}}{a-1}} \right\},$$

or

$$\begin{aligned} \frac{u}{\frac{p(v)}{p}} \frac{d\frac{p(v)}{p}}{du} &= (1+b) \left\{ \frac{h(u, a-1)^{-1}}{h(u, a+b)^{-1} - h(u, a-1)^{-1}} - \frac{h\left(\frac{u}{v}, a-1\right)^{-1}}{h\left(\frac{u}{v}, a+b\right)^{-1} - h\left(\frac{u}{v}, a-1\right)^{-1}} \right\} \\ &= (1+b) \left\{ \frac{h(u, a+b)}{h(u, a-1) - h(u, a+b)} - \frac{h\left(\frac{u}{v}, a+b\right)}{h\left(\frac{u}{v}, a-1\right) - h\left(\frac{u}{v}, a+b\right)} \right\}. \end{aligned}$$

After further calculations, we get

$$\frac{u}{p(v)/p} \frac{dp(v)/p}{du} = (1+b) \frac{\frac{h(u, a+b)}{h(u, a-1)} - \frac{h\left(\frac{u}{v}, a+b\right)}{h\left(\frac{u}{v}, a-1\right)}}{\frac{h(u, a-1) - h(u, a+b)}{h(u, a-1)} \frac{h\left(\frac{u}{v}, a-1\right) - h\left(\frac{u}{v}, a+b\right)}{h\left(\frac{u}{v}, a-1\right)}},$$

which is equation (48) in the text. This last term is negative by Property 4 (iii) in Section A.5.

A.6.4 Comparative statics of the wage-income covariance

Dividing (45) by (46) we get

$$\frac{w}{y} \frac{dy(p)}{dw} = \frac{1+b - [h(u, a-1) - h(u, a+b)] - \frac{\sigma-1}{\sigma} [h(u, c) - h(u, 1+b+c)]}{\frac{1+b}{\sigma} - [h(u, a-1) - h(u, a+b)]}, \quad (75)$$

which simplifies to equation (51) in the text (see calculations below). To compare the above with the corresponding term in the neoclassical model note that in that model

$$w = f(k) - kf'(k) \implies \frac{k}{w} \frac{dw}{dk} = -\frac{k^2 f''(k)}{f(k) - kf'(k)} = -\frac{k}{1 - \alpha(k)} \frac{kf''(k)}{f(k)}.$$

Or

$$\frac{k}{w} \frac{dw}{dk} = -\frac{k}{f(k)} \frac{k f''(k)}{1 - \alpha(k)} = \frac{k}{f(k)} \frac{f'(k)}{\sigma(k)} = \frac{\alpha(k)}{\sigma}.$$

Also, for the neoclassical model

$$y = f(k) \implies \frac{k}{y} \frac{dy}{dk} = \alpha(k).$$

Consequently, we have for the neoclassical model that

$$\frac{k}{y} \frac{dy}{dk} = \sigma \frac{k}{w} \frac{dw}{dk} \implies \frac{w}{y} \frac{dy}{dw} = \sigma.$$

Manipulating (75) we get

$$\begin{aligned} \frac{w}{y} \frac{dy(p)}{dw} &= \sigma \frac{1 + b - [h(u, a - 1) - h(u, a + b)] - \frac{\sigma - 1}{\sigma} [h(u, c) - h(u, 1 + b + c)]}{1 + b - \sigma [h(u, a - 1) - h(u, a + b)]} \\ &= \sigma \frac{1 + b - \sigma [h(u, a - 1) - h(u, a + b)]}{1 + b - \sigma [h(u, a - 1) - h(u, a + b)]} \\ &\quad + \sigma \frac{(\sigma - 1) [h(u, a - 1) - h(u, a + b)] + \frac{1 - \sigma}{\sigma} [h(u, c) - h(u, 1 + b + c)]}{1 + b - \sigma [h(u, a - 1) - h(u, a + b)]} \\ &= \sigma + \sigma \frac{1 - \sigma h(u, c) - h(u, 1 + b + c) - \sigma [h(u, a - 1) - h(u, a + b)]}{\sigma [1 + b - \sigma [h(u, a - 1) - h(u, a + b)]]} \end{aligned}$$

Now by Property 4 (ii) in Section A.5, if $\sigma < 1$

$$\frac{w}{y} \frac{dy(p)}{dw} = \sigma + (1 - \sigma) \frac{h(u, c) - h(u, 1 + b + c) - \sigma [h(u, a - 1) - h(u, a + b)]}{1 + b - \sigma [h(u, a - 1) - h(u, a + b)]} > \sigma.$$

If we divide numerator and denominator of the second term on the right hand side by σ we get equation (51).

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