Inference in Structural VARs with External Instruments

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- Structural VAR Identification Problem: Sims (1980)

- "External" Instrument Solution: Romer and Romer (1989)

- Weak Instruments: Staiger and Stock (1997)
  Andrews-Moreira-Stock (2006)

Notation

Reduced form VAR: $Y_t = A(\mathrm{L})Y_{t-1} + \eta_t$;

$$A(\mathrm{L}) = A_1\mathrm{L} + \dots + A_p\mathrm{L}^p;$$

Y is $r \times 1$

Structural Shocks: $\eta_t = H\varepsilon_t = \begin{bmatrix} H_1 & \cdots & H_r \end{bmatrix}\begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{rt} \end{pmatrix}$, H is non-singular.

Structural VAR: $Y_t = A(\mathrm{L})Y_{t-1} + H\varepsilon_t$

Structural MA: $Y_t = [\mathrm{I}-A(\mathrm{L})]^{-1}H\varepsilon_t = C(\mathrm{L})H\varepsilon_t$

$C(\mathrm{L})H$ is structural impulse response function (dynamic causal effect)

SVAR estimands (focus on shock 1)

Partitioning notation:

$$\eta_t = H\varepsilon_t = \begin{bmatrix} H_1 & \cdots & H_r \end{bmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{rt} \end{pmatrix} = \begin{bmatrix} H_1 & H_\bullet \end{bmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{\bullet t} \end{pmatrix}$$

SMA for $Y_t = \sum_k C_k H_1 \varepsilon_{1t-k} + \sum_k C_k H_\bullet \varepsilon_{\bullet t-k}$

where $[\mathrm{I} - A(\mathrm{L})]^{-1} = C_0 + C_1\mathrm{L} + C_2\mathrm{L}^2 + \ldots$

SVAR estimands:

Write SMA for $Y_t = \sum_k C_k H_1 \varepsilon_{1t-k} + \sum_k C_k H_\bullet \varepsilon_{\bullet t-k}$

Impulse Resp: $IRF_{j,k} = \dfrac{\partial Y_{jt}}{\partial \varepsilon_{1t-k}} = C_{j,k} H_1$, where $C_{j,k}$ is the j'th row of C_k

Historical Decomposition: $HD_{j,k} = \sum_{l=0}^{k} C_{j,l} H_1 \varepsilon_{1t-l}$

Variance Decomposition: $\mathrm{VD}_{j,k} = \dfrac{\mathrm{var}\left[\sum_{l=0}^{k} C_{j,l} H_1 \varepsilon_{1t-l}\right]}{\mathrm{var}\left[\sum_{l=0}^{k} C_{j,l} \eta_{t-l}\right]}$

Two approaches for structural VAR identification problem: $\eta = H\varepsilon$

1. Internal restrictions: Short run restrictions (Sims (1980)), long run restrictions, identification by heteroskedasticity, bounds on IRFs)

2. External information ("method of external instruments"): Romer and Romer (1989), Ramey and Shapiro (1998), …

   Selected empirical papers

   - **Monetary shock**: Cochrane and Piazzesi (2002), Faust, Swanson, and Wright (2003. 2004), Romer and Romer (2004), Bernanke and Kuttner (2005), Gürkaynak, Sack, and Swanson (2005)
   - **Fiscal shock**: Romer and Romer (2010), Fisher and Peters (2010), Ramey (2011)
   - **Uncertainty shock**: Bloom (2009), Baker, Bloom, and Davis (2011), Bekaert, Hoerova, and Lo Duca (2010), Bachman, Elstner, and Sims (2010)
   - **Liquidity shocks**: Gilchrist and Zakrajšek's (2011), Bassett, Chosak, Driscoll, and Zakrajšek's (2011)
   - **Oil shock**: Hamilton (1996, 2003), Kilian (2008a), Ramey and Vine (2010)

The method of external instruments: Identification

Methods/Literature

- Nearly all empirical papers use OLS & report (only) first stage

- However, these "shocks" are best thought of as instruments (quasi-experiments)

- Treatments of external shocks as instruments:

  Hamilton (2003)

  Kilian (2008 – *JEL*)

  Stock and Watson (2008, 2012)

  Mertens and Ravn (2012a,b) – same setup as here executed using strong instrument asymptotics

An Empirical Example: *(Stock-Watson 2012) Dynamic Factor Model*

Dynamic factor model:

$$X_t = \Lambda F_t + e_t$$

(X_t contains 200 series, $F_t = r = 6$ factors, e_t = idiosyncratic disturbance)

$[\mathrm{I}-A(\mathrm{L})]F_t = \eta_t$ (factors follow a VAR)

$\eta_t = H\varepsilon_t$ (Invertible)

U.S., quarterly data, 1959-2011Q2

ε-shocks and Instruments

1. Oil Shocks
    a. Hamilton (2003) net oil price increases
    b. Killian (2008) OPEC supply shortfalls
    c. Ramey-Vine (2010) innovations in adjusted gasoline prices
2. Monetary Policy
    a. Romer and Romer (2004) policy
    b. Smets-Wouters (2007) monetary policy shock
    c. Sims-Zha (2007) MS-VAR-based shock
    d. Gürkaynak, Sack, and Swanson (2005), FF futures market
3. Productivity
    a. Fernald (2009) adjusted productivity
    b. Gali (1999) long-run shock to labor productivity
    c. Smets-Wouters (2007) productivity shock

ε-shocks and Instruments, ctd.

4. Uncertainty
   a. VIX/Bloom (2009)
   b. Baker, Bloom, and Davis (2009) Policy Uncertainty

5. Liquidity/risk
   a. Spread: Gilchrist-Zakrajšek (2011) excess bond premium
   b. Bank loan supply: Bassett, Chosak, Driscoll, Zakrajšek (2011)
   c. TED Spread

6. Fiscal Policy
   a. Ramey (2011) spending news
   b. Fisher-Peters (2010) excess returns gov. defense contractors
   c. Romer and Romer (2010) "all exogenous" tax changes.

Identification of SVAR estimands (IRF, HD, VD):

Z_t is a $k\times 1$ vector of external instruments

- $\eta_t = [1-A(\mathrm{L})]Y_t$ and $A(\mathrm{L})$ are identified from reduced form
  - $Y_t = C(\mathrm{L})\eta_t$ … $C(\mathrm{L})$ is identified from reduced form

- Express IRF, HD, VD as functions of $\Sigma_{\eta\eta}$, Σ_{ZZ}, $\Sigma_{\eta Z}$

Identifying Assumptions:

$$\text{(i) } E\left(\varepsilon_{1t} Z_t'\right) = \alpha' \neq 0 \text{ (relevance)}$$

$$\text{(ii) } E\left(\varepsilon_{jt} Z_t'\right) = 0, j = 2, \ldots, r \text{ (exogeneity)}$$

$$\text{(iii) } E\left(\varepsilon_{1t} \varepsilon_{jt}\right) = 0 \text{ for } j \neq 1$$

Identification of IRF$_{j,k}$ = $C_{j,k}H_1$

$$\Sigma_{\eta Z} = E(\eta_t Z_t') = E(H\varepsilon_t Z_t') = \begin{bmatrix} H_1 & \cdots & H_r \end{bmatrix} \begin{pmatrix} E(\varepsilon_{1t} Z_t') \\ \vdots \\ E(\varepsilon_{rt} Z_t') \end{pmatrix} = \begin{bmatrix} H_1 & \cdots & H_r \end{bmatrix} \begin{pmatrix} \alpha' \\ 0 \\ 0 \end{pmatrix}$$

$$= H_1 \alpha'$$

<u>Normalization: T</u>he scale of H_1 and $\sigma^2_{\varepsilon_1}$ is set by a normalization, The normalization used here: a unit positive value of shock 1 is defined to have a unit positive effect on the innovation to variable 1, which is u_{1t}. This corresponds to:

(iv) $H_{11} = 1$ (unit shock normalization)

where H_{11} is the first element of H_1

Identification of IRF$_{j,k}$ = $C_{j,k}H_1$, ctd

$$\Sigma_{\eta Z} = H_1\alpha', \text{ so } H_1 = \Sigma_{\eta Z}\alpha/(\alpha'\alpha)$$

Impose normalization (iv):

$$\Sigma_{\eta Z} = H_1\alpha' = \begin{pmatrix} H_{11} \\ H_{1\bullet} \end{pmatrix}\alpha' = \begin{pmatrix} 1 \\ H_{1\bullet} \end{pmatrix}\alpha', \text{ so } \alpha' = \Sigma_{\eta_1 Z}$$

$$\text{and } H_1 = \Sigma_{\eta Z}\Sigma_{Z\eta_1}/(\Sigma_{\eta_1 Z}\Sigma_{Z\eta_1})$$

If Z_t is a scalar ($k = 1$): $\quad \mathbf{H_1} = \dfrac{E\eta_t Z_t}{E\eta_{1t}Z_t}$

Identification of HD = $\sum_{k=0}^{h} C_{k,j} H_1 \varepsilon_{1t-j}$ requires identification of $H_1\varepsilon_{1t}$

Proj(Z_t | η_t) = Proj(Z_t | ε_t) = Proj(Z_t | ε_{1t}) = $b\varepsilon_{1t}$ where $b = \dfrac{\alpha}{\sigma^2_{\varepsilon_1}}\varepsilon_{1t}$

$$H_1\varepsilon_{1t} = \text{Proj}(\eta_t \mid \varepsilon_{1\text{t}}) = \text{Proj}(\eta_t \mid b\varepsilon_{1t})$$

$$= \text{Proj}(\eta_t \mid \text{Proj}(Z_t \mid \eta_t)) = \text{Proj}(\eta_t \mid \Sigma_{Z\eta}\Sigma_{\eta\eta}^{-1}\eta_t)$$

$$= \Lambda\Sigma_{\eta\eta}^{-1}\ \eta_t,$$

where $\Lambda = \Sigma_{\eta Z}(\Sigma_{Z\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta Z})^{+}\Sigma_{Z\eta}$

(Note $\Sigma_{Z\eta} = \alpha' H_1$ has rank 1, so pseudo inverse is used)

Identification of VD = $$\frac{\text{var}\left[\sum_{l=0}^{k} C_{j,l} H_1 \varepsilon_{1t-l}\right]}{\text{var}\left[\sum_{l=0}^{k} C_{j,l} \eta_{t-l}\right]}$$

Note this requires identification of var($H_1\varepsilon_{1t}$), which from last slide is

$$\text{var}(\Lambda\Sigma_{\eta\eta}^{-1}\ \eta_t) = \Lambda\Sigma_{\eta\eta}^{-1}\Lambda'.$$

Overidentifying Restrictions

(1) Multiple Z's for one shock: $\Sigma_{Z\eta} = \alpha' H_1$ has rank 1. Reduced rank "regression" of Z onto η.)

(2) Z_1 identifies ε_1, Z_2 identifies ε_2, and ε_1 and ε_2 are uncorrelated.

This implies that Proj($Z_1 \mid \eta$) is uncorrelated with Proj($Z_2 \mid \eta$) or

$$\Sigma_{Z_1\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta Z_2} = 0$$

Estimation:

GMM: Note A, $\Sigma_{\eta\eta}$, and $\Sigma_{\eta Z}$ are exactly identified, so concentrate these out of analysis. Focus on $\Sigma_{\eta Z}$ and SVAR estimands.

$\Sigma_{\eta Z} = E(\eta_t Z_t')$, so $\text{vec}(\Sigma_{\eta Z}) = E(Z_t \otimes \eta_t)$

or $\Sigma_{\eta Z} = H_1\alpha'$ so that $\text{vec}(\Sigma_{\eta Z}) = (\alpha \otimes H_1)$

High level assumption (assume throughout)

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left([Z_t \otimes \eta_t] - vec(\Sigma_{\eta Z})\right) \xrightarrow{d} \text{N}(0,\Omega)$$

GMM Estimation:

(Ignore estimation of VAR coefficients A and $\Sigma_{\eta\eta}$ − these are straightforward to incorporate).

Efficient GMM objective function: $J(\Sigma_{\eta Z})$

$$= \left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left((\eta_t \otimes Z_t) - vec(\Sigma_{\eta Z})\right)\right)' \Omega^{-1} \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left((\eta_t \otimes Z_t) - vec(\Sigma_{\eta Z})\right)$$

where, $\Sigma_{\eta Z} = H_1\alpha'$. (Similarly when more than one shock is identified).

- $k = 1$ (exact identification): $\hat{\Sigma}_{\eta Z} = T^{-1}\sum_{t=1}^{T}\eta_t Z_t$
- $k > 1$(Homo): $\hat{\Sigma}_{\eta Z}$ can be computed from reduced rank regression estimator of Z onto η.

Estimation of H_1 (k = 1)

$$\Sigma_{\eta Z} = H_1 \alpha = \begin{pmatrix} \alpha \\ \alpha H_{1\bullet} \end{pmatrix},$$

so GMM estimator solves, $$T^{-1}\sum_{t=1}^{T} \eta_t Z_t = \begin{pmatrix} \hat{\alpha} \\ \hat{\alpha}\hat{H}_{1\bullet} \end{pmatrix}$$

GMM estimator: $$\hat{H}_1 = \frac{T^{-1}\sum_{t=1}^{T} \eta_t Z_t}{T^{-1}\sum_{t=1}^{T} \eta_{1t} Z_t}$$

IV interpretation: $$\eta_{jt} = H_{1j}\eta_{1t} + u_{jt},$$

$$\eta_{1t} = \Pi_j' Z_t + v_{jt}$$

4. Strong instrument asymptotics

$\sqrt{T} vec(\hat{\Sigma}_{\eta Z} - \Sigma_{\eta Z}) \xrightarrow{d} N(0,V)$ and asymptotic distributions of all statistics of interest follow from usual delta- method calculations.

- Overidentified case ($k > 1$):
  - usual GMM formula
  - J-statistics, etc. are standard textbook GMM

5. Weak instrument asymptotics: $k = 1$

(a) Distribution of $\hat{H}_1 = \dfrac{T^{-1}\sum_{t=1}^{T}\eta_t Z_t}{T^{-1}\sum_{t=1}^{T}\eta_{1t} Z_t}$

Weak IV asymptotic setup – local drift (limit of experiments, etc.):

$$\alpha = \alpha_T = a/\sqrt{T}\text{, so } \Sigma_{\eta Z} = H_1 a'/\sqrt{T} = \Gamma/\sqrt{T}$$

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left((Z_t \otimes \eta_t) - \Sigma_{\eta Z}\right) \xrightarrow{d} \mathrm{N}(0,\Omega) \qquad (*)$$

becomes

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(Z_t \otimes \eta_t\right) \xrightarrow{d} \mathrm{N}(\Gamma,\Omega) \qquad (*\text{-weakIV})$$

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(Z_t \otimes \eta_t\right) \xrightarrow{d} \mathrm{N}(\Gamma, \Omega)$$

Weak instrument asymptotics for H_1, ctd

$$\hat{H}_1 = \frac{T^{-1/2}\sum_{t=1}^{T}\eta_t Z_t}{T^{-1/2}\sum_{t=1}^{T}\eta_{1t} Z_t}$$

Standardize (*):

$$\sigma_Z^{-1}\sigma_{\eta_1}^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(Z_t \otimes \eta_t\right) \Rightarrow \lambda + z, \qquad (**)$$

where $\lambda = \sigma_Z^{-1}\sigma_{\eta_1}^{-1}\Gamma$ and $z \sim \mathrm{N}(0,\Omega/(\sigma_Z^2\sigma_{\eta_1}^2))$,

Thus, in $k = 1$ case, $\hat{H}_1 = \dfrac{T^{-1}\sum_{t=1}^{T}\eta_t Z_t}{T^{-1}\sum_{t=1}^{T}\eta_{1t} Z_t} \Rightarrow \dfrac{\lambda + z}{\lambda_1 + z_1} = H_1^*$

Weak instrument asymptotics for H_1, ctd

$$\hat{H}_1 = \frac{T^{-1}\sum_{t=1}^{T}\eta_t Z_t}{T^{-1}\sum_{t=1}^{T}\eta_{1t} Z_t} \Rightarrow \frac{\lambda + z}{\lambda_1 + z_1} = H_1^*$$

<u>Comments</u>

1. In the no-HAC case, convergence to strong instrument normal is governed by

   $$\lambda_1^2 = a^2 / \sigma_{\eta_1}^2 \sigma_Z^2 \;= \text{noncentrality parameter of first-stage } F$$

   For the HAC case, see Montiel Olea and Pflueger (2012)

Weak instrument asymptotics for H_1, ctd

$$\hat{H}_1 = \frac{T^{-1}\sum_{t=1}^{T}\eta_t Z_t}{T^{-1}\sum_{t=1}^{T}\eta_{1t} Z_t} \Rightarrow \frac{\lambda + z}{\lambda_1 + z_1} = H_1^*$$

<u>Comments</u>

2. Consider unidentified case: $a = 0$ so $\lambda = 0$ so

$$\hat{H}_{1j} = \frac{T^{-1}\sum_{t=1}^{T}\eta_{jt} Z_t}{T^{-1}\sum_{t=1}^{T}\eta_{1t} Z_t} \Rightarrow \frac{z_j}{z_1} \sim \int N(\delta_j, \frac{\tau_j^2}{z_1^2}) dF_{z_1^2}$$

where δ_j = plim of OLS estimator in the regression, $\eta_{jt} = \delta_j \eta_{1t} + v_{jt}$

- $\hat{H}_1$ is median-biased towards $\delta = E(\eta_t \eta_{1t})/\sigma_{\eta_1}^2$ = the first column of the Cholesky decomposition with η_{1t} ordered first

Weak instrument asymptotics for structural IRFs

Structural IRF: $C(\mathrm{L})H_1$

where $C(\mathrm{L}) = [\mathrm{I}-A(\mathrm{L})]^{-1} = C_0 + C_1\mathrm{L} + C_2\mathrm{L}^2 + \dots$

Effect on variable j of shock 1 after h periods: $C_{h,j}H_1$

<u>Weak instrument asymptotic distribution of IRF</u>

$$\sqrt{T}(\hat{A} - A) = O_p(1) \text{ (asymptotically normal)}$$

so

$$\hat{C}(L)\hat{H}_1 \Rightarrow C(\mathrm{L})H_1^*$$

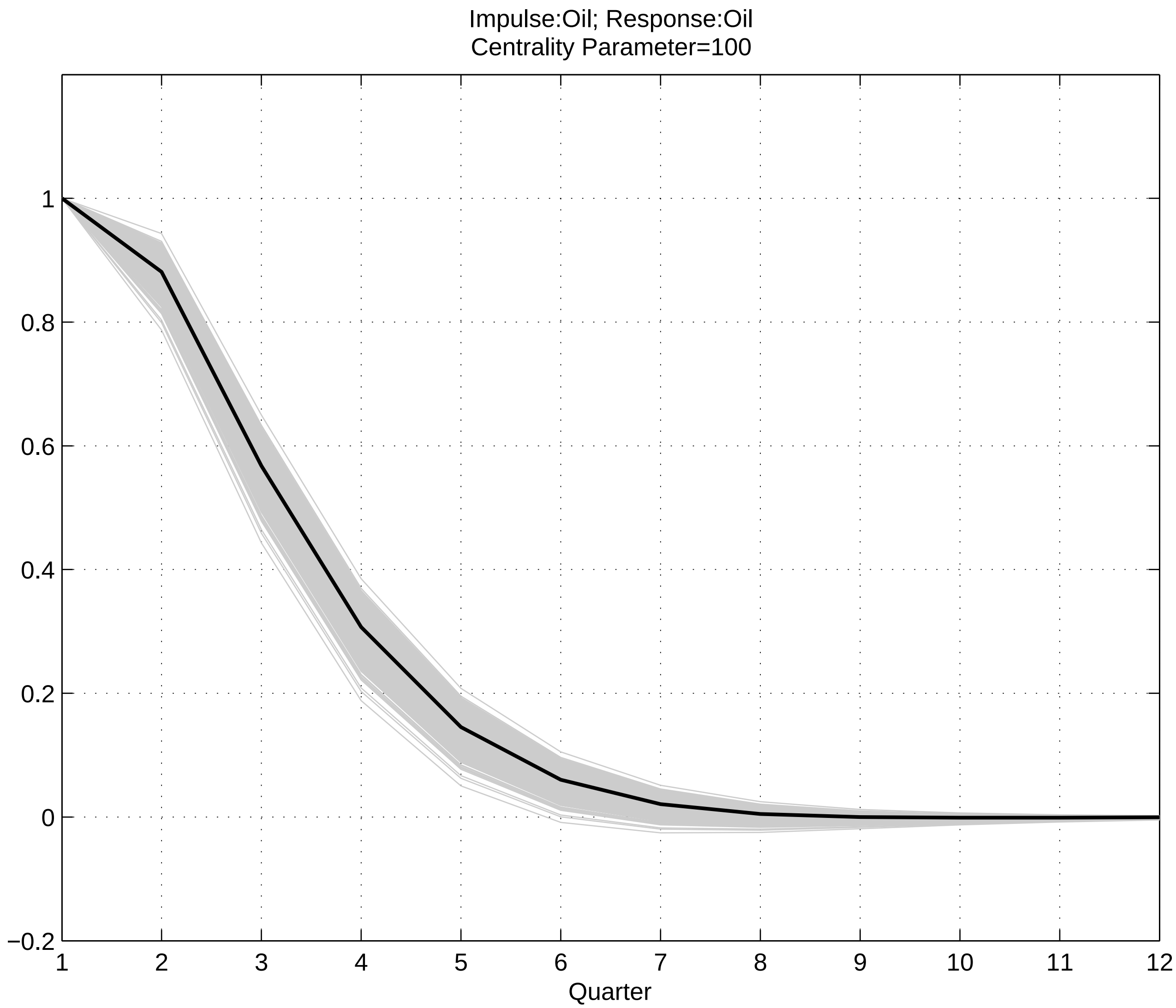
Estimator of h-step IRF on variable j: $\hat{C}_{h,j}\hat{H}_1 \Rightarrow C_{h,j}H_1^*$

- This won't be a good approximation in practice – need to incorporate $O_p(T^{-1/2})$ term …

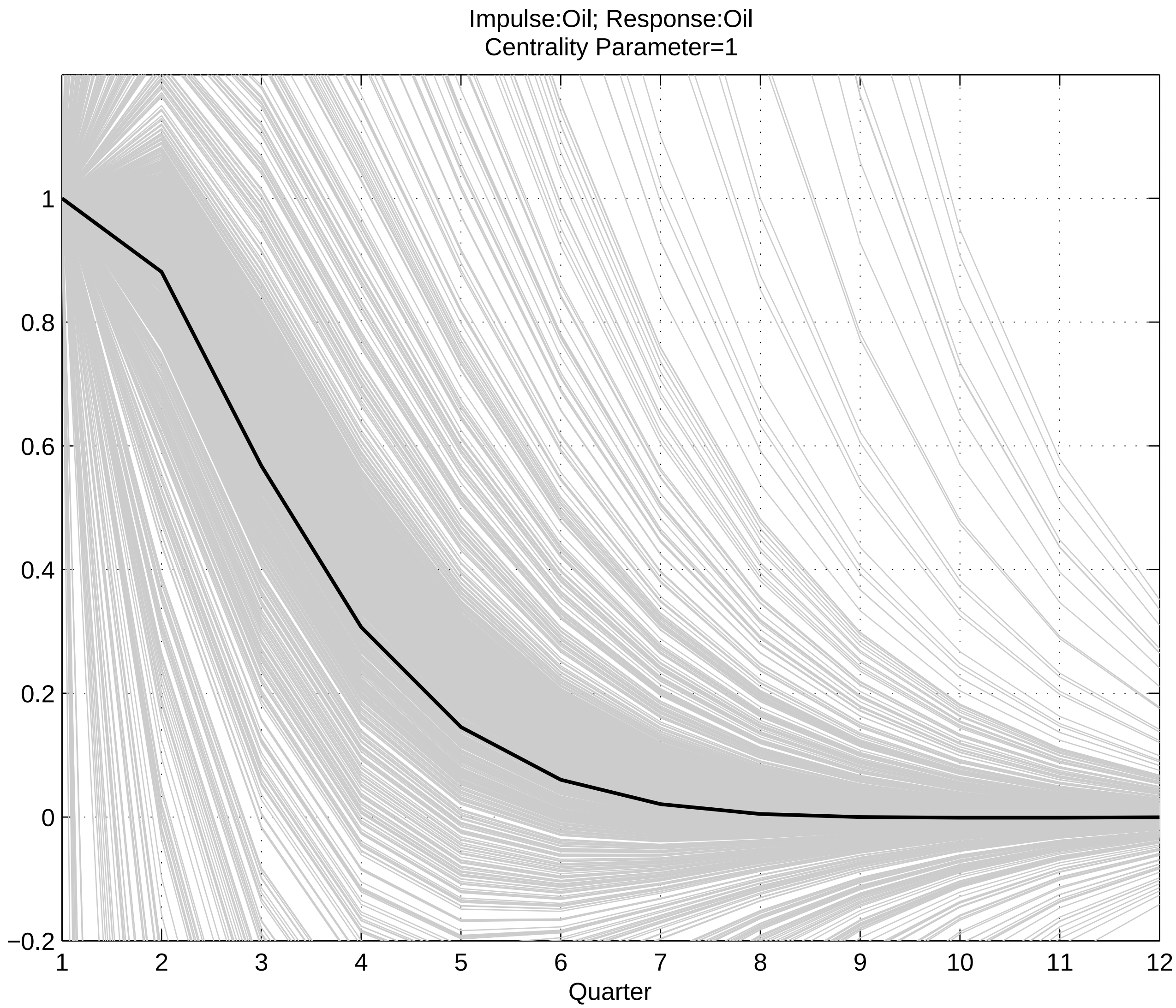
Numerical results for IRFs – asymptotic distributions

DGP calibration: $r = 2$

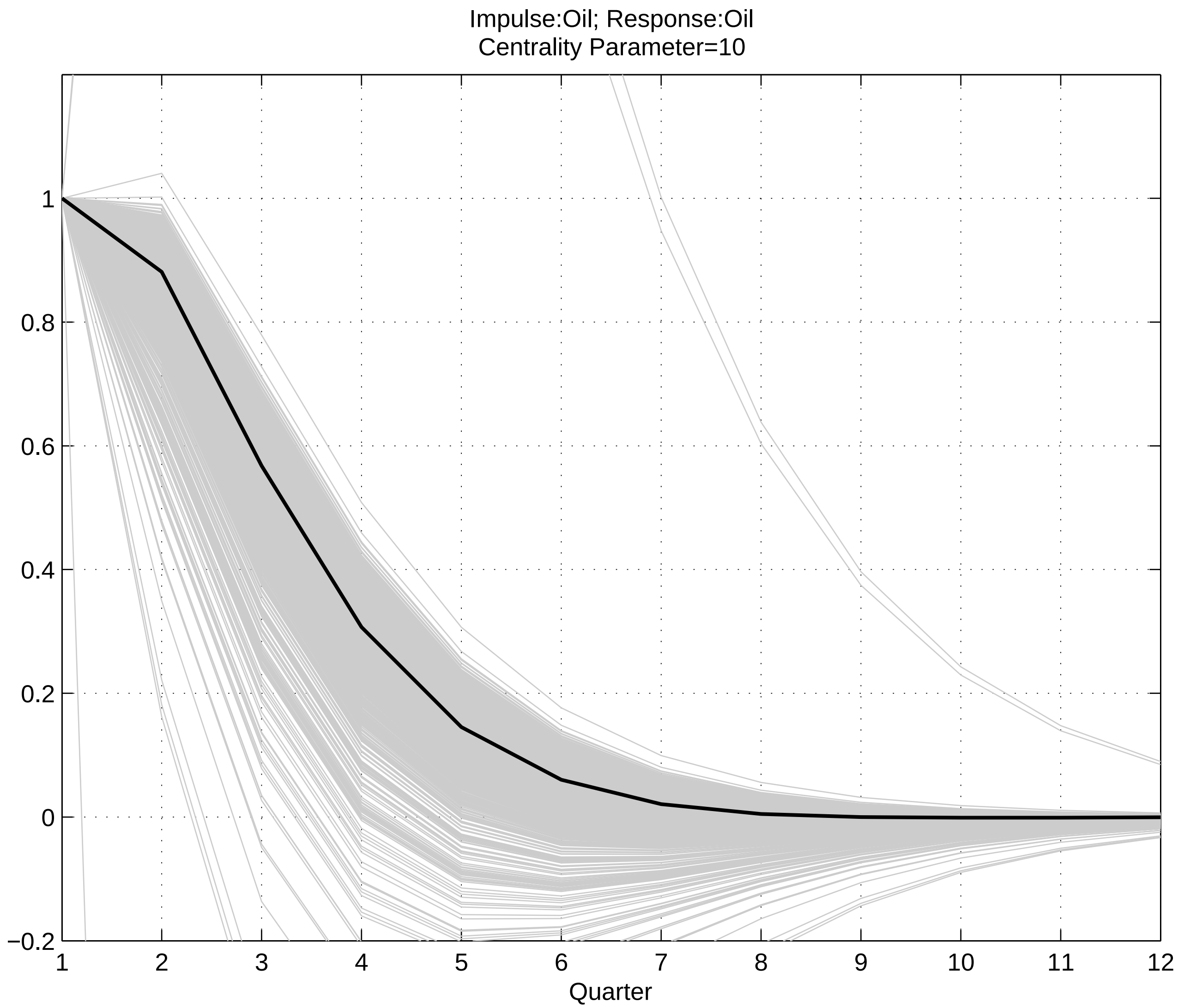
- $Y = (\Delta \ln POIL_t, \Delta \ln GDP_t)$, US, 1959Q1-2011Q2
- Estimate $\Phi(\mathrm{L})$, $\Sigma_{\eta\eta}$, and H_1, then fix throughout
  - o $A(\mathrm{L})$, $\Sigma_{\eta\eta}$: VAR(2)
  - o H_1: estimated using Z_t = Kilian (2008 – *REStat*) OPEC supply shortfall (available 1971Q1-2004Q3)



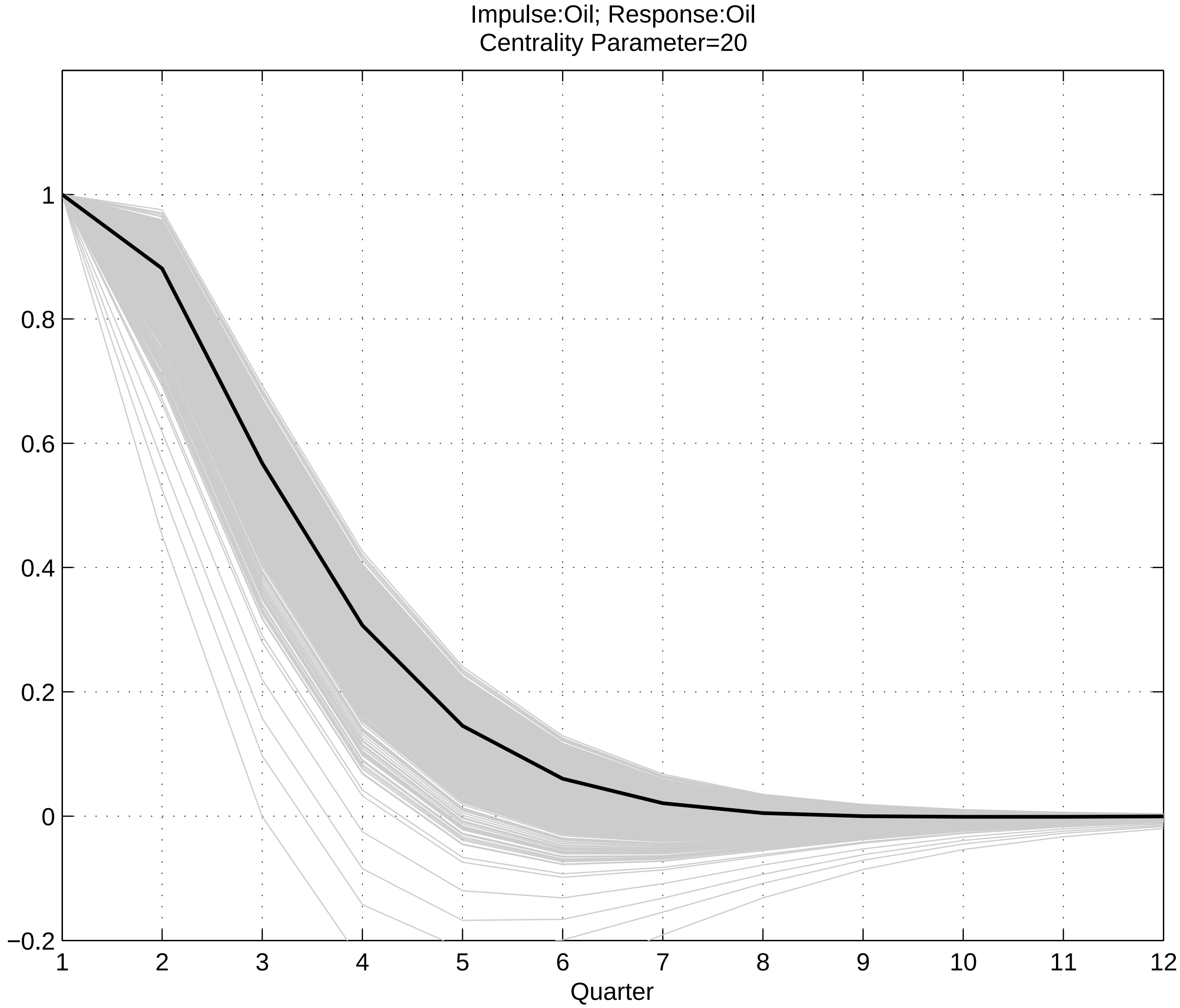
Effect of oil on oil growth: $\lambda_1^2 = 100$



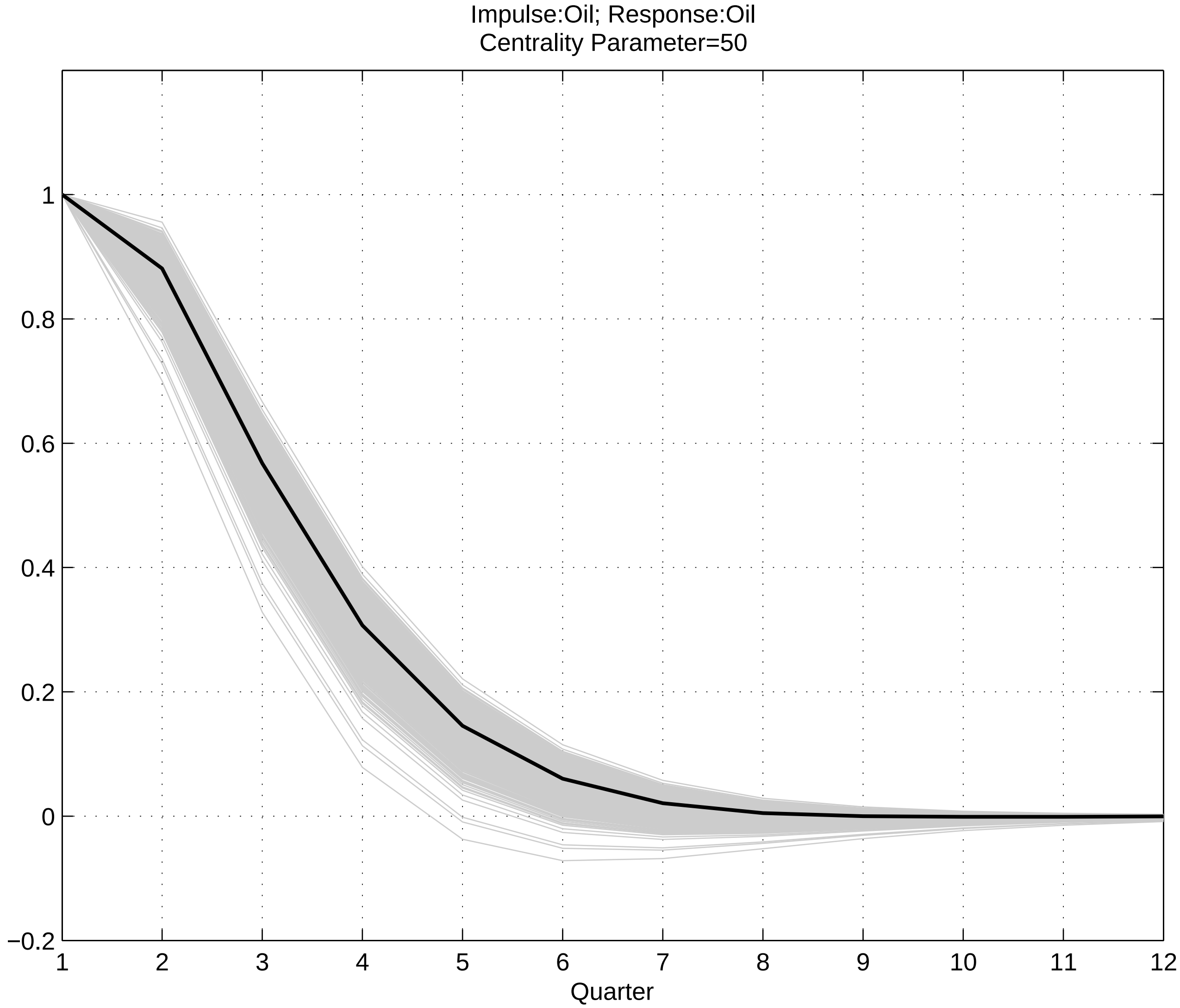
Effect of oil on oil growth: $\lambda_1^2 = 1$



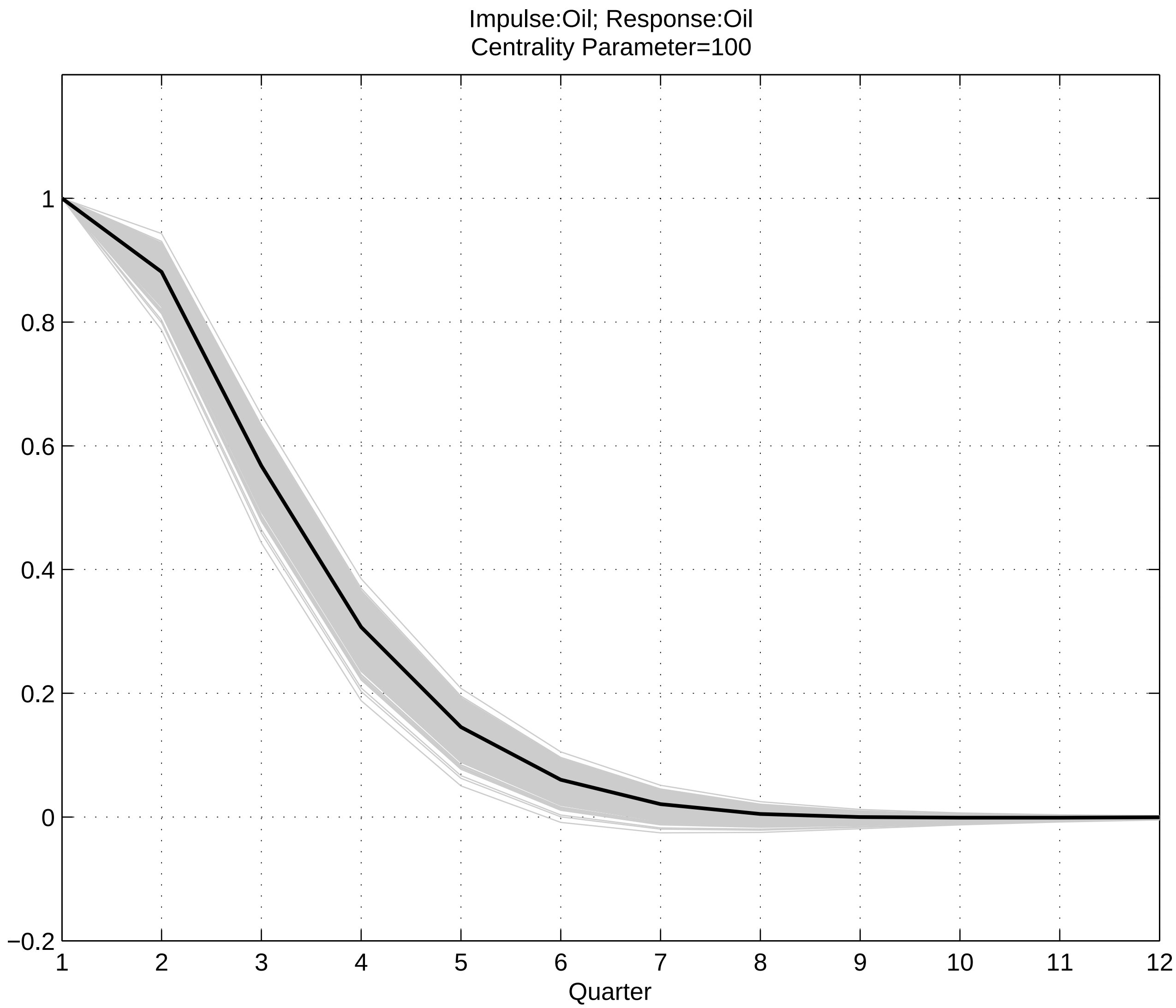
Effect of oil on oil growth: $\lambda_1^2 = 10$



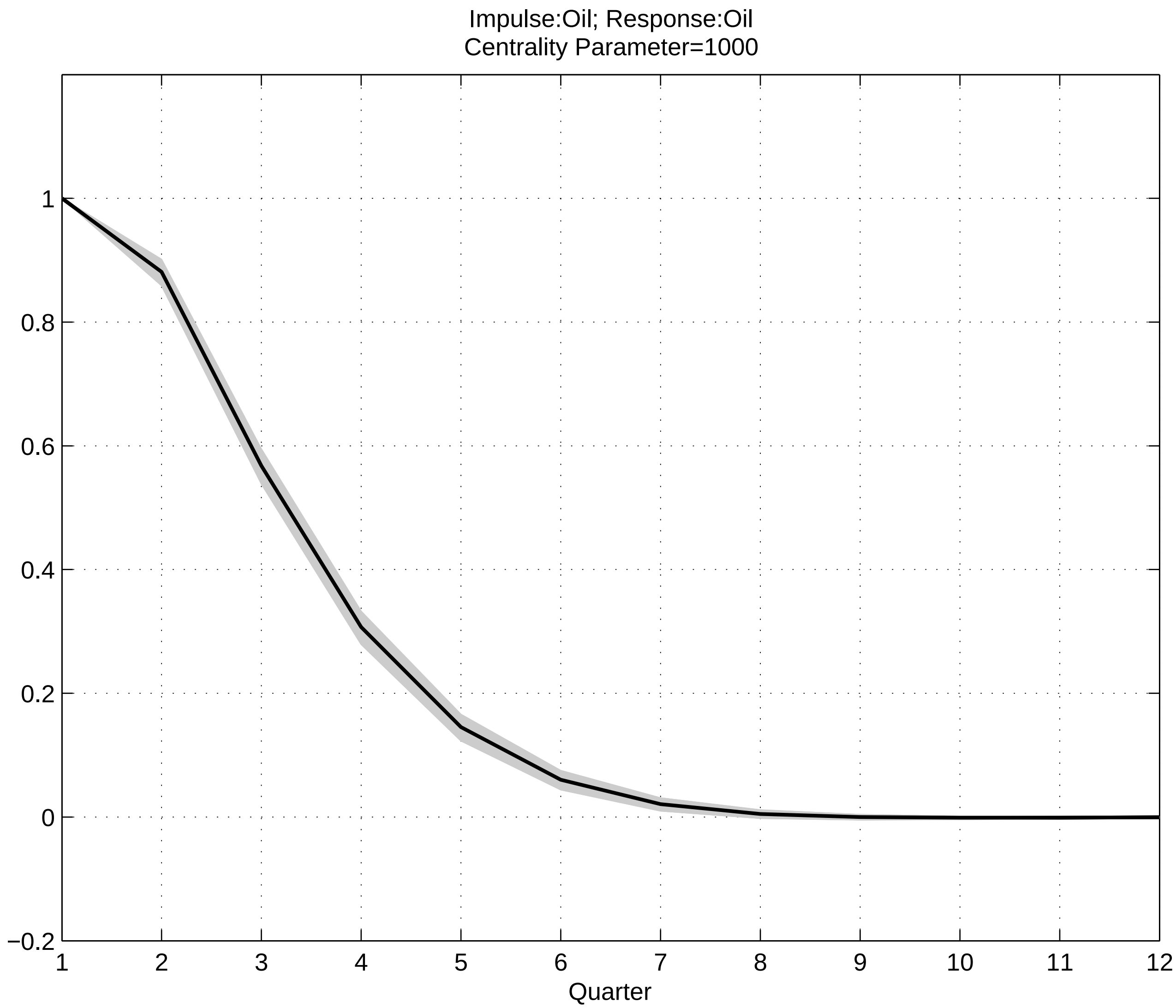
Effect of oil on oil growth: $\lambda_1^2 = 20$



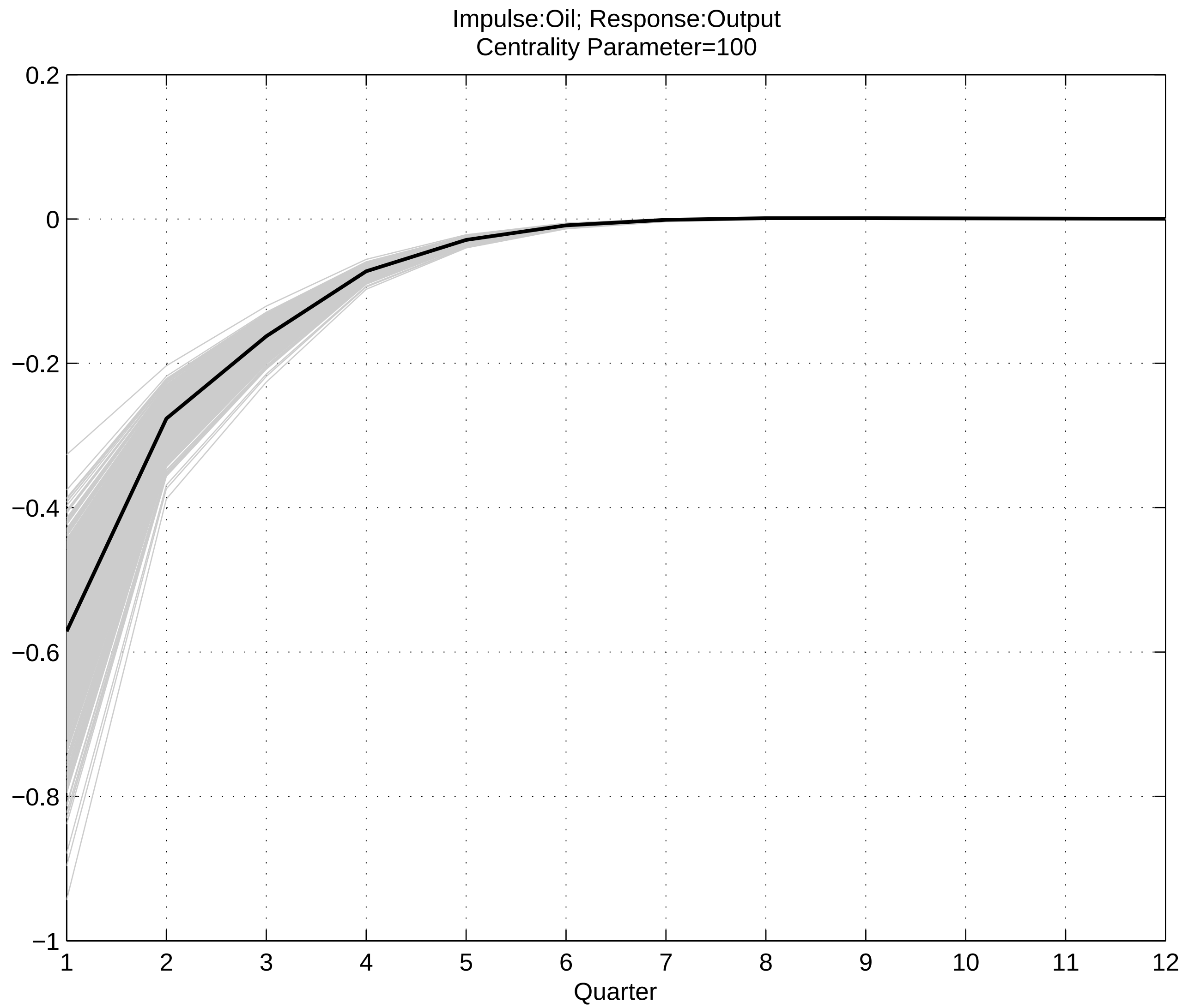
Effect of oil on oil growth: $\lambda_1^2 = 50$



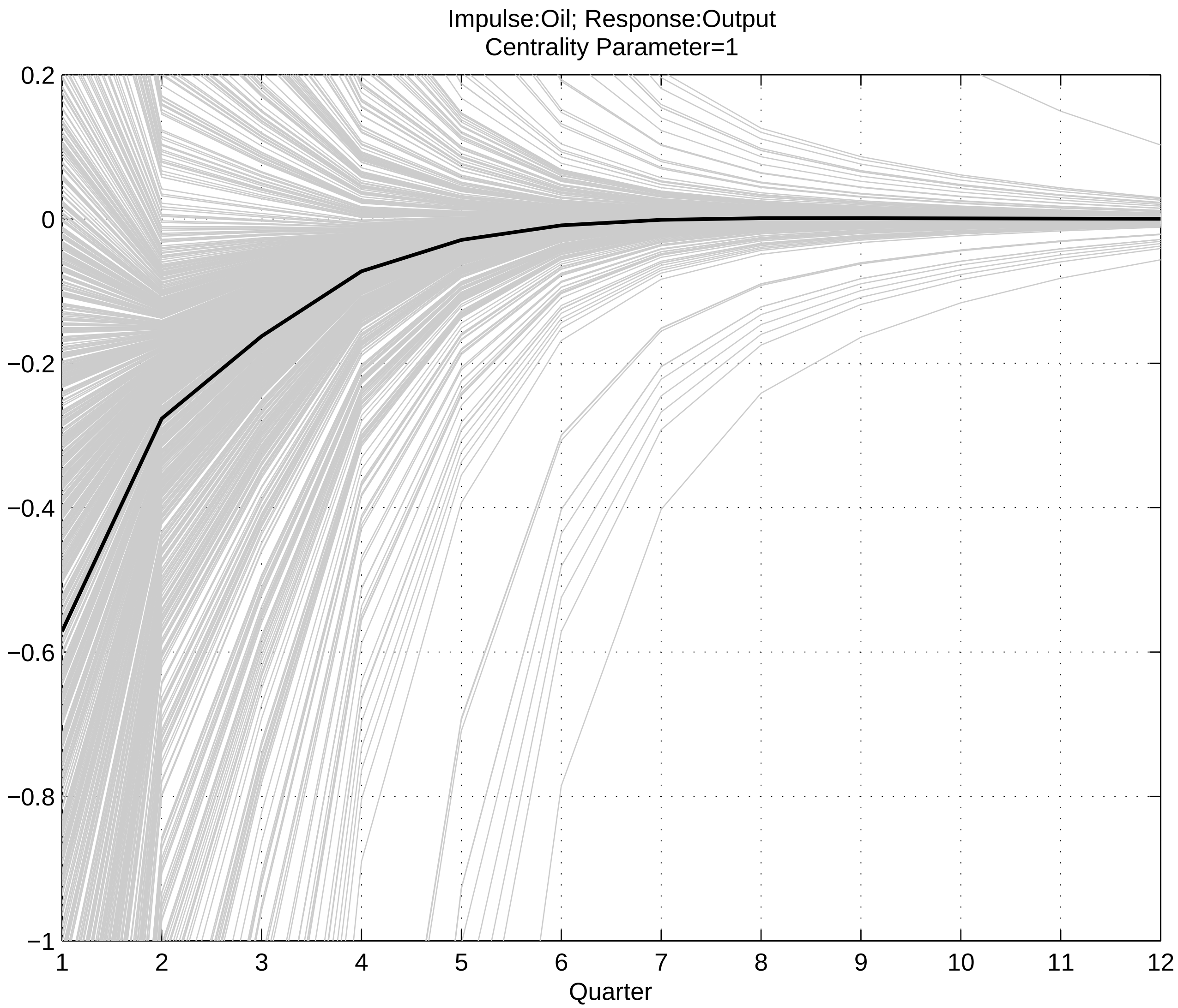
Effect of oil on oil growth: $\lambda_1^2 = 100$



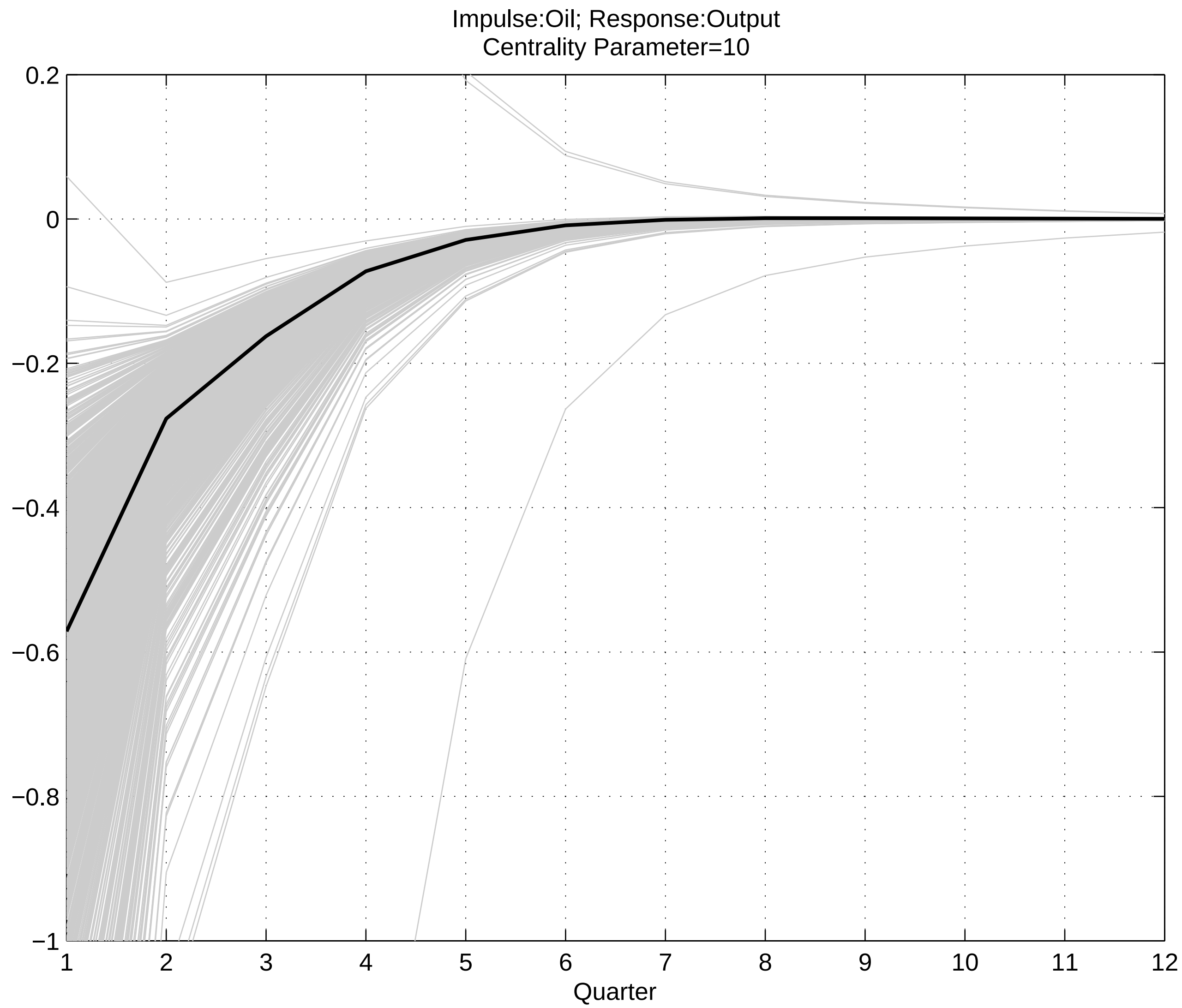
Effect of oil on oil growth: $\lambda_1^2 = 1000$



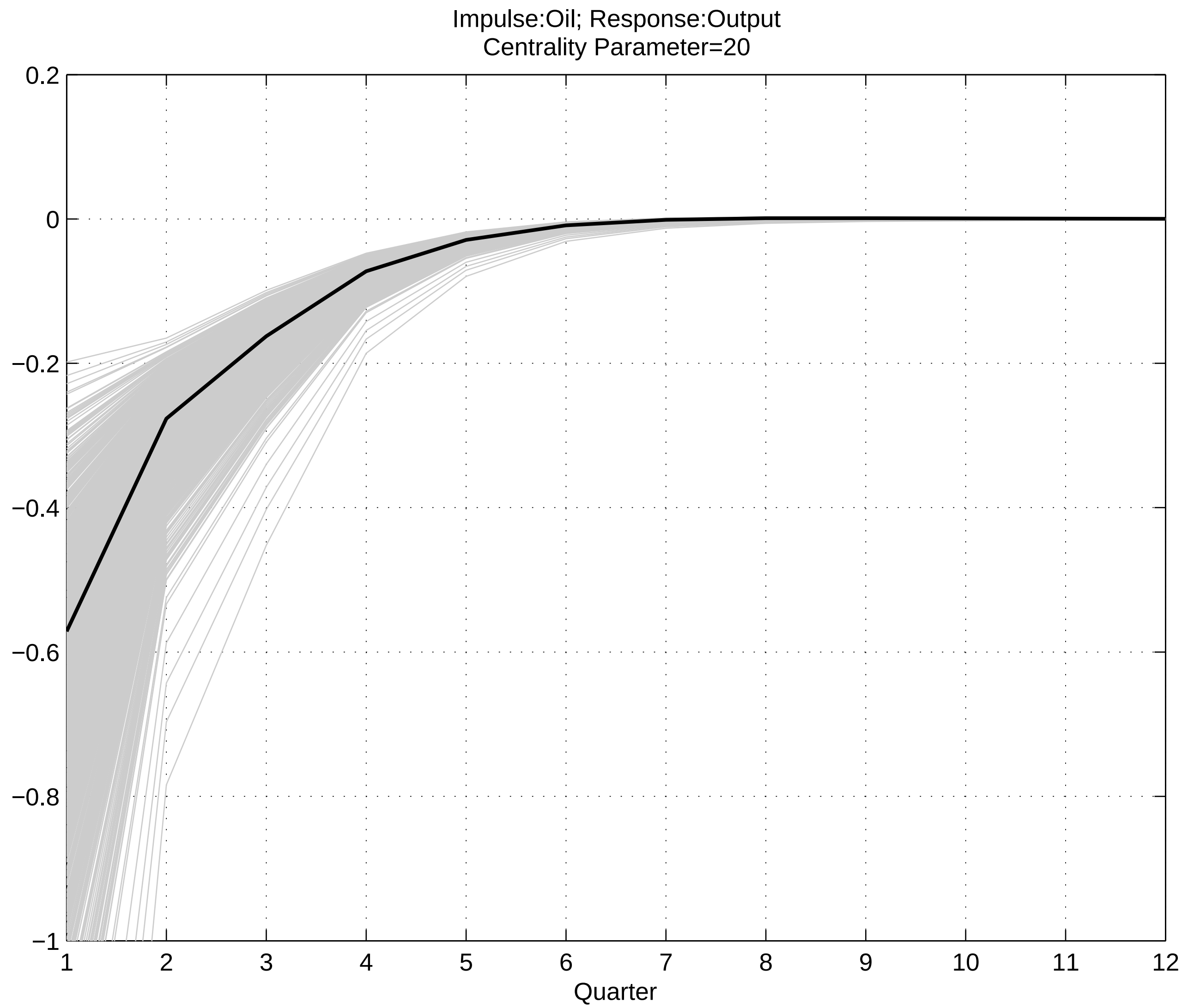
Effect of oil on GDP growth: $\lambda_1^2 = 100$



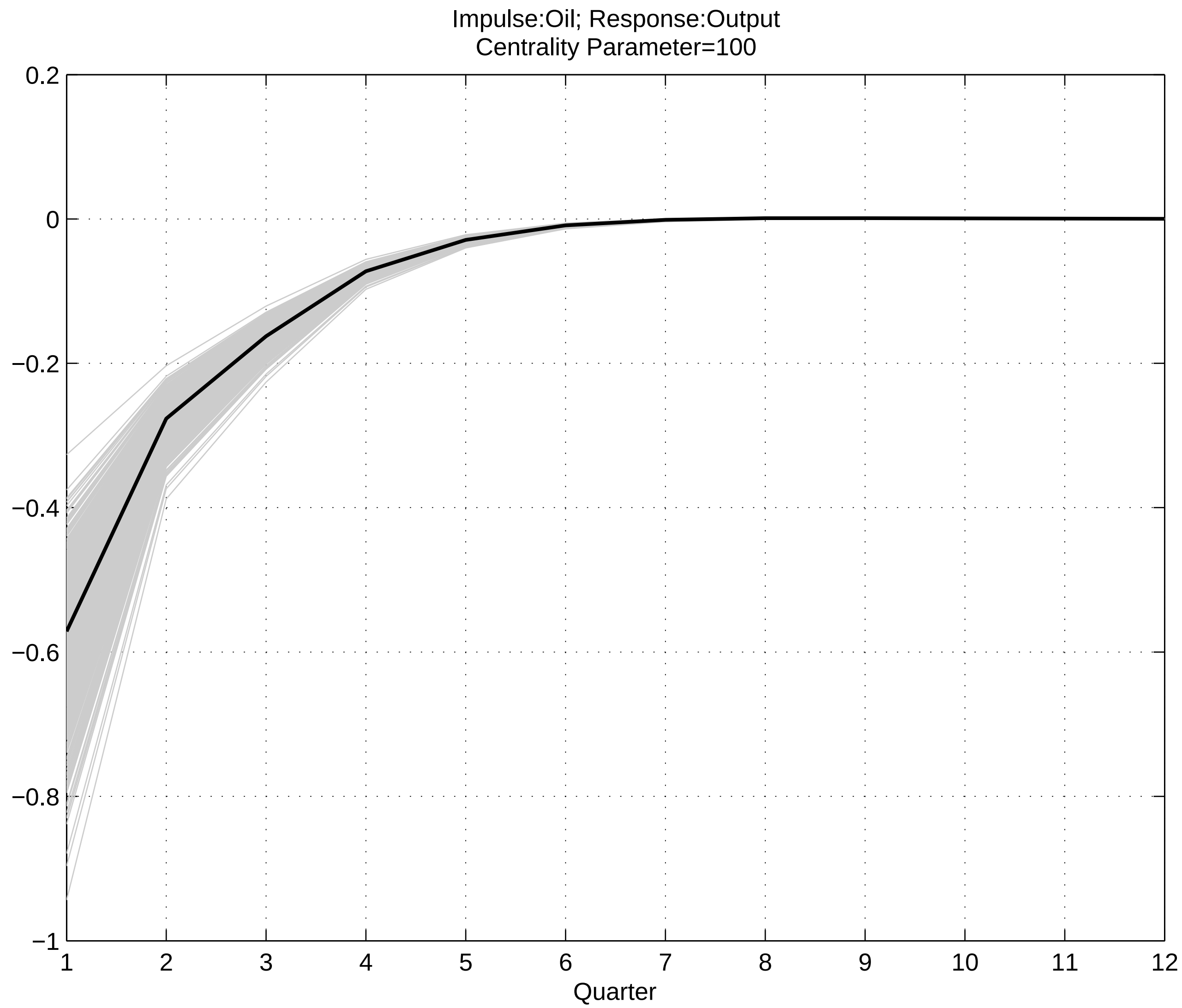
Effect of oil on GDP growth: $\lambda_1^2 = 1$



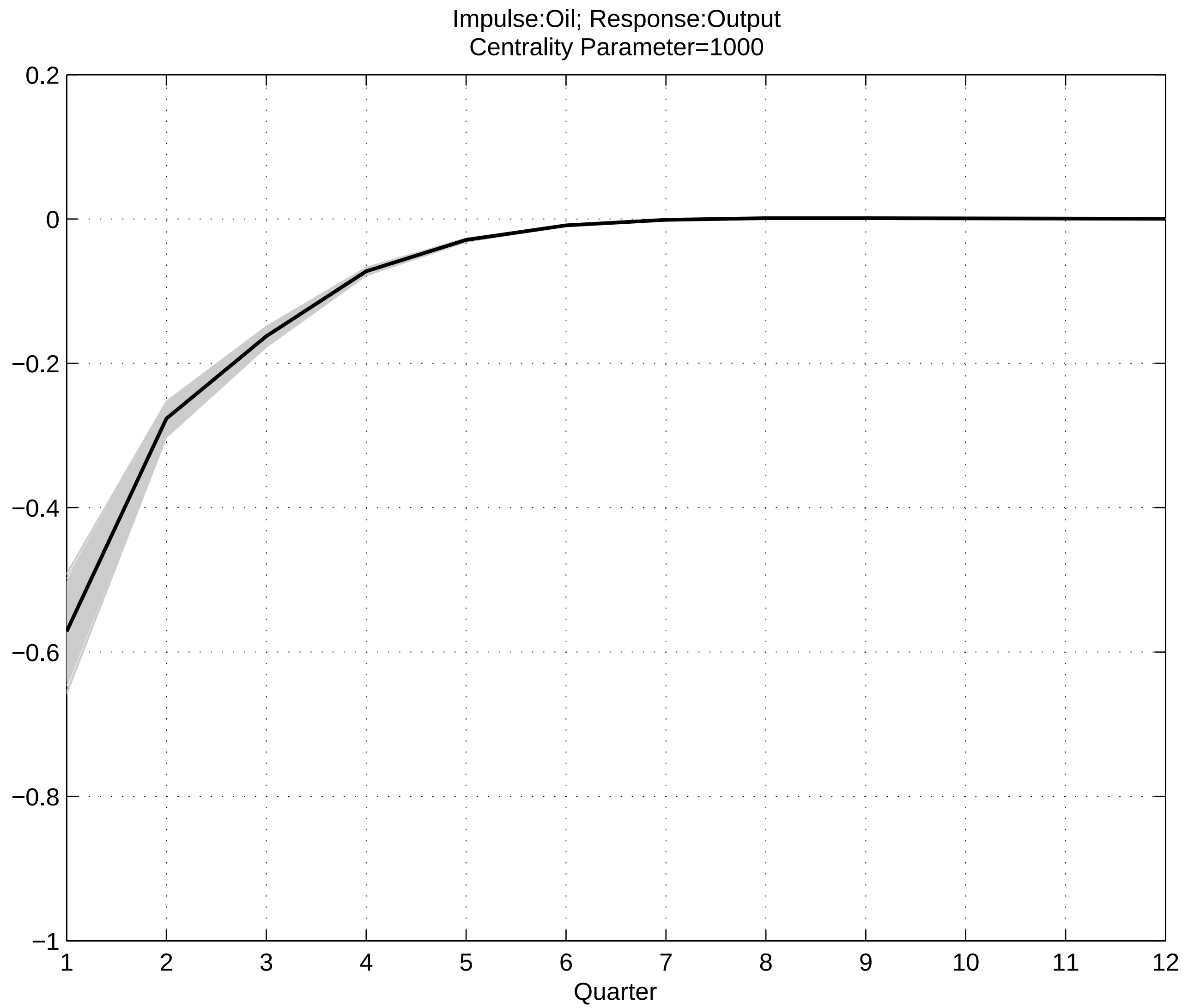
Effect of oil on GDP growth: $\lambda_1^2 = 10$



Effect of oil on GDP growth: $\lambda_1^2 = 20$



Effect of oil on GDP growth: $\lambda_1^2 = 100$



Effect of oil on GDP growth: $\lambda_1^2 = 1000$

Weak instrument asymptotics for HD and VD

Let $\Lambda = \Sigma_{\eta Z}(\Sigma_{Z\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta Z})^{-1}\Sigma_{Z\eta}$

HD: $H_1\varepsilon_{1t} = \Lambda\Sigma_{\eta\eta}^{-1}\ \eta_t$

VD: $\text{var}(H_1\varepsilon_{1t}) = \Lambda\Sigma_{\eta\eta}^{-1}\Lambda$

$$\hat{\Lambda} = \hat{\Gamma}(\hat{\Gamma}'\Sigma_{\eta\eta}^{-1}\hat{\Gamma})^{-1}\hat{\Gamma}'$$

where $\hat{\Gamma} = T^{-1/2}\sum_{t=1}^{T}\eta_t Z_t$ so that $vec(\hat{\Gamma}) \xrightarrow{d} N(vec(\Gamma),\Omega)$

$\hat{\Lambda} \xrightarrow{d}$ Function of noncentral Wishart r.v.s (Anderson & Girshick (1944))

Empirical Results: Example 2: Dynamic factor model:

$$X_t = \Lambda F_t + e_t \quad , \ [\mathrm{I}-A\,(\mathrm{L})]F_t = \eta_t \ , \ \ \eta_t = H\varepsilon_t$$

"First stage": F_1: regression of Z_t on η_t, F_2: regression of η_{1t} on Z_t

Structural Shock	***F*** $_1$	***F*** $_2$
1. Oil		
Hamilton	2.9	**15.7**
Killian	1.1	1.6
Ramey-Vine	1.8	0.6
2. Monetary policy		
Romer and Romer	4.5	**21.4**
Smets-Wouters	9.0	5.3
Sims-Zha	6.5	**32.5**
GSS	0.6	0.1
3. Productivity		
Fernald TFP	**14.5**	**59.6**
Smets-Wouters	7.0	**32.3**

Structural Shock	***F*** $_1$	***F*** $_2$
4. Uncertainty		
Fin Unc (VIX)	**43.2**	**239.6**
Pol Unc (BBD)	**12.5**	73.1
5. Liquidity/risk		
GZ EBP Spread	4.5	**23.8**
TED Spread	**12.3**	**61.1**
BCDZ Bank Loan	4.4	4.2
6. Fiscal policy		
Ramey Spending	0.5	1.0
Fisher-Peters Spending	1.3	0.1
Romer-Romer Taxes	0.5	2.1

Correlations among selected structural shocks

	O_K	M_{RR}	M_{SZ}	P_F	U_B	U_{BBD}	S_{GZ}	B_{BCDZ}	F_R	F_{RR}
O_K	1.00									
M_{RR}	**0.65**	1.00								
M_{SZ}	0.35	0.93	1.00							
P_F	0.30	0.20	0.06	1.00						
U_B	-0.37	-0.39	-0.29	0.19	1.00					
U_{BBD}	0.11	-0.17	-0.22	-0.06	0.78	1.00				
L_{GZ}	-0.42	-0.41	-0.24	0.07	**0.92**	**0.66**	1.00			
L_{BCDZ}	0.22	0.56	0.55	-0.09	**-0.69**	-0.54	-0.73	1.00		
F_R	**-0.64**	**-0.84**	**-0.72**	-0.17	0.26	-0.08	0.40	-0.13	1.00	
F_{RR}	0.15	**0.77**	**0.88**	0.18	0.01	-0.10	0.02	0.19	-0.45	1.00

$\text{Oil}_{\text{Kilian}}$	oil – Kilian (2009)
M_{RR}	monetary policy – Romer and Romer (2004)
M_{SZ}	monetary policy – Sims-Zha (2006)
P_F	productivity – Fernald (2009)
U_B	Uncertainty – VIX/Bloom (2009)
U_{BBD}	uncertainty (policy) – Baker, Bloom, and Davis (2012)
L_{GZ}	liquidity/risk – Gilchrist-Zakrajšek (2011) excess bond premium
L_{BCDZ}	liquidity/risk – BCDZ (2011) SLOOS shock
F_R	fiscal policy – Ramey (2011) federal spending
F_{RR}	fiscal policy – Romer-Romer (2010) federal tax

Weak instrument asymptotics for cross-shock correlation

Correlation between two identified shocks: Let Z_{1t} and Z_{2t} be scalar instruments that identify ε_{1t} and ε_{2t}:

$$\text{Cor}(\varepsilon_{1t}\varepsilon_{2t}) = \rho_{12} = \frac{\Sigma_{Z_1\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta Z_2}}{\sqrt{\Sigma_{Z_1\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta Z_1}}\sqrt{\Sigma_{Z_2\eta}\Sigma_{\eta\eta}^{-1}\Sigma_{\eta Z_2}}}$$

$$\begin{bmatrix} T^{-1/2}\sum Z_{1t}\eta_t \\ T^{-1/2}\sum Z_{2t}\eta_t \end{bmatrix} = \begin{bmatrix} \hat{\Gamma}_1 \\ \hat{\Gamma}_2 \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \right)$$

$$r_{12} = \frac{\hat{\Sigma}_{Z_1\eta}\Sigma_{\eta\eta}^{-1}\hat{\Sigma}_{\eta Z_2}}{\sqrt{\hat{\Sigma}_{Z_1\eta}\Sigma_{\eta\eta}^{-1}\hat{\Sigma}_{\eta Z_1}}\sqrt{\hat{\Sigma}_{Z_2\eta}\Sigma_{\eta\eta}^{-1}\hat{\Sigma}_{\eta Z_2}}} = \frac{\hat{\Gamma}_1\Sigma_{\eta\eta}^{-1}\hat{\Gamma}_2}{\sqrt{\hat{\Gamma}_1\Sigma_{\eta\eta}^{-1}\hat{\Gamma}_1}\sqrt{\hat{\Gamma}_2\Sigma_{\eta\eta}^{-1}\hat{\Gamma}_2}}$$

under null, $\Gamma_1'\Gamma_2 = 0$

$$\begin{bmatrix}\hat{\Gamma}_1\\ \hat{\Gamma}_2\end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix}, \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{bmatrix}\right), \text{ (hom } \Omega = \Sigma_{ZZ} \otimes \Sigma_{\eta\eta})$$

$$r_{12} = \frac{\hat{\Gamma}_1 \Sigma_{\eta\eta}^{-1} \hat{\Gamma}_2}{\sqrt{\hat{\Gamma}_1 \Sigma_{\eta\eta}^{-1} \hat{\Gamma}_1}\sqrt{\hat{\Gamma}_2 \Sigma_{\eta\eta}^{-1} \hat{\Gamma}_2}} \Rightarrow \frac{(\gamma_1 + \zeta_1)'(\gamma_2 + \zeta_2)}{\sqrt{(\gamma_1 + \zeta_1)'(\gamma_1 + \zeta_1)}\sqrt{(\gamma_2 + \zeta_2)'(\gamma_2 + \zeta_2)}}$$

$$\gamma_1 = \Sigma_{\eta\eta}^{-1/2}\Gamma_1/\sigma_{Z_1}^2 \text{ , } \gamma_2 = \Sigma_{\eta\eta}^{-1/2}\Gamma_2/\sigma_{Z2}^2$$

$$\zeta = \begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \sim \mathrm{N}(0, \bar{\Sigma} \otimes \mathrm{I}), \quad \bar{\Sigma} = \begin{bmatrix} 1 & corr(Z_1, Z_2)\\ corr(Z_1, Z_2) & 1\end{bmatrix}$$

Weak instrument asymptotics for cross-shock correlation, ctd.

$$r_{12} \Rightarrow \frac{(\gamma_1+\zeta_1)'(\gamma_2+\zeta_2)}{\sqrt{(\gamma_1+\zeta_1)'(\gamma_1+\zeta_1)}\sqrt{(\gamma_2+\zeta_2)'(\gamma_2+\zeta_2)}}$$

Comments

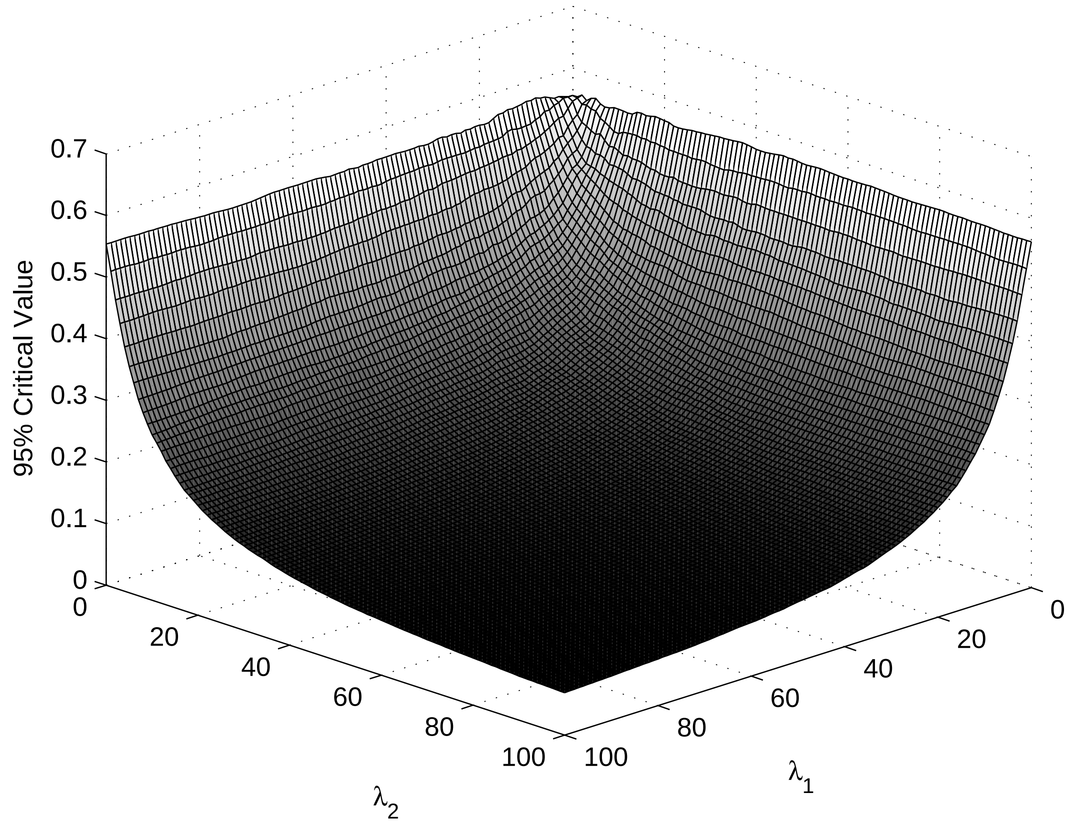
1. Nonstandard distribution – function of noncentral Wishart rvs
2. Normal under null as $\gamma_1'\gamma_1$ and $\gamma_2'\gamma_2 \to \infty$
3. Strong instruments under alternative: $r_{12} \xrightarrow{p} \frac{\gamma_1'\gamma_2}{\sqrt{\gamma_1'\gamma_1}\sqrt{\gamma_2'\gamma_2}}$

Weak instrument asymptotics for cross-shock correlation, ctd.

Numerical results: Asymptotic null distribution is a function of

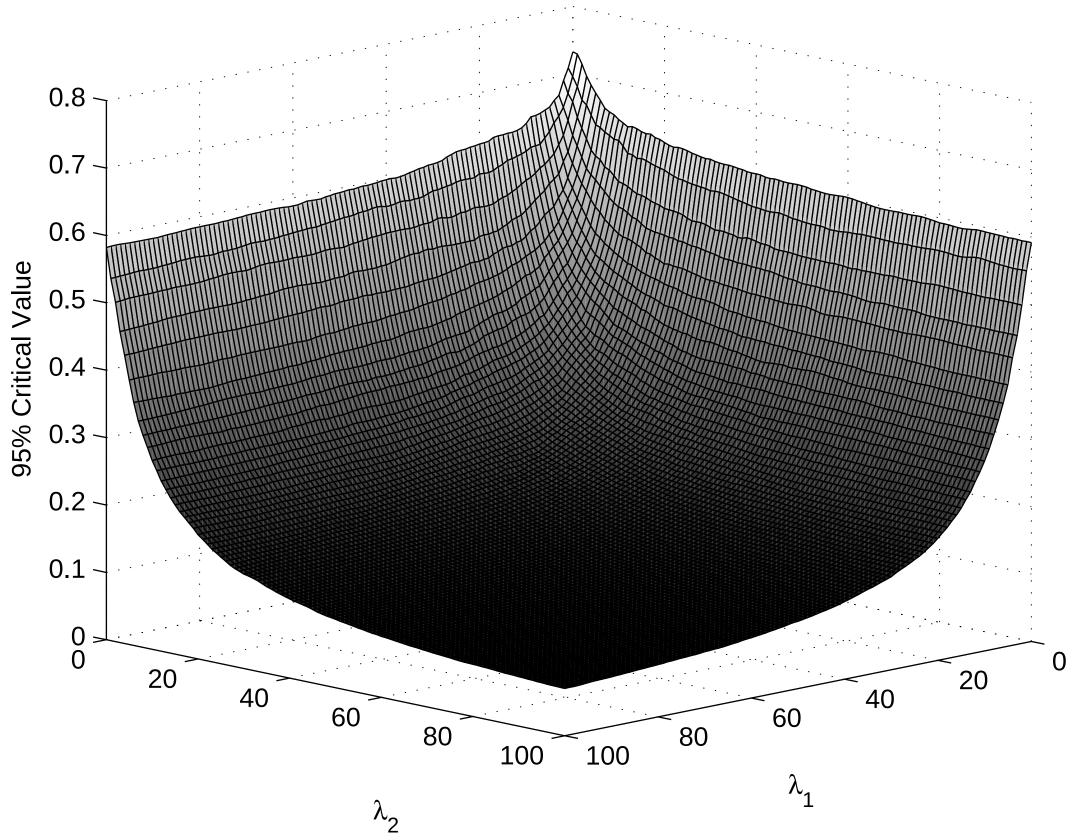
$$\gamma_1'\gamma_1 = \lambda_1,\ \gamma_2'\gamma_2 = \lambda_2 \text{ and } \operatorname{corr}(Z_1, Z_2)$$

Weak instrument asymptotics for cross-shock correlation, ctd.



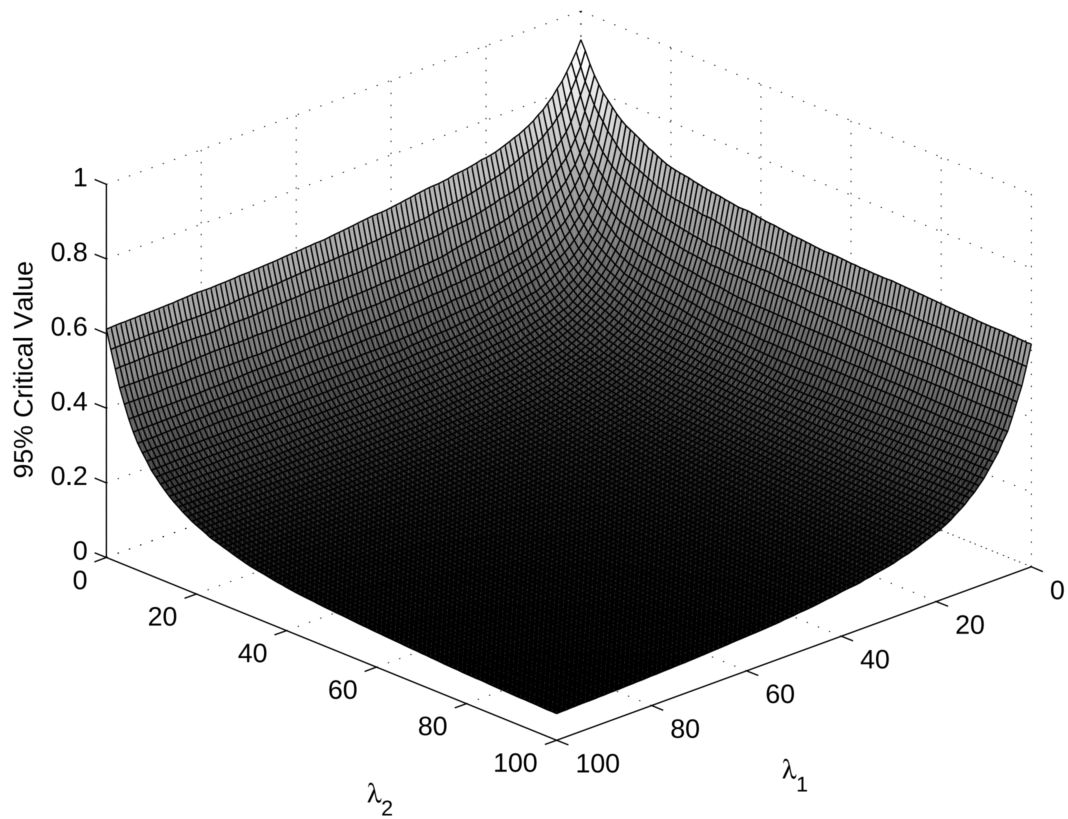
Weak instrument asymptotic null distribution of r_{12}: $|\mathrm{Corr}(Z_1,Z_2)| = 0$

Weak instrument asymptotics for cross-shock correlation, ctd.



Weak instrument asymptotic null distribution of r_{12}: $|\mathrm{Corr}(Z_1,Z_2)| = 0.4$

Weak instrument asymptotics for cross-shock correlation, ctd.



Weak instrument asymptotic null distribution of r_{12}: $|\mathrm{Corr}(Z_1,Z_2)| = 0.8$

Weak instrument asymptotics for cross-shock correlation, ctd.

Sup critical values (worst case over $\gamma_1'\gamma_1$ and $\gamma_2'\gamma_2$):

$\lvert\mathrm{corr}(Z_1, Z_2)\rvert$	**95 % critical value**
0	.5705
.2	.6253
.4	.7327
.6	.8406
.8	.9231

… go back to empirical results

Weak instrument asymptotics for reduced rank restriction

Let Z_{1t} and Z_{2t} be scalar instruments that identify ε_{1t}:

$\Sigma_{\eta Z} = H_1 \alpha'$ has rank 1

$$\begin{bmatrix} T^{-1/2}\sum Z_{1t}\eta_t \\ T^{-1/2}\sum Z_{2t}\eta_t \end{bmatrix} = \begin{bmatrix} \hat{\Gamma}_1 \\ \hat{\Gamma}_2 \end{bmatrix} \xrightarrow{d} N\left(\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \right)$$

where $\Gamma_2 = b\Gamma_1$

Non-HAC case: $\operatorname{var}\left(vec\left(T^{-1/2}\sum_{t=1}^{T}\eta_t Z_t{}'\right)\right) = \Sigma_{\eta\eta}\otimes\Sigma_{ZZ}$

LR = $\sum_{i=2}^{k}\theta_i$ where θ_i are the eigenvalues of

$$\left(\Sigma_{ZZ}^{-1/2}\left(T^{-1/2}\sum Z\eta'\right)\Sigma_{\eta\eta}^{-1}\left(T^{-1/2}\sum \eta Z'\right)\Sigma_{ZZ}^{-1/2}{}'\right)$$

Weak instrument limit

$$\left(\Sigma_{ZZ}^{-1/2}\left(T^{-1/2}\sum Z\eta'\right)\Sigma_{\eta\eta}^{-1}\left(T^{-1/2}\sum \eta Z'\right)\Sigma_{ZZ}^{-1/2}{}'\right)\Rightarrow\left((\zeta+\lambda)'(\zeta+\lambda)\right)$$

where vec(ζ) ~ N(0, $I_{r\times k}$) and $\lambda = \Sigma_{\eta\eta}^{-1/2}\Gamma\Sigma_{ZZ}^{-1/2}{}'$

$$\left(\Sigma_{ZZ}^{-1/2}\left(T^{-1/2}\sum Z\eta'\right)\Sigma_{\eta\eta}^{-1}\left(T^{-1/2}\sum \eta Z'\right)\Sigma_{ZZ}^{-1/2}{}'\right)\Rightarrow\left((\zeta+\lambda)'(\zeta+\lambda)\right)$$

Limiting distribution of OID test depends on vec(λ)'vec(λ).

vec(λ)'vec(λ) large, OID $\xrightarrow{d} \chi^2_{n-1}$

vec(λ)'vec(λ).= 0, OID = sum of n-1 smallest eigenvalues of $\zeta'\zeta$.

$n = 3$

(vec(λ)'vec(λ))$^{1/2}$	**95% CV**
100	7.8 (= χ^2_3 cv)
50	7.8
10	7.8
1	6.0
0	4.2

6. Weak-instrument robust inference

(1) All objects of interest are functions of $\Sigma_{\eta Z}$ ($= \Gamma/\sqrt{T}$)

$$\hat{\Gamma} = T^{-1/2}\sum \eta_t Z_t{}' \quad \text{and} \qquad vec(\hat{\Gamma}) \xrightarrow{d} N(vec(\Gamma), \Omega)$$

Construct Conf. Set for Γ :

$$\text{CS}(\Gamma) = \left(\Gamma \mid (vec(\hat{\Gamma}) - vec(\Gamma))'\Omega^{-1}(vec(\hat{\Gamma}) - vec(\Gamma)) < cv\right)$$

Joint CS for IRF(Γ), VD(Γ), HD(Γ), etc. determined by CS(Γ)

(2) Some objects have distributions that depend on, say vec(Γ)'vec(Γ).
Bonferroni.

(3) Best unbiased tests for a single IRF:

$$IRF = C_{h,j}H_1$$

Consider null hypothesis $IRF = C_{h,j}H_1 = \kappa_0$ with a single Z. Then $H_1 = \Gamma/\Gamma_1$, so null hypothesis is $C_{h,j}\Gamma - \kappa_0\Gamma_{11} = 0$. A single linear restriction on Γ.

With $\hat{\Gamma} \xrightarrow{d} N(\Gamma, \Omega)$, the best unbiased test in limiting problem rejects for large values of

$$|t^{\text{stat}}| = \frac{|C_{h,k}\hat{\Gamma} - \kappa_0 \hat{\Gamma}_{11}|}{SE(C_{h,k}\hat{\Gamma} - \kappa_0 \hat{\Gamma}_{11})}$$

Which can be inverted to find CS for *IRF* (κ).

Comments

- This is one degree of freedom test

- Conf. int. inversion can be done analytically (ratio of quadratics)

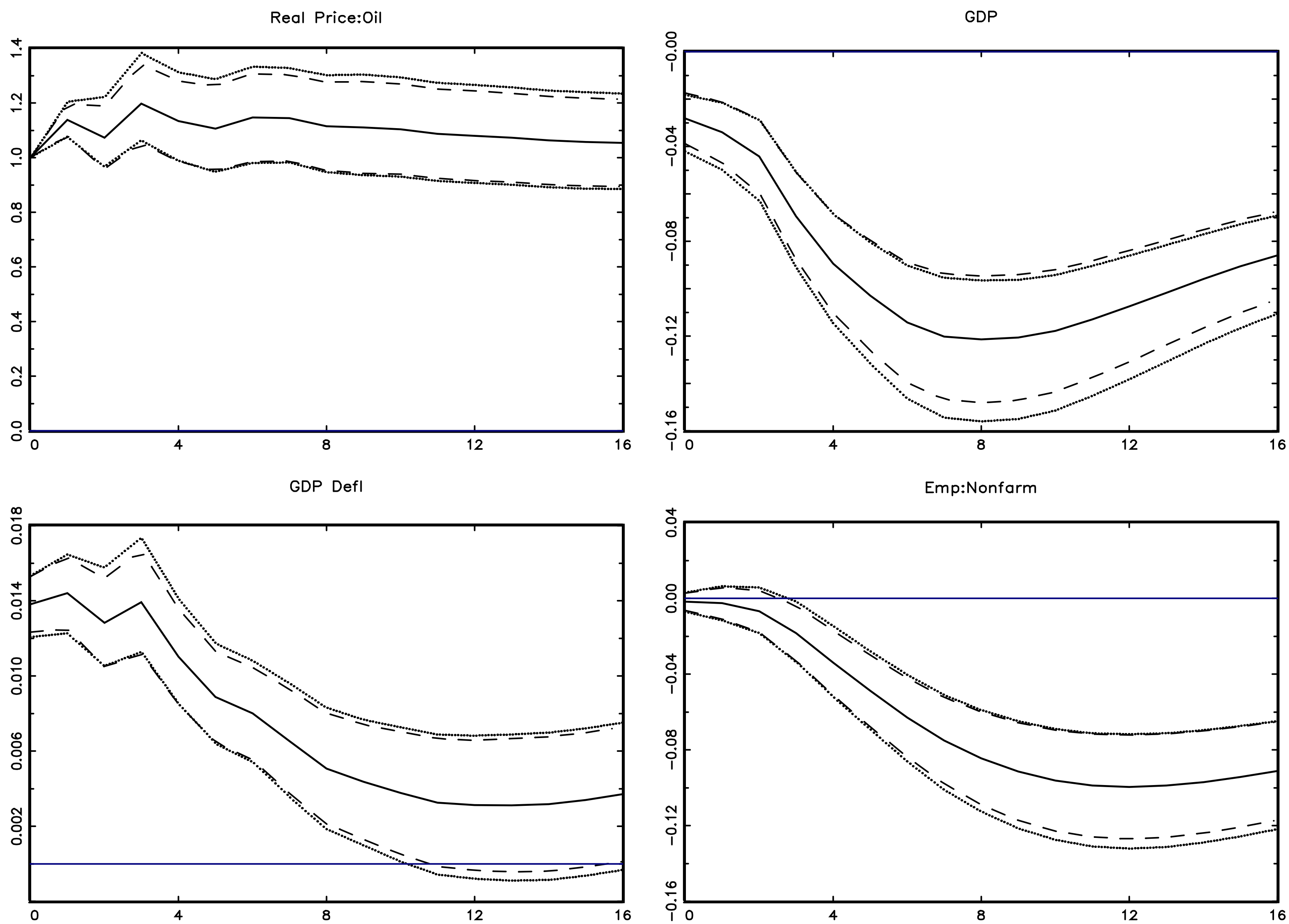
- Strong-instrument efficient (asy equivalent to standard GMM test)

- Multiple Z: The testing problem of H_0: $\kappa = \kappa_0$ can be rewritten as H_0: $\beta = \beta_0$ in the standard IV regression form,

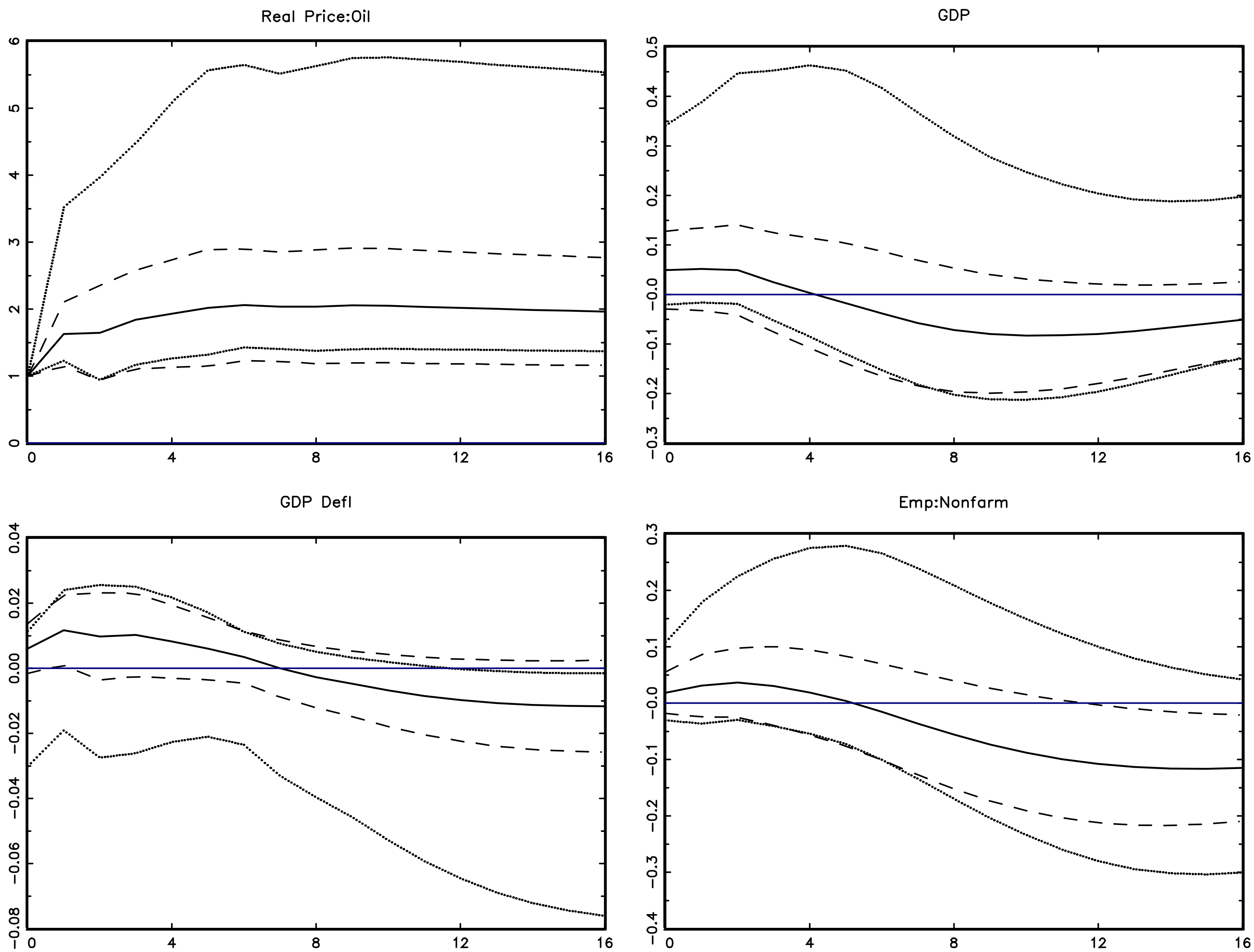
$$C(\kappa_0)'\eta_t = \beta_0\,\eta_{1t} + u_t$$
$$\eta_{1t} = \pi Z_t + v_t$$

so for multiple Z_t the Moreira-CLR confidence interval can be used. (Working on efficiency improvements)

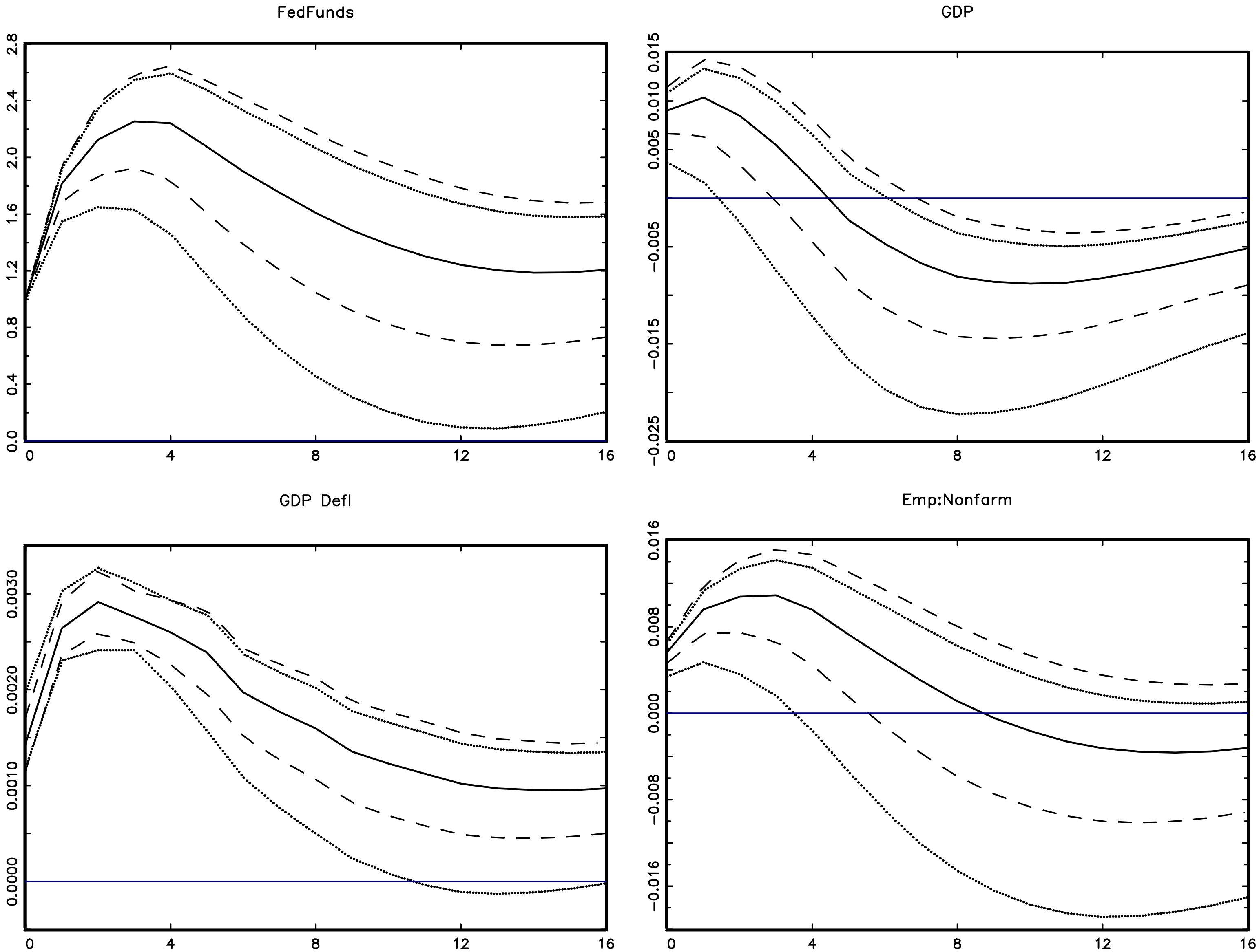
Examples: IRFs: strong-IV (dashed) and weak-IV robust (solid) pointwise bands



Hamilton (1996, 2003) oil shock ($F_2 = 15.7$)



Kilian (2008) oil shock ($F_2 = 1.6$)



Romer and Romer (2004) monetary policy shock ($F_2 = 21.4$)

Conclusions

<u>Work to do includes</u>

- Inference on correlations and on tests of overID restrictions in general
- Efficient inference for $k > 1$ (beyond Moreira-CLR confidence sets) – exploit equivariance restriction to left-rotations (respecify SVAR in terms of linear combination of Y's – this should reduce the dimension of the sufficient statistics in the limit experiment)
- Inference in systems imposing uncorrelated shocks
- Formally taking into account "higher order" ($O_p(T^{-1/2})$) sampling uncertainty of reduced-form VAR parameters
- HAC (non-Kronecker) case: (a) robustify; (b) efficient inference?