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Interval observer for uncertain time-varying SIR-SI model of vector-borne disease*

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Abstract— **The issue of state estimation is considered for an SIR-SI model describing a vector-borne disease such as dengue fever, with seasonal variations and uncertainties in the transmission rates. Assuming continuous measurement of the number of new infectives in the host population per unit time, a class of interval observers with estimate-dependent gain is constructed, and asymptotic error bounds are provided. The synthesis method is based on the search for a common linear Lyapunov function for monotone systems representing the evolution of the estimation errors.**

I. Introduction and presentation of the model

Vectors are living organisms that can transmit infectious diseases between humans or from animals to humans. Many of them are bloodsucking insects, which ingest disease-producing microorganisms during a blood meal from an infected host and inject it later into a new host during a subsequent blood meal. Vector-borne diseases account for more than 17% of all infectious diseases, causing more than 1 million deaths annually. As an example, more than 2.5 billion people in over 100 countries are at risk of contracting dengue, a vector-borne disease transmitted by some mosquito species of the genus *Aedes* [12]. Several of the arboviroses transmitted by the latter (Zika fever, chikungunya, dengue) have no satisfying vaccine or curative treatment so far, and prevention of the epidemics is a key to the control policy. In particular, the knowledge of the stock of susceptible individuals constitutes, among others, an important information to evaluate the probability of occurrence of such events.

The transmission dynamics of dengue is a quite complex issue, due to the role of cross-reactive antibodies for the four different dengue serotypes [1]. We will disregard here this multiserotype aspect, and will focus on a basic model for the evolution of a vector-borne disease [3], [4]. The latter is an SIR-SI [7] compartmental model with vital dynamics. It describes the evolution of the relative proportions of three classes, namely the susceptibles S, capable of contracting the disease and becoming infective; the infectives I, capable of transmitting the disease to susceptibles; and the recovered R, permanently immune after healing. Using index h for the host population and v for the vectors, the model writes

$$\dot{S}_h = \mu_h - \beta_h S_h I_v - \mu_h S_h \tag{1a}$$

$$\dot{I}_h = \beta_h S_h I_v - (\mu_h + \gamma) I_h \tag{1b}$$

$$\dot{R}_h = \gamma I_h - \mu_h R_h \tag{1c}$$

$$\dot{S}_v = \mu_v - \beta_v S_v I_h - \mu_v S_v \tag{1d}$$

$$\dot{I}_v = \beta_v S_v I_h - \mu_v I_v \tag{1e}$$

Notice that there is no recovery for the vectors, whose life duration is short compared to the disease dynamics. The parameters μ_h and μ_v represent birth and death rates for hosts and vectors. They are here assumed constant, and the disease is supposed not to induce supplementary mortality for the infected individuals of each populations. The parameters β_h and β_v, visible in the source terms for the evolution of the infected, are the transmission rates (between infective vectors and susceptible hosts, and between infective hosts and susceptible vectors). Last, γ is the host recovery rate, also assumed constant.

By construction, one has $\dot{S}_h + \dot{I}_h + \dot{R}_h = \dot{S}_v + \dot{I}_v = 0$, as the proportions verify $S_h + I_h + R_h = S_v + I_v \equiv 1$. This allows to remove one compartment of each population from the model, obtaining the following simplified system, which is the one studied in the sequel:

$$\dot{S}_h = \mu_h - \beta_h S_h I_v - \mu_h S_h, \tag{2a}$$

$$\dot{I}_h = \beta_h S_h I_v - (\mu_h + \gamma) I_h, \tag{2b}$$

$$\dot{I}_v = \beta_v (1 - I_v) I_h - \mu_v I_v, \tag{2c}$$

When the parameters are constant, the evolution of the solutions of system (2) depends closely upon the ratio $\mathscr{R}_0 := \frac{\beta_h \beta_v}{(\mu_h+\gamma)\mu_v}$. The disease-free equilibrium, defined by $S_h = 1$, $I_h = I_v = 0$ (and therefore $R_h = 0$, $S_v = 1$), always exists. When $\mathscr{R}_0 < 1$, it is the only equilibrium and it is globally asymptotically stable. It becomes unstable when $\mathscr{R}_0 > 1$, and an asymptotically stable endemic equilibrium then appears [4]. In practice, when the parameters are time-varying, complicated dynamics may occur. This is especially the case due to seasonal changes of the climatic conditions, alternating between periods that are favorable and unfavorable to the occurrence of epidemic bursts.

The conditions of transmission depend essentially on the climate conditions, and we will assume in the sequel that time-dependent lower and upper estimates of the two transmission rates are available in real-time. On the other hand, the only measurement of the state components typically available through Public Health Services is the *host*

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incidence, i.e. the number of new infected hosts per unit time $y(t) := \beta_h(t)S_h(t)I_v(t)$. *Our aim in this paper is to estimate at each moment the amount of the three involved populations* $S_h(t), I_h(t), I_v(t)$, *based on the measured incidence.*

An observer for this system has been proposed in the case of constant, known, parameters [11] (assuming the availability of measures of S_h and I_h), and this is up to our knowledge the unique contribution on this subject. We are interested in this paper in estimating the repartition of the host and vector populations in time-varying, uncertain, conditions. Our contribution is a class of *interval observers* [5] providing lower and upper estimates for the state components. Associated error estimates are found, that show, in particular, fast convergence towards the true values, in absence of uncertainties on the transmission rates.

Central use is made of the theory of *monotone systems* [6], [10] to synthesize interval observers (in the spirit e.g. of [9] and related contributions). The observer convergence is ensured by the search for *common linear Lyapunov functions* [8]. More precisely, the observer gains are chosen in order to maximize the convergence speed of the latter Lyapunov functions towards zero. This last point is, up to our knowledge, an original feature in such framework. Notice that related ideas have been applied to an SIR infection model [2]. Similarly to the situation presented in this reference, the epidemiological system studied here is *unobservable* in absence of infected, and the observers synthesized here attempt to exploit the epidemic bursts to speed up the state estimation.

The paper is organized as follows. The hypotheses on the model are presented in Section II, and some qualitative results on the latter are provided. The considered class of observers is given in Section III, together with some *a priori* estimates, and is proved to constitute a class of interval observers. The main result is provided in Section IV, where the asymptotic error corresponding to adequate gain choice is quantified. Several proofs have been gathered in the Appendix.

II. Model assumptions and properties

The host mortality rate (corresponding typically to several tens of years for humans) is assumed very small when compared to both the recovery and the vector mortality rate (both corresponding typically to some weeks). Our first hypothesis is therefore:

$$\mu_h \ll \gamma, \quad \mu_h \ll \mu_v. \tag{3}$$

The transmission rates β_h and β_v are supposed to be subject to seasonal variations, and hence both parameters will be functions of time. We assume moreover that their exact value is not known, but that they are bounded by lower and upper estimates $\beta_h^-(t), \beta_h^+(t)$ and $\beta_v^-(t), \beta_v^+(t)$, available in real-time: for any $t \geq 0$,

$$\beta_h^-(t) \leq \beta_h(t) \leq \beta_h^+(t), \qquad \beta_v^-(t) \leq \beta_v(t) \leq \beta_v^+(t). \tag{4}$$

System (2) is ("precisely") known when the previous relations hold with equalities.

Our first result shows that the solutions of system (2) respect expected boundedness results.

Lemma 1 (Properties of the model (2)*):* If the solution of system (2) verifies $S_h(0) \geq 0, I_h(0) \geq 0, I_v(0) \geq 0$, and $S_h(0) + I_h(0) \leq 1, I_v(0) \leq 1$, then the same properties are valid for any $t \geq 0$. The same is true for strict inequalities. □

See in Appendix a proof of Lemma 1.

III. Observers: definition and monotonicity properties

We now introduce the two following systems, and show in the sequel that, under appropriate conditions, they constitute interval observers for system (2):

$$\dot{S}_h^- = \mu_h(1 - S_h^-) - y + k_S^-(y - \beta_h^+ S_h^- I_v^+) \tag{5a}$$
$$\dot{I}_h^+ = y - (\mu_h + \gamma)I_h^+ \tag{5b}$$
$$\dot{I}_v^+ = \beta_v^+(1 - I_v^+)I_h^+ - \mu_v I_v^+ + k_v^+(y - \beta_h^- S_h^- I_v^+) \tag{5c}$$

$$\dot{S}_h^+ = \mu_h(1 - S_h^+) - y + k_S^+(y - \beta_h^- S_h^+ I_v^-) \tag{6a}$$
$$\dot{I}_h^- = y - (\mu_h + \gamma)I_h^- \tag{6b}$$
$$\dot{I}_v^- = \beta_v^-(1 - I_v^-)I_h^- - \mu_v I_v^- + k_v^-(y - \beta_h^+ S_h^+ I_v^-) \tag{6c}$$

As can be seen, output injection (the output is y) is used in this synthesis. The time-varying gains $k_S^\pm, k_v^\pm$ in (5b), (6b) are still to be defined.

Lemma 2 (Nonnegativity of the estimates): Suppose that for some ε_1, $\varepsilon_2 > 0$, the gains $k_S^\pm, k_v^\pm$ are chosen in such a way that, for any $t \geq 0$:

$$\mu_h + (k_S^\pm(t) - 1)y(t) \geq \varepsilon_1, \quad \text{whenever } S_h^\pm(t) \leq \varepsilon_2 \tag{7}$$

Then the solutions of system (5) are such that: if $S_h^-(t), I_h^+(t), I_v^+(t) \geq 0$ for $t = 0$, then the same remains true for any $t > 0$. Analogously, the solutions of system (6) are such that: if $S_h^+(t), I_h^-(t), I_v^-(t) \geq 0$ for $t = 0$, then the same remains true for any $t > 0$. □

Proof: Observe that, whenever S_h^- is close to zero, one has $\dot{S}_h^- \sim \mu_h + (k_S^- - 1)y \geq \varepsilon_1 > 0$, due to (7). One also has that $\dot{I}_h^+$ is nonnegative whenever I_h^+ is in a neighborhood of 0. The same holds for I_v^+, and this concludes the proof of Lemma 2 for system (5). The proof for system (6) is analogous. ■

The next result shows that, under sufficient conditions on the gain choice, (5) and (6) constitute interval observer for (2): they provide upper and lower estimates of the three state components.

Theorem 3 (Interval observer property): Assume that the gains $k_S^\pm, k_v^\pm$ are chosen in such a way that, for any $t \geq 0$, $k_S^\pm(t) \geq 0, k_v^\pm(t) \geq 0$ and condition (7) is verified. Suppose that the solutions of (5), (6) are such that $0 \leq S_h^-(t) \leq S_h(t) \leq S_h^+(t)$, $0 \leq I_h^-(t) \leq I_h(t) \leq I_h^+(t)$ and $0 \leq I_v^-(t) \leq I_v(t) \leq I_v^+(t)$ for $t = 0$. Then the same holds for any $t \geq 0$. □

To demonstrate Theorem 3, an instrumental decomposition result is now given, whose proof can be found in Appendix. Lemma 4 indicates in particular that the errors obey some linear positive systems: based on this remark, the proof of Theorem 3 is straightforward.

Lemma 4 (Observer errors dynamics): The observer errors attached to (5), (6), defined by

$$X_{(5)} := \begin{pmatrix} e_S^- \\ e_h^+ \\ e_v^+ \end{pmatrix} := \begin{pmatrix} S_h - S_h^- \\ I_h^+ - I_h \\ I_v^+ - I_v \end{pmatrix} \tag{8a}$$

$$X_{(6)} := \begin{pmatrix} e_S^+ \\ e_h^- \\ e_v^- \end{pmatrix} := \begin{pmatrix} S_h^+ - S_h \\ I_h - I_h^- \\ I_v - I_v^- \end{pmatrix} \tag{8b}$$

fulfill, for any $t \geq 0$, the equations

$$\dot{X}_{(5)}(t) = A_{(5)}(t)X_{(5)}(t) + b_{(5)}(t) \tag{9a}$$

$$\dot{X}_{(6)}(t) = A_{(6)}(t)X_{(6)}(t) + b_{(6)}(t) \tag{9b}$$

where $A_{(5)}(t), A_{(6)}(t)$ and $b_{(5)}(t), b_{(6)}(t)$ are defined in (10).

Moreover, for any $t \geq 0$, the matrices $A_{(5)}(t), A_{(6)}(t)$ are Metzler, and the vectors $b_{(5)}(t), b_{(6)}(t)$ are nonnegative, and null in the absence of uncertainties (that is when $\beta_h^-(t) = \beta_h(t) = \beta_h^+(t)$ and $\beta_v^-(t) = \beta_v(t) = \beta_v^+(t)$). □

Observe that the matrices $A_{(5)}(t), A_{(6)}(t)$ and vectors $b_{(5)}(t), b_{(6)}(t)$ defined in the statement depend upon time through the values of the transmission rates and their upper and lower estimates, but also through components of the initial system (2) and of the observers (5) and (6).

As a last remark, notice that it would be natural to introduce gains $k_h^\pm(t)$ in (5b), (6b). However, easy computations not reproduced here establish that the Metzler property in Lemma 4 requires their nullity.

IV. OBSERVERS: CONVERGENCE PROPERTIES

We now consider the issue of ensuring fast convergence of the estimates towards zero in the absence of uncertainties. As may be noticed, the dynamics of the (nonnegative) errors $e_h^\pm = |I_h - I_h^\pm|$ (see the second line of (9a) and (9b)) is not modified by the choice of the gains: the estimates of the proportion of infective hosts I_h converge at a constant rate $\mu_h + \gamma$ that essentially depends upon the recovery rate γ, as $\mu_h \ll \gamma$.

Observe that canceling the gains $k_S^\pm(t)$ and $k_v^\pm(t)$ yields converging estimates, see the matrices in (10a), (10b). But the fact that the host birth/death rate μ_h is very small makes it impracticable to settle for such slow convergence. We provide in the next result a way to cope with this issue and accelerate the estimation.

Theorem 5 (Convergence property): Assume that the assumptions of Theorem 3 are fulfilled and that, for *fixed positive* scalar $\rho_{(5)}, \rho_{(6)}, \omega_{(5)}, \omega_{(6)}, \varepsilon_{(5)}, \varepsilon_{(6)}$, the gains $k_S^\pm(t), k_v^\pm(t)$ are chosen in such a way that

$$k_S^-(t)\beta_h^+(t) - \omega_{(5)}k_v^+(t)\beta_h^-(t) = \xi_{(5)}(t), \tag{11a}$$

$$k_S^+(t)\beta_h^-(t) - \omega_{(6)}k_v^-(t)\beta_h^+(t) = \xi_{(6)}(t), \tag{11b}$$

where

$$\xi_{(5)}(t) := \min\left\{\gamma - \varepsilon_{(5)}; \frac{\omega_{(5)}(\mu_v - \mu_h + \beta_v^+(t)I_h^+(t))}{S_h^+(t) + I_v^+(t)}\right\}, \tag{12a}$$

$$\xi_{(6)}(t) := \min\left\{\gamma - \varepsilon_{(6)}; \frac{\omega_{(6)}(\mu_v - \mu_h + \beta_v^-(t)I_h^-(t))}{S_h^+(t) + I_v^-(t)}\right\}. \tag{12b}$$

Then, along any trajectories of (5), (6) one has, for all $t \geq 0$,

$$0 \leq V_{(5)}(t) \leq e^{-\int_0^t \delta_{(5)}(s)ds}\, V_{(5)}(0) + \int_0^t e^{-\int_s^t \delta_{(5)}(\tau)d\tau} F_{(5)}(s)ds, \tag{13a}$$

$$0 \leq V_{(6)}(t) \leq e^{-\int_0^t \delta_{(6)}(s)ds} V_{(6)}(0) + \int_0^t e^{-\int_s^t \delta_{(6)}(\tau)d\tau} F_{(6)}(s)ds, \tag{13b}$$

where

$$V_{(5)} = (S_h - S_h^-) + \rho_{(5)}(I_h^+ - I_h) + \omega_{(5)}(I_v^+ - I_v), \tag{14a}$$

$$V_{(6)} = (S_h^+ - S_h) + \rho_{(6)}(I_h - I_h^-) + \omega_{(6)}(I_v - I_v^-), \tag{14b}$$

are positive definite functions,

$$\delta_{(5)}(t) := \mu_h + \xi_{(5)}(t)I_v^+(t), \tag{15a}$$

$$\delta_{(6)}(t) := \mu_h + \xi_{(6)}(t)I_v^-(t), \tag{15b}$$

and

$$F_{(5)} := \omega_{(5)}\Big(k_v^+(\beta_h^+ - \beta_h^-)S_h I_v + (\beta_v^+ - \beta_v^-)I_h(1 - I_v)\Big) + k_S^-(\beta_h^+ - \beta_h^-)S_h I_v \tag{16a}$$

$$F_{(6)} := \omega_{(6)}\Big(k_v^-(\beta_h^+ - \beta_h^-)S_h I_v + (\beta_v^+ - \beta_v^-)I_h(1 - I_v)\Big) + k_S^+(\beta_h^+ - \beta_h^-)S_h I_v \tag{16b}$$

□

When the assumptions of Theorem 5 are fulfilled, inequalities (13) provide guaranteed bounds on the error estimates. In the absence of uncertainties, the functions in (16) are identically null (see Lemma 4), and only the first terms remain in the right-hand sides of (13). In such a case, the errors converge towards zero exponentially. An important point is that this convergence may occur with quite slow pace: when the estimates on the infectives are small, the convergence occurs at the natural rates of system (2). In fact, the speed of convergence is increasing with these estimates: the observers take the opportunity of the epidemic bursts to provide tighter estimates more rapidly. Notice that by construction $\xi_{(5)}(t), \xi_{(6)}(t) \leq \gamma$: $\delta_{(5)}(t), \delta_{(6)}(t)$ are therefore limited to take the maximal value $\mu_h + \gamma$, and the interest of the gain choice defined by (11)-(12) is to aim for this performance. As a final remark, notice that the system is unobservable in absence of infected: the observers try to speed up the estimate convergence during the epidemic bursts.

$$A_{(5)} = \begin{pmatrix} -\mu_h - k_S^- \beta_h^+ I_v^+ & 0 & k_S^- {\beta_h}^+ S_h \\ 0 & -(\mu_h+\gamma) & 0 \\ k_v^+ {\beta_h}^- I_v^+ & \beta_v^+(1-I_v) & -k_v^+ {\beta_h}^- S_h - \beta_v^+ I_h^+ - \mu_v \end{pmatrix},\ b_{(5)} = \begin{pmatrix} k_S^-(\beta_h^+ - \beta_h) S_h I_v \\ 0 \\ k_v^+(\beta_h - {\beta_h}^-) S_h I_v + (\beta_v^+ - \beta_v) I_h (1-I_v) \end{pmatrix} \tag{10a}$$

$$A_{(6)} = \begin{pmatrix} -\mu_h - k_S^+ \beta_h^- I_v^- & 0 & k_S^+ {\beta_h}^- S_h \\ 0 & -(\mu_h+\gamma) & 0 \\ k_v^- {\beta_h}^+ I_v^- & \beta_v^-(1-I_v) & -k_v^- {\beta_h}^+ S_h - \beta_v^- I_h^- - \mu_v \end{pmatrix},\ b_{(6)} = \begin{pmatrix} k_S^+(\beta_h - {\beta_h}^-) S_h I_v \\ 0 \\ k_v^-(\beta_h^+ - \beta_h) S_h I_v + (\beta_v - \beta_v^-) I_h (1-I_v) \end{pmatrix} \tag{10b}$$

APPENDIX

A. Proof of Lemma 1

Assume that $S_h(0) \geq 0, I_h(0) \geq 0, I_v(0) \geq 0$. Observe that whenever $S_h = 0$, one has $\dot{S}_h = \mu_h > 0$. Hence, $S_h(t) \geq 0$. Let us rewrite (2b)-(2c) in the more convenient form

$$\begin{pmatrix} \dot{I}_h \\ \dot{I}_v \end{pmatrix} = \begin{pmatrix} -(\mu_h+\gamma) & \beta_h S_h \\ \beta_v & -(\beta_v+\mu_v) \end{pmatrix} \begin{pmatrix} I_h \\ I_v \end{pmatrix} \tag{17}$$

Since S_h remains always nonnegative, system (17) above turns out to be monotone [6], [10] and, consequently, $I_h(t) \geq 0, I_v(t) \geq 0$ whenever starting from nonnegative initial values.

On the other hand, if $S_h + I_h = 1$ then $\frac{d}{dt}(S_h + I_h) = -\gamma I_h \leq 0$. Also, whenever $I_v = 1$, one has $\dot{I}_v = -\mu_v < 0$. Hence, $S_h + I_h$ and I_v remain under 1 if their initial values are located under 1.

In order to prove the same properties for the strict inequalities, first integrate (2) to obtain, for any $t \geq 0$,

$$S_h(t) = S_h(0) e^{-\int_0^t (\mu_h + \beta_h(s) I_v(s)) ds} + \mu_h \int_0^t e^{-\int_s^t (\mu_h + \beta_h(\tau) I_v(\tau)) d\tau} ds,$$

$$I_h(t) = I_h(0) e^{-\int_0^t (\mu_h+\gamma) ds} + \int_0^t e^{-(\mu_h+\gamma)(t-s)} \beta_h(s) S_h(s) I_v(s) ds$$

and

$$I_v(t) = I_v(0) e^{-\int_0^t (\mu_v + \beta_v(s) I_h(s)) ds} + \int_0^t e^{-\int_s^t (\mu_v + \beta_v(\tau) I_h(\tau)) d\tau} \beta_v(s) I_h(s) ds.$$

From this we can easily deduce that all three variables remain strictly positive if starting from strictly positive initial values, using the nonnegativity of the three components, that has been previously proved.

Finally, note that, for any $t \geq 0$,

$$(1 - S_h(t) - I_h(t)) = (1 - S_h(0) - I_h(0)) e^{-\int_0^t \mu_h ds} + \int_0^t e^{-\int_s^t \mu_h d\tau} \gamma I_h(s) ds$$

and

$$(1 - I_v(t)) = (1 - I_v(0)) e^{-\int_0^t \beta_v(s) I_h(s) ds} + \int_0^t e^{-\int_s^t \beta_v(\tau) I_h(\tau) d\tau} \mu_v I_v(s) ds.$$

Thus if $S_h(0) + I_h(0) < 1$, then $S_h + I_h$ remains strictly under 1. The same holds for I_v. This completes the proof of Lemma 1. ∎

B. Proof of Lemma 4

Let us first establish (9a). From (2), (5) and the fact that $y = \beta_h S_h I_v$, we get:

$$\begin{aligned} \dot{e}_S^- &= [\mu_h - \beta_h S_h I_v - \mu_h S_h] \\ &\quad -[\mu_h(1 - S_h^-) - y + k_S^-(y - \beta_h^+ S_h^- I_v^+)] \\ &= \mu_h(S_h^- - S_h) - k_S^-(y - \beta_h^+ S_h^- I_v^+) \\ &= -\mu_h e_S^- - k_S^-(y - \beta_h^+ S_h^- I_v^+) \end{aligned} \tag{18}$$

$$\begin{aligned} \dot{e}_h^+ &= [y - (\mu_h+\gamma) I_h^+] - [\beta_h S_h I_v - (\mu_h+\gamma) I_h] \\ &= -(\mu_h+\gamma)(I_h^+ - I_h) = -(\mu_h+\gamma) e_h^+ \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{e}_v^+ &= [{\beta_v}^+(1 - I_v^+) I_h^+ - \mu_v I_v^+ + k_v^+(y - {\beta_h}^- S_h^- I_v^+)] \\ &\quad -[\beta_v(1 - I_v) I_h - \mu_v I_v] \\ &= [\beta_v^+ I_h^+ - \beta_v^+ I_v^+ I_h^+ - \mu_v I_v^+ + k_v^+(y - {\beta_h}^- S_h^- I_v^+)] \\ &\quad -[\beta_v I_h - \beta_v I_v I_h - \mu_v I_v] \end{aligned}$$

By adding and substracting $({\beta_v}^+ I_h + \beta_v^+ I_v I_h + \beta_v^+ I_v I_h^+)$ on the right-hand side of the previous formula, we get

$$\begin{aligned} \dot{e}_v^+ &= -(\beta_v - \beta_v^+) I_h - \beta_v^+(I_h - I_h^+) + (\beta_v - \beta_v^+) I_v I_h \\ &\quad +\beta_v^+ I_v (I_h - I_h^+) + \beta_v^+(I_v - I_v^+) I_h^+ \\ &\quad -\mu_v e_v^+ + k_v^+(y - \beta_h^- S_h^- I_v^+) \\ &= (\beta_v^+ - \beta_v) I_h + \beta_v^+ e_h^+ - (\beta_v^+ - \beta_v) I_v I_h - \beta_v^+ I_v e_h^+ \\ &\quad -\beta_v^+ e_v^+ I_h^+ - \mu_v e_v^+ + k_v^+(y - \beta_h^- S_h^- I_v^+) \end{aligned} \tag{20}$$

We now treat the terms $y - \beta_h^+ S_h^- I_v^+$ in (18) and $y - \beta_h^- S_h^- I_v^+$ in (20). Since $y - {\beta_h}^+ S_h^- I_v^+ = \beta_h S_h I_v - \beta_h^+ S_h^- I_v^+$, one has

$$y - {\beta_h}^+ S_h^- I_v^+ = (\beta_h - \beta_h^+) S_h I_v - {\beta_h}^+ S_h e_v^+ + {\beta_h}^+ e_S^- I_v^+ \tag{21a}$$

and similarly

$$y - {\beta_h}^- S_h^- I_v^+ = (\beta_h - \beta_h^-) S_h I_v - {\beta_h}^- S_h e_v^+ + {\beta_h}^- e_S^- I_v^+ \tag{21b}$$

Insering (21a) in (18) and (21b) in (20), we get

$$\begin{aligned} \dot{e}_S^- &= -\mu_h e_S^- - k_S^-((\beta_h - \beta_h^+) S_h I_v - {\beta_h}^+ S_h e_v^+ + {\beta_h}^+ e_S^- I_v^+) \\ \dot{e}_v^+ &= (\beta_v^+ - \beta_v) I_h + \beta_v^+ e_h^+ - (\beta_v^+ - \beta_v) I_v I_h - \beta_v^+ I_v e_h^+ \\ &\quad -\beta_v^+ e_v^+ I_h^+ - \mu_v e_v^+ \\ &\quad +k_v^+((\beta_h - \beta_h^-) S_h I_v - \beta_h^- S_h e_v^+ + \beta_h^- e_S^- I_v^+) \end{aligned}$$

which finally yields (9a), together with (10a). The proof is the same for system (9b).

Last, the fact that the matrices $A_{(5)}(t), A_{(6)}(t)$ are Metzler (i.e. that their off-diagonal components are nonnegative), and that the vectors $b_{(5)}(t), b_{(6)}(t)$ are nonnegative and null in the absence of uncertainties, comes directly from the formulas previously proved and the estimates in Lemma 1 and 2. This achieves the proof of Lemma 4. ■

C. Proof of Theorem 5

We demonstrate here the claimed property for $V_{(5)}$, the case of $V_{(6)}$, being analogous, will not be treated. *Throughout this proof we remove the index* (5) *from the variables and parameters in order to simplify the notation.*

Defining the vector $u := \begin{pmatrix}1 & \rho & \omega\end{pmatrix}^\intercal$, the (state) function $V(t)$ defined in (14a) writes $V(t) = u^\intercal X(t)$ (recall that the error vector X has been given in (8a)). Therefore, writing as usual $\dot{V}$ its derivative along the trajectories, one has, due to formula (9a) in Lemma 4,

$$\dot{V} + \delta(t)V(t) = u^\intercal(A(t) + \delta(t)I)X(t) + u^\intercal b(t) \tag{22}$$

where I denotes de identity matrix. Now notice that, with F defined in (16a), one has $u^\intercal b(t) \leq F(t)$. We will show next that

$$u^\intercal(A(t) + \delta(t)I) \leq 0 \tag{23}$$

Given these two facts, it will be possible to deduce from (22) that

$$\dot{V} + \delta(t)V(t) \leq F(t)$$

which gives (13a) by Gronwall's lemma and thus achieves the proof. It therefore remains to show (23) in order to complete the proof of Theorem 5.

We now demonstrate that (23) holds when the gains are chosen as prescribed in the statement. Equation (23) can be written as the following system of inequalities, valid for any $t \geq 0$:

$$\delta \leq \mu_h + k_S^- \beta_h^+ I_v^+ - \omega k_v^+ \beta_h^- I_v^+, \tag{24a}$$

$$\delta \leq \mu_h + \gamma - \frac{\omega}{\rho}\beta_v^+(1 - I_v), \tag{24b}$$

$$\delta \leq \mu_v + k_v^+ \beta_h^- S_h + \beta_v^+ I_h^+ - \frac{k_S^-}{\omega}\beta_h^+ S_h. \tag{24c}$$

Note that if we set the gains k_S^- and k_v^+ to zero and choose $\frac{\omega}{\rho}$ small enough, we can see that δ can be at least equal to μ_h, due also to the fact that $\mu_v > \mu_h$. That is, μ_h is a lower bound of the convergence speed.

On the other hand, since the last term in the right-hand side of (24b) is nonpositive, δ cannot be greater or equal than $\mu_h + \gamma$. For this reason, one has to choose δ in (15a) that satisfies, for any $t \geq 0$,

$$\mu_h \leq \delta(t) < \mu_h + \gamma.$$

Set

$$\omega := 1, \quad \rho := \sup_{t \geq 0} \frac{\beta_v^+(t)(1 - I_v^-(t))}{\varepsilon}. \tag{25}$$

Using (11a) and (12a) in (24), yields the following equivalent set of inequalities:

$$\xi I_v^+ \leq I_v^+ \xi, \tag{26a}$$

$$\xi I_v^+ \leq \gamma - \frac{1}{\rho}\beta_v^+(1 - I_v), \tag{26b}$$

$$\xi I_v^+ \leq \mu_v - \mu_h - \xi S_h + \beta_v^+ I_h^+. \tag{26c}$$

Observe that (26a) is trivially verified. Also, by the fact that ξ obeys (12a), one has $\xi \leq \gamma - \varepsilon$. Therefore,

$$\frac{1}{\gamma - \xi I_v^+} \leq \frac{1}{\gamma - \xi} \leq \frac{1}{\varepsilon}, \tag{27}$$

where the first inequality holds since $I_v^+ \leq 1$. Using (27) and (25) yields, for any $t \geq 0$,

$$\frac{\beta_v^+(t)(1 - I_v^-(t))}{\gamma - \xi I_v^+} \leq \frac{\beta_v^+(t)(1 - I_v^-(t))}{\varepsilon} \leq \rho,$$

which implies (26b).

Last, by the definition of ξ given in (12a) we have, for any $t \geq 0$,

$$\xi(t) \leq \frac{\mu_v - \mu_h + \beta_v^+(t) I_h^+(t)}{S_h^+(t) + I_v^+(t)},$$

which implies (26c) since $S_h^+ \geq S_h$.

Then, the inequalities in (23) are verified for the chosen values of the parameters. This achieves the proof of Theorem 5. ■

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