Pricing Options Embedded in Debentures with Credit Risk*

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Abstract

In this article, we develop a strategy to simultaneously extract a yield curve and price call options embedded in debentures subject to credit risk. The implementation is based on a combination of two methods: term structure estimation adopting the Nelson-Siegel model sequentially followed by the use of the spread-curve (term structure of debentures minus local inter-bank risk-free rate) to calibrate a trinomial tree for short-term interest rates making use of the Hull and White model (1993). The proposed methodology allows us to price embedded options making debentures with and without embedded options comparable on a common basis. As a consequence, since a large number of the existing Brazilian debentures contain embedded options, our methodology increases the number of debentures available to estimate a term structure for Brazilian local fixed income bonds. We illustrate the method by pricing a call option for a debenture issued by the company "Telefonica Brasil".

Keywords: Embedded options, Term structure of interest rates, Debentures, Hull & White model.

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1. Introduction

Debentures are traditional instruments used worldwide by companies as an efficient mechanism for fundraising, offering numerous opportunities for financial engineering. In particular, the debentures in the Brazilian market acquired peculiar characteristics, usually assuming creative and flexible roles with the goal of serving its issuer in terms of financial management techniques.

In recent years, the Brazilian debenture market showed a rapid growth, both in terms of emissions as well as in terms of the amounts of funds raised by them. Its development proved even more vigorous at the time that the government reduced its role as an economy lead investor encouraging the private sector to carry out these duties. However, despite the fact that the primary market of debentures proved to be quite heated, the secondary market still lacks development, bearing in mind its low turnover.

The appearance of a large number of different types of debentures available in the market created a demand, from both market participants and regulatory and self-regulatory agencies, for the design of a model to extract a specific yield curve for the debenture market. This curve is expected to give greater transparency to the market providing various benefits to it, among them, a reliable price discovery, bid-ask spreads reductions, an increase in market liquidity, and a larger access of companies to the capital markets.

It is important to note that in order to extract such a curve, there is a direct need to find the prices of these assets. At this point, one of the important issues that appear when pricing securities with credit risk and illiquidity, which is precisely the case of Brazilian debentures, is the existence of embedded call options. Embedded call options give the issuer of the debenture the right to repurchase it in the future. Usually when these securities present some sort of repurchasing clause, most of these clauses end up being American options with time-varying strikes, which are not trivial to be priced.[1]

The problem of pricing options embedded in corporate bonds was studied by Berndt (2004) and Jarrow, Li, Liu, and Wu (2010), among others. However, taking into account the specificities of the existing Brazilian debentures, up to our knowledge, the methodology proposed in this paper is new.

In fact, motivated by the lack of a formal methodology to price debentures with the particular peculiarities that appear in the Brazilian market, we propose a combined strategy that first estimates the term structure of interest rates for debentures with no embedded options, and after that, prices options on the debentures that contain embedded options.

[1]The early repurchase mechanism is a protection for the debenture issuers as it allows the debt rescheduling in the case of more favorable economic scenarios in the future, compared to the scenario in which the paper was issued. In fact, this early repurchase allows the company to reduce the amount spent on interest payments and/or change the debt profile by promoting, in most cases, a debt extension.

Here, we present a brief description of our methodology. First, we estimate the term structure of interest rates for the debenture market adopting only debentures without embedded options. We follow the idea proposed by Almeida, Duarte and Fernandes (2000) and take into account the different ratings of debentures available, adopting a variation of the parametric model proposed in their work. While in Almeida, Duarte and Fernandes (2000) term structure movements are represented by Legendre polynomials, we make use of exponentials that appear in the Nelson-Siegel model (1987), which appear to be more stable than the polynomials. On a second step, we use the obtained term structure[2] to calibrate a trinomial tree for a one-factor dynamic arbitrage-free term structure model (Hull and White, 1993, 1994a, 1994b) to price the options embedded in the remaining debentures not used in the first step. Therefore, once we have the prices of these embedded options, we are able to identify the prices of all the debentures on a common basis, that is, as if they all did not have embedded options. At this stage, it would be possible (although not done in this paper) to re-estimate the term structure including all the debentures, both with and without embedded options.

We apply our methodology building the term structure of interest rates for a class of debentures indexed by the CDI index (one day inter-bank deposit rate), and illustrate how to price an embedded option by using a debenture issued by the company "Telefonica Brasil".

2. Related Literature

In order to be able to price defaultable securities with embedded options we need to both identify the default risk and propose a way to price the existing embedded option. We start this section by describing how the literature on the pricing of default risk has developed and follow presenting the different methodologies used to price embedded options.

Recent methods for capturing default risk are based on either structural or reduced-form models. Structural models assume that the value of the firm follows a certain stochastic process and the default event is triggered when the firm asset value falls below a certain critical threshold, which is often endogenous to the model. Among the pioneer authors in the use of the structural approach are Black and Scholes (1973), and Merton (1974) whose seminal work inspired several generalizations with more realistic structural models (see for instance, Duffie and Lando, 2001). On the other hand, reduced-form models are based on a predetermined exogenous default intensity process and usually treat a default event as a jump of a counting (Poisson) process with stochastic intensity. This is the approach followed by Pye (1974), Jarrow and Turnbull (1995), Jarrow, Lando and Turnbull (1997), Duffie and Singleton (1999), among others. The analysis of losses

[2] To be more precise, we use a spread derived from this curve. See the details of the methodology in Section 3.

conditional on the occurrence of a default varies a lot even within the classes of reduced-form and structural models.

In what regards the optimal strategy to exercise embedded options, we can distinguish primarily between models that use techniques of partial differential equations (PDE) and those using the martingale approach.

The seminal contribution on the pricing of embedded options with defaultable securities using PDE techniques is again credited to Merton (1974). In this paper he argued that the price of a defaultable security with embedded call option should come from the solution of a partial differential equation with boundary conditions that simultaneously describe the default event and optimal exercise of the embedded option. Under these models, closed-form solutions usually do not exist and finite difference methods have to be adopted instead. Subsequently, Kim, Ramaswamy and Sundareasan (1993) extended Merton's work by allowing stochastic interest rates. More recently, Sarkar (2000) allows for imperfections in the capital structure, e.g., sunk costs, taxes, bankruptcy costs, which change the optimal policy for the exercise of the embedded call option.

Alternatively, default models that use the martingale approach generate significant simplifications in the calculations of the prices of securities with embedded call options. For instance, Duffie and Singleton (1999) adopt the martingale approach to price defaultable bonds with embedded options assuming that the issuers decide to exercise the embedded call option seeking to strategically minimize their market value. Acharya and Carpenter (2002) analyze options embedded in defaultable securities making the following simplification: They consider those options to be American options whose underlying is a fixed income bond with no default risk and fixed coupon payments. In a related article, Guntay (2002) proposes a double-risk environment to price a defaultable security that pays coupons and presents an embedded call option. In this model, the risk of exercising the option and the default risk are two correlated processes. Guntay (2002) allows taxes and restitution costs to affect the intensity rate of the process that triggers the option exercise and also that firm's characteristics affect the intensity rate of the default process. Peterson and Stapleton (2003), in a recent contribution to the literature on the pricing of options on credit-sensitive bonds, built a three-factor model for the joint processes of the term structure of default-free rates and corresponding credit spreads. They price Bermudan options on defaultable securities adopting a recombining log-binomial tree method.

Generally, there are no analytical solutions to price bonds with embedded call options. In such cases, the Monte Carlo simulation method is appropriate, in particular in environments with a large number of parameters or with stochastic parameters. Seminal papers on Monte Carlo simulation to price derivatives were written by Bossaerts (1989) and Boyle, Broadie and Glasserman (1997), among others. The literature on simulation-based methods involves the parameterization of the border decision (Garcia, 2003), reduction of dimensionality or nonparametric

representation of early exercise region (Barraquand and Martineau, 1995; Clewlow and Strickland, 1998), and the value function approach. The approximation of the value function can be based on decision trees (Broadie and Glasserman, 1997), stochastic methods (Broadie and Glasserman, 2004), regression methods (Carrière, 1996; Tisitsiklis and Roy, 1999; Longstaff and Schwartz, 2001; Clèment, Lamberton and Protter, 2002), or dual methods (Rogers, 2002). Fu, Laprise, Madan, Su and Wu (2001) empirically test and compare the performance of some of these algorithms based on simulation.

3. Methodology

In this paper we propose a methodology in three stages for the simultaneous extraction of the price of an embedded option and a parametric yield curve for debentures with credit risk in the Brazilian market.

To price the call option embedded in debentures we propose a variation of the Hull and White model (1994a). For this procedure, a trinomial tree for the short-term interest rate is calibrated in order to capture the current term structure of interest rates in the market and price the derivative in question. However, the applicability of this model lies in the choice of which yield curve to use to calibrate the model. Thus, a relevant question is: Which yield curve should be used?

One naive possibility would be the future interbank deposit (ID) curve. However, we know that this curve does not reflect the credit risk embedded in debentures and we would be underestimating the term structure that should be captured by the short-term rate. Bearing in mind this problem, we should then add a default component as in Duffie and Singleton (1999). A feasible alternative is to determine/estimate a first approximation of the term structure of interest rates specifically for the debenture market, segregating securities according to their ratings, so that it could be used as information to calibrate the Hull and White trinomial tree (1994a).

Our three-stage methodology can be summarized as follows. Initially, we estimate a preliminary yield curve by using debentures with no embedded options, divided into groups according to their ratings in order to take into account the credit risk. The second step consists of pricing the call option embedded in debentures with such characteristic. Finally, by using a non-arbitrage condition,[3] we compute the debenture price minus the option price and re-estimate the yield curves for this market.

Finally, the only gap that remains to be filled is the choice of the model to be used to estimate the yield curve. In the present paper, we propose a model based on the Nelson-Siegel model (1987), which will be presented in the following section.

[3] Callable Bond = Noncallable Bond − Call Option

3.1 The Nelson and Siegel Model

We know that the price of a debenture is given by the present value of its cash flows discounted by the yield curve of this security:

$$P_{model}(0) = \sum_{j=1}^{m} \frac{FC_j}{1 + r(\tau_j)}, \tag{1}$$

where m is the number of cash flows of the debenture.

It is of fundamental importance to define the functional form of $r(\tau)$, the term structure. In their article, Nelson and Siegel (1987) propose a parsimonious way of modeling the yield curve, which is employed in this paper, given by the following equation:

$$r(\tau_j) = \beta_1 + \beta_2 \left(\frac{1 - e^{-\lambda \tau_j}}{\lambda \tau_j} \right) + \beta_3 \left(\frac{1 - e^{-\lambda \tau_j}}{\lambda \tau_j} - e^{-\lambda \tau_j} \right). \tag{2}$$

One of the aspects that have led to the widespread use of the Nelson and Siegel model is its very flexible functional form, which is able to generate a variety of curves, including upward- and downward-sloping curves, with either upward or downward curvature. By closely observing Equation (2), note that the format of the model's yield curve is determined by the three factors that multiply the betas. The first factor assumes the value 1 (constant) and it can be interpreted as the long-term level of the interest rate since the factors of β_2 and β_3 vanish as the time to maturity tends to infinity. The second factor converges to 1 when the time to maturity approaches zero, and it converges to zero when the time to maturity increases indefinitely. Therefore, this component mostly affects short-term interest rates. The third factor behaves similarly to the second factor, *i.e.*, it converges to 1 when the time to maturity approaches zero, and it converges to zero when the time maturity tends to infinity, but it is concave on τ. For this reason, this component is associated with medium-term interest rates. The parameter λ determines the position(s) of the curve's inflexion point(s). Finally, these factors have the usual interpretation of the yield curve in terms of level, slope and curvature.

Once we have the ANBIMA's indicative prices for computing the model parameters β_1, β_2, β_3 and λ, we can use a simple minimization procedure with a quadratic loss function as follows:

$$\sum_{i=1}^{N} \frac{(P_{model,i} - P_{ANBIMA,i})^2}{duration_i^2}. \tag{3}$$

Note that the model described above is able to generate a yield curve for any class of assets that fits into its framework. In this paper, we propose a variation of the aforementioned model since, for the market of debentures in particular, it

is interesting to estimate distinct curves for securities with similar risk characteristics, as suggested by Almeida, Duarte and Fernandes (2000). Thus, bearing in mind that the credit risk plays an important role in the pricing of these securities, we propose to segregate the debentures according to their ratings and then to estimate a curve for each subgroup (AAA, AA and A). An interesting aspect that will be explored in the modeling is the relationship between the curve of the assets with a given rating and the CDI curve, which is obtained by interpolating the vertices of the future DI on the date in question. Specifically, we will concentrate our analysis on the difference between these two curves, which is called here the spread over the CDI of each particular rating.

Finally, it is important to note that, in order to apply our methodology and compute the yield curves associated with each subgroup (rating) in the first stage, we need a minimum amount of debentures with no embedded options. Otherwise, we would have an identification problem in the model.[4]

3.2 The Hull and White Model

Once we have the approximation for the term structure of interest rates in the debenture market, we are able to clarify the methodology that will be employed to price the embedded options. First, we will describe the Hull and White model (1994a), which will be used to model the interest rate's dynamics:

$$dr = [\theta(t) - ar] + \sigma dz. \tag{4}$$

Hull and White built a trinomial tree to represent the movements in r by using time intervals of length Δt and considering that, at every step in time, the interest rate is given by $r_0 + k\Delta r$, where k is a positive or a negative integer and r_0 is the initial value of r. In the trinomial model of Hull and White, the tree's branches can assume three different forms. These forms are described in the Appendix C.

The first stage is to build a preliminary tree for r by setting $\theta(t) = 0$ and the initial value of $r = 0$. For this process, $r(t+\Delta t) - r(t)$ is normally distributed. For the purpose of tree construction, we define r as the continuously compounded rate associated with a period Δt. We will denote the expected value of $r(t+\Delta t) - r(t)$ as $r(t)M$ and its respective variance as V.

First of all, we must choose the time step size, Δt. Once we have the time step, we are able to define the size of the increment of the interest rate at every period of the tree, Δr, as[5]

$$\Delta r = \sqrt{3V}. \tag{5}$$

The first objective is to build a tree with evenly spaced nodes in both the dimension of r and the size of t. To do this, it is necessary to resolve which

[4] We thank one of the Referees for the comment to clarify this point.

[5] The choice of this value is based on theoretical works in the field of numerical procedures.

branching form will apply to each node of the tree. Once we have done this, we will be able to calculate the probabilities at each node.

Let (i, j) be the node for which $t = i\Delta t$ and $r = j\Delta r$. Denote by p_u, p_m and p_d the probabilities of the highest, middle and lowest branches, respectively, emanating from a certain node. These probabilities are chosen to match the average change in r, $r(t)M$, and the variance of this change in the next time interval, Δt. As the probabilities must add up to one, we have three equations, one for each type of branching. When r is at node (i, j), the average change in the next time interval is $j\Delta rM$, and the variance of this change in r is V.

The next stage in the construction of the tree is the introduction of a time-varying bias correction term. To do this, it is necessary to displace the nodes at time $i\Delta t$ by an amount α_i, constructing a new tree. The value of r at node (i, j) in the new tree is equal to the value of r at node (i, j) in the old tree, plus the value of α_i. The probabilities remain the same in the new tree. The values of α_i are chosen so that the tree is able to price all the discounted securities consistently with the term structure observed in the market. The consequences of moving from one tree to another are equivalent to changing the process being modeled from

$$dr = -ar + \sigma dz \tag{6}$$

to

$$dr = [\theta(t) - ar] + \sigma dz. \tag{7}$$

Define $Q_{i,j}$ as the present value of an asset that pays off 1 unit if node (i, j) is reached, and zero otherwise. The values of α_i and $Q_{i,j}$ are calculated using forward induction. More formally, assume that the values of $Q_{i,j}$ have been determined for $i \leq m$ $(m \geq 0)$. The next step is to determine the value of α_m so that, at time 0, the tree correctly prices a discount security with maturity at $(m + 1)\Delta t$. The interest rate at node (m, j) is $\alpha_m + j\Delta r$ so that the price of a discount security with maturity at $(m + 1)\Delta t$ is given by

$$P(0, m + 1) = \sum_{-n_m}^{n_m} Q_{m,j} \exp\left[-(\alpha_m + j\Delta r)\Delta t\right], \tag{8}$$

where n_m is the number of nodes outside the central node at time $m\Delta t$. The solution of this equation can be obtained by using any numerical procedure aimed at finding roots of an equation. Once having determined α_m, we can find $Q_{i,j}$ for $i = m + 1$ using the formula

$$Q_{m+1,j} = \sum_k Q_{m,k}\, q(k, j) \exp\left[-(\alpha_m + k\Delta r)\Delta t\right], \tag{9}$$

where $q(k, j)$ is the probability of moving from node (m, k) to node $(m + 1, j)$, with this sum being taken over all values of k for which $q(k, j)$ is different from zero.

Here, we finish the first stage of the proposed methodology for pricing embedded options. To summarize, after the construction of the tree, we will have a structure for the evolution of the interest rates, which is going to be used as the discount rates for the debentures' cash flows. The second stage concerns how, from the interest rates tree, we can build an equivalent tree for pricing debentures. We will see that, with a small modification in the pricing tree for debentures, it will be possible to price a call option for this asset. In order to do this, we will follow the approach suggested by the Black-Derman-Toy model (1990), which explains how to price a bond that pays coupons.

3.3 From the Yield Curve to the Pricing of the Embedded Option

By using the information arising from the curve of debentures with no embedded options and from the tree for modeling the term structure of interest rates, we can build a model for pricing the embedded options.

In a simplified way, the heart of the matter of modeling the price of an embedded call option can be synthesized in the answer of the following question: When is it advantageous for the issuer to redeem debentures in advance of the maturity date? Unfortunately, this answer is not straightforward. In order to try to answer this question, we need to make certain assumptions, some more restrictive than others.

The first assumption concerns the company's intention to roll over its debt. The second assumption deals with the feasibility of the redemption process, that is, the existence of sufficient cash. This factor may not be always important since the company can make a programmed roll over of its debt, *i.e.*, condition a new issue on the success of the redemption of a former issue. This fact should be mentioned because it can limit the scope of our analysis. The third assumption concerns the absence of reissuing costs. Undoubtedly, this is the strongest hypothesis but, unfortunately, this kind of information cannot be accessed.

Given these assumptions, we can formulate the answer for the following question: Will it be interesting for the issuer company to redeem its debentures when the market funding rate is lower than the interest rate that the company is paying on its issues? Remember that we are interested in debentures whose yields are linked to the CDI, more specifically, debentures that pay a spread over the CDI. We assume that the market funding rate for a company, at a certain point in time, is the spread-curve over the CDI of the rating to which that company belongs. The intuition is quite simple: for every rating, this spread-curve represents the market expectations of the future difference between the funding rate of an issuer with credit risk, described by that level of rating, and the funding rate of an issuer with no credit risk.

The vast majority of debentures with early redemption clauses include the payment of a premium to the debenture holder if early redemption occurs. This premium can assume two forms. In the first form, as a pro rata of the number

of days from the first day of redemption up to the maturity date, in business or calendar days. In the second form, this premium is fixed throughout the exercise period. The redemption premium should be taken into account when we define the market funding rate of a company since, in practice, this premium represents an additional cost for reissuing. Thus, the market funding rate for a given period, which is described by the spread over the CDI for that period, should be added to the redemption premium for the same period. In addition, if information about issuing costs is available, these costs should also be incorporated into the spread in the definition of the company's market funding rate. The impact of both the redemption premium and the reissuing cost is to reduce the feasibility of early redemption by the issuer.

3.4 Model Operationalization

In this section we will discuss how, in practice, we can price embedded options. We must bear in mind the required inputs for this purpose: the spread-curve over the CDI of the rating to which the issuer company belongs, the CDI forward rates consistent with the construction of the tree, and the characteristics of that debenture. The required characteristics are the issuing value (notional), the issuing fees, and the formula for calculating the premium that is paid when early redemption occurs, as well as the date from which the redemption can be made. For simplicity, the explanation will focus on non-amortizing debentures, as the extension of this procedure to other types of debentures is quite straightforward.

The starting point is the construction of a tree that simulates the evolution of the spread over the CDI. For this tree, we arbitrarily set the number of equally-spaced periods from the analysis date to the maturity date equals to eight. Once we have done this, it is possible to calculate the forward rates for the CDI between every period of the tree.

In order to find the price of the embedded call option, we can implement a backward induction process. In the last period ($t = 8$), the debenture matures and the issuer will have to pay exactly the notional amount plus the interest accrued between the penultimate date of capitalization and the maturity date. Thus, there is no gain for exercising this option at maturity as there is no option for not redeeming. Thus, the tree that describes the evolution of the price of the embedded call option will be a mirror of our spread tree with one period too few, *i.e.*, seven periods.

In the penultimate period, we analyse every node in order to verify whether or not it is interesting to exercise the option by comparing the spread at that particular node plus the premium to be paid at that date with the costs of the debenture issuing. Denoting by $C(7, j)$ the option price on the seventh period and j-th node, this price can be represented as

$$C(7,j) = \max\{0, Notional \times (TIR_{issue} - (Spread(7,j) + \pi(7)))\}, \tag{10}$$

where $Spread(7,j)$ is the spread of the corresponding element in the tree over the CDI of the seventh period, and $\pi(7)$ is the redemption premium due on the seventh period.

Once we have calculated the option prices on the seventh period, we can proceed retroactively to the previous period, obtaining what would be the prices of the seventh period when discounted to the sixth period by the CDI forward rate between the sixth and the seventh periods. Bearing in mind that, for every node of the CDI tree, there are probabilities p_u, p_m and p_d associated with the trajectory for the next period, we can assign what would be the prices of the seventh period to every node of the sixth period:

- if at node $(6,j)$ the branching process is of the form 1A:

$$C_{pres}(6,j) = \frac{p_u C(7,j+1) + p_m C(7,j) + p_d C(7,j-1)}{(1+Fwd(6))^{\Delta t}}, \tag{11}$$

- if at node $(6,j)$ the branching process is of the form 1B:

$$C_{pres}(6,j) = \frac{p_u C(7,j) + p_m C(7,j+1) + p_d C(7,j+2)}{(1+Fwd(6))^{\Delta t}}, \tag{12}$$

- if at node $(6,j)$ the branching process is of the form 1C:

$$C_{pres}(6,j) = \frac{p_u C(7,j) + p_m C(7,j-1) + p_d C(7,j-2)}{(1+Fwd(6))^{\Delta t}}. \tag{13}$$

This would be the option value at that node when there was no possibility of redemption at that period. When redemption is possible, we should compare the value above with the option value when redemption occurs directly in the sixth period, *i.e.*, $C(6,j) = \max\{C_{pres}(6,j), \max\{0, Notional \times (TIR_{issue} - (Spread(6,j) + \pi(7)))\}\}$. An explanation of why we consider the maximum of two values lies in the fact that, if the amount saved by the company when it redeems earlier were greater, it would not make any economic sense to wait until the following period to redeem.

This iterative process should be repeated retroactively until we reach the option price on the analysis day.

4. Empirical Application

In this section we illustrate the aforementioned methodology proposed to price the option for early redemption of the debenture TSPP12. After we have priced the option for this security, it is possible to apply the same procedure to the universe of debentures employed in the estimation of the yield curves.

Table 1
Characteristics of the debenture TSPP12

Issuing date	01/05/2005
Expiration date	01/05/2015
Redemption date	01/05/2005
Nominal issuing value	R$ 10.000
Yield to maturity	120 % of the DI rate
Premium	1,022 $\times DD$

In Table 1, we summarize the relevant information about the asset in question. DD represents the number of calendar days elapsed up to the date of renegotiation, inclusive, counted from the fixed date for redemption.

Note that, in accordance with the above procedure for calculating the price of the embedded call option in this debenture, we need two main ingredients: the spread over the CDI and the value of the premium to be paid in each one of the eight periods considered (remember that we use a trinomial tree with eight periods to price the option). Figures (1) and (2) present these inputs for calculating the price of the embedded call option on 24/12/2009.

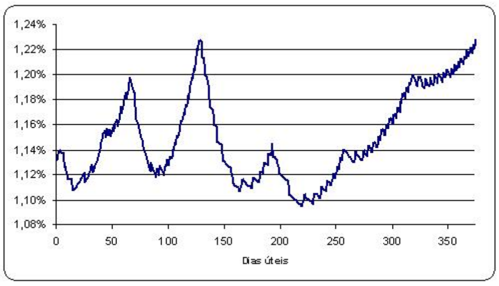

Figure 1
Spread over the CDI curve

0,14%	0,12%	0,10%	0,08%	0,06%	0,04%	0,02%	0,00%

Figure 2
Value of the premium in each period (in percentage)

Once we have these preliminary data, by using the methodology described in Section 3, we can build the tree that models the evolution of the spread of the debenture in question. The results for such procedure are reported in Figure (3).

			2,63%	3,34%	4,12%	4,10%
		3,01%	1,73%	2,20%	2,71%	2,72%
	1,63%	1,98%	1,15%	1,45%	1,78%	1,80%
1,14%	1,08%	1,31%	0,76%	0,96%	1,18%	1,20%
	0,72%	0,86%	0,50%	0,64%	0,78%	0,80%
		0,57%	0,33%	0,42%	0,51%	0,53%
			0,22%	0,28%	0,34%	0,35%

Figure 3
Trinomial tree for the evolution of the spread over the CDI

Once we know the spread tree, we are able to calculate the price of the embedded call option by modeling its pricing tree following the methodology proposed by Hull and White, which was described in Section 3.2.

			12,42	7,70	4,36	1,69	0,00
		43,19	30,23	18,69	12,35	6,01	0,00
	88,57	88,24	89,02	60,21	40,68	33,15	23,42
116,99	119,98	123,70	127,80	109,48	90,00	89,77	87,38
	142,75	147,93	153,43	142,01	129,96	129,87	129,50
		164,51	170,37	163,51	156,29	156,51	157,28
			181,59	177,73	173,68	174,22	175,63

Figure 4
Trinomial tree for the evolution of the debenture price

To check the plausibility of the exercise of this option, we repeat the above procedure for all the remaining days from the date of renegotiation of the debenture. In this way, at the end of the exercise, we will have the daily price of the call option in question. In Figure (5), we present a series of prices computed for this option.

As it can be observed, the price presents a growing trend over time, which can be interpreted as an increase in the possibility of exercise. Indeed, corroborating with this analysis, the debenture was redeemed in advance by the issuer on 02/02/2011.

Finally, note that, according to the proposed methodology, once we have the series of daily prices of the embedded option in this debenture, we can incorporate it in the estimation of the yield curve for this class of assets. As a large number of the existing Brazilian debentures contain embedded options, our methodology allows us to increase the number of assets available to estimate the term structure.

Figure 5
Price evolution of the embedded option on the debenture

5. Conclusion

When we take into account the investment needs in long-term projects in the Brazilian economy and the inability of the Brazilian State to bear all this amount alone, it is expected an increasing participation of the stock market as a long-term supplier of funds. In this context, we expect an increasing number of debenture issues for this intended purpose, which makes necessary a sound framework on aspects concerning the pricing of these securities.

In this paper, we presented a methodology for pricing the embedded call option in debentures in parallel with the estimation of a yield curve for this group of assets. It is important to highlight that this device (call options) works as a protection against a possible change in the credit market when compared to the moment when the fundraising was made.

The proposed methodology contains three stages. First, we estimate the term structure of interest rates for the debenture market based on the model of Nelson-Siegel. After that, we build a trinomial tree for the evolution of the spread between the curve obtained previously and the interpolated curve for the CDI index making use of the Hull-White model. Finally, we build a second tree that estimates the fair price of the embedded call option using backward induction.

In order to illustrate the proposed methodology, we estimated a series of prices for an embedded option in a particular debenture. As a result, we observed that our model provided increasing prices for this call option. This would be a strong indicator that this security would be redeemed by the issuer, what, in fact, did happen.

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Appendix

A. Estimating parameters of the dynamics of the short-term rate

A.1 Estimating the volatility parameter

Once we have estimated the curves for securities with no embedded call options daily, we are able to use, for each date t, the differences between these curves as the input for the options pricing tree.

Thus, assuming that these curves have been estimated for N days, we have a series of length N for the initial value of the curve, $C_0(t)_{t=1}^{N}$. From this series, we build another series of length $N-1$, $D(t)_{t=1}^{N-1}$, where $D(t) = C_0(t+1) - C_0(t)$. By using that the variance V of $r(t+\Delta t) - r(t)$ is given by $\sigma^2 \Delta t$, we estimate the volatility parameter as

$$\hat{\sigma} = \sqrt{\hat{V}}, \tag{14}$$

where $\hat{V}$ is the estimated variance for the series of first differences.

A.2 Estimation of the mean-reversion parameter

Once we know the estimated value of the volatility parameter, we are able to estimate the mean-reversion parameter. The idea is quite intuitive: we have the modeling of the interest rate of the first tree, the one that still does not consider the current term structure, given by

$$dr = -ar dt + \sigma dz. \tag{15}$$

Remembering that $\Delta t = 1$, a trivial discretization of this process would be given by

$$r(t) - r(t-1) = -ar(t-1) + \epsilon(t). \tag{16}$$

Then, in order to obtain the value of $\hat{\rho}$, we can run the following regression:

$$r(t) = (1-a)r(t-1) + \epsilon(t) = \rho r(t-1) + \epsilon(t). \tag{17}$$

Thus, the estimated value for the mean-reversion parameter will be given by

$$\hat{a} = 1 - \hat{\rho}. \tag{18}$$

B. Considerations about Debentures

The word debenture is derived from the Late Middle English *debentur* which, in turn, is originated from the Latin *debere*, which means obligation or what should be paid. As its name suggests, the debenture is a loan certificate given by a company as evidence of debt. Thus, the debenture is an instrument issued by a company to its holders, who are the creditors of the company, representing a fraction of a loan. Each debenture entitles its holder credit rights against the issuer, and these rights are set forth in the issuing deed.

Regarding the debentures issuing, the investor lends to the issuer company the amount corresponding to the value of the securities that were acquired, with the promise of receiving, at the end of the contract, the principal plus interest as defined in the issuing deed. The purpose of this type of funding is to meet, in the most cost-effective way, the financial needs of the company, thereby avoiding the constant and costly short-term operations. Therefore, joint-stock companies have at their disposal the necessary facilities to raise funds from the public, with longer maturities and lower interest rates, with or without monetary adjustment, and redemptions with either a fixed deadline or by random selection, according to the needs of the companies, in the way that best fits their cash flows.

The Extraordinary General Meeting or the Board of Directors of the issuer, as applicable, according to what is stipulated in the Statute, will define the loan characteristics by setting the issuing conditions such as amount, number of debentures, expiration date, issuing date, yield (including a premium or a discount on the issue), amortizations or scheduled redemptions, convertibility or not into stocks, monetary adjustment, and whatever else that may be necessary, deliberating about it.

Finally, regarding the remuneration paid to the debenture holder, the debentures are divided basically between those indexed to the CDI index, paying as a compensation over the face value a percentage of the cumulative CDI or the cumulative CDI plus a spread previously defined, or those which pay a correction of the face value over inflation, which may be the cumulative IGP-M or IPCA for that period plus an interest rate previously defined.

Simply put, the value of a debenture is the present value of its expected cash flows. This procedure seems trivial: we just need to compute the cash flows and then discount them by an appropriate discount rate. In practice, however, there are two reasons that make this procedure not as trivial as it might seem. First, although we ignore the possibility that the debenture issuer does not honour its commitments, *i.e.*, he defaults, it is not easy to compute the cash flows of debentures with embedded call options. This happens because the exercise of the option is subject to the decision of the issuer, and this decision may be based on the evaluation of one or more economic variables, or it can occur in a discretionary way, for example, through the decision made at a shareholders' meeting. Thus, the debenture issuer can modify the investor's cash flow when he exercises the

embedded call option.

A further complication concerns the interest rates that will be adopted to discount the cash flows. The starting point is the government's yield curve. From this yield curve, we add a spread that reflects the additional risks to which the investor is exposed. The computation of this spread is not simple, and we also lack a way to model it. The *ad hoc* process to price a debenture with no embedded options is to discount all the cash flows by the same discount rate, which is equal to the expected internal rate of return of that security.

Thus, the standard approach used to price a debenture is to discount its cash flows by its respective zero-coupon rate. If the call option embedded in the debenture is exercised by the issuer, for example, this cash flow can be interrupted.

B.1 Evolution of the Brazilian Debenture Market

No one knows for sure when debentures first appear, even though it is widely known that there was a security in England with similar characteristics for at least 500 years. In Brazil, the debentures constitute one of the oldest forms of fundraising through securities. The origin of its regulation dates back to the Empire (Law 3,150 and Decree 8821, both dated 1882).

Until the early 60s, the debenture market was less pronounced because there were no mechanisms to protect the long-term applications from the inflation effects. The Law number 4728 of 1965, issued in the midst of the changes that restructured the national financial system, introduced important innovations in the debentures, emphasizing the possibility that they could be converted into shares, and the monetary adjustment. Later, through the enactment of the Law of Stock Corporations, Law number 6.404/76 (subsequently amended by Law number 10.303, of October 31st, 2001), the debentures assumed the current form. At the same time, the creation of the Securities and Exchange Commission of Brazil (CVM), through the Law number 6,385/76, brought discipline and regulations to the capital markets, providing more safety to investors.

The introduction of the National System of Debentures (SND) in 1988 by ANDIMA allowed the registration, custody, and settlement of securities, what contributed to an increase in the transparency of the market. In the same year, the acquisition of debentures was encouraged by financial institutions and, after that, the *Plano Verão* allowed the use of a wide range of indexes, responding to an increasing demand from investors.

With the advent of the *Plano Real*, the debenture market gained a fresh impetus, benefiting from the monetary stability and the gradual lengthening of maturities of debt securities, the patrimonial and financial restructuring of companies, the recovery of the economic growth, and the process of privatization. In this process, the debentures have become an important fundraising instrument for leasing, management and participation, public services, trading, and intermediate inputs companies.

Recently, two CVM instructions brought a number of innovations to the debenture market. The first one, the CVM Instruction number 400 of November 29th, 2003, consolidated several norms about public offerings of securities and it also guided a series of current market practices, such as bookbuilding. This instruction also introduced some common practices in other countries that had been demanded by the participants of the Brazilian market, such as the shelf registration (*Programa de Distribuição*), green shoe (supplementary lot distribution option), and the possibility of increasing the offer in 20 percent without changing the prospectus.

The CVM Instruction number 404 of February 13th, 2004, introduced the standardized debentures and the conditions for the simplified registration procedure of debenture issues. The standardized debentures are securities that have scriptures with uniform clauses and should be negotiated in special environments with market makers that provide a minimum liquidity for these securities. These debentures can contribute significantly to the development of a more dynamic market for debt securities issued by publicly-traded companies and provide investors with a security that, due to its simplicity and uniformity, do not demand neither complex contractual interpretations, nor sophisticated calculations for negotiations. It is expected that the standardization of the clauses significantly reduces the time that investors and intermediaries will have to devote to the reading and understanding of the scriptures.

C. Probabilities from Hull and White Model

If at node (i, j) the branching process is of the form 1A, the equations for the probabilities are given by

$$p_u = \frac{1}{6} + \frac{j^2M^2 + jM}{2}, \tag{19}$$

$$p_m = \frac{2}{3} - j^2M^2, \tag{20}$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 - jM}{2}. \tag{21}$$

If at node (i, j) the branching process is of the form 1B, the equations for the probabilities are given by

$$p_u = \frac{1}{6} + \frac{j^2M^2 - jM}{2}, \tag{22}$$

$$p_m = -\frac{1}{3} - j^2M^2 + 2jM, \tag{23}$$

$$p_d = \frac{7}{6} + \frac{j^2M^2 - 3jM}{2}. \tag{24}$$

If at node (i, j) the branching process is of the form 1C, the equations for the probabilities are given by

$$p_u = \frac{7}{6} + \frac{j^2M^2 + 3jM}{2}, \tag{25}$$

$$p_m = -\frac{1}{3} - j^2M^2 - 2jM, \tag{26}$$

$$p_d = \frac{1}{6} + \frac{j^2M^2 + jM}{2}. \tag{27}$$

Most time, the branching process of the form 1A is the most appropriate. When $a > 0$, it is necessary to switch to the branching form 1C when j is larger. This must be done to ensure that the probabilities p_u, p_m and p_d are all positive. Similarly, when $a < 0$, it is necessary to switch from the branching form 1A to the branching form 1B when j is smaller (*i.e.*, negative and large in magnitude). We define j_{max} as the value of j where we switch from the branching form 1A to the branching form 1C, and j_{min} as the value of j where we switch from the branching form 1A to the branching form 1B. It can be shown that p_u, p_m and p_d are always positive if j_{max} is chosen as an integer between $-\frac{0{,}184}{M}$ and $-\frac{0{,}816}{M}$. Note that when $a > 0$, $M < 0$. In practice, the authors suggest that it is most efficient to choose j_{max} as the smallest integer greater than $-\frac{0{,}184}{M}$ and j_{min} equals to $-j_{max}$.