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ABSTRACT

In this paper we consider strictly convex monotone continuous complete preorderings on $\mathbb{R}^n_+$ that are locally representable by a concave utility function. By Alexandroff's (1939) theorem, this function is twice differentiable almost everywhere.

We show that if the bordered hessian determinant of a concave utility representation vanishes on a null set, then demand is countably rectifiable, that is, except for a null set of bundles, it is a countable union of C^1 manifolds. This property of consumer demand is enough to guarantee that the equilibrium prices of a pure exchange economy will be locally unique, for almost every endowment. We give an example of an economy satisfying these conditions but not the Katzner (1968) - Debreu (1970, 1972) smoothness conditions.

1. Introduction

In the past twenty years economic theorists have devoted a considerable effort to the issue of local uniqueness of equilibrium prices. In a seminal paper Debreu (1970) showed that a pure exchange economy with m consumers whose demand functions are smooth is such that, for almost every endowment, equilibrium prices are locally unique. If, in addition, the norm of the demand functions goes to $+\infty$ as the boundary of the price simplex is approached, then the set of equilibrium prices associated with each endowment is compact, and therefore, local uniqueness implies finiteness.

Smoothness of demand is a very strong condition. Katzner (1968) considered strictly quasiconcave twice-continuously differentiable utility functions, for which the marginal utilities are always strictly positive and the indifference hypersurfaces do not intersect the boundary of the positive orthant. In this setting, he showed that demand will be smooth at a value x iff the bordered hessian of the utility function does not vanish at x. Under Katzner's assumptions, this condition is satisfied on an open dense subset of consumption bundles and on a corresponding open dense subset of full measure of prices and incomes. Katzner's condition is equivalent to saying that the indifference hypersurface has nonvanishing Gaussian curvature at x. A more general result by Debreu (1972) established that for preferences of class C^2, demand is smooth at x iff the indifference hypersurface has nonzero Gaussian curvature at x (see Debreu (1976) also, where it is shown that smoothness of demand does not require the existence of a C^2 utility representation with no critical point).

Differentiability almost everywhere is therefore the most that can be expected under very general assumptions. However this is an interesting

property only if one can use it to infer the nature of demand from observed reactions to price changes. Therefore we need to guarantee that most quantity variations do not occur on a negligible set of prices. This argument motivated Rader (1973) to introduce the condition that demand should map null sets of prices and incomes into null sets of quantities - condition (N). He established that differentiability almost everywhere together with condition (N) guarantee that, for almost all endowments, equilibrium is locally unique (although the set of endowments having infinite equilibria might be dense).

Rader (1973) showed also that demand is differentiable almost everywhere in prices if the utility function is concave and demand is continuous and locally Lipschitzian in income. He also established that if demand is uniformly Lipschitzian income and the incremental ratio of prices to quantity is bounded from below, then condition (N) is satisfied. In a later paper, Radner (1979) proved that demand is differentiable almost everywhere if the utility function is twice differentiable and condition (N) holds if utility is analytic (derivatives of all orders exist and are locally subject to a common bound).

Kleinberg (1980) studied the issue of generic finiteness of equilibria using the weaker concept of approximate differentiability of demand. He considered strictly quasi-concave utility functions of class C^1 for which ∇u is approximately differentiable. In addition, he imposed some continuity and boundedness conditions on the approximate derivative of ∇u to guarantee that the demand function is approximately differentiable on the complement of an at most countable set of prices and incomes. Kleinberg showed that for this class of demand functions equilibria is locally unique for almost every endowment.

The purpose of this paper is to examine what can be the contribution of concavifiability of preferences to the issue of local uniqueness of equilibrium prices. An assumption on local concavifiability is not too strong and is already implied by Katzner's (1968) conditions. In fact, as Mas-Colell (1985) showed (Proposition 2.6.4), if a strictly convex C^2 preference has indifference hypersurfaces with nonvanishing Gaussian curvature and can be representable by a C^2 utility function with no critical point, then for any compact convex set K there is a C^2 utility representation with no critical point where the restriction to K is differentiability strictly concave (i.e., has a negative definite hessian matrix on K).

We consider monotone continuous complete preorderings that are locally representable by a concave utility function u, which might not be differentiable throughout the interior of the effective domain. However, it is well known that the function u is differentiable almost everywhere on the interior of its effective domain and that the partial functions admit everywhere one-sided derivatives. Much is also known about the second derivative of a concave function. Alexandroff (1939) showed that a concave function admits almost everywhere a 2^{nd} order expansion. Moreover, he introduced extended partial derivatives which lie within the one-sided derivatives of the partial functions and showed that these extensions are almost everywhere differentiable, with remainder converging uniformly on the choice of the extensions and with derivatives given by the matrix of the quadratic form in the 2^{nd} order expansion. This result was our main motivation to write this paper.

If, in addition to local concavifiability we assume that the bordered hessian matrix of the local utility representation is nonsingular almost

everywhere, then, we can claim that, from the point of view of geometric measure theory, the range of the demand function behaves like a C^1 manifold. In fact, except for a null set of bundles, demand is a countable union of C^1 submanifolds. In the terminology of geometric measure theory, the demand range is countably rectifiable. We show that this property of demand is enough to guarantee that, in a pure exchange economy, equilibrium prices are locally unique, for almost every endowment.

We give an example of a 2×2 pure exchange transferable utility economy satisfying our assumption, but not the Katzner-Debreu smoothness conditions, and for which the set of equilibrium prices is finite, for almost every endowment.

It is interesting to notice that the property of rectifiability of demand that we have established is weaker than Rader's (1973) conditions for generic local uniqueness. In fact, under rectifiability the condition that demand map null sets into null sets is satisfied but demand is not differentiable in almost every price and income. The comparison of our results with Kleinberg's (1980) is more subtle, since he requires the demand function to be only approximately differentiable except on a countable set and his assumptions on preferences do not involve local concavifiability. However, Kleinberg (1980) assumes differentiability of the utility function, which is an assumption that we have dispensed with.

2. RECTIFIABILITY OF DEMAND

2.1 Preliminarities

The consumption set is $\mathbb{R}^n_+$ and a preference relation R is a complete preordering on $\mathbb{R}^n_+$. Let $x \sim y$ if $x\,R\,y$ and $y\,R\,x$ and let $x\,P\,y$ if

$x \, R \, y$ but not $y \, R \, x$. We say that a preference relation R is (i) continuous if it graph is closed on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ (ii) strictly convex if $x \, R \, y$ implies $\lambda x + (1-\lambda) y \, P \, y$, for any $0 < \lambda \leq 1$ and (iii) monotone if $x >> y$ implies $x \, P \, y$. A function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is a utility representation for R when $x \, R \, y$ if and only if $u(x) \geq u(y)$.

Given a strictly convex preference relation R the demand function for R is a function d mapping $\text{int}\, \Delta^{m-1} \times \mathbb{R}_{++}$ into $\mathbb{R}^n_+$ (where Δ^{n-1} is the n-1 dimensional simplex), satisfying $p \, d(p,y) = y$ and $p \, z \leq y \Rightarrow d(p,y) \, R \, z$. The demand set is $d(\text{int}\, \Delta^{n-1} \times \mathbb{R}_{++})$.

A preference relation R is locally (strictly) concavifiable if for any compact convex set $K \subseteq \mathbb{R}^n_+$ there is a (strictly) concave representation u_K for R. By Alexandroff's (1939) theorem, this function u_K is twice-differentiable almost everywhere and the hessian matrix, where it exists, is a negative semi-definite matrix. We will next recall the precise statement of this theorem.

2.1.2. Alexandroff's theorem

For technical reasons we want to think of all concave functions as defined throughout $\mathbb{R}^n$ and taking the value $-\infty$ outside of the effective domain. The extended real-valued function obtained this way is called a proper concave function. Denote by dom u the effective domain of the function u and by int(dom u) the respective interior.

A proper concave function u on $\mathbb{R}^n$ is differentiable on a dense subset D of int(dom u) and the complement of D on int(dom u) is a set of measure zero; moreover, u is actually continuously differentiable on D (see Rockafellar (1970) 25.5)

Alexandroff (1939) established that a proper concave function u on $\mathbb{R}^n$ is also twice-differentiable almost everywhere on int(dom u). Since the domain D of the first derivative may have an empty interior, we should be more precise and recall the exact statement of Alexandroff's theorem.

Let $u(x_i;\bar{x}_{-i}) : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be the proper concave function obtained by setting all variables but x_i equal to $\bar{x}_{-i} \in \mathbb{R}^{n-1}$. The right derivative $u'_+(x_i;\bar{x}_{-i})$ and the left derivative $u'_-(x_i;\bar{x}_{-i})$ are well defined throughout the effective domain of this function and $u'_-(x_i;\bar{x}_{-i}) \le u'_+(x_i;\bar{x}_{-i})$ (see Rockefellar (1970) 23). Alexandroff (1939) defined an extended partial derivative u_i of the u as any function satisfying the inequality $u'_-(x_i;\bar{x}_{-i}) \le u_i(x) \le u'_+(x_i;\bar{x}_{-i})$, where $x = (x_i;\bar{x}_{-i})$. Note that $u_i(x)$ coincides with the partial derivative $\frac{\partial u}{\partial x_i}(x)$ when it exists.

<u>Theorem</u> (Alexandroff): the extended partial derivative u_i is differentiable almost everywhere on int(dom u) and at any point x of differentiability, for any direction $y \in \mathbb{R}^n$, we have

$$\left| u_i(x + sy) - u_i(x) - \nabla u_i(x)\cdot sy \right| \le \varepsilon(s)$$

where $\varepsilon(s)/s$ converges to zero uniformly on the directions y and also independently of the choice of the extension u_i. Moreover, the matrix $H(x) = [\nabla u_1(x)\ldots\nabla u_n(x)]$ is uniquely determined, independent of the choice of the extensions and is a symmetric negative semidefinite matrix.

Actually, the matrix $H(x) = [h_{ik}(x)]$ exists only at points where u is once differentiable (see Alexandroff (1939) pp. 5 and 6) but not necessarily

at any such point. Furthermore, for any direction given by a normalized vector y, the quadratic for $\frac{1}{2} y' H(x) y$ coincides with the directional second-derivative given by $\lim_{s \to 0} (u(x + sy) - u(x) - \sum_{i=1}^{n} u_i(x) sy_i)/s^2$. By a result due to Iessen (1929), in the case of a concave function, a directional second-derivative is a usual second-derivative, that is, $u(x+sy)-u(x)- \sum_{i=1}^{n} u_i(x)sy_i - \frac{s^2}{2} \sum_{i=1}^{n} \sum_{K=1}^{n} h_{iK}(x)\, y_i y_k = \varepsilon s^2$ where $\varepsilon \to 0$ as $s \to 0$, uniformly on all directions y. Then, any concave function admits at almost every point x a second-order Taylor expansion and the matrix of the quadratic form is the matrix $H(x)$.

Busemann and Feller (1935) had shown that a concave function in $\mathbb{R}^2$ admits directional second-derivatives almost everywhere in its effective domain. Alexandroff (1939) extended their result to an arbitrary finite dimension (in part 2 of his paper) and established the relation between the directional second-derivatives and the derivatives of the extended partial derivatives (in part 3 and 4 of his paper).

2.1.3. Rectifiability

Recall that a function $f : A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is said to be locally Lipschitzian if for any $a \in A$ there exist an open ball $B_\varepsilon(a)$ and a constant M such that $\|f(x) - f(y)\| \leq M \|x-y\|$, for any $x,y \in B_\varepsilon(a)$.

Denote by L^n the Lebesgue measure in $\mathbb{R}^n$. We say that a set $E \subseteq \mathbb{R}^n$ is M-rectifiable if $L^n(E) < \infty$ and L^n - almost all of E is contained in the union of the images of countably many locally Lipschitzian functions from $\mathbb{R}^m$ to $\mathbb{R}^n$. A set $E \subseteq \mathbb{R}^n$ is said to be countably n-rectifiable if L^n - almost all of E is contained in the union of countably many rectifiable sets. As we will see next, from the point of view of geometric measure

theory, rectifiable sets behave like C^1 manifolds.

Geometric measure theory can be described as differential geometry generalized through measure theory to deal with maps and surfaces that are not smooth.

Denote by L^n the Lebesgue measure on $\mathbb{R}^n$ and by $\bar{B}_\delta(a)$ the closed ball around $a \in \mathbb{R}^n$ with radius δ. Let A be a measurable subset of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$; the set A is said to have density zero at a if $\forall_{\varepsilon>0} \exists_{\bar{\delta}>0}$ such that $L^n(B_\delta(a) \cap A) < \varepsilon\, L^n(B_\delta(a))$ for any $\delta < \bar{\delta}$.

Consider an extended real valued measurable function f defined on a measurable set $A \subseteq \mathbb{R}^n$. The <u>approximate lim sup</u> of f at $a \in \mathbb{R}^n$ is defined as

$$\operatorname{ap}\limsup_{x \to a} f(x) = \inf B, \quad \text{where}$$

$$B = \{t \in \mathbb{R}: \{x \in A: f(x) > t\} \text{ has density zero at } a\}$$

From the definition, if the set A has density zero at a then $\operatorname{ap}\limsup_{x \to a} f(x) = -\infty$. It is immediate that if lim sup exists then it is equal to the approximate lim sup.

Similarly, $\operatorname{ap}\liminf f(x) = \sup \{t \in \mathbb{R}: \{x \in A: f(x) < t\}$ has density zero at $a\}$.

Recall that a function f from $A \subseteq \mathbb{R}^n$ is said to be pointwise Lipschitzian at $a \in A$ if $\limsup_{x \to a} \|f(x)-f(a)\|/\|x-a\| < \infty$. Similarly, a function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be <u>approximately pointwise Lipschitzian</u> at $a \in A$ if $\operatorname{ap}\limsup_{x \to a} \|f(x)-f(a)\|x-a\| < \infty$.

The following lemmas summarize the most important facts about approximately Lipschizian functions.

Lemma 1 (Federer): if a function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is approximately pointwise Lipschitzian on A then A is the countable union of measurable sets such that the restriction of f to each set is Lipschitzian and moreover f is approximately differentiable a.e. on A: that is, there exists a linear map $L: \mathbb{R}^n \to \mathbb{R}^m$ with

$$\operatorname*{ap\,lim}_{x \to a} \|f(x) - f(a) - L(x-a)\|/\|x-a\| = 0$$

(for a proof see Federer (1969) 3.1.8).

Remark

Then, the range of a function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ which is approximately pointwise Lipschitzian is a countably rectifiable set. In fact, one can take a countable cover of $f(A)$ by bounded sets and each one is a rectifiable set, since the Lipschitzian restriction of f can be extended to the whole $\mathbb{R}^n$ preserving the Lipschitzian constant (see Federer (1969)). Moreover, if $g: B \subseteq \mathbb{R}^n \to \mathbb{R}^n$ admits an approximately pointwise Lipschitzian restriction $f: A \subseteq B \to \mathbb{R}^n$ such that $f(A)$ has full measure in $g(B)$, then $g(B)$ is also countably rectifiable through Lipschitzian restrictions of f.

Lemma 2: a function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ that is approximately pointwise Lipschitzian on A maps null sets into null sets, provided $n \leq m$.

In fact, by lemma 1, $A = \bigcup_{K=1}^{\infty} C_K$ such that $f|_{C_K}$ is Lipschitzian and if $B \subseteq A$ is null set, then $f(B \cap C_K)$ is also null and $f(B) = \bigcup_{K=1}^{\infty} f(B \cap C_K)$.

Now we will establish a result on the rectifiability of the inverse of a differentiable function.

<u>Proposition 1</u>: let f and g be functions defined on an open subsets of $\mathbb{R}^n$ and taking values in $\mathbb{R}^n$; suppose that the restriction of f to a measurable subset D admits the restriction of g to f(D) as an inverse function. Let a be a limit point of D. If f is differentiable at a and f'(a) is nonsingular, then

$$\operatorname*{ap\,lim\,sup}_{\substack{s \to 0 \\ g|f(D)}} \frac{\|g(b+s)-g(b)\|}{\|s\|} \le \|(f'(a))^{-1}\|$$

Moreover, if D is a full measure subset of dom f then the above inequality holds as an equality.

<u>Proof</u>: Inverting the incremental ratio we have,

$$\operatorname*{ap\,lim\,sup}_{s \to 0} \|g(b+s) - g(b)\|/\|s\| =$$

$$= \inf \{t \in \mathbb{R}: \{x \in (D-a): \frac{\|f(a+x)-f(a)\|}{\|x\|} < \frac{1}{t}\} \text{ has density zero at } a\}$$

$$= 1/\sup \{t \in \mathbb{R}: \{x \in (D-a): \frac{\|f(a+x)-f(a)\|}{\|x\|} < t\} \text{ has density zero at } a\}$$

$$= 1/\operatorname*{ap\,lim\,inf}_{\substack{x \to 0 \\ f|D}} \frac{\|f(a+x)-f(a)\|}{\|x\|} \le 1/\liminf_{x \to 0} \frac{\|f(a+x)-f(a)\|}{\|x\|}$$

when the last inequality follows from the following set inclusion:

$$\{t \in \mathbb{R}: \{x \in (D-a): \frac{\|f(a+x)-f(a)\|}{\|x\|} < t\} \text{ has density zero at } a\} \supseteq$$

$$\{t \in \mathbb{R}: \{x \in (\text{dom } f - a): \frac{\|f(a+x) - f(a)\|}{\|x\|} < t\} \text{ has density zero at } a\}$$

If D has full measure in dom f then these two sets are equal and the above inequality holds as an equality.

To finish the proof recall that for a mapping h with nonsingular derivative Dh at a point a, the lim inf of the incremental ratio of h at a is equal to $\|(Dh(a))^{-1}\|^{-1}$ (see Federer (1969) p. 209).

Finally, we present the results that allow us to think of a rectifiable set almost as a countable union of C^1 manifolds. The Lipschitzian functions in the definition of a rectifiable set can be replaced by C^1 functions due to the following lemma.

Lemma 3: (Federer): a function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ that is a.e. in A approximately pointwise Lipschitzian is such that for $\varepsilon > 0$ there exists a C^1 map $g: \mathbb{R}^n \to \mathbb{R}^m$ identical to f on a subset of A of measure greater than $1-\varepsilon$ (for a proof see Federer (1969) 3.1.16).

Using this lemma, Federer established the following characterization of rectifiability.

Lemma 4: in the definition of a rectifiable set E one can take the Lipschitzian functions to be C^1 diffeomorphisms on compact domains with disjoint images whose union coincides with E almost everywhere.
(see Federer (1969) 3.2.18 and 3.2.29 and Morgan (1988) 3.11)

2.2. The result

2.2.1.

Theorem 1: if a continuous, monotone, strictly convex preference relation R on $\mathbb{R}^n_+$ is locally concavifiable and for each representation u_K the bordered hessian matrix is singular only in a set of measure zero, then the demand function d is approximately pointwise Lipschitzian for almost every value. That is, $\operatorname*{ap\,lim\,sup}_{(p,y)\to(p^0,y^0)} d(p,y) < \infty$ for almost every $x^0 = d(p^0,y^0) \in d(\text{int}\,\Delta^{n-1} \times \mathbb{R}_{++})$.

In other words, by lemma 1 and the remark in 2.1.3 demand is countably rectifiable and, by lemma 4 in 2.1.3, except for a null set of bundles, it is the union of countably many pairwise disjoint compact C^1 submanifolds.

2.2.2 Proof of the Theorem

Consider a countable cover of $\mathbb{R}^n_+$ by closed cubes. For each closed cube $K \subseteq \mathbb{R}^n_+$, R is representable by a concave utility function u with no critical point, by monotonicity of R. For simplicity, we assume that u is a proper concave function with effective domain K. We assume that K has nonempty interior, since otherwise K would be irrelevant for the purpose of proving that demand is approximately pointwise Lipschitzian for almost every value. For the same reason, we will consider only the optimal interior solutions in K, since the boundary of K is a n-null set.

Let N be the null subset of int K when u is not twice-differentiable. Let $\delta(x)$ be the bordered hessian determinant of u at $x \in \text{int}\,K\backslash N$, given by

$$\delta(x) = \begin{vmatrix} u_{11}(x) & \dots & u_{1n}(x) & u_1(x) \\ u_{1n}(x) & \dots & u_{nn}(x) & u_n(x) \\ u_1(x) & \dots & u_n(x) & 0 \end{vmatrix}$$

Let CP be the null subset of int K\N where the bordered hessian determinant δ of u vanishes. Let RP be the complement of CP on int K\N.

Recall Debreu's (1972) decomposition of the demand function and adopt it to the case where dom Du is just a full measure subset of int K. Let $g: x \to \nabla u(x)/\|\nabla u(x)\|$ and $\hat{g}(x)$ be the vector of the first n-1 components of g(x). The restriction $\tilde{d}$ of the demand function d to d^{-1}(dom Du) is the inverse of the function $f: x \to (p,y) = (\hat{g}(x), g(x)\cdot x)$. This function f is the composition of $\alpha: x \to (g(u), u(x))$ and $\beta: (\bar{p},v) \to (\bar{p},\bar{m}(v))$ when $p \in \text{int } \Delta^{n-1}$ and $m(v) = \min_{u(z) \geq v} p.z.$

Let $\tilde{g}$, $\tilde{\alpha}$ and $\tilde{f}$ be any extensions of g, α and f, respectively obtained using Alexamdroff's extended partial derivatives.

The Jacobian determinant of $\tilde{\alpha}$ at $x \in$ int K\N is $J\,\tilde{\alpha}(x) = -\delta(x)\, g^n(x)/\|\nabla u(x)\|^n$, where $g^n(x)$ is the n^{th} component of g(x) (that is, $g^n(x) = u_n(x)/\|Du(x)\|$). Then $J(\alpha)(x) \neq 0$ on RP. The function β has the same Jacobian determinant as m, which is the inverse of the function v given by $s \to \max_{pz \leq s} u\,(z)$. Now $\nabla v(s) = \|\nabla u(x)\|$, for $x = d(p,s)$. Therefore $J\,\tilde{f}(x) = J(\beta o \tilde{\alpha}) = -\,\delta(x)u_n(x)/\|\nabla u(x)\|^{n+2}$ and $J\,f(x) \neq 0$ if and only if $x \in$ RP.

Now, by Proposition 1 in 2.1.3 above, for $x^0 \in RP$, let $(p^0,y^0) = f(x^0)$ and we have $\underset{(p,y)\to(p^0,y^0)}{\text{ap lim sup}}\ \tilde{d}(p,y) < \infty$, where $\tilde{d} = d|d^{-1}(\text{dom } Du)$. Since dom Du has full measure in int K we also have $\underset{(p,y)\to(p^0,y^0)}{\text{ap lim sup}}\ d(p,y) < \infty$, for $(p^0,y^0) = f(x^0)$ and $x^0 \in RP$. We have proven that the demand function is approximately pointwise Lipschitzian on the inverse image of RP and RP is a full measure subset of int K. To complete the proof, notice that the countable union of the null complements of each set RP in the respective cube K is a null subset of $\mathbb{R}^n_+$. Q.E.D.

Notice that the result still holds for a strictly quasi concave function u which is differentiable almost everywhere and such that the derivative ∇u admits an extension f to the whole domain of u so that (i) f is itself differentiable almost everywhere, (ii) f' exists whenever ∇u exists and (iii) f' is nonsingular almost everywhere. This observation is interesting in the case when u admits almost-everywhere a second-order expansion and the matrix in this quadratic form coincides with f', almost verywhere (as it is the case with a concave function).

3. Local Uniqueness

Consider a exchange economy with m consumers and n goods. Let d_j: int $\Delta^{n-1} \times \mathbb{R}_{++} \to \mathbb{R}^n_+$ be the demand function and $w_j \in \mathbb{R}^n_{++}$ be the endowment of the j^{th} consumer. We keep the demand functions fixed and parameterize an economy by an endowment vector $w = (w_1,\ldots,w_m)$. Given $w \in \mathbb{R}^{nm}_{++}$, an element $p \in \text{int } \Delta^{n-1}$ is an equilibrium price vector of the economy w if $\sum_{j=1}^{m} d_j(p,p\cdot w_j) = \sum_{j=1}^{m} w_j$. Let E(w) be the set of equilibrium price vectors of the economy w. We say that equilibrium prices are locally unique

if all elements in $E(w)$ are isolated points of this set.

<u>Theorem 2</u>: If the demand functions $d_1,\ldots,d_m$ of all consumers are approximately pointwise Lipschitzian for almost every demand bundle, then the equilibrium prices are locally unique, for almost every endowment vector.

<u>Proof</u>: Let $U = \operatorname{int} \Delta^{n-1} \times \mathbb{R}_{++} \times \mathbb{R}_{++}^{n(m-1)}$ and define the function $F\colon U \to \mathbb{R}^{nm}$ as in Debreu (1970) by

$$e = (p, y_1, w_2, \ldots, w_m) \to F(e) = (F_1(e), \ldots, F_m(e))$$

where $F_1(e) = d_1(p,y_1) + \sum_{i=2}^{m} d_i\ (p, p\cdot w) - \sum_{i=2}^{m} w_i$

$$F_j(e) = w_j \quad \text{for } j \neq i.$$

For $e \in u$, $p\cdot F_1(e) = y_1$. Also, p is an equilibrium price vector iff $(p,y_1,w_2,\ldots,w_m) \in F^{-1}(w)$. We want to show that for a.e. ω, every e in $F^{-1}(\omega)$ is locally isolated. Let us start by claiming that the range of F is a $m\cdot n$-countably rectifiable set (through Lipschitzian restrictions of F)

Let M_j be the full measure subset of $\mathbb{R}_+^n$ which is the countable union $\bigcup_{K=1}^{\infty} M_{jk}$ of disjoints images of C^1 diffeomorphic restrictions of d_j. Denote by N_j its null complement in $\mathbb{R}_+^n$.

Define $\tilde{N}_1 = \{w \in \mathbb{R}_+^{nm}\colon w = F(e),\ e = (p,y_1,w_2,\ldots,w_m)$ and $d_1(p,y_1) \in N_1\}$

$\tilde{N}_j = \{w \in \mathbb{R}_+^{nm} : w = F(e),\ e = (p,y_1,w_2,\ldots,w_m)$ and $d_j(p,p\cdot wj) \in N_j\}$, $j \neq 1$.

We want to show that $\bigcup_{j=1}^{m} \tilde{N}_j$ is a null set. Now $\tilde{N}_1 = \mathbb{R}_+^n \backslash \tilde{M}_1$ and $\tilde{N}_j = \mathbb{R}_+^n \backslash \tilde{M}_j$ where

$$\tilde{M}_1 = \{w \in \mathbb{R}_+^{nm}: w \in F(e),\ e = (p, y_1, w_2, \ldots, w_m) \text{ and } d_1(p, y_1) \in M_1\}$$

$$\tilde{M}_j = \{w \in \mathbb{R}_+^{nm}: w \in F(e),\ e = (p, y_1, w_2, \ldots, w_m) \text{ and } d_j(p, p \cdot w_j) \in M_j\}\ ,\ j \neq 1$$

We need to establish that $\tilde{M}_j (j = 1, \ldots, m)$ is of full measure in $\mathbb{R}_+^{nm}$. Now $w \in \tilde{M}_1$ iff $d_1(p, y_1) \in M_1$ for any $(p, y_1, w_2, \ldots, w_m) \in F^{-1}(w)$, that is, $(p, y_1) \in d_1^{-1}(M_1) \cap \gamma_1(F^{-1}(w)) = B_1$ where γ_j is the j^{th} projection mopping (applied in this case to $\underset{j=1}{\overset{m}{X}} \mathbb{R}^n$). Similarly, $w \in \tilde{M}_j (j \neq 1)$ iff $(p, p \cdot w_j) \in d_j^{-1} (M_j) \cap \gamma_j(F^{-1}(w)) = B_j$.

Let $\tilde{B}_j^i = \{x \in \mathbb{R}^n : x = (p, p \cdot w_i),\ p \in j_1(B_j),\ w_i \in \mathbb{R}_+^n\}$

and $\tilde{B}_j^1 = \{x \in \mathbb{R}^n: x = (p, y_1),\ p \in j_1 (B_j),\ y_1 \in \mathbb{R}_+\}$.

Then $\tilde{M} = \underset{i=1}{\overset{m}{X}} A_i^j$ where $A_1^j = M_j + \sum_{i \neq j} d_i(\tilde{B}_j^i) - \sum_{i \neq 1} A_i^j$ and $A_i^j = \mathbb{R}_+^n$, for $i \neq 1$.

By the Brunn-Minkowski theorem (see Federer (1969), 3.2.4)), for any nonempty subsets A and B of $\mathbb{R}^n$ (not necessarily of finite measure), we have $L^n(A+B)^{1/n} \geq L^n(A)^{1/n} + L^n(B)^{1/n}$. Then $L^n(A_1^j)^{1/n} \geq L^n(M_j)^{1/n} + L^n(\sum_{i \neq j} d_i(\tilde{B}_j^i) - \sum_{i \neq 1} A_i^j)^{1/n}$, where, on the right-hand-side, at least the first term is $+\infty$. So $\tilde{M}_j$ is of full measure in $\mathbb{R}_+^{nm}$ and therefore the range of F is $m \cdot n$ - countably rectifiable through F.

To complete the proof of the theorem, notice that the set $F(U)$ can be regarded as the union of countably many disjoint $m \cdot n$-rectifiable sets R_K (by taking appropriate set differences). Now, by lemma 5 in 2.1.3 (from Federer (1969)), one can take the countably many Lipschitzian functions in the definition of each rectifiable set R_K to be C^1 diffeomorphisms D_{iK} on

compact domains with disjoint images, whose union coincides with the rectifiable set almost everywhere. Let $N = F(\cup) \setminus \bigcup_{K=1}^{\infty} \bigcup_{i=1}^{m}$ range D_{ik}, which is a $m \cdot n$-null set.

Now, any endowment vector on $F(\cup) \setminus N$ is such that $E(w)$ has only isolated points. In fact, by the C^1-inverse function theorem, any element $w \in F(\cup) \setminus N$ and any element $e \in F^{-1}(w)$ have neighborhoods 0_w and 0_e, respectively, that are homeomorphic under the restriction to 0_e of some C^1 map D_{iK}. Moreover, for each $w \in F(\cup) \setminus N$, this map D_{iK} is uniquely determined, because the rectifiable sets are disjoint and the D_{iK} maps have disjoint images. Then, for any $w \in F(\cup) \setminus N$, any price vector p in $E(w)$ has a neighborhood 0_p where there are no other elements of $E(w)$ (this neighborhood 0_p is induced by 0_e through the one-to-one correspondence between $E(w)$ and $F^{-1}(w)$).

Q.E.D.

4. AN EXAMPLE

Here, we give an example of a pure exchange economy where utility functions are concave, but not differentiable, and we still have finiteness of equilibria, for almost every endowment. In this example the set of endowments generating infinite equilibria is dense.

Consider an economy with two consumers with the same concave utility function on $\mathbb{R}^2_+$, which is constructed so that an indifference curve is differentiable only at irrational points.

For each $q_n \in Q$, let f_n be the characteristic function of the set $\{y \in \mathbb{R}_+ : y \geq q_n\}$. Notice that f_n is nondecreasing and has a jump at q_n. Let $\sum_{n=1}^{\infty} 1/2^n f_{q_n}$; now f is nondecreasing, at any $q_n \in Q$ it has a jump of

magnitude $1/2^n$ and it is differentiable except at the rationals, with zero derivative.

Now integrate f to obtain a convex increasing function. Since f has only a countable set of discontinuities it is Riemann integrable. Let g be such that.

$$g(x) = {}^{x}\!\int_0 f(t)dt$$

The derivative g' exists at every continuity point of f, that is, at any irrational x. Furthermore, at any rational q_n we have $g'(q_n) = f(q_n^-)$ and $g'_+(q_n) = f(q_n^+) = f(q_n)$. Notice that g is strictly convex:

$$g(\alpha x + (1-\alpha)y) \leq \int_0^{\alpha x} f(t)dt + \int_0^{(1-\alpha)y} f(t)dt,$$

since f is nondecreasing and for $0 < \alpha < 1$ we have

$$\int_0^{\alpha x} f(t)dt < \alpha\int_0^{x} f(t)dt$$

Now let $h: \mathbb{R}_{++} \to \mathbb{R}_{++}$ be defined by $h(x) = g(1/x)$. The function h is decreasing, convex, differentiable only at irrational points; it will be used as an indifference curve. Let the utility function be $u: \mathbb{R}^2_{++} \to \mathbb{R}$ given by $u(x_1,x_2) = x_2 - h(x_1)$.

The preferences in this example satisfy the assumptions of theorems 1 and 2 but not the smoothness conditions of Katzner (1968) or Debreu (1972)-(1976). In fact the bordered hessian determinant of u, at $(x_1,x_2) \in (\mathbb{R}\backslash Q) \times \mathbb{R}$, is

$$\begin{vmatrix} -h''(x_1) & 0 & -h'(x_1) \\ 0 & 0 & 1 \\ -h'(x_1) & 1 & 0 \end{vmatrix} = h''(x_1)$$

and $h''(x_1) = \frac{1}{x_1^2} g''(^1/x_1) - \frac{2}{x_1^3} g'(^1/x_1)$. Now g'' is the derivative of the Alexandroff's extended partial derivative $g'_+ = f$ and therefore g'' vanishes identically on its domain $\mathbb{R}\backslash Q$. Then $h''(x_1) = -\frac{2}{x_1^3} f(^1/x_1) > 0$, for any $x_1 \in \mathbb{R}\backslash Q$ and the bordered hessian determinant is nonzero on the full measure set $(\mathbb{R}\backslash Q) \times \mathbb{R}$.

Let us examine the demand functions and the equilibria of this economy. If $(x_1, x_2) \in (\mathbb{R}_{++}\backslash Q) \times \mathbb{R}_{++}$ then $-h'(x_1) = p_1/p_2$, implying $x_1 = (h')^{-1}(-p_1/p_2)$ except at prices associated with points where h is not differentiable and these prices are elements of the subdifferentials

$\partial h(q_n) = \{p_1/p_2 \in \mathbb{R}_{++}$:

$-h'_+(q_n) \leq p_1/p_2 \leq -h'_-(q_n)\}$ and generate demand $x_2 = \frac{y - p_1 q_n}{p_2}$. We have determined completely the form of the demand function.

It is easy to see that when the endowment vector (w_1, w_2) is such that w_1 is irrational, then the equilibrium price ratio is equal to $-h'(w_{1/2})$. For w_1 rational, the set of equilibrium price ratios is the interval $[-h'_+(w_{1/2}), -h'_-(w_{1/2})]$.

That is, almost every endowment generates finite equilibrium prices but the set of endowments giving rise to infinite equilibrium prices is dense.

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