Escola de Pós-Graduação em Economia - EPGE
Fundação Getúlio Vargas

ESSAYS ON THE MONETARY ASPECTS OF THE TERM STRUCTURE OF NOMINAL INTEREST RATES

Tese submetida à Escola de Pós-Graduação em Economia da Fundação Getúlio Vargas como requisito de obtenção do Título de Doutor em Economia

Aluno: Ricardo Dias de Oliveira Brito

Professor Orientador: Renato Galvão Flôres Jr.

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Rio de Janeiro, 5 de Setembro de 2001

Ricardo Dias de Oliveira Brito

À Márcia e Maurício Brito,

melhores amigos e maiores mestres.

Índice

Essays on the Monetary Aspects of the Term Structure of Nominal Interest Rates

Ricardo D. Brito

October 4, 2001

1 Introduction

Interest rates are key economic variables to much of finance and macroeconomics, and an enormous amount of work is found in both fields about the topic. Curiously, in spite of their common interest, finance and macro research on the topic have seldom interacted, using different approaches to address its main issues with almost no intersection. Concerned with interest rate contingent claims, finance term structure models relate interest rates to lagged interest rates; concerned with economic relations and macro dynamics, macro models regress a few interest rates on a wide variety of economic variables. If models are true though simplified descriptions of reality, the relevant factors should be captured by both the set of bond yields and that of economic variables. Each approach should be able to address the other field concerns with equal efficiency, since the economic variables are revealed by the bond yields and these by the economic variables.

However, models are also approximations that choose a subset of states to imperfectly capture the full set of factors. The set of yields and the set of economic variables do not capture the same factors nor give the same relative importance to each factor. Indeed, financial models have not accomplished to properly incorporate macro variables, despite the belief that their changes are major sources of changes in the shape of the yield curve. Macro models, on the other hand, have not accomplished to properly incorporate the yield curve, despite the belief that their shape is an indicator of economic conditions.

Many stylized facts still challenging established theory may find explanation in this largely unexplored complementarity. Some that deserve mention are:

(i) the pro-cyclical pattern of the level of nominal interest rates (Fama & French [33]);

(ii) the countercyclical pattern of the term spread[1], as well as the low sensitivity of long yields to monetary policy changes (Fama & French [33] and Thornton [66]);

(iii) the pro-cyclical pattern of the curvature of the term structure;

(iv) the lower predictability of the slope of the middle of the yield curve (Campbell, Lo & MacKinlay [18]);

(v) the negative correlation of changes in real rates and expected inflation at short horizons (Campbell & Ammer [17] and Barr and Campbell [10]);

(vi) almost all variance of the yields can be explained by three unknown factors that impact on the level, slope and curvature of the term structure

[1] The term spread is defined as the difference between the yield-to-maturities of a long and a short term bond.

(Litterman & Scheinkman [43]);

(vii) the high sensitivity of the volatility of the short-rate change to levels (Chan *et al.* [19]);

(viii) the low predictability of the short-rate changes (Chan *et al.* [19]);

(ix) the overreaction of the short-term rate;

(x) the leptokurtosis of the density of the yield changes at high-frequency data.

This thesis intends to be a contribution to the integration of the finance and the macro theories of interest rates. The three essays address monetary aspects of the term structure of nominal interest rates. Along them, the discontinuous setting of the nominal short-rate by the monetary authority plays a crucial role. Either explicitly or implicitly, the monetary authority forces the left-end of the term structure to match an exogenously specified level through discontinuous changes of the short-term nominal interest rate. Given that the monetary authority is constrained to keep inflation close to zero, future changes in the controlled rate can be forecasted by looking at the dynamics of the expected inflation and may be incorporated into the shape of the term structure. In the remaining of this section we briefly summarize the three papers.

1.1 Stochastic Growth and Monetary Policy: the impacts on the term structure of interest rates

This paper builds a simple intertemporal optimization model with staggered prices *a la* Fuhrer & Moore [35], investment costs and a policy rule for setting short-rates as a function of the past inflation, which is used to explain the

cyclical patterns of the term structure of nominal interest rates.

Neutral interest rate is formally defined and the net disinvestment on bonds is proposed as a microeconomics foundation for the excess demand term that appears in the Fuhrer & Moore's [35] sticky inflation equation. Fixed by the policy rule:

$$i_{t+1} = i_t + v_t,$$

where i_t is the nominal short-rate, $v_t = \left\{ \begin{array}{l} 0 \; with \; probability \; (1 - \varsigma \left|\pi_{t-1}\right|) \\ \\ e\frac{\pi_{t-1}}{\left|\pi_{t-1}\right|} \; with \; probability \; \varsigma \left|\pi_{t-1}\right| \end{array} \right\}$ is the monetary policy shock, π_{t-1} is the last period inflation, and e and ς are positive constants, the nominal short-rate process combines with the inflation stickiness assumption to break monetary policy neutrality. The resulting model is able to explain the following stylized facts:

(i) pro-cyclical pattern of the level of nominal interest rates;

(ii) countercyclical pattern of the term spread;

(iii) pro-cyclical pattern of the curvature of the term structure;

(iv) lower predictability of the slope of the middle of the yield curve; and

(v) negative correlation of changes in expected inflation and real rates at short horizons.

The model extends Balduzzi, Bertola & Foresi's [8], Rudebusch's [62], Tice & Webber's [67] and Piazzesi's [60] analyses of the monetary policy impacts on the term structure in the sense that it is done in an intertemporal optimization framework and allows the joint explanation of more stylized facts.

1.2 A Jump-Diffusion Yield-Factor Model of Interest Rates

This paper builds a two yields-factor model of interest-rate derivatives pricing, which is applied to the U.S. Treasury nominal zero-coupon bonds. The two yields chosen are the Federal Fund Rate Target and the one-year yield, what allows to separate the monetary policy risk from the remaining macroeconomic risk.

The Federal Fund Rate Target is shown to be a Poisson process, given the expectations of the macroeconomic conditions incorporated in the one-year yield. The one-year yield is parametrized to result in a consistent model, in accordance with Duffie and Kan [28].

Given the two states, the model adjusts well to the U.S. average yield curve and to the actual yield curve from January 1990 to December 2000. Furthermore, (vi) almost all variance of yields can be described in terms of the one-year yield, the Federal Fund Rate Target and their spread, respectively shown to impact on the level, the slope and the hump of the term structure of nominal interest rates.

The model is a simple alternative to Babbs & Webber [6], Attari [5], El-Jahel *et al.* [30], Duffie *et al.* [29], Piazzesi [58] and Das [25].

1.3 What the short-term interest rate target can do for mean-reverting modelling

This paper shows that the bad fit of univariate constant-mean-reverting continuous diffusion processes to Federal Fund Rate data (the U.S. overnight rate)

is due to the omission of its target, discontinuously set by the Federal Open Market Committee. With a constant long run mean, neither the *level volatility* models like Chan *et al.* [19], nor the *stochastic level volatility* models suggested by Brenner *et al.* [13] are able to explain the rate changes or the squared rate changes. The variable-mean-reverting diffusion process that combines the rate target as the long-run mean with the *stochastic level volatility* results in an improved fit and forecasts a considerable proportion of both changes.

The puzzling (vii) Chan *et al.*'s [19] high sensitivity of the volatility of the short-rate change to levels shrinks; while (viii) the predictability of the short-rate changes improves. (ix) The apparent Federal Fund Rate overreaction is also solved. Because the rate target is generated by a jump process and the volatility seems to be of the stochastic level kind, (x) the density of yield changes are not normally distributed and fatter tails are justified. Furthermore, weak evidences are produced that the rate target might outperform the Federal Fund Rate as a short-term factor in term structure models.

The model extends Chan *et al.* [19], Brenner *et al.* [13] and Hamilton [37] specifications.

2 Conclusion and Extensions

Understanding the monetary aspects involved in the term structure of nominal interest rates seems to payoff in improved ability to explain the interest rate dynamics as the three essays of this thesis have shown. Represented by the short-rate target, the monetary policy played an important role at different

levels of abstraction: in a complete economy model like *Stochastic Growth and Monetary Policy*, in a more specific asset pricing model like *A Jump-Diffusion Yield-Factor Model of Interest Rates*, or in a specific parametrization of the stochastic process followed by the short-rate like *What the short-term interest rate target can do for mean-reverting modelling.*

The usual list of extensions necessary to give research some credibility apply to all three papers. Other empirical experiments may show how good are the proposed models to fit various sets of data, in and out of the sample. The relative importance of the individual hypotheses to generate the model results deserve explicit study and the testable hypotheses should be checked.

Each paper also suggests particular extensions. In *Stochastic Growth and Monetary Policy*, different sets of parameters can be simulated and, alternatively, the set of parameters can be fully estimated. The model's speed of adjustment and length of the cycles should be compared with the observed ones. The usual confrontation of the model generated dynamics with the dynamics implied by an unrestricted VAR also deserves mention. Given that expansions are longer than recessions, the introduction of some asymmetry may improve the actual fits.

In *A Jump-Diffusion Yield-Factor Model of Interest Rates*, the comparison with other bond pricing models may give an idea of the model relative efficiency. How good are out of the sample predictions is also an important complementary exercise.

In *What the short-term interest rate target can do for mean-reverting modelling* other GARCH processes may be experimented, the likeness of the density

of the proposed process to the empirical one guiding the choice. However, rather than local improvements as alternative GARCH specifications, it is thought that the results in this and the previous papers point out the clear need of abandoning the univariate set-up and using multivariate diffusions models to address the issues of this thesis.

Chapter 1

Stochastic Growth and Monetary Policy: the impacts on the term structure of interest rates

October 5, 2001

Abstract

This paper builds a simple, empirically-verifiable rational expectations model for term structure of nominal interest rates analysis. It solves a stochastic growth model with investment costs and sticky inflation, susceptible to the intervention of the monetary authority following a policy rule. The model predicts several patterns of the term structure which are in accordance to observed empirical facts: (i) pro-cyclical pattern of the level of nominal interest rates; (ii) countercyclical pattern of the term spread; (iii) pro-cyclical pattern of the curvature of the yield curve; (iv) lower predictability of the slope of the middle of the term structure; and (v) negative correlation of changes in real rates and expected inflation at short horizons.

JEL classification: E32; E43; E52

Keywords: Controlled Short Rate; Discontinuous Changes; Nominal Yield Curve Cyclical Patterns; Expectation Hypothesis Failure

1 Introduction

This paper provides an answer to two apparently unrelated questions:

- How can an intertemporal equilibrium model adequately fit an arbitrary exogenous term structure of interest rates?
- What is the role of monetary policy in determining the term structure of interest rates?

On the first issue, intertemporal general equilibrium modelling of interest rates still leaves many questions unanswered. As an example, scalar time-homogenous affine equilibrium models[1], that provide tractable and rich analytic results with terse description of an equilibrium economy, are intrinsically incapable of fitting an arbitrary exogenous term structure because of their constant level of reversion. Worse, when tested against more general scalar specifications, they are usually rejected, suggesting either the existence of nonlinearity or of omitted variables (Chan et al. [19] or Aït-Sahalia [3]).

On the second issue, despite the belief that changes in the monetary policy impact on asset returns in general[2] and are a major source of changes in the shape of the yield curve[3], micro-financial models have not accomplished to properly incorporate it yet. The neglect to deal with macro links leaves unexplained, certain stylized facts like the pro-cyclical nominal interest rate levels, the countercyclical term spread (Fama & French [33]), or the negative short-run

[1] The univariate version of Cox, Ingersoll and Ross [24] can be seen as the most important member of the class.

[2] For example, Thorbecke [65] and Patelis [56] document the existence of a monetary risk premium and show the role of monetary policy in the predictability of the asset returns.

[3] See Mankiw & Miron [46].

correlation between expected inflation and the expected future real interest rate (Campbell & Ammer [17] and Barr & Campbell [10]).

As macro links, omitted variables and constant reversion levels seem to be the weak points of the scalar time-homogeneous equilibrium models, an attempt is made here to incorporate a macro monetary policy variable into an intertemporal equilibrium model. The goal is to get a simple, empirically-verified rational expectations model for the term structure of nominal interest rates. A model which allows great flexibility in the changes of the yield curve, in response to changes in the macroeconomic environment.

We portray the character of fluctuations in the term structure of nominal interest rates, inflation and aggregate output with staggered price contracts and investment costs, subject to technology shocks and expectational errors by price bargainers. We end up solving a stochastic growth model, subject to investment costs and sticky inflation similar to Fuhrer [34], but susceptible to the intervention of an external authority. The intertemporal optimization implies a complete description of the multi-period expected returns, and the model allows the derivation of a nominal term structure which incorporates the effects of monetary policy. Through discontinuous changes of the short-term nominal interest rate, the Central Bank forces the left-end of the term structure to match an exogenously specified level. This implies a non-zero net supply of nominal riskless bonds and adds the possibility of jumps in all forward-looking variables. Given that the monetary authority is constrained to keep inflation close to zero, future changes in the controlled rate can be forecasted by looking at the dynamics of the expected inflation and may be incorporated into the

shape of the term structure.

The resulting model extends Balduzzi, Bertola & Foresi's [8], Rudebusch's [62], McCallum's [47], Tice & Webber's [67] and Piazzessi's [60] analyses of the monetary policy impacts on the term structure in the sense that, in an intertemporal equilibrium framework, it allows the joint explanation of more stylized facts. Indeed, with a relatively simple model it is shown that the monetary policy has real effects. We eventually explain: (i) the pro-cyclical pattern of the level of nominal interest rates; (ii) the countercyclical pattern of the term spread[4] (as well as the low sensitivity of long yields to monetary policy changes); (iii) the pro-cyclical pattern of the curvature of the term structure; (iv) the lower predictability of the slope of the middle of the yield curve; and (v) the negative correlation of changes in real rates and expected inflation at short horizons. Though empirical evidence on these facts is abundant in the literature (see for example, Campbell, Lo & MacKinlay [18], Fama & French [33], Rudebusch [62] and Barr and Campbell [10]) no simple model exists taking simultaneously into account all them. Moreover, implications of the here developed model can be explored in a bond pricing context.

The paper has the following structure. Section 2 presents the empirical patterns, while section 3 reviews the term structure pattern implied by the plain Real Business Cycle model and points out its nominal indeterminacy. Both act as a motivation to section 4, where the proposed model is explained in a representative agent framework. Examples and simulations are performed in

[4]The term spread is defined as the difference between the yield-to-maturities of a long and a short term bond.

section 5, and section 6 concludes. The equivalence between the representative agent and the competitive formulation of the model is fully shown in Appendix 1; Appendix 2 explains the numerical method used in the simulations.

2 Some Stylized Facts

This section presents empirical evidences on the movements of the term structure of nominal interest rates, inflation and output, to which the numerical predictions of the theoretical models will be subsequently compared. The empirical pattern of the term structure is reproduced below using the interest rate data available at the FED of Saint Louis' web site (www.stls.frb.org/fred), which are taken from the H.15 Release by the Board of Governors. The seven rates chosen were: 3-Month Treasury Bill Rates (TB3m), 6-Month Treasury Bill Rates (TB6m), 1-Year Treasury Bill Rates (TB1), 3-Year Treasury Constant Maturity Rate (CM3), 5-Year Treasury Constant Maturity Rate (CM5), 7-Year Treasury Constant Maturity Rate (CM7), 10-Year Treasury Constant Maturity Rate (CM10). T-Bills are secondary market rates on Treasury securities and the CM rates are constant maturity yields.[5].

For B_t^j, the nominal price at t of the pure discount $j - period$ bond (or the zero coupon bond that matures in j periods from t), the yield-to-maturity, y_t^j, is the per period interest rate accrued during the j periods:

[5]The results to be presented below hold for the Fama & Bliss data set as well, that uses only fully taxable, non-callable bond. The monthly data contain one to five years-to-maturity bonds and cover the period from July 1952 to January 1998, providing 547 observations. The Fama and Bliss data set was constructed by Fama and Bliss [32] and was subsequently updated by the Center for Research in Security Prices (CRSP). The results can be made available upon request.

$$B_t^j = \left(1 + y_t^j\right)^{-j};$$

what means the yield-to-maturity is the average return on the bond held until maturity.

Because B_t^j is known at time t, y_t^j is the $j-period$ riskless nominal rate prevailing at time t for repayment at $t+j$. The $one-period$ riskless nominal rate prevailing at time t for repayment at $t+1$ deserves special notation, i_t:

$$B_t^1 = (1 + i_t)^{-1},$$

and is denoted the spot interest rate. For $j > l$, the $l-period$ nominal holding return of the $j-period$ bond between t and $t+l$, $h_{t+l,\ t}^j$, is the per period interest rate accrued during the l periods:

$$\frac{B_{t+l}^{j-l}}{B_t^j} = \left(1 + h_{t+l,\ t}^j\right)^l.$$

Given the consumer price index at t, P_t, and the inflation between t and $t+l$, $\pi_{t+l,\ t} = \frac{P_{t+l}}{P_t} - 1$, the $l-period$ real holding return of the $j-period$ nominal bond, $r_{t+l,\ t}^j$, can be analogously defined as:

$$\left(1 + r_{t+l,\ t}^j\right)^l = \frac{B_{t+l}^{j-l}}{B_t^j}\frac{P_t}{P_{t+l}} = \frac{\left(1 + h_{t+l,\ t}^j\right)^l}{1 + \pi_{t+l,\ t}}.$$

Note that both $r_{t+l,\ t}^j$, $\pi_{t+l,\ t}$ and $h_{t+l,\ t}^j$ only become known at $t+l$.

The published data are bond-equivalent yields (r_{BEY}) or discount rates

(r_D). They were transformed to yield-to-maturity by respectively: $y^j = (1 + r_{BEY} * \frac{j}{100})^{\frac{1}{j}} - 1$ and $y^j = (1 - r_D * \frac{j}{100})^{-\frac{1}{j}} - 1$, where j is time-to-maturity in years. All yields below will be expressed in annualized form.

2.1 Pro-cyclical nominal interest rate levels and countercyclical term spread

The evolution of the yields-to-maturity of the three-month and of the ten-year bonds are plotted in Figure 1 with shades added to mark the business cycles. Every white period points one expansion cycle from trough to peak, as classified by the NBER. The gray periods mark the contraction periods from peak to trough. The (i) pro-cyclical pattern of the level of interest rates is clear: the level increases during expansion and decreases during contraction. This may be related to the pro-cyclical pattern of the inflation level, as shown in Figure 2.

Figure 3 shows the evolution of the slope and the curvature of the yield curve [6]. The (ii) term spread presents a countercyclical pattern: the slope of the yield curve is big at the trough and decreases during the cycle to become small at the peak. (iii) Curvature seems to decrease along contractions (shades) and to increase during expansions.

From (i), (ii) and (iii), it results that the mean term structure at the trough is a positive sloped, relatively steeper, concave curve, while the mean term structure at the peak is a negative sloped, relatively flatter, convex curve.

[6] The slope of the yield curve is nothing more than the term spread $(CM10 - TB3m)$. The curvature is defined as $(CM10 - 2 \cdot CM5 + TB3m)$.

2.2 Lower predictability of slope of the medium term rates

In the analysis of the term structure, the many versions of the Expectation Theory of the term structure of interest rates have played an important role. Loosely stating, the Expectation Hypothesis says that the expected excess returns on long-term bonds over short term bonds (the term premiums) are constant over time. This means the term premium can depend on the maturity of the bonds but not on time: $E_t\left[h^j_{t+l,\ t} - h^k_{t+l,\ t}\right] = f\left(j,k,l\right)$, with $\frac{\partial f}{\partial t} = 0\ \forall\ j > k > l$ [7]. In its Pure version (the Pure Expectation Hypothesis, PEH), it imposes the term premium to be zero.

If any version of the Expectation Hypothesis holds, the slope of the yield curve is able to forecast interest rate moves, and this predictability is uniform along all maturities. For example, to test such a predictability for the "one-period return", the PEH reduces to check whether the slope, b, of the regression:

$$\left(y^{l-1}_{t+1} - y^l_t\right) = a + b\frac{1}{l-1}\left(y^l_t - y^1_t\right) + e_t, \tag{1}$$

is significant. Indeed, the above hypothesis implies that $b = 1$ for every l.

Using monthly zero-coupon bond yields over the period 1952:1 to 1991:2 [8], Campbell, Lo & MacKinlay [18] estimated equations similar to (1) for 2 to 120 months and got the results shown in Table 1.

Besides the b's being statistically different from 1, the stylized fact that their results bring to scene is the U-shaped pattern of these slope coefficients: the

[7] The Expectation Hypothesis can be stated in real or in nominal terms.
[8] Campbell, Loo & MacKinlay [18] use the data from McCulloch and Know [48].

forecasting power diminishes from the one month to the one year case and then increases up to the ten years case. This means that (iv) the predictability of the middle of the yield curve is lower than those of the edges.

2.3 Principal component analysis

Are the previous four stylized facts the result of some identifiable factors? In this regard, principal component analysis might point at least how many factors are relevant for empirical term structure motion. Table 2 shows factors with a pattern similar to the one uncovered by Litterman & Scheinkman's [43].

The first factor has the same sign in all bonds but, different from Litterman & Scheinkman, its impact is higher on the shorter ones. This gives a different interpretation, that the first factor causes moves in the levels and in part of the slope changes. The second factor changes sign from the short end to the long end of the maturities, which means it causes the changes in slope. Finally, the third factor, which has more impact at the short and long ends of the term structure, is interpreted as the curvature factor.

Table 3 shows the proportion of total variance explained by the three factors.

In the FRED data, the first two factors explain most of the movements and almost nothing is left to factors 3 and further [9].

Using the FRED 1969-2000 sample and varying frequency, we have performed other principal component analyses (not shown) and obtained that,

[9] Litterman & Scheinkman [43] used weekly observations, from January 1984 to June 1988, of maturities 6-month, 1, 2, 5, 8, 10, 14 and 18-year. They got averages of 89.5 %, 8.5 % and 2 % for the proportion of the total explained variance by the first three factors. Their different result might have been caused by the different frequency and length of the time series, or span of the maturities.

once frequency is increased, the first factor loses explanatory power to the second and third ones. This is a weak evidence that the 2nd. and 3rd. factors are more important in explaining short run movements [10].

2.4 Negative correlation of changes in real rates and expected inflation at short horizons

Well known in the fixed income theory is the Fisher hypothesis that there is no correlation between the expected inflation and the real interest rates: nominal interest rates change to fully compensate for expected inflation variations.

However, this hypothesis is not verified once taken to data: (iv) there exist negative correlation between expected inflation and real interest rate at short horizons. This fact is shown for example by Barr and Campbell [10], who, working with U.K. data, find correlations of changes in real rates and expected inflation of -0.69, -0.06 and-0.08 for 1-year, 5-year and 10-year, respectively[11]. The significant negative correlation got at a short horizon is puzzling, since it is expected that investors increase (decrease) their asked nominal interest rates every time a higher (lower) inflation is expected.

[10] This is also an evidence that L&S different results might have been caused by the different length of the time series or span of the maturities.

[11] Campbell & Ammer [17] and Pennacchi [57] find the same evidence using respectively rational-expectations methodology applied to US data, and survey data.

3 A Simple Intertemporal Equilibrium Theory of the Term Structure with production

Because intertemporal optimization models imply a complete description of the multi-period expected returns, and the term structure of interest rates is merely the plot of these observed returns, they are suitable as the microfoundation of a term structure model, as shown by Cox Ingersoll and Ross [23].

In the Real Business Cycle (RBC) model with labor supplied inelastically, the representative agent maximizes:

$$E_t\left[\sum_{i=t}^{\infty}\beta^{i-t}u\left(c_i\right)\right] \tag{2}$$

with: $u'(.) \geq 0, \quad u''(.) < 0$; subject to the budget constraint:

$$\begin{aligned} & c_t + k_{t+1} + b_{t+1}^1 + \sum_{j=2}^{\infty} b_{t+1}^j \\ = \quad & \theta_t k_t^{\alpha} + (1-\delta)\,k_t + \frac{1}{(1+\pi_{t,t-1})}\left[(1+i_t)\,b_t^0 + \sum_{j=1}^{\infty}\frac{B_t^j}{B_{t-1}^{j+1}}b_t{}^j\right] - \tau_t; \end{aligned} \tag{3}$$

to the technology shock AR(1) dynamics:

$$\log\theta_t = \rho\log\theta_{t-1} + \varepsilon_t,\ \ \rho\in(0,1), \varepsilon_t \,\tilde{}\, N\left(0,\sigma_{\varepsilon}^2\right); \tag{4}$$

and the transversality conditions:

$$\lim_{t\to\infty}\beta^t k_t = 0; \tag{5}$$

$$\lim_{t\to\infty}\beta^t \sum_{j=1}^{\infty}\frac{B_t^j}{P_t}b_t^j = 0; \tag{6}$$

where:

c stands for real consumption;

k is the real capital stock;

θ is the productivity shock;

$0 < \alpha < 1$ is the capital elasticity[12];

δ is capital depreciation;

$(1 + \pi_{t,t-1}) = \frac{P_t}{P_{t-1}}$ is the inflation between $t - 1$ and t, with the price index P_t not known before t;

$(1 + i_t)$ is the nominal interest rate of the $one - period$ bond held between $t - 1$ and t, known at $t - 1$;

B_t^j is the nominal price of the $j - period$ bond;

b_t^j is the quantity of the bond the consumer carries from $t - 1$ to t, and j is the number of periods to maturity;

b_t^0 is the quantity of the bond redeemed at t;

and τ_t are real taxes.

Because labor is inelastically supplied, the production function is presented in terms of per-capita capital, and the above formulation couches the case of a constant return-to-scale production function. Also, to make presentation lighter, instead of the usual normalization of nominal unit price at maturity,

[12] The production function $f(k,\theta) = \theta_t k_t^{\alpha}$ presents the usual conditions:

$$f_1(.) \geq 0,\; f_2(.) > 0,\; f_{11}(.) \leq 0,\; f_1(0,.) = \infty,\; f_1(\infty,.) = 0;$$

$B_t^0 = 1 \; \forall \; t$, we assume that the next-to-mature bond costs one nominal unit and is worth $(1 + i_{t+1})$ nominal units at redemption.

From the above, the representative agent value function can be posed as:

$$
\begin{aligned}
& V\left(k_t; b_t^j; j > 0;\; \theta_t\right) && (7) \\
= \; & \max_{c,\, k,\, b} \left\{ \begin{array}{c} u\left(c_t\right) + \beta E_t V(k_{t+1}, b_{t+1}^{j>0}, \theta_{t+1}) \\ -\lambda_t \left[\begin{array}{c} c_t + \tau_t + k_{t+1} + b_{t+1}^1 + \sum_{j=2}^{\infty} b_{t+1}^j - \theta_t k_t^{\alpha} - (1-\delta)\, k_t \\ -\frac{1}{(1+\pi_{t,t-1})} \left[(1+i_t)\, b_t^0 + \sum_{j=1}^{\infty} \frac{B_t^j}{B_{t-1}^{j+1}} b_t{}^j \right] \end{array} \right] \end{array} \right\},
\end{aligned}
$$

and solved to result in the agent's optimal allocation rules:

$$u'\left(c_t\right) = \beta E_t \left\{ \left[\alpha \theta_{t+1} k_{t+1}^{\alpha-1} + (1-\delta) \right] u'\left(c_{t+1}\right) \right\} \tag{8}$$

$$\frac{u'\left(c_t\right)}{\left(1 + i_{t+1}\right)} = \beta E_t \left[\frac{1}{\left(1 + \pi_{t+1,t}\right)} u'\left(c_{t+1}\right) \right]; \tag{9}$$

and

$$u'\left(c_t\right) \frac{B_t^j}{P_t} = \beta E_t \left[\frac{B_{t+1}^{j-1}}{P_{t+1}} u'\left(c_{t+1}\right) \right] \quad \forall j; \tag{10}$$

taking prices as given.

Recursion on (10) and the law of iterated expectations implies the $l-period$ real holding return of the $j-period$ nominal bond $(r_{t+l,\,t}^j)$:

$$1=\beta^{l}E_{t}\left[\frac{B_{t+l}^{j-l}}{B_{t}^{j}}\frac{P_{t}}{P_{t+l}}\frac{u'(c_{t+l})}{u'(c_{t})}\right]=\beta^{l}E_{t}\left[\left(1+r_{t+l,\ t}^{j}\right)^{l}\frac{u'(c_{t+l})}{u'(c_{t})}\right]\quad\forall j\ and\ l>1, \tag{11}$$

and gives the whole real term structure implied by the model.

Inasmuch as the yield-to-maturity of every $l-period$ bond (y_t^l) is known for certainty at t, $\left(1+y_t^l\right)^l=\frac{B_{t+l}^0}{B_t^l}$, it can be taken out of the expectation operator, resulting in:

$$\frac{1}{\left(1+y_t^l\right)^l}=\frac{B_t^l}{B_{t+l}^0}=\beta^{l}E_{t}\left[\frac{1}{1+\pi_{t+l,\ t}}\frac{u'(c_{t+l})}{u'(c_{t})}\right]\quad\forall l; \tag{12}$$

that provides the whole nominal term structure.

It is trivial that, for $l=1$, $y_t^1=i_{t+1}$; and the above formula simplifies to:

$$1=\beta E_{t}\left[\left(1+r_{t+1,\ t}^{1}\right)\frac{u'(c_{t+1})}{u'(c_{t})}\right]=\beta E_{t}\left[\frac{1+i_{t+1}}{1+\pi_{t+1,\ t}}\frac{u'(c_{t+1})}{u'(c_{t})}\right]; \tag{13}$$

where the spot rate i_{t+1} can be put outside the expectation if desired.

From (12), again by use of the law of iterated expectations, we obtain:

$$\begin{aligned}&\frac{1}{\left(1+y_t^{2l}\right)^{2l}} \\ =\ &\frac{1}{\left(1+y_t^l\right)^l}E_t\left[\frac{1}{\left(1+y_{t+l}^l\right)^l}\right]+Cov_t\left(\frac{\beta^l}{1+\pi_{t+l,\ t}}\frac{u'(c_{t+l})}{u'(c_t)},\frac{1}{\left(1+y_{t+l}^l\right)^l}\right)\quad\forall l;\end{aligned} \tag{14}$$

which is a generalized version of the PEH, adjusted for the risk premium $Cov_t\left(\frac{\beta^l}{1+\pi_{t,\ t+l}}\frac{u'(c_{t+l}^*)}{u'(c_t^*)},\frac{1}{\left(1+y_{t+l}^l\right)^l}\right)$.

Equation (14) means the PEH holds only in the special cases where the risk premium is zero.

Also, working on (12), results in the generalized "one-period return" PEH:

$$\begin{aligned}&\left(1+y_t^1\right) \qquad\qquad (15)\\ = \;& \left(1+y_t^l\right)^l E_t\left[\frac{1}{\left(1+y_{t+1}^{l-1}\right)^{l-1}}\right] + \frac{Cov_t\left(\frac{1}{1+\pi_{t+1,\ t}}\frac{u'(c_{t+1})}{u'(c_t)}, \frac{1}{\left(1+y_{t+1}^{l-1}\right)^{l-1}}\right)}{E_t\left[\frac{1}{1+\pi_{t+1,\ t}}\frac{u'(c_{t+1})}{u'(c_t)}\right]} \quad \forall l;\end{aligned}$$

as called by Campbell, Lo and MacKinlay [18]. Again, only when the risk premium is zero, does the "one-period" PEH hold.

The agent's optimal conditions allow us to define:

$$M_{lt} = \beta^l \frac{u'\left(c_{t+l}\right)}{u'\left(c_t\right)} \qquad (16)$$

as the stochastic discount function (or the pricing kernel); which in the present model is equivalent to the intertemporal marginal rate of substitution in consumption.

3.1 Equilibrium without external intervention: inflation and nominal interest rate indeterminacy

An equilibrium sequence is defined as a set of stochastic vectors

$\left(\theta_t,\ k_{t+1},\ c_t,\ i_{t+1},\ \pi_{t,t-1},\ r_{t+l,t}^j,\ b_{t+1}^j,\ \tau_t\right)$ satisfying the f.o.c.'s and the market clearing conditions for every t.

Without external intervention, the exogenous supply of bonds is zero:

$$b_t^j = 0 \quad \forall\, j;$$

as well as taxes $\tau_t = 0$, and, given (4), the consumers' decision simplifies to split wealth between capital and consumption by obeying (8) and the simplified budget constraint:

$$c_t = \theta_t k_t^\alpha + (1-\delta)\, k_t - k_{t+1}, \tag{17}$$

for every t.

The initial capital stock, the technology dynamics (4), and the transversality condition (5) define the saddle path expected to be followed by $(k\ , c)$ in the system (8) and (17). Substitution of (17) into (8) defines a stochastic difference equation in k that, given the initial capital stock, initial technology and (5), obtains the optimal capital path (k^*) and provides the inputs to obtain the optimal consumption path (c^*) by (17). The above hypotheses are enough to guarantee that the distribution of optimum aggregate capital converges pointwise to a limit distribution when returns are decreasing: k is pushed to the level k_{ss} where the expected marginal productivity of capital equals the rate of time preference: $\alpha k_{ss}^{\alpha-1} - \delta = (1/\beta) - 1$. When returns-to-scale are constant, they are as well enough to guarantee that the rates of growth converge pointwise to a limit distribution[13].

The application of $\{c_t^*\}_{t=0}^{\infty}$ to (11) endogenously determines the expected

[13]See Brock [15] for the proof.

$l - period$ real returns on a $j - period$ nominal bonds from t to $t + l$:

$$1 = \beta^l E_t \left[\left(1 + r^j_{t+l,\, t}\right)^l \frac{u'\left(c^*_{t+l}\right)}{u'\left(c^*_t\right)} \right] \quad \forall\, j > l; \tag{18}$$

and gives the whole expected real term structure implied by this equilibrium.

We now define what we understand by neutral values.

Definition 1 **At any time t, the endogenous variables values are neutral, denoted $\left(k^N_{t+1},\ c^N_t,\ i^N_{t+1},\ h^{N\,j}_{t+1,t},\ \pi^N_{t+1,t},\ r^{N\,j}_{t+1,t},\ b^{N\,j}_{t+1t+1}\right)$, when the real stock of bonds is fully rolled over with no portfolio rebalance:**

$$b^{j+1}_{t+1} - b^j_t = 0\ \forall\ j.$$

This means that we qualify all interest rates as neutral when they are obtained without changes in the bonds' maturity profile. There is no net external intervention, in the sense that the debt-credit profile is kept constant. Thus, within a period, the neutral values are nothing more than those for which the private sector's net demand for every maturity bond is zero, what means people do not sell bonds to finance capital or the other way around.

Because there is a stochastic shock in the production function, the neutral real spot rate fluctuates around a trend defined by the optimal capital path. For example: if k_t is increasing along time and the production function presents decreasing returns-to-scale, the productivity trend is decreasing and real neutral rate is expected to decrease as the economy tends to the steady state.

Without an external intervention, the real interest rates, given by (18), are

completely defined by (4), (8), (17) and (5). Equation (13) is nothing more than the Fisher relation that defines next period inflation given the spot nominal interest rate, or the other way around. Because the expected spot real interest rate is completely determined by the real factors and is every time the expected marginal productivity of capital, expected inflation sensitivity to the level of the nominal interest rate is one, what means no correlation between nominal and real variables.

Although the inflation and nominal rate indeterminacies are a consequence of having more variables than equations, the inclusion of a cash-in-advance restriction or a fiscalist-theory type of reasoning does not change the above conclusions. Due to this one-to-one correspondence between i and π, there is no cyclical pattern (i) in the level of the nominal term structure, (ii) or in that of the term spread, (iii) or in that of the curvature. (iv) The predictability of the slope of the yield curve is good and equally credible for every maturity. Moreover, (v) there is no correlation between expected inflation and the real interest rate since the real interest rates vary with the marginal productivity of capital and the Fisher hypothesis holds.

Summing up, system (4), (8), (17) and (5) alone does not split the changes in the nominal rate into changes in the real rate and inflation, and is not of great use in explaining how monetary policy affects real activity and inflation. Basically, it assumes neutrality (and superneutrality) and thus thwarts the possibility that nominal interest rate and inflation vary independently. Quite unrealistic, inflation reduction to zero can be done in one painless down-move of the nominal rate to the expected marginal productivity level with no impact

on the real activity.

Notwithstanding, there exists one degree of freedom in the above model to couch an ad hoc assumption, and this is done, in conjunction with inflation stickiness, in section 4.

4 The Model

The proposed model describes a closed economy [14] with firms and capital accumulation, subject to investment cost and staggered price contracting, and susceptible to the intervention of a monetary authority. For presentation purposes, we develop the main ideas in the representative agent framework. The equivalence with a more detailed economy, where consumers and firms interact in a world of staggered price contracting, is shown in Appendix 1.

4.1 The Real Side with investment costs

The representative agent maximizes (2), subject to a budget constraint slightly different from (3):

$$\begin{aligned} & c_t + \left[k_{t+1} + \varphi \left(\frac{k_{t+1}}{k_t} - 1 \right)^2 \right] + b_{t+1}^1 + \sum_{j=2}^{\infty} b_{t+1}^j \\ = \; & \theta_t k_t^{\alpha} + (1-\delta) k_t + \frac{1}{(1+\pi_{t,t-1})} \left[(1+i_t) b_t^0 + \sum_{j=1}^{\infty} \frac{B_t^j}{B_{t-1}^{j+1}} b_t^j \right] - \tau_t, \end{aligned} \tag{19}$$

[14] As pointed in Meltzer [49] pp.50, in an open economy, the exchange rate would be just one more of the many relative prices in the transmission process, without altering the basic results.

and to (4), (5), (6); where: $\varphi\left(\frac{k_{+1}}{k}-1\right)^2$ is the cost of adjustment, and the other variables have the previous stated meaning.

Now, the representative agent value function can be posed as:

$$
\begin{aligned}
& V\left(k_t; b_t^j;\ j>0; \theta_t\right) && (20)\\
= & \max_{c,\ k,\ b}\left\{\begin{array}{c} u\left(c_t\right)+\beta E_t V(k_{t+1}, b_{t+1}^{j>0}, \theta_{t+1}) \\ -\lambda_t\left[\begin{array}{c} c_t+\tau_t+\left[k_{t+1}+\varphi\left(\frac{k_{t+1}}{k_t}-1\right)^2\right]+b_{t+1}^1+\sum_{j=2}^{\infty} b_{t+1}^j \\ -\theta_t k_t^\alpha-(1-\delta)\, k_t-\frac{1}{(1+\pi_{t,t-1})}\left[(1+i_t)\, b_t^0+\sum_{j=1}^{\infty}\frac{B_t^j}{B_{t-1}^{j+1}} b_t{}^j\right]\end{array}\right]\end{array}\right\};
\end{aligned}
$$

and the solution is similar to the one in section 3, except that:

$$
\begin{aligned}
& \left[1+2\varphi\left(\frac{k_{t+1}}{k_t}-1\right)\frac{1}{k_t}\right]u'\left(c_t\right) && (21)\\
= & \beta E_t\left\{\left[\alpha\theta_{t+1}k_{t+1}^{\alpha-1}+(1-\delta)-2\varphi\left(\frac{k_{t+2}}{k_{t+1}}-1\right)\frac{k_{t+2}}{k_{t+1}^2}\right]u'\left(c_{t+1}\right)\right\},
\end{aligned}
$$

replaces (8).

4.2 Contracting Specification and the Inflation Dynamics

Once accounted the investment costs, it is assumed that consumption and capital goods (c and k) are the same final good, which is the aggregation of two differentiated goods produced, consumed and invested together in a fixed proportion of half each. Although undesirable, the no substitutability between these (differentiated) component goods simplifies matters and buttresses a stag-

gered price contracting similar to Fuhrer & Moore [35]. In our paper, agents negotiate the nominal price contracts of the two final goods, that remain in effect for two periods. As the model hypothesizes that production, consumption and investment are split between these two goods, the aggregate price index at t is defined as the geometric mean of the contract prices:

$$P_t = X_t^{\frac{1}{2}} X_{t-1}^{\frac{1}{2}}; \tag{22}$$

where:

X_t is the contract price

and P_t is the aggregate price index at t.

Agents set nominal contract prices so that the current real contract price equals the average real contract price index expected to prevail over the life of the contract, adjusted for excess demand conditions:

$$\frac{X_t}{P_t} = E_t \left[\frac{X_{t+1}}{P_{t+1}} \right]^{\frac{1}{2}} \frac{X_{t-1}}{P_{t-1}}^{\frac{1}{2}} Y_t^{\gamma}; \tag{23}$$

where the excess demand term Y_t was parametrized as $Y_t = e^{y_t}$. With this, y_t is the excess demand which can be calculated from the budget constraint (19) as:

$$\begin{aligned} y_t &= c_t + \left[k_{t+1} + \varphi \left(\frac{k_{t+1}}{k_t} - 1 \right)^2 \right] - \left(\theta_t k_t^{\alpha} + (1 - \delta) k_t \right) \qquad (24) \\ &= -\left(b_{t+1}^1 + \sum_{j=2}^{\infty} b_{t+1}^j + \tau_t \right) + \left(\frac{1}{(1 + \pi_{t,t-1})} \left[(1 + i_t) b_t^0 + \sum_{j=1}^{\infty} \frac{B_t^j}{B_{t-1}^{j+1}} b_t{}^j \right] \right). \end{aligned}$$

Considering the expression after the first equality signal, the two first members describe total demand for goods, while the last one (the big expression between brackets) is the supply of goods. Thus, excess demand can be read as the private sector's net demand for bonds, and there is no excess demand ($y_t = 0$) when variables from t to $t+1$ are neutral (as stated in the Definition).

Equation (23) causes the inflation dynamics:

$$(1 + \pi_{t,t-1}) = (1 + \pi_{t-1,t-2})^{\frac{1}{2}} (1 + E_t [\pi_{t+1,t}])^{\frac{1}{2}} (Y_t Y_{t-1})^{\gamma} \Omega_t, \qquad (25)$$

where $\Omega_t = e^{\omega_t}$ is the expectational error orthogonal to date $t-1$ information set, and allows inflation stickiness in the present model. Note that if expressed in log terms, (25) gives an expression very similar to the one in Fuhrer & Moore [35], which will be used in the simulations in section 5 below.

4.3 The Monetary Authority Intervention and the Role Played by Money

Since we are interested on the study of moves in the yield curve, and not on the study of optimal monetary policy rules, we don't care about objective functions of the monetary authority and related issues. It is enough that the monetary authority be concerned about inflation, have funds to intervene in the bond market, and knows its dynamics is given by (25). This being the case, it is prone to control the one-period spot interest rate to fight inflation. Due to operating constraints, it is assumed, without loss of generality, that it uses the rule:

$$i_{t+1} = i_t + v_t, \tag{26}$$

where:

$$v_t = \left\{ \begin{array}{l} 0,\ with\ probability:\ (1 - \varsigma \left|\pi_{t-1}\right|) \\ e\frac{\pi_{t-1}}{\left|\pi_{t-1}\right|},\ with\ probability:\ \varsigma \left|\pi_{t-1}\right| \end{array} \right\};$$

and e and ς are positive constants [15].

In other words, the spot rate tends to remain constant from period to period, except for jumps whose probability is an increasing function of the inflation level. If inflation is positive the eventual jump is positive, and if inflation there is deflation the jump is negative. When inflation grows, the probability of jumps increases and so the expected value of the next period spot rate. Because inflation is persistent, policy only reverts when the inflation target has been mostly reached.

The key to our model is monetary authority behavior in the bond market. It acts buying or selling one-period bonds that pay riskless nominal interest rate $(1 + i_{t+1})$, but risky real interest rate:

$$\left(\frac{1 + i_{t+1}}{1 + \pi_{t+1,t}}\right),$$

revealed at $t+1$. Besides, the authority runs no deficit, what forces it to charge the individuals a lump sum tax to payoff the net interest:

[15] (26) implies the monetary authority inflation targeting is zero. This assumption can be relaxed by subtracting a constant (or a variable) from π_{t-1}.

$$\tau_t = \left[\frac{1+i_t}{1+\pi_{t,t-1}} - 1\right] b_t^a \ \forall\, t > 0, \tag{27}$$

where b_t^a stands for the per capita bond demand.

As individuals receive the full proceeds of bonds they hold and are charged lump sum, they choose to long or short the one-period bond once its real expected return diverges from the expected neutral rate. Thus, although lending to or borrowing from the monetary authority are just simple storage in the aggregate, non-zero net demand for one-period government bonds shows up due to the non-cooperative individual behavior induced by the tax system.

Not only the above rule makes it easy to forecast tomorrow's spot rate, but it also answers for the system stability as long as it guarantees that inflation does not explode, providing the long run level of the variables. Stability is the cause for the long rates' low sensitivity to monetary policy changes: given the parameters, long run values are implied, and they are the ones that weight most in the valuation of long term bonds.

No explicit cash motive has been couched; but, without the cash-in-advance restriction, why would society use money and bear the costs of monetary policy? Like Woodford [69], it is assumed that modelling the fine details of the payments system and the sources of money demand is inessential to explain how money prices are determined or to analyze the effects of alternative policies on the inflation path or on other macro variables.

Though buttressing the use of money is not a goal of this paper, we point out a simple fact of life: money allows specialization, what causes productivity

gains, and that is why society copes with the monetary authority and its effects. The economic system is enormously more efficient with than without money and the monetary authority. Loosely modelling, at the real side, there exist storable goods and two possible production systems. The monetary system, f^M, makes use of money, allows specialization and is thus much more productive than the other, f^B, non-monetary, non-specialized system: $f^M(k, \theta) \gg f^B(k, \theta) \quad \forall k$. Although storage is also possible, it is greatly inefficient: production always generates net goods, even after accounting for all sort of costs and when $\theta = \theta_{\mathsf{inf}}$, while storage just returns the amount stored back.

It is just being assumed here that the gain from being a monetary economy is discrete and independent of the inflation level, up to an inflation upper bound above which the economy retraces to the non-monetary system (f^B). The dread to bear such a retrace is what justifies the external authority concern about the inflation level. Due to system stability, it will always be assumed that inflation is below the upper bound and $f = f^M$.

Since real balance effects do not appear in the inflation dynamics (25), nor the monetary authority controls the money supply [16], the inflation level determination does not depend upon money demand. The key to analyze the determination of the inflation level without explicit reference to money is to model inflation as a function of the level of the real interest rate, and nominal interest rate as a function of past inflation. This makes real quantities dependent upon the level of inflation and allows the introduction of the monetary authority and its policy effects.

[16] When the monetary authority controls interest rates, money becomes endogenous.

4.4 Equilibrium with Intervention Possibility

Equation (19) can be simplified a bit. Because the Central Bank only intervenes in the *one−period* bond market, only b_t^0 and b_{t+1}^1 can be different from zero and the exogenous supply of the bonds longer than one period is zero: $b_t^j = 0 \, \forall \, j > 1$. In the representative agent world, equilibrium means:

$$b_t^a = b_t^0;$$

by the intervention policy (27), the economy budget constraint (19) becomes:

$$c_t = \theta_t k_t^\alpha + (1-\delta) k_t + b_t^0 - \left[k_{t+1} + \varphi \left(\frac{k_{t+1}}{k_t} - 1 \right)^2 \right] - b_{t+1}^1 \qquad (28)$$

and the excess demand (24):

$$\begin{aligned} y_t &= c_t + \left[k_{t+1} + \varphi \left(\frac{k_{t+1}}{k_t} - 1 \right)^2 \right] - \left(\theta_t k_t^\alpha + (1-\delta) k_t \right) \qquad (29) \\ &= -b_{t+1}^1 + b_t^0. \end{aligned}$$

with c_t, k_{t+1}, b_{t+1}^1 optimally given by (21) and (9).

Inflation dynamics simplifies to:

$$(1 + \pi_{t,t-1}) = (1 + \pi_{t-1,t-2})^{\frac{1}{2}} \left(1 + E_t \left[\pi_{t+1,t}\right]\right)^{\frac{1}{2}} \exp\left\{ \gamma \left(-b_{t+1}^1 + b_{t-1}^0 \right) \right\} \Omega_t, \qquad (30)$$

The economy equilibrium sequence $\left(\theta_t,\ i_{t+1},\ c_t,\ k_{t+1},\ b_{t+1}^1,\ \pi_{t,t-1},\ r_{t,t-1}^j\right)$ is now given by the system of six simultaneous equations (28), (4), (30), (26),

(21) and (13), and the transversality conditions (5) and (6), given the initial values for $\pi_{0,-1}$, b_1^0, k_1 and i_1.

4.5 Understanding the model dynamics

The monetary transmission mechanisms are Tobin's Q theory of investment and the wealth effects on consumption: the spot rate change sponsors consumption and portfolio responses with real effects.

Although in the representative agent framework, we are able to argue in terms of the Q-theory of investment. It is possible to get the evolution of marginal Tobin's Q:

$$Q = 1 + 2\varphi \left(\frac{k_{t+1}^*}{k_t^*} - 1 \right) \frac{1}{k_t^*}; \tag{31}$$

for the optimal capital sequence $\{k_t^*\}_{t=0}^{\infty}$.

To illustrate the implications of the model, we can make use of phase diagrams to look at the implied dynamics and the evolution of the term structure along time. Figure 4 shows the saddle path for the pair (Q, k). Q is above unit for increasing k and is below unit for decreasing k.

The steady state is the point where the effective output equals the potential one, and there is no excess demand ($y_t = 0$). In this case, at every technology shock that improves (worsens) efficiency, $\Delta Q = 0$ moves northeast (southwest). The effect is similar in the case of monetary interventions that lower (rise) the real interest rate. However, as these last interest changes are transitory, a backwards move in the $\Delta Q = 0$ curve is expected to take place sometime in the

future.

The variety of term structure shapes and dynamics allowed makes comprehensive illustration unfeasible, but intuition can be gained in the analysis of simple cases. For example, without inflation, the left diagram in Figure 5 shows the dynamics of Q and K, and the right diagram shows the implied dynamics of the real term structure. It is the case without intervention of an economy's growth path.

From equation (18) it can be inferred that the real term structure becomes flatter as the economy comes close to the steady-state ($t = s.s.$), since the ratios of two different time consumptions approach unity (and the real yields approach β^{-1} for every maturity). y_t^j is given by:

$$y_t^j = \frac{1}{\beta}\left(E_t\left[\frac{u'(c_{t+j})}{u'(c_t)}\right]\right)^{\frac{-1}{j}} - 1, \quad \forall j,$$

and at $t = 0$ (k_0 below k_{ss}), the real term structure is downward sloping since c_t is expected to grow at decreasing rates. The just described expansion path contrasts the initial negative slope of the real term structure with the empirical initial positive slope of the nominal term structure shown in section 2. This stress our that plain RBC models, or the univariate version of Cox, Ingersoll and Ross [24], aren't good enough to explain the nominal term structure. Something practitioners in the financial markets are well aware of.

Figure 6 shows what happens when a temporary increase in the real spot interest rate is expected at a certain date and for a certain period, due to a

tight of the Central Bank to fight increasing inflation[17]: once the tight becomes expected, Q jumps down and K begins to decrease up to the time when the change happens (at T). Between the effective tight and the time policy is again loosened, Q increases, while K first decreases, to increase after Q reaches unit. (Q, K) changes happen so that when policy reverts to loose again (at T'), the pair is over the original saddle and goes to the steady-state.

Figure 7, on the other hand, shows what happens when the time of the target is uncertain. Once the change becomes justifiable by "high" inflation, Q jumps to an intermediary saddle path, located in accordance with the probability of change. While the change does not happen, inflation is increasing and the intermediary saddle moves southwest (due to the increasing probability), bringing together the pair (Q, K). Once the tight takes place (at T), Q jumps again to a point that depends on the expected future monetary policy.

The combination of the real spot interest rate with the inflation dynamics allows to obtain all sort of shapes for the term structure.

4.6 Explanation of the stylized facts

The five stylized facts can be explained by our model.

With the spot-rate exogenously fixed, sticky inflation and adjustment costs, the Fisher hypothesis of constant real interest rates can't hold and the expected real spot interest rate strays from the expected marginal product of capital for a while. A positive (negative) inflation shock not accompanied by a spot-rate

[17]This is an unrealistic exercise with didactical purposes only. Central Bank's interventions are uncertain as well as their duration.

jump lowers (raises) the real interest rate below (above) the present capital productivity level and sponsors capital investment (disinvestment). But, due to increasing investment costs, capital does not adjust instantaneously.

Inasmuch as the expected inflation is pro-cyclical, (i) the nominal interest rates level is high in the peak and low in the trough of the business cycle.

Pro-cyclical nominal rates means existing bonds are expected to lose (gain) value during the expansion (contraction) as the rates increase (decrease). The negative of the modified duration of the bond, defined as:

$$-M.Duration = \frac{\partial B}{\partial y}\frac{1}{B} = -j\frac{1}{(1+y)}$$

shows that longer bonds are relatively more affected by the expected future change in the level of the term structure. Thus, (ii) the countercyclical pattern of the term spread can be explained as a "level upside-move risk" that is proportional to the bond duration. Due to system stability, people believe there are upper and lower bounds for the expected inflation and the probability of a monetary authority action against inflation is increasing with inflation itself. When the economy begins an expansion, the nominal interest rates and inflation levels are low, and inflation is expected to grow. Spot rate jumps in the near future will have positive signs, this meaning lower bond prices and capital losses for the long maturities bond holders, who charge their borrower for that. As expansion takes place, inflation increases, followed by the spot-rate. Since there is a perceived upper limit for the inflation, the 'level upside-move risk' decreases along this path, and the reduction in the term spread is consistent.

The description of the recession goes along the same lines.

Convexity, defined as:

$$Convexity = \frac{\partial^2 B}{\partial y^2}\frac{1}{B} = j\,(j+1)\,\frac{1}{(1+y)^2},$$

shows that the (iii) pro-cyclical curvature is explained by the same 'level upside-move risk'.

The way nominal spot interest rate is modified gives rise to a (iv) negative short-run correlation between expected inflation and expected future real interest rate, since inflation innovations are not instantaneously transmitted to the nominal spot rate.

The monetary authority operating procedure, together with inflation stickiness and the system stability seem enough to justify (v) the better predictability of the slope of the yield curve at the short- and at the long-ends respectively (or the worse predictability of the slope of the middle of the yield curve). The monetary authority operating procedure and inflation stickiness imply the persistency of monetary policy and that inflation lasts for a while, explaining the good predictability of the slope at the short-end of the term structure. At the long-end, because the system is stable, long-term bond yields are mainly defined by the long run values, and shocks have a transitory and small impact. Investors have reasonable certainty about inflation and the spot rate in the near future, as well as in the long run given the system is stable. However, due to the same inflation stickiness and operating procedures, people is uncertain about how long it takes for a policy to reach its goal and when it is going to be

reverted, these being the causes for increased middle term uncertainty.

In the context of the present model, we have three shocks that can be decomposed into orthogonal factors, but not interpreted as a factor itself. Our structural shocks are not orthogonal: technology shocks may cause expectational errors and inflation, and inflation may cause spot rate jumps. Factor 1 for example, which affects all yields with the same sign but affects long yields less, might have considerable weight on the technology, ε, and expectational error shock, ω, since both impact more short rates and die out with time. We thus let factor interpretation for further research.

5 Model Solution, Simulations and Predictions

Equations (4), (13), (25), (26), (21) and (28) form a non-linear stochastic difference system with rational expectations that can be numerically solved according to Novales et al.[55] by use of Sims [63] method described in the Appendix 2.

Numerical exercises reported below used the following set of parameters: $\alpha = 0.4$ and $\delta = 0.025$ are standard calibration parameters for quarterly frequency data. Values for $\sigma = 2$ and $\beta = 0.995$ are in accordance with Fuhrer's [34] similar model. A $\varphi = 380$ seems reasonable in view of the existing literature (see Dixit & Pindyck [26]). Finally, $\rho = 0.9$ and $\gamma = 0.024$ were estimated from data. The procedure performed to estimate ρ was close to Cooley&Prescott [20]: first assuming capital does not vary from quarter to quarter, we have $\log \theta_t - \log \theta_{t-1} = (\log Y_t - \log Y_{t-1})$, an expression which allows building up the θ_t series, where Y is the gap between GNP and potential GNP; then, with

the obtained $\theta' s$, ρ is estimated. The γ was estimated by instrumental variables using CPI inflation seasonally adjusted and the negative of the System Open Market Accounting Holdings (per-capita and discounted a trend).

5.1 Experiments

Figures 8 and 9 illustrate the dynamics of two experiments: (i) a disinflation experiment, when inflation and capital start above the steady state (Figure 8), and (ii) an expansion experiment, when capital as well as inflation start below the steady state level (Figure 9).

As shown in Figure 8, the level of the nominal interest rates are initially high, but the short real interest rate is expected to increase and inflation to decrease. The evolution of the term structure is illustrated in the figure.

In Figure 9, capital and consumption increase along time, while the real interest rates decreases.

In both cases the impulse response functions seem to describe real data[18].

5.2 Simulation with the U.S. data

It is worth asking if the numerical predictions of the theoretical model present patterns similar to the stylized facts in section 2.

In a attempt to test whether the model reproduces the data pattern, we have performed the following Monte Carlo exercise: given date t states $\pi_{t-1,t-2}$, b_t^0, k_t, and i_{t+1}, to build the joint expectation conditional on the available informa-

[18] More rigorous tests are certainly desirable; comparision with an unrestricted VAR seeming the natural candidate.

tion set, 500 random paths of the model's variables were obtained by simulating the system 10 years ahead, using shocks got from a bivariate normal random vector with covariance matrix $\begin{pmatrix} 00725 & \\ -.002 & .00579 \end{pmatrix}$, which is the estimated matrix from the above residual series. With the joint expectation of the model variables calculated, the nominal term structure on t was then defined by the yields of the many maturity bonds:

$$y_t^j = \frac{1}{\beta}\left(E_t\left[\frac{1}{1+\pi_{t+j,t}}\frac{u'(c_{t+j})}{u'(c_t)}\right]\right)^{\frac{-1}{j}} - 1, \quad \forall j = 1, ..., 40.$$

To move from t to $t+1$, and calculate the term structure on $t+1$ as just described, we assumed the realized shock to be the residual shock $(\widehat{\varepsilon}_t,\ \widehat{\omega}_t)$ estimated from equations (4) and (25) from 1969:1 to 2000:4. The $\widehat{\varepsilon}$ was as the residual of the equation for estimating ρ. The $\widehat{\omega}$ was the residual of the equation for estimating γ (see Section 5 introduction).

The results are sensible and "close" to the qualitative pattern documented in Section 2. Tables 4 and 5 below show the relative importance of the factors and their respective eingenvectors.

The simulation also reproduces the correlation between expected inflation and real interest rate. Table 6 shows the obtained values, which are close to the U.K. empirical ones.

6 Conclusion

The simple macro model developed in this paper is able to fit the empirical term structure of interest rates in different situations. It doesn't focus on the behavior of some instantaneous spot rate process, derived from a particular equilibrium model, to obtain the term structure, as usual in the literature. Instead, it sees the spot-rate as an instrument of the monetary authority, who controls it to match the goal of low price variation. A key behavioral rule introduces the needed flexibility in linking macro variables changes to movements in the yield curve. This being the case, the long run levels of the state variables may be forecasted with a high degree of accuracy, as well as the future changes in the spot rate. To obtain the term structure, people does take into account the current drift of the inflation and what future monetary policy actions it implies.

Simulations produced results qualitatively close to several stylized facts: (i) pro-cyclical pattern of the level of nominal interest rates; (ii) countercyclical pattern of the term spread (as well as low sensitivity of long yields to monetary policy changes); (iii) pro-cyclical pattern of the curvature of the term structure; (iv) lower predictability of the slope of the middle of the yield curve; and (v) negative correlation of changes in real rates and expected inflation at short horizons. Other empirical experiments may show how good is the proposed model to fit various empirical sets of data. From a theoretical viewpoint, new and probably more accurate, bond pricing mechanisms can be developed from it.

Appendices

A The Competitive Problem

The equivalence of the representative consumer with a competitive economy is shown below. As usual in the competitive framework, consumers and firms maximize their objective function taking prices as given. Without loss of generality, it is assumed that the firms are the owners of capital and are all equity financed[19].

A.1 Consumers

The consumers budget constraint is given by:

$$
\begin{aligned}
& c_t + q_t z_{t+1} + b_{t+1}^1 + \sum_{j=2}^{\infty} b_{t+1}^j \\
= \; & \left(q_t + d_t\right) z_t + w_t l_t + \frac{1}{\left(1+\pi_{t,t-1}\right)} \left[\left(1+i_t\right) b_t^0 + \sum_{j=1}^{\infty} \frac{B_t^j}{B_{t-1}^{j+1}} b_t^j \right] - \tau_t;
\end{aligned}
\tag{A.1}
$$

and the transversality conditions (6) and:

$$
\lim_{t \to \infty} \beta^t \left(q_t + d_t\right) z_t = 0; \tag{A.2}
$$

where: q is the real stock price; z is the quantity of the stocks; d is the real dividends; w_t denotes the real wage; and l_t is the amount of labor.

This results in the consumers' optimal allocation rules:

[19] For the firms decision between equity and debt in a framework similar as ours, see Brock and Turnovsky [16]. Notice that they deal with such decision in a perfect foresight situation.

$$1 = \beta E_t \left[\frac{q_{t+1} + d_{t+1}}{q_t} \frac{u'(c_{t+1})}{u'(c_t)} \right]; \tag{A.3}$$

and (9), (11), taking prices as given.

Given that consumers do not enjoy leisure, $l_t = 1 \; \forall \; t$.

A.2 Firms

At the firm's side, the law of motion for its capital stock is:

$$K_{t+1} = K_t^d + I_t^d = (1 - \delta) K_t + I_t^d;$$

where:

K_{t+1} is the capital stock to be used next period;

K_t^d stands for used capital demanded for use next period; and

I_t^d is the real investment on new capital.

Define the gross profits to be given by:

$$\Pr ofits_t = f(K_t, l_t, \theta_t) - w_t l_t.$$

Assuming firms are all equity financed, the following identity holds:

$$\Pr ofits_t = RE + d_t z_t,$$

and the ex-dividend relation is:

$$q_t z_{t+1} = p_{k,\, t} K_{t+1};$$

with RE for retained earnings and $p_{k,\ t}$ being the real price for used capital.

Financing of new and used capital obeys:

$$p_{k,\ t}K_t^d + I_t^d + \varphi\left(\frac{K_{t+1}}{K_t} - 1\right)^2 = RE + q_t\left(z_{t+1} - z_t\right) + (1-\delta)\, p_{k,\ t}K_t;$$

and the net cash flow is defined as:

$$\begin{aligned} N_t &= f\left(K_t, l_t, \theta_t\right) + (1-\delta)\, p_{k,\ t}K_t - w_t l_t - p_{k,\ t}K_t^d \\ &\quad - \left[\left(K_{t+1} - (1-\delta) K_t\right) + \varphi\left(\frac{K_{t+1}}{K_t} - 1\right)^2\right] \\ &= d_t z_t + q_t\left(z_t - z_{t+1}\right) \end{aligned}$$

Thus, the firm problem can be posed as:

$$W\left(K_t\right) = \max_{k,\ l}\left\{N_t + E_t\left[M_{1t}W\left(K_{t+1}\right)\right]\right\},$$

with M_{1t} treated parametrically by firms[20], and gives the first order conditions:

$$w_t = f\left(K_t, l_t, \theta_t\right) - f_1\left(K_t, l_t, \theta_t\right) K_t;$$

$$p_{k,\ t} = E_t\left[M_{1t}W_1\left(k_{t+1}\right)\right]; \tag{A.4}$$

[20] As noted above, in equilibrium, it depends on the consumers' behavior.

$$\left[1+2\varphi\left(\frac{K_{t+1}}{K_t}-1\right)\frac{1}{K_t}\right]=E_t\left[M_{1t}W_1\left(K_{t+1}\right)\right]; \tag{A.5}$$

The envelope is:

$$W_1\left(k_t\right)=f_1\left(K_t,l_t,\theta_t\right)-2\varphi\left(\frac{K_{t+1}}{K_t}-1\right)\frac{K_{t+1}}{K_t^2}+\left(1-\delta\right)p_{k,\ t}. \tag{A.6}$$

Substituting the envelope forwarded one period into (A.4) as well as (A.4) into (A.5) results the firms' optimal decision rules:

$$\begin{aligned} & p_{k,\ t} \\ = \ & E_t\left[M_{1t}\left\{f_1\left(K_{t+1},l_{t+1},\theta_{t+1}\right)-2\varphi\left(\frac{K_{t+2}}{K_{t+1}}-1\right)\frac{K_{t+2}}{K_{t+1}^2}+\left(1-\delta\right)p_{k,\ t+1}\right\}\right] \end{aligned} \tag{A.7}$$

and

$$\left[1+2\varphi\left(\frac{K_{t+1}}{K_t}-1\right)\frac{1}{K_t}\right]=p_{k,\ t};$$

taking prices as given.

It is worth pointing that by (A.3) and (32):

$$\begin{aligned} & \left[\frac{q_{t+1}+d_{t+1}}{q_t}M_{1t}\right] \\ = \ & \left[\frac{\left\{f_1\left(K_{t+1},l_{t+1},\theta_{t+1}\right)-2\varphi\left(\frac{K_{t+2}}{K_{t+1}}-1\right)\frac{K_{t+2}}{K_{t+1}^2}+\left(1-\delta\right)p_{k,t+1}\right\}}{p_{k,t}}M_{1t}\right]; \end{aligned}$$

and $p_{k,\ t}$ is equal to Tobin's marginal Q is given by:

$$p_{k,\ t} = 1 + 2\varphi \left(\frac{K_{t+1}}{K_t} - 1 \right) \frac{1}{K_t} = Q;$$

A.3 Competitive Economy Equilibrium

An equilibrium is defined as a set of stochastic processes

$\left(r^j_{t,t+l},\ q_t,\ p_{k,t},\ z_t,\ b^j_{t+1},\ k_{+1},\ c_t,\ l_t \right)$ satisfying the f.o.c.'s and the market clearing conditions.

Because $l_t = 1\ \forall\ t$, we can argue in terms of per capita capital k_t.

To make things simpler, assume there is no issue of new shares and the firm finances itself by retained earnings:

$$z_{t+1} = z_t = 1,$$

which implies, by the ex-dividend relation and the market clearing that:

$$q_t = p_{k,t} k_{t+1} \quad \forall t.$$

The economy resources constraint is thus:

$$f(k_t, \theta_t) = c_t + \left[(k_{t+1} - (1-\delta) k_t) + \varphi \left(\frac{k_{t+1}}{k_t} - 1 \right)^2 \right], \tag{A.8}$$

and given the model parameters, the economy equilibrium conditions become:

$$p_{k,t} = \beta E_t \left[\left\{ f_1 (k_{t+1}, 1, \theta_{t+1}) - 2\varphi \left(\frac{k_{t+2}}{k_{t+1}} - 1 \right) \frac{k_{t+2}}{k^2_{t+1}} + (1-\delta) p_{k,t+1} \right\} \frac{u'(c_{t+1})}{u'(c_t)} \right]$$

and

$$\left[1+2\varphi\left(\frac{K_{t+1}}{K_t}-1\right)\frac{1}{K_t}\right]=p_{k,\ t},$$

with c_t given by (A.8). A system of two simultaneous equations that can be solved for the two unknowns p_k and k (or Q and k).

B Numerical Solution

The non-linear stochastic difference system with rational expectations (4), (13), (25), (26), (21) and (28) can be linearized around the steady and solved by some linear solution methods with reasonable precision, as shown in Novales et al. [55].

We have chosen to use Sim's [63] method to solve our model. The procedure consists of dealing with each conditional expectation and the associated expectational error as additional variables, adding to the system an equation that defines the expectational error. In our case, we have defined the variables:

$$W_{1t}=E_t\left\{\left[\alpha\theta_{t+1}k_{t+1}^{\alpha-1}+(1-\delta)-2\varphi\left(\frac{k_{t+2}}{k_{t+1}}-1\right)\frac{k_{t+2}}{k_{t+1}^2}\right]u'\left(c_{t+1}\right)\right\},$$

$$W_{2t}=E_t\left[\frac{1}{(1+\pi_{t+1,t})}u'\left(c_{t+1}\right)\right],$$

$$W_{3t}=E_t\left[\pi_{t+1,t}\right];$$

and the respective expectational errors η_{1t}, η_{2t}, η_{3t}.

The resulting linearized system is then written as::

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t$$

where:

$y_t = (c_t - c_{ss}, k_{t+1} - k_{ss}, b_{t+1} - b_{ss}, W_{1t} - W_{1ss}, W_{2t} - W_{2ss}, i_{t+1} - i_{ss}, \log\theta,$

$\pi_{t,t-1}, W_{3t}, b_t - b_{ss})$ is the vector of variables determined within the model (inclusive W, but except η) and Γ_0 and Γ_1 are the matrices containing the system linearized coefficients.

If Γ_0 is invertible (and it is):

$$y_t = \Gamma_0^{-1}\Gamma_1 y_{t-1} + \Gamma_0^{-1}\Psi z_t + \Gamma_0^{-1}\Pi\eta_t = \widetilde{\Gamma_1} y_{t-1} + \widetilde{\Psi} z_t + \widetilde{\Pi}\eta_t$$

and $\widetilde{\Gamma_1}$ has a Jordan decomposition $\widetilde{\Gamma_1} = P\Lambda P^{-1}$.

Defining $w_t = P^{-1}y_t$, we obtain:

$$w_t = \Lambda w_{t-1} + P^{-1}\left(\widetilde{\Psi} z_t + \widetilde{\Pi}\eta_t\right)$$

where, for every eigenvalue λ_j of $\widetilde{\Gamma_1}$ we have an equation:

$$w_{jt} = \lambda_{jj}\Lambda w_{j,t-1} + P^{j\cdot}\left(\widetilde{\Psi} z_t + \widetilde{\Pi}\eta_t\right)$$

where $P^{j\cdot}$ denotes the j-th row of P^{-1}.

As the state variables and the shadow prices are assumed to grow less than $\beta^{-\frac{1}{2}}$, for w_j, with $|\lambda_{jj}| > \beta^{-\frac{1}{2}}$, we need have:

$$w_{jt} = P^{j\cdot} y_t = 0, \quad \forall t$$

which provides the system stability conditions.

Using the notation $P = [P_{\bullet S}, P_{\bullet U}]$ and $P^{-1} = \begin{bmatrix} P^{S\bullet} \\ P^{U\bullet} \end{bmatrix}$, where U stands for "unstable", the system can be written as:

$$w_{S,t} = \Lambda_S w_{S,t-1} + \begin{bmatrix} I & -\Phi \end{bmatrix} P^{-1} \Psi z_t; \tag{B.1}$$

where: $\Phi = P^{S\bullet} \Pi \eta \left(P^{U\bullet} \Pi \eta \right)^{-1}$.

To arrive at an equation in y, we use $y = Pw$ to transform (B.1) into:

$$y_t = P_{\bullet S} \Lambda_S P^{S\bullet} y_{t-1} + \left(P_{\bullet S} P^{S\bullet} - P_{\bullet S} \Phi P^{U\bullet} \right) \Psi z_t,$$

which can be solved after imposing:

$$w_{U,0} = P^{U\bullet} y_0 = 0.$$

Figures and Tables

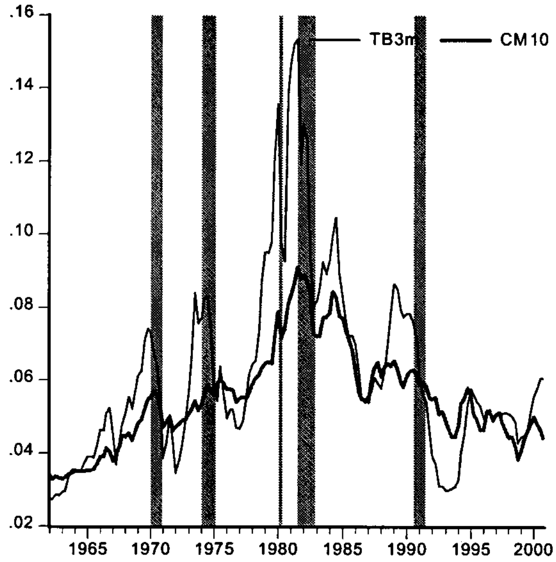

Figure 1: Evolution of U.S. nominal yields from 1962:01 to 2000:04

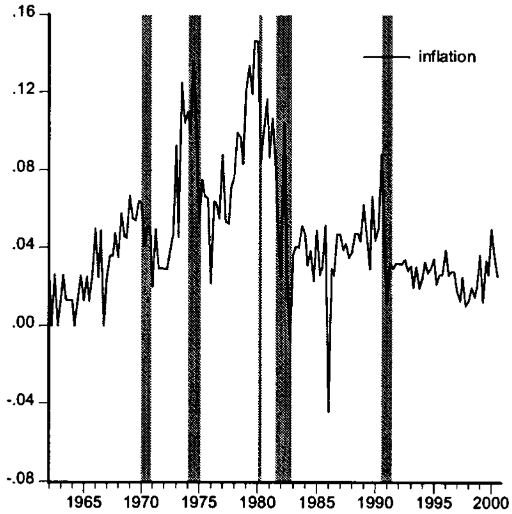

Figure 2: U.S. quarterly inflation from 1962:1 to 2000:4

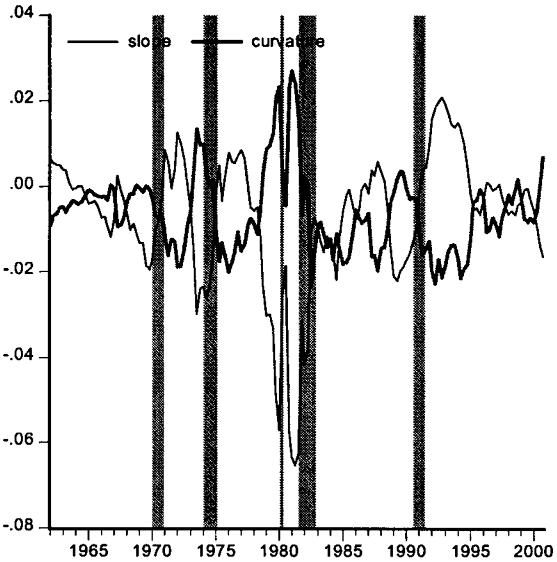

Figure 3: Evolution of U.S. slope and curvature of the term spread from

1962:03 to 2000:04

	2	3	6	12	24	48	120
b	.502	.467	.320	.272	.363	.442	1.402
(s.e.)	(.096)	(.148)	(.146)	(.208)	(.223)	(.384)	(.142)

Table 1: b estimates by Campbell, Loo and MacKinlay

Maturity	1st. P.C.	2nd. P. C.	3rd. P. C.
TB3m	0.516	-0.525	0.598
TB6m	0.521	-0.235	-0.251
TB1	0.502	0.054	-0.618
CM3	0.317	0.433	-0.004
CM5	0.237	0.447	0.177
CM7	0.185	0.404	0.259
CM10	0.140	0.340	0.314

Table 2: Empirical First three principal components (or eigenvectors whith largest eigenvalues) - quartely U.S. nominal yields from 1969:03 to 2000:04

Maturity	Total Variance Explained by Factor1+Factor2	Proportion of Total Explained Variance Accounted for by Factor 1	Factor 2	Factor 3
TB3m	99.1	92.0	7.1	0.8
TB6m	99.6	98.1	1.5	0.2
TB1	98.9	98.8	0.1	1.0
CM3	99.6	87.4	12.2	0.0
CM5	99.4	78.5	20.9	0.3
CM7	98.7	72.8	25.9	1.0
CM10	95.8	66.6	29.3	2.3
Average	**98.7**	**84.9**	**13.8**	**0.8**

Table 3: Empirical Relative Importance of the Empirical Factors (%) - quartely U.S. nominal yields from 1969:03 to 2000:04

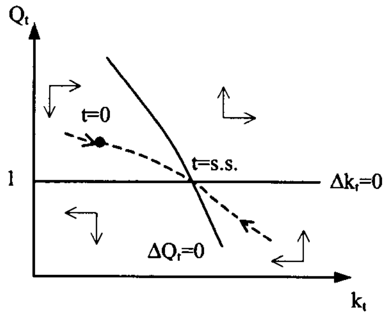

Figure 4: Q x k phase diagram

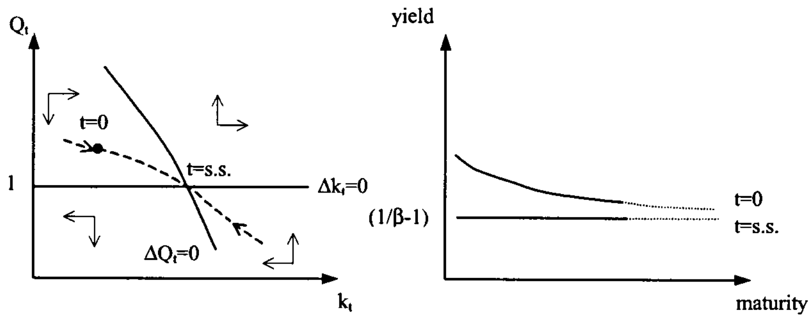

Figure 5: Expansion path and the real term structure without inflation

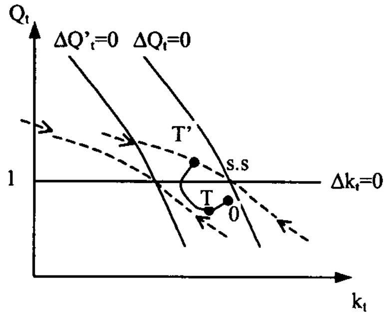

Figure 6: Certain transitory tight

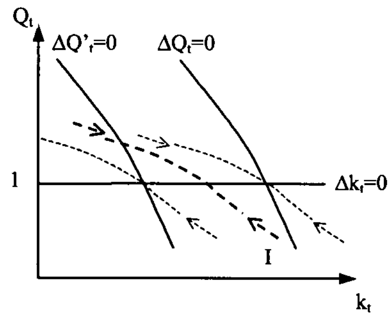

Figure 7: Uncertain transitory tight

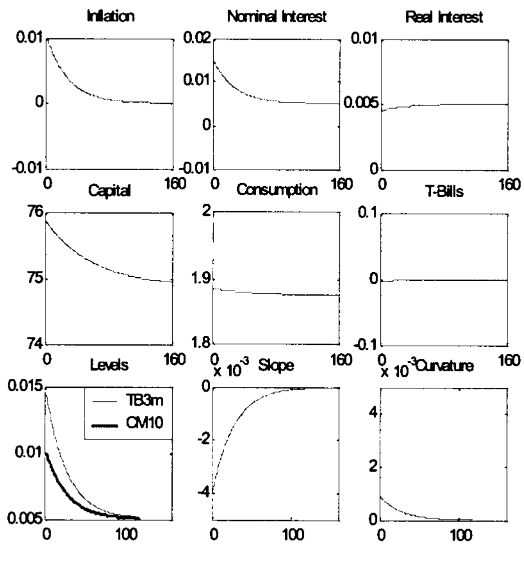

Figure 8: Contraction path

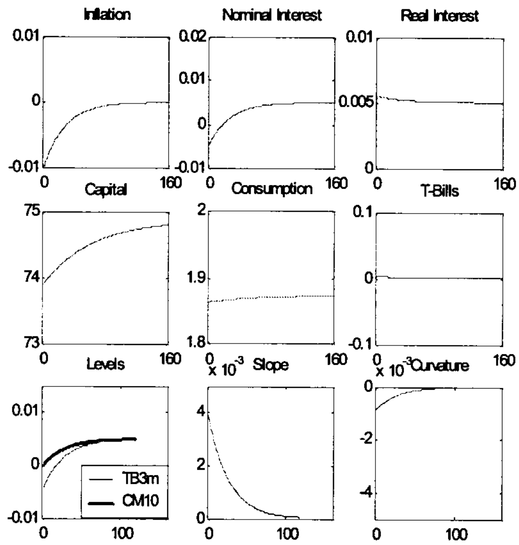

Figure 9: Expansion path

	1st. P.C.		2nd. P. C.		3rd. P. C.	
Maturity	Simul.	Empir.	Simul.	Empir.	Simul.	Empir.
TB3m	**0.600**	0.516	**-0.678**	-0.525	**0.413**	0.598
TB6m	**0.499**	0.521	**-0.065**	-0.235	**-0.730**	-0.251
TB1	**0.424**	0.502	**0.337**	0.054	**-0.243**	-0.618
CM3	**0.291**	0.317	**0.371**	0.433	**0.245**	-0.004
CM5	**0.241**	0.237	**0.340**	0.447	**0.242**	0.177
CM7	**0.204**	0.185	**0.299**	0.404	**0.269**	0.259
CM10	**0.162**	0.140	**0.282**	0.340	**0.217**	0.314

Table 4: Simulated and Empirical First three principal components (or eigenvectors whith largest eigenvalues) - quartely U.S. nominal yields from 1969:03 to 2000:04

Maturity	Total Variance Explained by Factor1+Factor2		Proportion of Total Explained Variance Accounted for by Factor 1		Factor 2		Factor 3	
	Simul.	Empir.	Simul.	Empir.	Simul.	Empir.	Simul.	Empir.
TB3m	**97.3**	99.1	**76.8**	92.0	**20.5**	7.1	**2.7**	0.8
TB6m	**84.8**	99.6	**84.5**	98.1	**0.3**	1.5	**13.4**	0.2
TB1	**91.0**	98.9	**80.4**	98.8	**10.6**	0.1	**2.0**	1.0
CM3	**91.4**	99.6	**68.3**	87.4	**23.1**	12.2	**3.6**	0.0
CM5	**92.0**	99.4	**64.9**	78.5	**27.0**	20.9	**4.9**	0.3
CM7	**87.0**	98.7	**60.1**	72.8	**26.9**	25.9	**7.7**	1.0
CM10	**84.2**	95.8	**51.7**	66.6	**32.5**	29.3	**6.8**	2.3
Average	**89.7**	98.7	**69.5**	84.9	**20.1**	13.8	**5.9**	0.8

Table 5: Simulated and Empirical Relative Importance of the Empirical Factors (%) - quartely U.S. nominal yields from 1969:03 to 2000:04

Maturity	Correlations Simul.	Empir.
3-month	-0.87	
6-month	-0.60	
1-year	-0.40	-0.69
3-year	-0.35	
5-year	-0.30	-0.06
7-year	-0.22	
10-year	-0.06	-0.08

Table 6: Simulated correlations between the same maturities real rate and expected interest rate

Chapter 2

A Jump-Diffusion Yield-Factor Model of Interest Rates

October 2, 2001

Abstract

In this paper the Federal Fund Rate Target and the one-year T-Bill are the two yield factors explaining the movements of the term structure. Using Duffie & Kan [28] approach, the two rates are consistent modeled and an affine model of the term structure results that is able to incorporate the Federal Open Market Committee discontinuous moves. An explicit formula for the zero-coupon bond price is derived and the model presents good fit to the January 1990 - December 2000 Monthly U.S. Treasury term structure. The factors level, slope and curvature are respectively identified as the 1-year T-Bill, the Federal Fund Rate Target and their spread.

JEL classification: E43; E52; G13

Keywords: Exponential affine model, Discontinuous short-rate target; 1-year yield; Monetary policy risk; Zero-coupon bond price

1 Introduction

As Merton [52] and [51] pointed out, the variation in a general asset price process can be decomposed into two types of changes: the "normal" diffusion due to new information that arrives continuously and causes marginal changes in the asset level (modeled with constant probability and size proportional to the square root of observation frequency); and the "rare" jumps, due to important new information, that arrives at discrete points in time and has more than a marginal effect (modeled with probability proportional to the observation frequency and constant size).

The active trade at interest rate markets favors the full incorporation of all small pieces of information at the time they become public and supports the continuous diffusion models like Vasicek [68], Cox Ingersoll & Ross [24], Heath, Jarrow & Morton [40] and others. However, there exist single news items - like inflation forecast releases and the monetary authority actions - that do carry important information for the market and may cause interest rates to jump, but have not deserved much attention. The first few papers considering discontinuities, like Ahn & Thompson [1] and Babbs & Webber [6], have got neither academic nor market recognition; and, although the Federal Reserve has started setting a discontinuous target for the Federal Fund Rate (the U.S. overnight rate), recent works like Attari [5], El-Jahel et al. [30], Duffie et al. [29], Piazzesi [58] and Das [25] are still marginal.

This paper presents a continuous-time two-yield factor model of the term structure of interest rates which incorporates such FED discontinuous action.

Since Federal Fund Rate Target moves are sponsored by the expectations of inflation and activity level, both incorporated in the medium-term yields of the nominal term structure [1], we assume the Federal Fund Rate Target and one "representative" term structure yield are enough to describe the full system dynamics. After estimating a simple (but compatible) parametrization for the bivariate process that is not rejected by the U.S. Treasury data, we use Duffie & Kan [28] approach to derive an affine model of the term structure of the nominal interest rates that is able to incorporate the FED action risk.

The reasoning goes along the following lines. Based on the review of the theory and on evidences of jump, we propose a parsimonious joint specification of the Federal Fund Rate Target and the 1-year T-Bill (the rate chosen as the representative) that is shown to fit the data. The monetary authority's discontinuous moves of the short-rate target cause the term structure to move discontinuously. The intensity of the moves is a function of the difference between the long- and the short-rate, what means the authority's action can be anticipated. The spread evolves countercyclically, being big when rates' levels are low (or small when levels are high), and provides different fundamentals for the observed mean reversion.

From the proposed two yield-factor process, we derive an explicit exponential-affine bond price formula able to mimic verified data patterns. The model produces a reasonable fit to the U.S. Treasury term structure from January 1990 to December 2000. The results allow to identify the level, the slope and the

[1] The short-term interest rate dependence of the business condition variables is explicit in the Taylor rule literature (see Taylor [64]). That the FED is mainly reacting to information contained in the yield curve is shown by Piazzesi [59].

curvature factors respectively as the representative rate, the Target and their spread. The implications for actual interest rate forecasting are discussed.

2 The data

This study covers the period from January, 1990 to December, 2000. The monthly nominal interest rate data used is from the H.15 Release by the Board of Governors (available at the FED of Saint Louis' web site: www.stls.frb.org/fred) and from the Fama-Bliss Discount Bonds File available on the Center of Research in Security Prices tape, observed at the end of the month. The H.15 Release data are bond-equivalent yields (r_{BEY}) or discount rate (r_D), and were transformed to annualized continuously-compounded yield-to-maturity by respectively: $r_{YTM} = \frac{1}{M}\ln(1+r_{BEY}*\frac{M}{100})$ and $r_{YTM} = -\frac{1}{M}\ln(1-r_D*\frac{M}{100})$, where M is time-to-maturity in years. The Fama-Bliss data are monthly continuously-compounded yield-to-maturity, and were multiplied by 12 to obtain the annualized continuously-compounded yield-to-maturity. The nine rates chosen were: the Federal Fund Rate Target (r^0), the Federal Funds Rate (FF), the 3-Month Treasury Bill Rate (3-m) and the 6-Month Treasury Bill Rate (6-m) from the H.15 Release; and the 1-Year to 5-Year Treasury Rates (1-y, 2-y, 3-y, 4-y, 5-y) from the Fama-Bliss File. The T-bills are secondary market rates on Treasury securities actually observed. The Fama-Bliss File uses only fully taxable, non-callable, non-flower bonds. The Federal Funds Rate and the Federal Funds Rate Target are also actually observed and seem to incorporate a premium over Treasury lending that poses a problem in presenting a monotone upward slop-

ing average yield curve. Furthermore, the Federal Fund Rate varies too much along the month what may cause the end of month observation to be biased. Because in the following work we are not using the Federal Fund Rate, but the Target only, we have subtracted a constant premium equal to 0.0038, enough to generate a smooth monotone average yield curve, when including the Target.

Tables 1.a and 1.b below present some descriptive statistics for the chosen rates level and difference, while Figures 1.a and 1.b respectively show the yield curve surface and the average yield curve for the period from January 1990 to December 2000 for the Federal Fund Rate Target and the Treasury yields. The statistics for the levels shown in Table 1.a make clear the monotonicity of the average yield curve. The Augmented Dickey-Fuller test statistic rejects nonstationarity at 10% significance level for all rates. Table 1.b presents descriptive statistics for the rates differences. The Federal Fund Rate Target presents huge kurtosis, with the Jarque-Bera statistics rejecting normality. The first order autocorrelations are significant for all models and maturities[2] and the Ljung-Box Q-statistics (not reported) reject the null hypothesis of no autocorrelation up to order $k \leqslant 12$ for series on the short-end of the yield curve and up to 2-years maturity.

[2] Given the number of observations (T) is 131, autocorrelations smallers than 0.17 in absolute value ($\pm 2/\sqrt{T}$) are not significantly different from zero at (approximately) 5% significance level.

3 The Federal Fund Rate Target and its role in the yield curve modeling

In this section we first estimate a discrete process for the Federal Fund Rate difference which takes the yield curve into account. Then, we step up parametrizing and estimating a bivariate process for the target and the 1-year yield which will buttress the exponential-affine model developed in the following section.

3.1 A parametric process for the Federal Fund Rate Target

Figure 2 shows that the Federal Fund Rate Target process (r^0) is a pure discontinuous process, what justifies to parametrize its difference dr^0 as a discrete process that jumps.

Given the Federal Open Market Committee does not chose policy goals and routes on daily basis, but only after accumulate a reasonable amount of information, and given longer nominal interest rates (r^1) are widely accepted leading indicators that incorporate expected future inflation and activity levels, we take them as a factor sponsoring target moves and parametrize dr^0 to be:

$$dr_t^0 = dJ_t \tag{1}$$

where:

$$dJ_t = \begin{cases} \delta \frac{(\theta + r_t^1 - r_t^0)}{\left|\theta + r_t^1 - r_t^0\right|} \quad \textit{with probability } \lambda \left|\theta + r_t^1 - r_t^0\right| dt \\ 0 \quad \textit{with probability } 1 - \lambda \left|\theta + r_t^1 - r_t^0\right| dt \end{cases},$$

and r^1 is the longer rate proxing for expected future inflation and activity level.

Because dJ is not zero mean, it needs reformulation to continue having the interpretation of unexpected variation and we shall rewrite dr^0 as:

$$dr_t^0 = \delta\lambda\left(\theta + r_t^1 - r_t^0\right)dt + \left[dJ_t - \delta\lambda\left(\theta + r_t^1 - r_t^0\right)dt\right]; \tag{2}$$

what provides a non-standard foundation for the mean-reverting drift: it is the result of a non-zero mean discontinuous shock.

Conditional on the information available at t, the above process is a Poisson and presents the following two discrete time first moments[3]:

$$E_t\left[\Delta r_t^0\right] = \delta\lambda\left(\theta + r_t^1 - r_t^0\right)\Delta t \tag{3}$$

$$E_t\left[\left(\Delta r_t^0 - E_t\left[\Delta r_t^0\right]\right)^2\right] = \delta^2\lambda\left|\theta + r_t^1 - r_t^0\right|\Delta t. \tag{4}$$

The FED procedure to move the rate by multiples of 25 basis points, since November 1990, allows to restrict $\delta = .0025$ [4]. The choice of the r^1 can be based on best fit to data.

We have estimated (3) and (4) for r^1 =$\{$3-m, 6-m, 1-y, 2-y, 3-y, 4-y, 5-$y\}$ by GMM, with a constant, r_{t-1}^0 and r_{t-1}^1 as instruments. Table 2 below present the estimates for the models, together with Hansen's overidentifying restriction statistics and p-values, for the period between January 1990 and December 2000.

[3] Derived in Apendix A.

[4] In this context, changes of 50 or 75 basis points are taken as the simultaneous realization of two or three jumps. See Thornton [66] for the historical evidence.

None of the estimated models can be rejected at 5%. Because the 1-year yield is the rate that provides the best fit, besides being a middle-term rate, we chose it as the representative rate of the whole term structure, r^1 = 1-y, in the study to follow. From now on, references to r^1 in a empirical context are intended to mean the 1-year yield.

3.2 A bivariate process for the Target and the term structure

Given a set of factors X, that satisfy the stochastic differential system:

$$dX_t = \mu\left(X_t\right) dt + \sigma\left(X_t\right) dW_t + dZ\left(X_t\right), \tag{5}$$

where: $\mu(\cdot)$ is a drift function, $\sigma(\cdot)$ is a volatility function, W_t is a standard Brownian motion,

$$dZ\left(X_t\right) = \begin{cases} z \quad \textit{with probability } \lambda\left(X_t\right) dt \\ 0 \quad \textit{with probability } 1 - \lambda\left(X_t\right) dt \end{cases};$$

and $f\left(X_t, T - t\right)$, the nominal price at t of a nominal bond maturing at T; (f, μ, σ) are compatible if we have:

$$f\left(X_t, T - t\right) = E\left[\exp\left(-\int_t^T R\left(X_u\right) du\right) \mid X_t\right], \tag{6}$$

with E the expectation under risk neutral probabilities [5].

[5] In this context, the spot rate is defined as:

$$R(x) = \lim_{\tau \downarrow 0} \frac{-\log f(x, \tau)}{\tau}.$$

Figure 3.a plots the Target (r^0) and some yields of different maturities, showing that the Target and the term structure seem to present the same trend. To make the picture clear, Figure 3.b plots the Target (r^0) and the 1-year yield (1-y). The apparent relation makes us wonder if there are gains in including the Target in the spot rate processes, or better, given last subsection results, if there are gains on modeling a compatible bivariate process for the Target and one representative rate of the term structure.

Duffie & Kan [28] show that a necessary condition for (f, μ, σ) to be a compatible affine yield-factor model of interest rates:

$$f\left(X_t, T-t\right) = \exp\left\{A\left(T-t\right) + B\left(T-t\right)\cdot X_t\right\}, \tag{7}$$

is that dX_t can be written as:

$$dX_t = \left(a \cdot X_t + b\right) dt + \Sigma \cdot \begin{pmatrix} \sqrt{v_1\left(X_t\right)} & 0 & \dots & 0 \\ 0 & \sqrt{v_2\left(X_t\right)} & & \\ & & & \\ 0 & & & \sqrt{v_n\left(X_t\right)} \end{pmatrix} dW_t + dZ\left(X_t\right), \tag{8}$$

with $v_i\left(x\right) = \alpha_i + \beta_i \cdot x$, and $Z\left(X_t\right)$ a pure jump component with fixed probability distribution and affine arrival intensity function $\lambda\left(X_t\right)$, under the risk neutral probabilities[6].

The above means that, to model an affine compatible bivariate process of the Target (r^0) and the representative yield (r^1), it is necessary that the proposed

[6]This is a necessary, but not sufficient condition.

model be nested in (8). Taking (2) as given, the most general process for the representative yield allowed by (8) has the form:

$$dr^1 = \left(\mu_0 + \mu_1 r^0 + \mu_2 r^1\right) dt + \sigma\sqrt{(v_0 + v_1 r^0 + v_2 r^1)}dW + dz^1; \qquad (9)$$

where:

$$dz_t^1 = \begin{cases} \rho\delta\frac{(\theta + r_{t-}^1 - r_{t-}^0)}{\left|\theta + r_{t-}^1 - r_{t-}^0\right|} & \textit{with probability } \lambda\left|\theta + r_{t-}^1 - r_{t-}^0\right| dt \\ 0 \quad \textit{with probability } 1 - \lambda\left|\theta + r_{t-}^1 - r_{t-}^0\right| dt \end{cases} .$$

And if the v_1, v_2 as well as the ρ are not significant, what is shown to be case, the specific process left is:

$$dr^1 = \left(\mu_0 + \mu_1 r^0 + \mu_2 r^1\right) dt + \sigma dW; \qquad (10)$$

which can be jointly estimated with (2).

We have estimated the system (2)-(10) by GMM, with a constant, r_{t-1}^0 and r_{t-1}^1 as instruments, using the two first discrete-time moments for both processes. That means, the moments (3) and (4) of the Target and the Ito-Taylor approximation of the representative rate continuous-time moments [7]:

$$E_t\left[\Delta r_t^1\right] = \left(\mu_0 + \mu_1 r_t^0 + \mu_2 r_t^1\right)\Delta t + \mu_1 \Delta r_t^0 \Delta t + \mu_2\left(\mu_0 + \mu_1 r_t^0 + \mu_2 r_t^1\right)(\Delta t)^2, \qquad (11)$$

[7] Derived in Apendix A.

and

$$E_t\left[\left(\Delta r_t^1 - E_t\left[\Delta r_t^1\right]\right)^2\right] = \sigma^2 \Delta t + \frac{1}{3}\left(\mu_2 \cdot \sigma\right)^2 (\Delta t)^3 + (\mu_2)^2 \cdot \delta^2 \lambda \left|\theta + r_t^1 - r_t^0\right| (\Delta t)^3 . \tag{12}$$

Estimates presented in Table 3, together with Hansen's overidentifying restriction statistic and p-value show interesting results. First, the p-value of 0.35 disallow the proposed model rejection. Second, the magnitudes of λ and θ do not change in relation to Table 2 estimates, when the Target was estimated alone.

However, the most interesting results from Table 3 are the signs and magnitudes of $\mu's$. Contrary to the standard mean-reverting parametrizations, the μ_2 estimate presents positive sign [8]. Does that imply the r^1 process is explosive?

The answer is no, that doesn't. Indeed, the estimated parameters verify the stylized fact that the term spread $\left(r^1 - r^0\right)$ is big and decreasing when rates' levels are low, and $\left(r^1 - r^0\right)$ is small and increasing when rates' levels are high; this meaning the short end of the term structure varies more along the cycle than the long end. By rewriting the system (2)-(10) with zero expected errors,

[8] Standard mean-reverting spot rate processes have the form:

$$dr_t = \left(\beta_t - \alpha r_t\right) dt + \sigma\left(r_t\right) dW;$$

with $\alpha > 0$. This implies that even if β_t is allowed to vary along time, r is expected to decrease if $\alpha r_t > \beta_t$ and is expected to increase if $\alpha r_t < \beta_t$.

we get:

$$\begin{bmatrix} dr_t^0 \\ dr_t^1 \end{bmatrix} = \left\{ \begin{bmatrix} -\delta\lambda & \delta\lambda \\ \mu_1 & \mu_2 \end{bmatrix} \begin{bmatrix} r_t^0 \\ r_t^1 \end{bmatrix} + \begin{bmatrix} \delta\lambda\theta \\ \mu_0 \end{bmatrix} \right\} dt$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix} dW_t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} dN_t \tag{13}$$

$$= \{A \cdot r_t + B\} dt + V \cdot dW_t + \iota_1 \cdot dN_t;$$

where:

$$dN_t = \begin{cases} \delta \frac{(\theta + r_{t-}^1 - r_{t-}^0)}{|\theta + r_{t-}^1 - r_{t-}^0|} - \delta\lambda \left(\theta + r_{t-}^1 - r_{t-}^0\right) dt & \textit{with probability } \lambda \left|\theta + r_{t-}^1 - r_{t-}^0\right| dt \\ -\delta\lambda \left(\theta + r_{t-}^1 - r_{t-}^0\right) dt & \textit{with probability } 1 - \lambda \left|\theta + r_{t-}^1 - r_{t-}^0\right| dt \end{cases},$$

has $E_t[dN_t] = 0$.

The above system is well behaved if the trace of A is smaller than zero, $tr(A) < 0$, and its determinant is bigger than zero, $|A| > 0$. It is the case for the above estimated coefficients: $tr(A) = -4.64 < 0$ and $|A| = 0.56$. The estimated system is stable, and the long run levels of r^0 and r^1 are respectively given by $\overline{r^0} = \frac{\mu_0 - \theta\mu_2}{-\mu_1 - \mu_2} = 0.0532$ and $\overline{r^1} = \frac{\mu_0 + \theta\mu_1}{-\mu_1 - \mu_2} = 0.0583$ [9]. The estimated drifts:

$$dr_t^0 = \left(-0.0119 - 2.2875 \cdot r_t^0 + 2.2875 \cdot r_t^1\right) dt,$$

and

$$dr_t^1 = \left(0.0025 - 2.2757 \cdot r_t^0 + 2.0309 \cdot r_t^1\right) dt,$$

[9] Both value higher than the sample means calculated in Table 1.

mean that when levels are relative low, the spread $(r^1 - r^0)$ is relative big, and both rates are expected to increase, but r^0 increases more than r^1. Furthermore, kept the spread constant, the r^1 drift decreases with the level.

The resulting dynamics are in accordance with the nominal term structure stylized facts, described in Fama & French [33] for example, that yield levels and the term spread evolve in opposite direction, resulting in wider fluctuations of shorter rates along the cycle.

4 The Zero-Coupon Bond Price Formula

Following Duffie & Kan [28], we consider the zero-coupon bond price process for a fixed maturity date T:

$$F(x,t) = f(x, T-t).$$

By Ito's lemma:

$$dF(X_t,t) = D^*F(X_t,t)\,dt + F_x(X_t,t)\,\sigma(x)\,dW_t + [F(X_t+dZ,t) - F(X_t,t)];$$

where $D^*$is the infinitesimal generator, defined by:

$$D^*F(x,t) = DF(x,t) + \lambda(x)\int [F(x+z,t) - F(x,t)]\,dv(z), \qquad (14)$$

with:

$$DF\left(x,t\right)=F\left(x,t\right)\left[\begin{array}{c}-A'\left(T-t\right)-B'\left(T-t\right)\cdot x-B\left(T-t\right)\cdot\mu\left(x\right)\\ +\frac{1}{2}\sum_i\sum_j B_i\left(T-t\right)B_j\left(T-t\right)\sigma_i\left(x\right)\sigma_j\left(x\right)^T\end{array}\right],$$

$\mu(x)$ being the risk-neutral drift and $\sigma(x)$ being the risk-neutral volatility.

The price function at time t of a T-maturity zero-coupon bond $F(x,t)$ solves the Partial Differential-Difference Equation (PDDE):

$$D^{*}F\left(x,t\right)-R\left(x\right)\cdot F\left(x,t\right)=0,$$

with initials:

$$A\left(0\right)=B_0\left(0\right)=B_1\left(0\right)=0,$$

and boundary conditions:

$$F\left(x,T\right)=1.$$

Since the estimated process under real probabilities is given by [10]:

$$\begin{bmatrix} dr_t^0 \\ dr_t^1 \end{bmatrix}=\left\{\begin{bmatrix} 0 & 0 \\ \mu_1 & -\mu_3 \end{bmatrix}\begin{bmatrix} r_t^0 \\ r_t^1 \end{bmatrix}+\begin{bmatrix} 0 \\ \mu_0 \end{bmatrix}\right\}dt+\begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}dW_t+\begin{bmatrix} 1 \\ 0 \end{bmatrix}dJ_t;$$

we can adapt Duffie & Kan [28] general formulas to our specific infinitesimal

[10] Note this expression have not been written to have zero expected error like (13).

generator:

$$
\begin{aligned}
D^*F\left(r^0,r^1,t\right) &= DF\left(r^0,r^1,t\right)+\lambda\left|\theta-r_t^0+r_t^1\right|\cdot\left[F\left(r^0+z^0,r^1+z^1,t\right)-F\left(r^0,r^1,t\right)\right] \\
&= DF\left(r^0,r^1,t\right) \\
&\quad +\lambda\left|\theta-r_t^0+r_t^1\right|\cdot F\left(r^0,r^1,t\right)\left[\exp\left\{B_0\left(T-t\right)z^0_{+(-)}\right\}-1\right],
\end{aligned}
$$

with:

$$
DF\left(r^0,r^1,t\right)=F\left(r^0,r^1,t\right)\left[\begin{array}{c}
-A'\left(T-t\right)-B_0'\left(T-t\right)\cdot r^0-B_1'\left(T-t\right)\cdot r^1 \\
+B_0\left(T-t\right)\cdot\left(-\pi^0\left(\theta-r^0+r^1\right)\right) \\
+B_1\left(T-t\right)\cdot\left(\mu_1 r^0-\mu_3 r^1+\mu_0-\pi^1\right) \\
+\frac{1}{2}B_1^2\left(T-t\right)\sigma^2
\end{array}\right];
$$

and:

$$
z^0_{+(-)}=\left\{\begin{array}{ll}
\delta & if\ \ \theta+r^1_{t-}>r^0_{t-} \\
-\delta & if\ \ \theta+r^1_{t-}<r^0_{t-}
\end{array}\right.,
$$

where the market price of risks of r^0 and r^1, respectively parametrized as $\pi_t^0=\pi^0\left(\theta-r_t^0+r_t^1\right)$ and $\pi_t^1=\pi^1$, have been subtracted from the real drifts to allow risk-neutral pricing [11]. The value of the parameters π_t^0 and π^1 will be chosen to minimize the squared deviations between a given yield curve and that implied by the model.

[11] The explanations for these parametrizations are provided in subsection 6.2.

Given $R(x_t) = r_t^0$:

$$D^* F(r,t) - r_t^0 \cdot F(r,t) = 0$$

and writing $T - t = \tau$, we have the non-linear system of ordinary differential equations:

$$\begin{aligned} A'(\tau) &= B_0(\tau) \cdot \left(-\pi^0 \theta\right) + B_1(\tau) \cdot \left(\mu_0 - \pi^1\right) + \frac{1}{2} B_1^2(\tau) \cdot \sigma^2 \\ &\quad \pm \lambda\theta \cdot \left[\exp\left\{B_0(\tau) z_\pm^0\right\} - 1\right], \end{aligned}$$

$$B_0'(\tau) = B_0(\tau) \cdot \pi^0 + B_1(\tau) \cdot \mu_1 \ \mp \ \lambda \cdot \left[\exp\left\{B_0(\tau) z_\pm^0\right\} - 1\right] - 1,$$

$$B_1'(\tau) = B_0(\tau) \cdot \left(-\pi^0\right) + B_1(\tau) \cdot \left(-\mu_3\right) \ \pm \ \lambda \cdot \left[\exp\left\{B_0(\tau) z_\pm^0\right\} - 1\right],$$

$$if \qquad \theta + r_t^1 \gtrless r_t^0;$$

with $z_+^0 = 0.0025$ and $z_-^0 = -0.0025$.

To make the solution of the above system easier, we chose to linearize it by use of the first order approximation:

$$\exp\left\{B_0(\tau) z_\pm^0\right\} \approx 1 + B_0(\tau) z_\pm^0,$$

which has good accuracy given the constants $z_\pm^0$ are small.

The linearized system is given by:

$$A'(\tau) = B_0(\tau) \cdot \left(\pm \lambda z_\pm^0 - \pi^0\right) \theta + B_1(\tau) \cdot \left(\mu_0 - \pi^1\right) + \frac{1}{2} B_1^2(\tau) \sigma^2, \qquad (15)$$

$$B_0'(\tau) = B_0(\tau)\left(\mp\,\lambda z_{\pm}^0 + \pi^0\right) + B_1(\tau)\cdot\mu_1 - 1, \tag{16}$$

$$B_1'(\tau) = B_0(\tau)\left(\pm\,\lambda z_{\pm}^0 - \pi^0\right) + B_1(\tau)(-\mu_3). \tag{17}$$

It is time to note that the two apparent different systems are indeed the same one, given the term $+\lambda z_+^0 = \lambda\cdot 0.0025$ implied by $\theta + r_t^1 > r_t^0$ is equal to the term $-\lambda z_-^0 = \lambda\cdot 0.0025$ implied by $\theta + r_t^1 < r_t^0$. Thus, we use $z^0 = +z_+^0 = -z_-^0$ and $-z^0 = -z_+^0 = z_-^0$ hereafter.

We solve for $B\prime s$ first. Given the system of first order differential equations:

$$\begin{aligned}\begin{bmatrix} B_0'(\tau) \\ B_1'(\tau)\end{bmatrix} &= \begin{bmatrix} -\,\lambda z^0 + \pi^0 & \mu_1 \\ +\,\lambda z^0 - \pi^0 & -\mu_3\end{bmatrix}\begin{bmatrix} B_0(\tau) \\ B_1(\tau)\end{bmatrix} + \begin{bmatrix} -1 \\ 0\end{bmatrix} \\ &= M\cdot B(\tau) + c,\end{aligned} \tag{18}$$

the system solution is stable if the trace of M is smaller than zero, $tr(M) < 0$, and the determinant of M is bigger than zero, $|M| > 0$. That means:

$$tr(M) = -\lambda z^0 + \pi^0 - \mu_3 < 0,$$

and:

$$|M| = \left(-\lambda z^0 + \pi^0\right)(-\mu_3) - \mu_1\left(\lambda z^0 - \pi^0\right) = \left(\lambda z^0 - \pi^0\right)(\mu_3 - \mu_1) > 0.$$

The first-order system (18) can be transformed to two second-order differ-

ential equations, given by:

$$B_0''(\tau) - tr(M) \cdot B_0'(\tau) + |M| \cdot B_0(\tau) = -\mu_3,$$

$$B_1''(\tau) - tr(M) \cdot B_1'(\tau) + |M| \cdot B_1(\tau) = \left(-\lambda z^0 + \pi^0\right),$$

and the $B\prime s$ solutions are given by:

$$B_0(\tau) = \Psi_{00} \cdot e^{s_0 \tau} + \Psi_{01} \cdot e^{s_1 \tau} + \overline{B}_0,$$

and

$$B_1(\tau) = \Psi_{10} \cdot e^{s_0 \tau} + \Psi_{11} \cdot e^{s_1 \tau} + \overline{B}_1;$$

where:

$$s_{0,1} = \frac{tr(M) \pm \sqrt{tr^2(M) - 4 \cdot |M|}}{2},$$

$$\overline{B}_0 = \frac{-\mu_3}{|M|}, \qquad \overline{B}_1 = \frac{\left(-\lambda z^0 + \pi^0\right)}{|M|},$$

and the Ψ's are constants defined by the initial and boundary conditions.

The boundary conditions $B_0(\tau_0) = -\tau_0$, $B_0(\tau_1) = 0$, $B_1(\tau_0) = 0$, and $B_1(\tau_1) = -\tau_1$, give:

$$\Psi_{00} = \frac{1}{e^{s_1(\tau_0 - \tau_1) + s_0 \tau_1} - e^{s_0 \tau_0}} \left[\tau_0 + \left(1 - e^{s_1(\tau_0 - \tau_1)}\right) \cdot \overline{B}_0\right],$$

$$\Psi_{01} = \frac{1}{e^{s_0(\tau_0 - \tau_1) + s_1 \tau_1} - e^{s_1 \tau_0}} \left[\tau_0 + \left(1 - e^{s_0(\tau_0 - \tau_1)}\right) \cdot \overline{B}_0\right],$$

$$\Psi_{10} = \frac{1}{e^{s_1(\tau_1 - \tau_0) + s_0 \tau_0} - e^{s_0 \tau_1}} \left[\tau_1 + \left(1 - e^{s_1(\tau_1 - \tau_0)}\right) \cdot \overline{B}_1\right],$$

$$\Psi_{11} = \frac{1}{e^{s_0(\tau_1-\tau_0)+s_1\tau_0} - e^{s_1\tau_1}} \left[\tau_1 + \left(1 - e^{s_0(\tau_1-\tau_0)}\right) \cdot \overline{B}_1\right],$$

and the initial conditions $B_0(0) = B_1(0) = 0$ will be fulfilled by choosing the suitable set of parameters.

From B_0 and B_1, equation (15) is integrated to give [12]:

$$\int_{\tau_0}^{\tau} A'(u)\, du = A(\tau) - A(\tau_0) = G(\tau) - G(\tau_0),$$

which, with conditions $A(0) = A(\tau_0) = A(\tau_1) = 0$, results in:

$$A(\tau) = G(\tau) - G(\tau_0),$$

with $G(u)$ equal to (derived in Appendix B):

$$\begin{aligned} G(u) \quad = \quad & \left(\left(\lambda z^0 - \pi^0\right)\theta\Psi_{00} + \left(\mu_0 - \pi^1\right)\Psi_{10}\right) \cdot \frac{e^{s_0 u}}{s_0} \\ & + \left(\left(\lambda z^0 - \pi^0\right)\theta\Psi_{01} + \left(\mu_0 - \pi^1\right)\Psi_{11}\right) \cdot \frac{e^{s_1 u}}{s_1} \\ & + \frac{1}{2}\sigma^2 \cdot \left(\begin{array}{c} \Psi_{10}^2 \cdot \frac{e^{2s_0 u}}{2s_0} + 2\Psi_{10}\Psi_{11} \cdot \frac{e^{(s_0+s_1)u}}{s_0+s_1} + 2\Psi_{10} \cdot \overline{B}_1 \cdot \frac{e^{s_0 u}}{s_0} \\ + \Psi_{11}^2 \cdot \frac{e^{2s_1 u}}{2s_1} + 2\Psi_{11} \cdot \overline{B}_1 \cdot \frac{e^{s_1 u}}{s_1} + \left(\overline{B}_1\right)^2 \cdot u \end{array} \right) \\ & + \left[\left(\lambda z^0 - \pi^0\right)\theta \cdot \overline{B}_0 + \left(\mu_0 - \pi^1\right) \cdot \overline{B}_1\right] \cdot u. \end{aligned}$$

Using (7) to write our specific $\tau - period$ zero-coupon bond price at time t

[12] Instead of integrating between (τ_0, τ), we could alternatively have integrated from $(0, \tau)$:

$$\int_0^{\tau} A'(u)\, du = A(\tau) - A(0) = G(\tau) - G(0).$$

as:

$$f\left(r^0, r^1, \tau, t\right) = \exp\left\{A(\tau) + B_0(\tau) r_t^0 + B_1(\tau) r_t^1\right\},$$

the yield at t on $\tau - period$ bond, y_t^τ, is defined by:

$$y_t^\tau = \frac{-\log f\left(r^0, r^1, \tau, t\right)}{\tau} = \frac{-\left\{A(\tau) + B_0(\tau) r_t^0 + B_1(\tau) r_t^1\right\}}{\tau}; \tag{19}$$

and we can check whether the model predicted yields reasonably track the observed yields. The issue to be addressed in the following section.

5 Real Data Fit

5.1 A preliminary test

If the proposed model is reasonable, every yield-to-maturity have formula (19) and the variation of any maturity yield is given by:

$$\Delta y_t^\tau = -\frac{B_0(\tau)}{\tau} \cdot \Delta r_t^0 - \frac{B_1(\tau)}{\tau} \cdot \Delta r_t^1.$$

This means regressions of the form:

$$\Delta y_t^\tau = c_0^\tau + c_1^\tau \cdot \Delta r_t^0 + c_2^\tau \cdot \Delta r_t^1, \tag{20}$$

should have high R^2 and not significant intercepts for every maturities. The OLS regression results are shown in Table 4 below [13]. Indeed, the estimated c_0's are not significant different from zero and the R^2 is not bad.

[13] These regressions are suggested in Balduzzi et al. [9].

5.2 Adjusting the model to data

We need now chose the set of parameters that fulfill the boundary condition $A(\tau_1) = 0$, the initials $B_0(0) = B_1(0) = A(0) = 0$ and that simultaneously provides best model adjustment to data.

Related to data fit, the choice of the parameters may obey one of the two possible objectives:

- to minimize the distance between the model yield curve and the average yield curve shown in Figure 1.b, like Aït-Sahalia [2] or Backus et al. [7];
- to minimize the sum of squared deviations between the model predicted yield surface and the realized yield surface presented in Figure 1.a.

The closeness of 0 and τ_0 makes things easier since, given $B_0(\tau_0) = -\tau_0$ and $B_1(\tau_0) = 0$, we only need $B_0'(\tau_{0-}) = -1$ and $B_1'(\tau_{0-}) = A'(\tau_{0-}) = 0$ to approximately guarantee the initials $A(0) = B_0(0) = B_1(0) = 0$.

We have first tried to fit the model to data without changing any estimated parameter from Table 3, but the pair (π_0, π_1) only. Unfortunately, the choice of only two parameters have not been enough to obtain neither good fit to the average yield curve (in Figure 1.b), nor to the surface (in Figure1.b)[14].

We have thus allowed the adjustment of parameter $\mu_1$together with (π_0, π_1), and indirectly obtained the triplet by full information maximum likelihood, given the other parameters estimates from Table 3, and the restriction that $A(\tau_1) = 0$. Table 5 below presents the obtained values for the triplet, which,

[14] Appendix C, at the end, presents the results for the "fully estimated" model that uses the parameters from Table 3 and the pair (π_0, π_1) obtained by maximum likelihood.

together with Table 3 estimates for the remaining parameters, provide the set of parameters used in the model application. Note the new $\mu_1 = -2.3932$ is not significantly different from the previous estimate (-2.2757).

As illustrated in Figures 4.a and 4.b, the above set of parameters generates functions $A(\cdot)$, $B_0(\cdot)$ and $B_1(\cdot)$ which obey the initial and boundary conditions.

6 Model Interpretation

6.1 Reasoning the functions $\mathsf{B}_0(\cdot)$ and $\mathsf{B}_1(\cdot)$

An interesting result of Figure 4.a is that $B_0(\tau) < 0$ for $\tau < 1$, and $B_0(\tau) > 0$ for $\tau > 1$. Thus, target increases cause certain loses in value to bonds with maturity shorter than one year ($f_{r^0}(r^0, r^1, \tau) = B_0(\tau) \cdot f(r^0, r^1, \tau) < 0$), and certain gains in value to bonds with maturity longer than one year ($f_{r^0}(r^0, r^1, \tau) = B_0(\tau) \cdot f(r^0, r^1, \tau) > 0$). Or in terms of nominal yields, target increases imply increases in short-term nominal yields and decreases in long-term nominal yields; the frontier between short- and long-term imposed to be 1 year. To understand the above, we need remember nominal rates are the composition of a relatively constant expected real rate and a relatively variable expected inflation rate, this meaning nominal rates fluctuations are mainly due to expected inflation variations. The rate target rises to fight inflation and, once increased, is expected to remain high for the time necessary to reduce inflation, which presents inertia. When the lower inflation level is reached, the target is reduced to compatible levels. That said, the expectation hypothesis rationalizes the above result: longer term yields are averages of expected future short rates,

and the net effect of a target increase on a specific maturity depends on how long the short rate is expected to remain high. The obtained $B_0(\cdot)$ means that the net effect is positive for yields shorter than one year and negative for yields longer than one year, implying disinflation policy succeed faster than one year [15].

6.2 Reasoning the market prices of risks

The negative π_1 means that the market price of risk of the representative yield (r_1) is negative and has the usual interpretation (see Cox,Ingersoll and Ross [24], for example). This means the covariance of the changes in the representative yield with changes in optimally invested wealth is negative. Given $f_{r^1}(r^0, r^1, \tau) = B_1(\tau) \cdot f(r^0, r^1, \tau) < 0 \;\forall\, \tau$, negative π_1 implies a positive bond price risk premium, $\pi_1 \cdot f_{r^1}(r^0, r^1, \tau)$, because the bond prices are high when wealth is high and have low marginal utility.

The market price of risk of the target equals $\pi_0(\theta + r^1 - r^0)$. The negative π_0 implies negative market price of risk when $(\theta + r^1 - r^0) > 0$ and positive market price of risk when $(\theta + r^1 - r^0) < 0$. This is a novelty due to the way we modeled the target process. Although unusual, this change in the sign of the market price of risk is intuitive, since target moves are perceived as having either upside risk if $(\theta + r^1 - r^0) > 0$, or downside risk if $(\theta + r^1 - r^0) < 0$, but never both. Given an asset can never have an always positive excess return, the market price of risk does the necessary adjustmente to this behavior of the state vari-

[15] Confronted to empirical evidence, this one-year division line is definitely low (see Thornton [66]). Notwithstanding, this $B_0(\cdot)$ pattern may guide the choice of the representative yield to be used as the second factor of our model.

able. If an asset can only go up, the market imposes a discount (price of risk is negative) and if an asset can only go down the market charges a premium (price of risk is positive). In the short-term bond context ($\tau < 1$), $\left(\theta + r^1 - r^0\right) > 0$ is understood as the target can increase or stay steady, meaning risk of capital loss ($f_{r^0}\left(r^0, r^1, \tau\right) < 0$), charged by the term $\pi_0\left(\theta + r^1 - r^0\right) \cdot f_{r^0}\left(r^0, r^1, \tau\right) > 0$. On the other hand, $\left(\theta + r^1 - r^0\right) < 0$ is understood as the target can decrease or stay steady, meaning possibility of capital gain ($f_{r^0}\left(r^0, r^1, \tau\right) < 0$), discounted by the term $\pi_0\left(\theta + r^1 - r^0\right) \cdot f_{r^0}\left(r^0, r^1, \tau\right) < 0$. For the long-term bonds ($\tau > 1$), $B_0\left(\tau\right) > 0$ inverts the logic.

Illustrated in Figures 5 and 6 are the model yield curve. Figure 5 shows that the fit with respect to the average yield curve is not bad. Figure 6.a presents the model generated surface, using the realized Target and the 1-year yield as input factors. For ease of comparison, Figure 6.b repeats the realized surface of Figure 1.a., and Figure 6.c plots the difference between the model predicted and the realized yield surface. The model results in better fit for short maturities and the yield deviations are never above 1,5% for any maturity.

7 An explanation for the factors level, slope and curvature

By applying principal component analysis to the U.S. term structure, Litterman & Scheinkman [43] identify three factors impacting on the level, slope and curvature of the yield curve. Figure 7 below illustrates some model yield curve shapes that result from different combinations of the target and 1-year T-bill.

Note that by reducing the 1-year yield (going right in the pictures) the level of the yield curve goes down, what justifies the level factor to be attributed to the representative yield (r^1). On the other hand, by reducing the target (going down in the pictures) the yield curve steeps up allowing the interpretation of the target as causing the slope factor. There remains the curvature factor, which can be attributed to the term spread $\left(r^1 - r^0\right)$. That is easily seen by gradually decreasing the term spread from the maximum (the southwest plot) to the minimum (the northeast plot) and noting the the curve turn from concave to convex.

8 Conclusion

We have developed a exponential affine bond pricing model with the Federal Fund Rate Target and a representative yield (the 1-year yield) as the explaining factors. The model has analytic solution and parameters can be estimated from actual data, providing reasonable yield prediction. The observed term structure shapes can be generated by the model and the causes of level, slope and curvature are respectively identified as the representative yield, the target and the term spread between them.

Future research should go further on the model adjustment to real economy parameters, and more applications should check out-of-sample predictions.

Figures and Tables

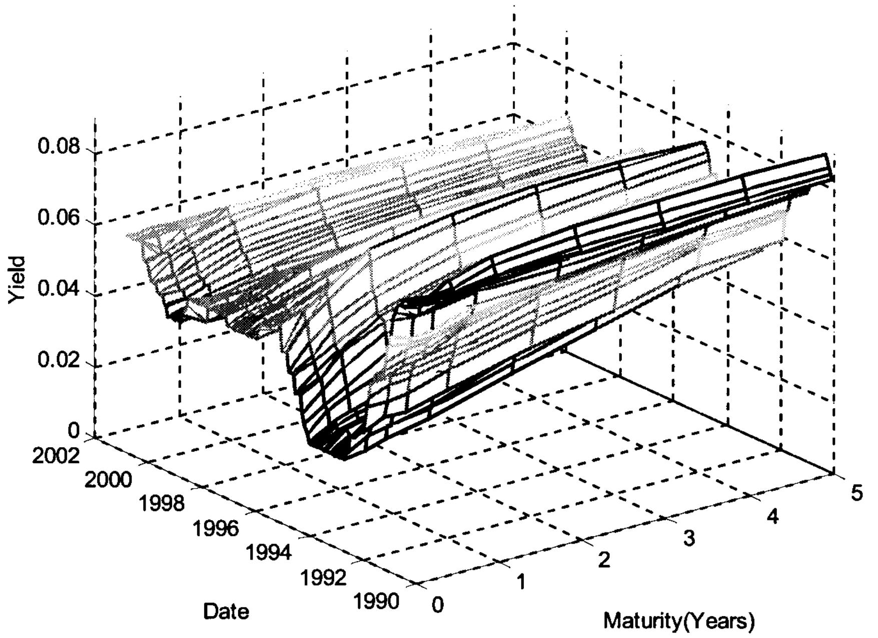

Figure 1.a

Realized U.S. Nominal Term Structure

This figure show the actual U.S Nominal Term Structure from Jan1990 to Dec200. It was constructed using: the Federal Fund Rate Target (r^o), the 3-month Treasury bill (3-m), the 6-month Treasury bill (6-m), the 1-year Treasury bill (1-y), the 2-year Treasury bill (2-y), the 3-year Treasury bill (3-y), the 4-year Treasury bill (4-y) and the 5-year Treasury bill (5-y) from the H.15 Release by the Board of Governors and from the Fama-Bliss file by the CRSP.

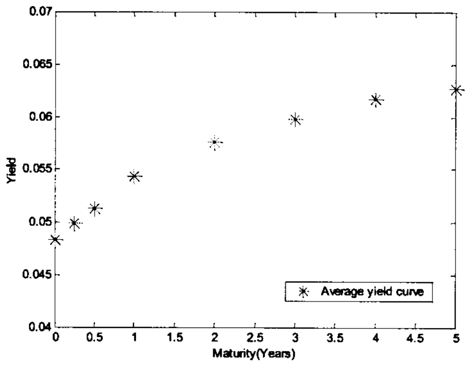

Figure 1.b

Average yield curve

This figure presents the U.S. Treasury average yield curve from January 1990 to December 2000. The chosen points are the Federal Fund Rate Target (r^o), the 3-month Treasury bill (3-m), the 6-month Treasury bill (6-m), the 1-year Treasury bill (1-y), the 2-year Treasury bill (2-y), the 3-year Treasury bill (3-y), the 4-year Treasury bill (4-y) and the 5-year Treasury bill (5-y) from the H.15 Release by the Board of Governors and from the Fama-Bliss file by the CRSP.

Table 1. a
Descriptive statistics of the monthly levels of the U.S. Term Structure from January 1990 to December 2000

Rates	r^0	FF	3-m	6-m	1-y	2-y	3-y	4-y	5-y
Mean	0.0484	0.0509	0.0497	0.0511	0.0542	0.0574	0.0597	0.0616	0.0626
Maximum	0.0787	0.0914	0.0788	0.0815	0.0854	0.0875	0.0884	0.0890	0.0888
Minimum	0.0262	0.0228	0.0270	0.0286	0.0311	0.0377	0.0420	0.0431	0.0434
Std. Dev.	0.0135	0.0138	0.0121	0.0119	0.0119	0.0110	0.0103	0.0101	0.0100
Skewness	0.18	0.19	0.28	0.24	0.20	0.38	0.50	0.54	0.54
Kurtosis	3.02	3.03	3.16	3.10	3.03	3.01	3.00	2.88	2.77
J-B	0.68	0.81	1.86	1.31	0.92	3.10	5.60	6.48	6.79
Probability	0.71	0.67	0.40	0.52	0.63	0.21	0.06	0.04	0.03
ADF Test Statistic	-2.79	-2.69	-3.19	-3.05	-3.02	-3.00	-3.02	-2.77	-2.58
H_0: Nonstationarity				Reject at 10% (critical value=-2.58)			Reject at 5% (critical value=-2.89)		

The descriptive statistics reported in Table 1 are for the Federal Fund Rate Target (r^O), the Federal Fund Rate (FF), the 3-month Treasury bill (3-m), the 6-month Treasury bill (6-m), the 1-year Treasury bill (1-y), the 2-year Treasury bill (2-y), the 3-year Treasury bill (3-y), the 4-year Treasury bill (4-y) and the 5-year Treasury bill (5-y) from the H.15 Release by the Board of Governors and from the Fama-Bliss file by CRSP. Each series statistics were computed using the 132 observations from January 1990 to December 2000. The Federal Fund Rate and Target where demeaned by 0.0038 to generate a monotone curve when the target is taken as the one-day point on the yield curve.

Table 1. b
Descriptive statistics of the monthly differences of the U.S. Term Structure from Jan.1990 to Dec.2000

Rates	dr^0	dFF	d3-m	d6-m	d1-y	d2-y	d3-y	d4-y	d5-y
Maximum	0.0075	0.0265	0.0054	0.0053	0.0079	0.0068	0.0067	0.0069	0.0068
Minimum	-0.0075	-0.0208	-0.0059	-0.0060	-0.0062	-0.0068	-0.0074	-0.0078	-0.0077
Std. Dev.	0.0019	0.0077	0.0020	0.0021	0.0024	0.0027	0.0028	0.0029	0.0028
Skewness	0.09	0.13	-0.22	-0.07	0.24	0.13	0.20	0.18	0.13
Kurtosis	7.40	4.00	3.57	3.30	3.28	2.68	2.73	2.79	2.84
J-B	106.07	5.86	2.87	0.61	1.66	0.95	1.32	0.99	0.48
Probability	0.00	0.05	0.24	0.74	0.44	0.62	0.52	0.61	0.79
Monthly ρ_1	0.30	-0.45	0.27	0.31	0.35	0.34	0.28	0.23	0.23
ρ_2	0.36	-0.12	0.16	0.22	0.08	0.08	0.04	0.01	0.02
ρ_3	0.40	0.27	0.25	0.17	0.09	0.01	-0.02	0.00	0.01
ρ_6	0.32	-0.09	0.24	0.19	0.08	0.04	0.00	0.00	-0.02
ρ_{12}	0.16	0.32	0.18	0.05	-0.01	-0.06	-0.08	-0.05	-0.10

The descriptive statistics reported in Table 1.b are for the differences of the series ($dy_t=y_{t+1}-y_t$): the Federal Fund Rate Target (dr^o), the Federal Fund Rate (dFF), the 3-month Treasury bill (d3-m), the 6-month Treasury bill (d6-m), the 1-year Treasury bill (d1-y), the 2-year Treasury bill (d2-y), the 3-year Treasury bill (d3-y), the 4-year Treasury bill (d4-y) and the 5-year Treasury bill (d5-y) from the H.15 Release by the Board of Governors. Each series statistics were computed using the 131 observations from January 1990 to November 2000.

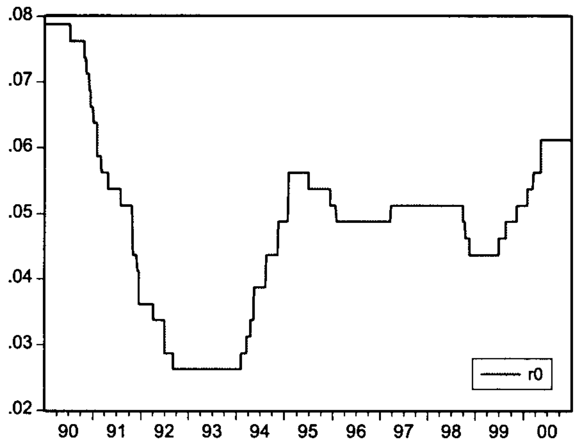

Figure 2

Monthly Federal Fund Rate Target level from January 1990 to December 2000

This figure shows evolution of the Federal Fund Rate Target level demeaned by 0.0038, as explained in section 2. The rate is continuously compounded, annualized, and expressed in decimal number.

Table 2
GMM estimates for the target process with alternative r^0

r^0	3-m	6-m	1-y	2-y	3-y	4-y	5-y
λ	4085.07	4301.40	940.30	500.48	360.76	275.82	230.23
(t-statistic)	(71.02)	(32.56)	(4.00)	(4.04)	(3.52)	(3.29)	(3.11)
θ	-0.0010	-0.0020	-0.0059	-0.0089	-0.0109	-0.0128	-0.0140
(t-statistic)	(-2.77)	(-7.07)	(-7.27)	(-5.92)	(-4.79)	(-4.13)	(-3.73)
χ^2 Test	6.3106	8.1330	4.6093	5.3667	6.1644	6.1644	6.9556
(p-value)	(0.1771)	(0.0868)	(0.3299)	(0.2517)	(0.1872)	(0.1872)	(0.1383)

The estimated parameters in Table 2 are from equation (2). The estimation horizon is from Jan.1990 to Dec.2000 (132 observations) and theparameters are estimated from the system (3)-(4). The χ^2 test statistic is reported with p-values in parentheses.

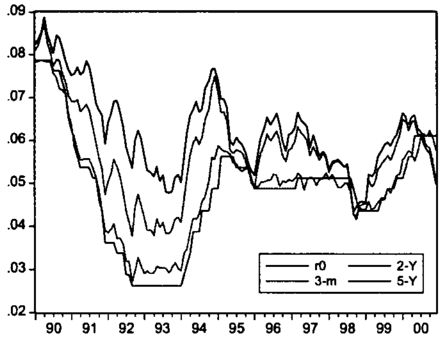

Figure 3.a

Monthly Federal Fund Rate Target, 3-Month, 2-Year yield and 5-Year yield levels from January 1990 to December 2000

This figure shows evolution of the Federal Fund Rate Target level (r^0) demeaned by 0.0038, as explained in section 2; and the 3-month yield (3-m), the 2-year yield (2-y), and the 5-year yield (5-y). The rates are continuously compounded, annualized, and expressed in decimal number.

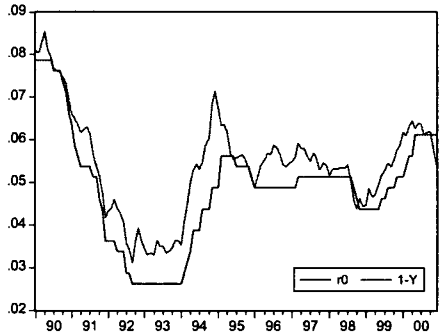

Figure 3.b

Monthly Federal Fund Rate Target and 1-Year yield levels from Jan.1990 to Dec2000

This figure shows evolution of the Federal Fund Rate Target level (r^0) demeaned by 0.0038, as explained in section 2, and the 1-year yield (1-y). The rates are continuously compounded, annualized, and expressed in decimal number.

Table 3
GMM joint-estimates for the target and the 1-year yield processes

Parameter	λ	θ	μ_0	μ_1	μ_2	σ
Estimate	915.0196	-0.0052	0.0025	-2.2758	2.0309	0.0072
(t-statistic)	(6.19)	(-7.43)	(0.23)	(-5.09)	(3.82)	(17.07)
χ^2 Test	6.7417		**(p-value)**	(0.3454)		

The estimated parameters in Table 3 are from system (3), (4), (11) and (12). The estimation horizon is from January 1990 to December 2000 (131 observations). The χ^2 test statistic is reported with p-values in parentheses.

Table 4
Regression of changes in yields against changes in two factors
Jan.1990-Dec.2000

Maturity (Years, τ)	c^{τ}_0	c^{τ}_1	c^{τ}_2	R^2	D.W.
0.25	-0.0001	0.0755	0.5286	0.51	2.11
	(-0.4825)	(2.2256)	(10.0652)		
0.50	0.0000	0.0607	0.7101	0.77	2.53
	(-0.4673)	(2.4502)	(18.5115)		
2	0.0000	-0.0175	1.0346	0.84	2.07
	(-0.1934)	(-0.6423)	(24.5885)		
3	0.0000	-0.0287	1.0510	0.78	2.12
	(-0.1281)	(-0.8733)	(20.6770)		
4	0.0000	-0.0311	1.0231	0.74	2.03
	(-0.1751)	(-0.8539)	(18.1518)		
5	0.0000	-0.0147	0.9632	0.68	2.01
	(-0.2676)	(-0.3693)	(15.6578)		

The estimated parameters reported in Table 4 (together with t-statistics in parantheses) are from equation 20. These parameters correspond to continuously compounded annualized rate, expressed as a decimal number.

Table 5
FIML values for μ_1, π_0 and π_1

Parameter	μ_1	π_0	π_1
Value	-2.3932	-40.1130	-0.0128

Table 5 presents the maximum likelihoofd values of the parameters μ_1, π_0 and π_1, for the period January 1990 to December 2000, when the other parameters are assumed to be

$\lambda = 915.02$, $\theta = -0.0052$, $\mu_0 = 0.0025$, $\mu_2 = -2.0309$, $\sigma = 0.0072$.

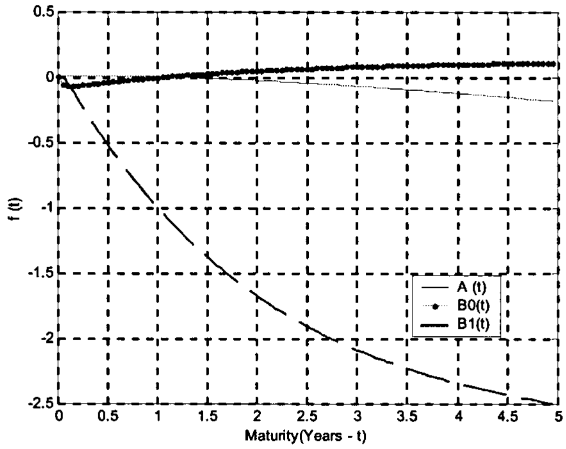

Figure 4.a

$A(t)$, $B_0(t)$, and $B_1(t)$ plot

This figure shows the evolution of the functions along maturity for the model with parameter values $\lambda = 915.02$, $\theta = -0.0052$, $\mu_0 = 0.0025$, $\mu_1 = -2.3932$, $\mu_2 = -2.0309$, $\sigma = 0.0072$, $\pi_0 = -40.1130$, $\pi_1 = -0.0128$. Note that the initial and boundary conditions are satisfied.

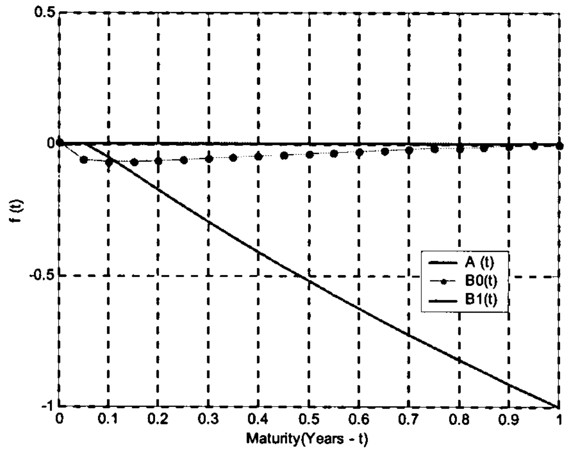

Figure 4.b

Zoom of $A(t)$, $B_0(t)$, and $B_1(t)$ plot

This figure shows the evolution of the functions along maturity for the model with parameter values $\lambda = 915.02$, $\theta = -0.0052$, $\mu_0 = 0.0025$, $\mu_1 = -2.3932$, $\mu_2 = -2.0309$, $\sigma = 0.0072$, $\pi_0 = -40.1130$, $\pi_1 = -0.0128$.. Note that the initial and boundary conditions are satisfied.

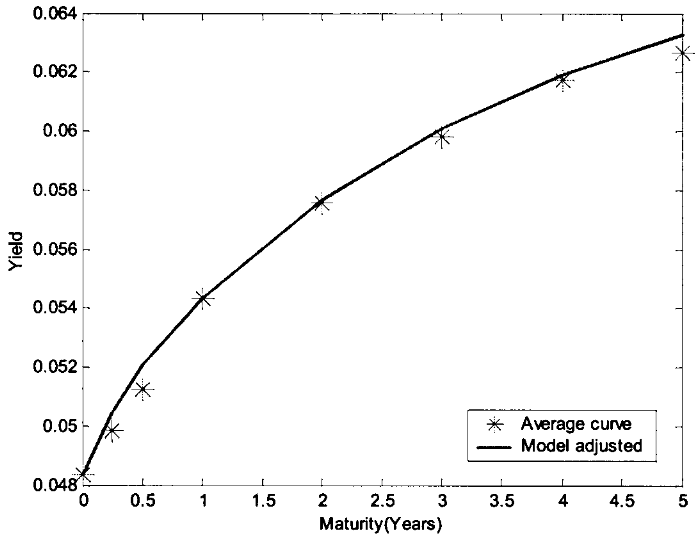

Figure 5

Average yield curve and model yield curve

This figure presents the U.S. average yield curve from January 1990 to December 2000 and the model yield curve that takes the average Federal Fund Rate Target and the average 1-year T-Bill as inputs, with parameter values $\lambda = 915.02$, $\theta = -0.0052$, $\mu_0 = 0.0025$, $\mu_1 = -2.3932$, $\mu_2 = -2.0309$, $\sigma = 0.0072$, $\pi_0 = -40.1130$, $\pi_1 = -0.0128$.

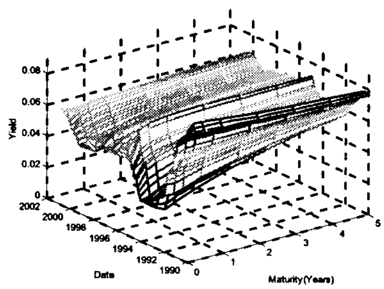

Figure 6.a: Model Predicted U.S.Term Structure

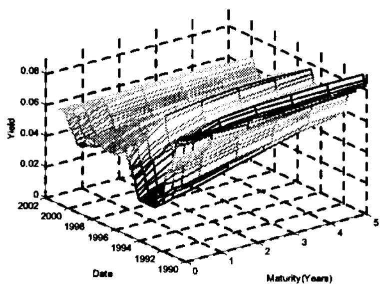

Figure 6.b: Actual U.S.Term Structure

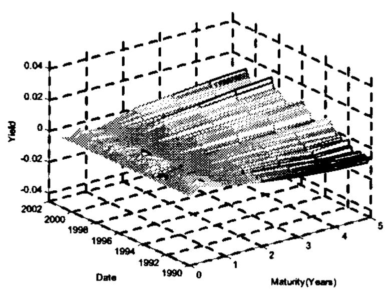

Figure 6.c: Difference between Model and Actual U.S.Term Structure

These figures show the U.S. Term Structure from Jan1990 to Dec200 generated by the model, with parameter values $\lambda = 915.02$, $\theta = -0.0052$, $\mu_0 = 0.0025$, $\mu_1 = -2.3932$, $\mu_2 = -2.0309$, $\sigma = 0.0072$, $\pi_0 = -40.1130$, $\pi_1 = -0.0128$, using the realized FFRTarget and the 1-year T-Bill as inputs (Figure a), the actual U.S. Term Structure (Figure b) and the difference between the model and the realized (Figure c). The sum of squared deviations in Figure c is equal to 2.7255.

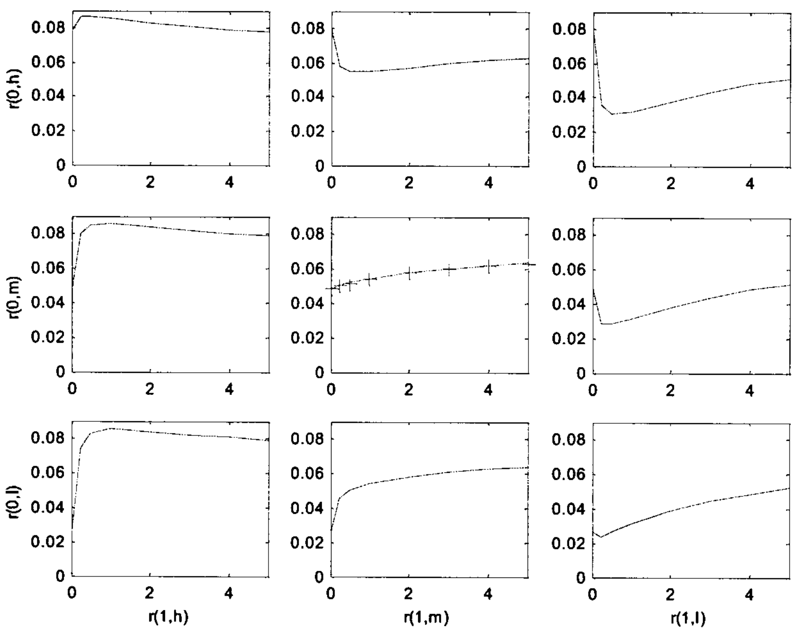

Figure 7

Term Structures Shapes (Maturity x Yield)

These plots ilustrate the model term structure shape that results from inputing specific target ($r^0 = r(0,\cdot)$) and 1-year T-bill ($r^1 = r(1,\cdot)$). The target varies along the lines: $r(0,h) = .0787$, $r(0,m) = .0484$, $r(0,l) = .0262$, ; while the 1-year T-bill varies along the columns:.$r(1,h) = .0854$, $r(1,m) = .0545$, $r(1,l) = .0311$. The h, m, l values were the maximum, mean and minimum values taken by both rates during the period Jan.1990-Dec.2000.

Appendices

Appendix A: Discretization

A.1 The Poisson case

The pure Poisson is a discrete process and does not demand a discretization. It is well known that for a process with jump sizes δ_t and intensity $\lambda_t dt$, the conditional mean is:

$$E_t \left[\Delta J_t \right] = \delta_t \lambda_t \Delta t$$

and the conditional variance is:

$$vaX_t \left[\Delta J_t \right] = \delta_t^2 \lambda_t \Delta t.$$

A.2 The diffusion case

The Ito-Taylor expansion (see Kloeden &Platen [42]) of the process:

$$dX = \mu \left(X \right) dt + \sigma_1 \left(X \right) dW, \tag{21}$$

or:

$$\int_{t_0}^{t} dX_s = \int_{t_0}^{t} \mu \left(X_s \right) dt + \int_{t_0}^{t} \sigma_1 \left(X_s \right) dW_s, \tag{22}$$

is given by:

$$
\begin{aligned}
X_t &= X_{t0} + \int_{t0}^{t} \left[\mu(X_{t0}) + \int_{t0}^{s} \left(\frac{\partial \mu(X_z)}{\partial X} \mu(X_z) + \frac{1}{2} \frac{\partial^2 \mu(X_z)}{\partial X^2} \sigma^2(X_z) \right) dz \right] ds + \\
&\int_{t0}^{t} \int_{t0}^{s} \frac{\partial \mu(X_z)}{\partial X} \sigma(X_z)\, dW_z ds + \qquad (23) \\
&\int_{t0}^{t} \left[\sigma(X_{t0}) + \int_{t0}^{s} \left(\frac{\partial \sigma(X_z)}{\partial X} \mu(X_z) + \frac{1}{2} \frac{\partial^2 \sigma(X_z)}{\partial X^2} \sigma^2(X_z) \right) dz \right] dW_s + \\
&\int_{t0}^{t} \int_{t0}^{s} \frac{\partial \sigma(X_z)}{\partial X} \sigma(X_z)\, dW_z dW_s,
\end{aligned}
$$

which results in:

$$
\begin{aligned}
X_t &= X_{t0} + \mu(X_{t0}) \Delta t + \left(\frac{\partial \mu(X_{t0})}{\partial X} \mu(X_{t0}) + \frac{1}{2} \frac{\partial^2 \mu(X_{t0})}{\partial X^2} \sigma^2(X_{t0}) \right) (\Delta t)^2 + \\
&\frac{\partial \mu(X_{t0})}{\partial X} \sigma(X_{t0}) \Delta W_{t0} \Delta t + \sigma(X_{t0}) \Delta W_{t0} \\
&+ \left(\frac{\partial \sigma(X_z)}{\partial X} \mu(X_z) + \frac{1}{2} \frac{\partial^2 \sigma(X_z)}{\partial X^2} \sigma^2(X_z) \right) \Delta t \Delta W_{t0} + \\
&\frac{\partial \sigma(X_z)}{\partial X} \sigma(X_z) \Delta W_z \Delta W_s + R. \qquad (24)
\end{aligned}
$$

where R is the rest.

In the above formula: $\int_{t0}^{t} \int_{t0}^{s} dW_z ds$ ~ $N(0, \frac{1}{3}\Delta^3)$ and $\int_{t0}^{t} \int_{t0}^{s} dz dW_s = \frac{1}{2} (\Delta t)^2$.

A.3 The variable mean reversion case

$$
dX = (\mu_0 + \mu_1 J + \mu_2 X)\, dt + \sigma dW; \qquad (25)
$$

$$
\begin{aligned}
X_t &= X_{t0} + (\mu_0 + \mu_1 J_{t0} + \mu_2 X_{t0}) \Delta t + \sigma \Delta W_{t0} + \mu_1 \Delta J_{t0} \Delta t \\
&- \mu_2 \cdot (\mu_0 + \mu_1 J_{t0} + \mu_2 X_{t0}) (\Delta t)^2 - \mu_2 \sigma \Delta W_{t0} \Delta t + R. \qquad (26)
\end{aligned}
$$

Appendix B: Derivation of $G(u)$

The $G(u)$ function is given by:

$$\begin{aligned} G(u) &= \left[\left(\lambda z^0 - \pi^0\right)\theta\right] \cdot \int B_0(u)\,du + \left(\mu_0 - \pi^1\right) \cdot \int B_1(u)\,du \\ &\quad + \frac{1}{2}\sigma^2 \cdot \int B_1^2(u)\,du, \end{aligned}$$

where:

$$\int B_0(u)\,du = \Psi_{00} \cdot \frac{e^{s_0 u}}{s_0} + \Psi_{01} \cdot \frac{e^{s_1 u}}{s_1} + \overline{B}_0 \cdot u,$$

$$\int B_1(u)\,du = \Psi_{10} \cdot \frac{e^{s_0 u}}{s_0} + \Psi_{11} \cdot \frac{e^{s_1 u}}{s_1} + \overline{B}_1 \cdot u,$$

and for:

$$\begin{aligned} B_1^2(u) &= \left(\Psi_{10} \cdot e^{s_0 u} + \Psi_{11} \cdot e^{s_1 u} + \overline{B}_1\right)^2 = \\ &\quad \Psi_{10}^2 \cdot e^{2 s_0 u} + 2\Psi_{10}\Psi_{11} \cdot e^{(s_0+s_1)u} + 2\Psi_{10} \cdot \overline{B}_1 \cdot e^{s_0 u} \\ &\quad + \Psi_{11}^2 \cdot e^{2 s_1 u} + 2\Psi_{11} \cdot \overline{B}_1 \cdot e^{s_1 u} + \left(\overline{B}_1\right)^2, \end{aligned}$$

$$\begin{aligned} \int B_1^2(u)\,du &= \Psi_{10}^2 \cdot \frac{e^{2 s_0 u}}{2 s_0} + 2\Psi_{10}\Psi_{11} \cdot \frac{e^{(s_0+s_1)u}}{s_0+s_1} + 2\Psi_{10} \cdot \overline{B}_1 \cdot \frac{e^{s_0 u}}{s_0} \\ &\quad + \Psi_{11}^2 \cdot \frac{e^{2 s_1 u}}{2 s_1} + 2\Psi_{11} \cdot \overline{B}_1 \cdot \frac{e^{s_1 u}}{s_1} + \left(\overline{B}_1\right)^2 \cdot u. \end{aligned}$$

Thus:

$$
\begin{aligned}
G(u) \;=\;& \left[\left(\lambda z^0-\pi^0\right)\theta\right]\cdot\left(\Psi_{00}\cdot\frac{e^{s_0u}}{s_0}+\Psi_{01}\cdot\frac{e^{s_1u}}{s_1}+\overline{B}_0\cdot u\right)\\
&+\left(\mu_0-\pi^1\right)\cdot\left(\Psi_{10}\cdot\frac{e^{s_0u}}{s_0}+\Psi_{11}\cdot\frac{e^{s_1u}}{s_1}+\overline{B}_1\cdot u\right)\\
&+\frac{1}{2}\sigma^2\cdot\left(\begin{array}{c}\Psi_{10}^2\cdot\frac{e^{2s_0u}}{2s_0}+2\Psi_{10}\Psi_{11}\cdot\frac{e^{(s_0+s_1)u}}{s_0+s_1}+2\Psi_{10}\cdot\overline{B}_1\cdot\frac{e^{s_0u}}{s_0}\\
+\Psi_{11}^2\cdot\frac{e^{2s_1u}}{2s_1}+2\Psi_{11}\cdot\overline{B}_1\cdot\frac{e^{s_1u}}{s_1}+\left(\overline{B}_1\right)^2\cdot u\end{array}\right),
\end{aligned}
$$

or:

$$
\begin{aligned}
G(u) \;=\;& \left(\left(\lambda z^0-\pi^0\right)\theta\Psi_{00}+\left(\mu_0-\pi^1\right)\Psi_{10}\right)\cdot\frac{e^{s_0u}}{s_0}\\
&+\left(\left(\lambda z^0-\pi^0\right)\theta\Psi_{01}+\left(\mu_0-\pi^1\right)\Psi_{11}\right)\cdot\frac{e^{s_1u}}{s_1}\\
&+\frac{1}{2}\sigma^2\cdot\left(\begin{array}{c}\Psi_{10}^2\cdot\frac{e^{2s_0u}}{2s_0}+2\Psi_{10}\Psi_{11}\cdot\frac{e^{(s_0+s_1)u}}{s_0+s_1}+2\Psi_{10}\cdot\overline{B}_1\cdot\frac{e^{s_0u}}{s_0}\\
+\Psi_{11}^2\cdot\frac{e^{2s_1u}}{2s_1}+2\Psi_{11}\cdot\overline{B}_1\cdot\frac{e^{s_1u}}{s_1}+\left(\overline{B}_1\right)^2\cdot u\end{array}\right)\\
&+\left[\left(\lambda z^0-\pi^0\right)\theta\cdot\overline{B}_0+\left(\mu_0-\pi^1\right)\cdot\overline{B}_1\right]\cdot u.
\end{aligned}
$$

Appendix C: Fully estimated model version

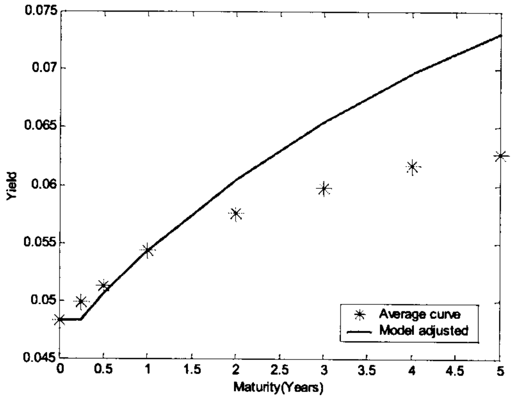

Figure C.1

Average yield curve and model yield curve

This figure presents the U.S. average yield curve from Jan1990 to Dec2000 and the model yield curve that takes the average Federal Fund Rate Target and the average 1-year T-Bill as inputs, with parameter values $\lambda = 915.02$, $\theta = -0.0052$, $\mu_0 = 0.0025$, $\mu_1 = -2.2757$, $\mu_2 = -2.0309$, $\sigma = 0.0072$, $\pi_0 = -24.8757$, $\pi_1 = -0.0128$.

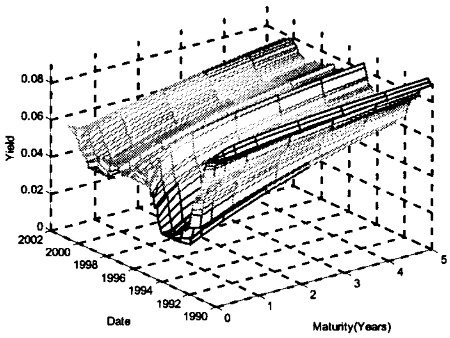

Figure C.2.a: Model Predicted U.S.Term Structure

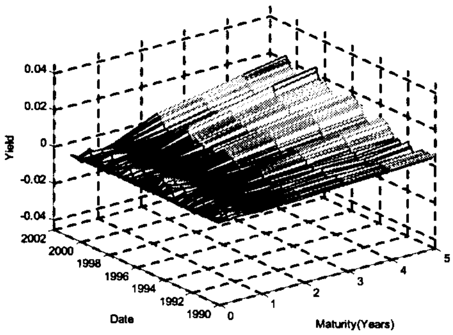

Figure C.2.b: Difference between Model and Actual U.S.Term Structure

These figures show the U.S. Term Structure from Jan1990 to Dec200 generated by the model, with parameter values $\lambda = 915.02$, $\theta = -0.0052$, $\mu_0 = 0.0025$, $\mu_1 = -2.2757$, $\mu_2 = -2.0309$, $\sigma = 0.0072$, $\pi_0 = -24.8757$, $\pi_1 = -0.0128$, using the realized FFRTarget and the 1-year T-Bill as inputs (Figure a), the actual U.S. Term Structure (Figure b) and the difference between the model and the realized (Figure c). The sum of squared deviations in Figure c is equal to 4.1132.

Chapter 3

What the short-term interest rate target can do for mean-reverting modelling

October 9, 2001

Abstract

This paper shows that the bad fit of the univariate mean-reverting continuous diffusion processes to Federal Fund Rate data (the U.S. overnight rate) is due to the omission of its target, set by the Federal Open Market Committee. With a constant long run mean, neither the **level volatility** models like Chan **et al.** [19], nor the **stochastic level volatility** models suggested by Brenner **et al.** [13] are able to explain the rate changes or the squared rate changes. The mean-reverting diffusion process that combines the rate target as the long-run mean with the **stochastic level volatility** results in an improved fit and forecasts a considerable proportion of both changes. The apparent Federal Fund Rate overreaction is solved and weak evidences are pointed that the rate target might be a better short-term factor on term structure models than the Federal Fund Rate. To close, a discrete process for the rate target is proposed.

JEL classification: C32; C51; E43; E52; G12

Keywords: Controlled Short Rate; Discontinuous Changes; Short-term interest rate; Rate target; Stochastic level volatility; GMM estimation; E-GARCH

1 Introduction

The short-term interest rate is a fundamental economic variable to much of finance and macroeconomics. Be it either on pricing derivatives, or on portfolio decisions, or on monetary policy issues or on business cycle analysis, the short-term interest rate plays a main role. However, in spite of the considerable effort to model and estimate its dynamics, no consensus has emerged.

Beginning with Merton [50], the common research strategy has consisted in examining specifications for the drift and the volatility functions of a univariate diffusion process:

$$dr_t = \mu(r_t)\,dt + \sigma_t(r_t)\,dW_t, \tag{1}$$

where: r is the interest rate, $\mu(\cdot)$ is a drift function, $\sigma_t(\cdot)$ is a volatility function that may depend on time, and W_t is a standard Brownian motion. In the progress, some appealing empirical features have oriented research. Vasicek's [68] mean-reverting drift $\mu(r_t) = (\alpha_0 + \alpha_1 r_t)$, with $\alpha_0 > 0$ and $\alpha_1 < 0$, soon got hegemony, due to the stationarity of nominal interest rates. Extending Vasicek's constant volatility process, the next improvements took two alternative routes to model $\sigma_t(\cdot)$, respectively guided by the volatility sensitivity to levels or by the volatility clustering. First the **level volatility** approach assumed the volatility was dependent only on the interest rate level, $\sigma_t(r_t) = \sigma r_t^{\gamma}$, as shown by Chan et al. [19]: Brennan & Schwartz [11], Dothan [27], Brennan & Schwartz [12] and Courtadon [21] with $\gamma = 1$, or Cox, Ingersoll & Ross [22] with $\gamma = \frac{3}{2}$, or Cox, Ingersoll & Ross [24] with $\gamma = \frac{1}{2}$ are some examples. Later, applying

the GARCH literature, the **stochastic volatility** approach specified volatility only as a function of unexpected shocks to the interest rate, $\sigma_t(r_t) = \sigma_t = \sigma\left(\{dW_{t-i}\}_{i=1}^{t}\right)$, what means σ_t is some function $\sigma(\cdot)$ of the sequence of past shocks $\{dW_{t-i}\}_{i=1}^{t}$, like in Engle **et al.** [31]. Recently, both approaches have been combined, as in Longstaff & Schwartz [45], Brenner **et al.** [13], Andersen & Lund [4], specifying volatility processes like $\sigma_t(r_t) = \sigma\left(\{dW_{t-i}\}_{i=1}^{t}\right) \cdot r_t^{\gamma}$.

In spite of their spread use in both academia and in industry, these models have practical difficulties: nonlinearity, regime switching and omitted variables are some of the problems raised when trying to fit the above specifications to bond data (see Chan **et al.** [19] and Aït-Sahalia [3], Hamilton [37]).

This paper reviews the fit of the listed processes to the Federal Fund Rate (the U.S. overnight rate) for the period between January, 1st 1990 to December, 29th 2000. It shows that the constant-mean-reversion processes, estimated either with the level volatility or with the stochastic level volatility, do not explain the rate changes nor the volatility changes. Moreover, their residuals present negative autocorrelation, indicative of overreaction. Then, we suggest improving on the drift, instead of on the volatility, and examine an alternative specification to the constant long-run mean, in which the Federal Fund Rate reverts to its discrete-moving target, set by the Federal Open Market Committee. This new proposed drift, together with level volatility or stochastic level volatility, is estimated and tested, showing better results than the constant long-run mean process. The reversion-to-the-target specification forecasts a considerable proportion of the changes in the rate level and of the changes in the volatility when combined with the stochastic level volatility diffusion. Furthermore,

autocorrelation becomes insignificant and the volatility exponent γ is not as important here as it is in Chan **et al.** [19]. In fact, the results qualify significantly those in Chan **et al.** [19], which might be not as encompassing as thought until now. It is also shown that the fluctuations of the Federal Fund Rate that are uncorrelated with the target have low correlation with the rest of the term structure. This suggests that the rate target might be a better short-term factor for term structure models than the Federal Fund Rate itself. Finally, given the influence of the target on the Federal Fund Rate in particular and on the term structure in general, a conditional Poisson process for it is proposed and estimated.

2 Continuous interest rate processes

The continuous-time theory of the short-term interest rate process departs from an Ito stochastic differential equation:

$$dr_t = \mu\left(r_t, X_t\right) dt + \sigma\left(r_t, X_t\right) dW_t \tag{2}$$

where: X is a vector of states other than r; $\mu(\cdot,\ \cdot)$ and $\sigma(\cdot,\ \cdot)$ are the drift and the diffusion functions respectively, implying the conditional mean and variance of changes in the short-term rate may depend on the rate level r as well as on the other states X; and dW is the differential increment of a univariate standard Brownian motion W.

2.1 The constant long-run mean

Although $\mu(r, \cdot)$ and $\sigma(r, \cdot)$ can depend on many states other than the interest rate and can take various parametric forms, the leading models parsimoniously hypothesize r is a univariate process. This assumed, the stationarity of the interest rates supports Vasicek's [68] constant-mean-reverting drift, and given:

$$dr_t = (\alpha_0 + \alpha_1 r_t)\, dt + \sigma_t(r_t)\, dW_t, \tag{3}$$

with constants $\alpha_0 > 0$ and $\alpha_1 < 0$ [1], most of the job to fit the interest rate process is left to the specification of volatility function $\sigma_t(\cdot)$.

Extending Vasicek [68], who assumes a constant volatility $\sigma_t(r_t) = \sigma$, the **level volatility** approach incorporates the non-negativity constraint and the apparent sensitivity of rates changes to interest level by use of the stochastic differential equation:

$$dr = (\alpha_0 + \alpha_1 r)\, dt + \sigma r^{\gamma} dW; \tag{4}$$

as shown by Chan **et al.** [19]. Each γ implies a different distribution, and (4) nests many of the popular parametrizations. Three processes nested in (4) are the Ornstein-Uhlenbeck process introduced by Vasicek [68] (Vas from here on):

$$dr = (\alpha_0 + \alpha_1 r)\, dt + \sigma dW; \tag{5}$$

[1] $\mu(r_t) = (\alpha_0 + \alpha_1 r_t)$, with constants $\alpha_0 > 0$ and $\alpha_1 < 0$, is not sufficient for r to be positive. However it implies r is stationary and greatly reduces the probability of negative r. Although the assumption of constant long run level $(-\frac{\alpha_0}{\alpha_1})$ is a reasonable one for the real interest rate, it is for the nominal interest rate that most of the works apply it.

the square-root process applied by Cox Ingersoll & Ross' [24] (CIR hereafter):

$$dr = (\alpha_0 + \alpha_1 r)\, dt + \sigma\sqrt{r} dW; \tag{6}$$

and the geometric process used by Brennan & Schwartz's [12] (BS from now on):

$$dr = (\alpha_0 + \alpha_1 r)\, dt + \sigma r dW; \tag{7}$$

whose quite simple specifications generate known distributions for the marginals and conditionals and provide closed form solutions for interest-rate contingent claims. (5) results in a Gaussian process, (6) gives a non-central chi-squared distribution and (7) is lognormally distributed. Some important applications of (5) are Jamshidian [41] and Gibson & Schwartz [36]; of (6) are Cox, Ingersoll & Ross [24] itself, Ramaswamy & Sundaresan [61] and Longstaff [44]; and of (7) are Brennan & Schwartz [12] itself and Courtadon [21], to quote some. Within (4), there is still the possibility to let γ free to adjust to data, as foreseen by Chan **et al.** [19] ($CKLS$ hereafter) in their important contribution.

Later, the **stochastic volatility** approach incorporates the volatility clustering feature[2] formalized in the GARCH literature, and specifies $\sigma_t\,(r_t)$ only as a function of unexpected shocks to the interest rate:

$$dr_t = (\alpha_0 + \alpha_1 r)\, dt + \sigma_t dW_t, \tag{8}$$

[2]Volatility clustering is the tendency for large swings in prices to be followed by large swings of random direction.

with σ_t some function of the sequence $\{dW_{t-i}\}_{i=1}^{t}$, like Engle **et al.**'s [31] ARCH-M parametrization:

$$\sigma_t^2 = \omega + \sum_{j=1}^{p} \phi_j \left(dW_{t-j}\right)^2$$

or like Nelson's [54] Exponential-GARCH:

$$\log\left(\sigma_t^2\right) = \omega + \beta \log\left(\sigma_{t-1}^2\right) + \phi \left| \frac{dW_{t-1}}{\sigma_{t-1}} - \sqrt{\frac{2}{\pi}} \right| + \eta \frac{dW_{t-1}}{\sigma_{t-1}}. \tag{9}$$

that allows for asymmetric shocks to volatility captured by η.

Recently, both level and stochastic volatility approaches have been combined as in Longstaff & Schwartz [45], Brenner **et al.** [13], Andersen & Lund [4], specifying processes like:

$$dr_t = \left(\alpha_0 + \alpha_1 r\right) dt + \sigma_t \cdot r_t^{\gamma} dW_t. \tag{10}$$

Below, we name the **stochastic level volatility** processes like (10) by their γ's. The model with $\gamma = 1$ is named SBS, the process with $\gamma = \frac{1}{2}$ is the $SCIR$, the name $SVas$ goes for the model with $\gamma = 0$, and the process that uses the $\widehat{\gamma}$ estimate from the level volatility equation is the $SCKLS$. Note that the stochastic level volatility process with $\gamma = 0$ is the same as the stochastic volatility one, the reason why we are only going to examine the level volatility and the stochastic level volatility models.

2.2 The variable long-run mean

Back to the drift specification, a less usual alternative to the constant-mean-reversion is to assume a variable-mean-reversion, where the rate level r reverts to some state X, say the self rate target r^0. In this case, $X = r^0$ and (2) can be stated as:

$$dr_t = \left[\alpha_0 + \alpha_1 \left(r_t^0 - r_t\right)\right] dt + \sigma_t \cdot r_t^\gamma dW_t, \qquad (11)$$

now with constants $\alpha_0 > 0$ and $\alpha_1 > 0$.

Since it has become common practice among monetary authorities to set a target for the short-term interest rate, and to enforce it as a policy measure, such a modeling seems reasonable. In the U.S. for example, the Federal Open Market Committee publicly sets a discontinuous target for the Federal Fund Rate.

In the following, the way to name the variable long-run mean models is the same as before. To the level volatility models, we apply VBS, $VCIR$, $VVas$ and $VCKLS$, respectively, to $\gamma = 1,\ 1/2,\ 0$ and data driven. To the stochastic level volatility models, we apply $VSBS$, $VSCIR$, $VSVas$ and $VSCKLS$, respectively, to $\gamma = 1,\ 1/2,\ 0$ and the previously found data driven value from the level volatility equation.

3 Estimation

It seems established to estimate the level volatility models by Hansen's [39] Generalized Method of Moments (GMM), while the stochastic volatility E-GARCH models are estimated by Maximum Likelihood [3]. To guarantee our estimates be comparable to the existing literature, we don't stray from this tradition. For a comparison between level and stochastic volatility models, we compute measures with the intuitive appeal of the coefficients of determination to be described below.

3.1 GMM Estimation of level models

The econometric approach chosen to estimate and test the level volatility processes is the Generalized Method of Moments, which for estimating a $q-dimensional$ vector of coefficients requires $k > q$ orthogonality conditions. Like Chan **et al.** [19], we chose the orthogonality conditions resulting from the product of the first two moments errors by a constant and the contemporaneous interest rate level.

In our case, the parameters of the continuous-time model (2) can be estimated using a discrete-time econometric specification that approximates its

[3] Although the ARCH models can be estimated by GMM as shown in Hamilton [38] chapter 21, we don't know any GMM estimation of the E-GARCH models.

continuous moments [4]:

$$E_t[\Delta r_t] = \mu(r_t, X_t)\,\Delta t \tag{12}$$

and

$$E_t\left[(\Delta r_t - E_t[\Delta r_t])^2\right] = \sigma^2 \cdot r_t^{2\cdot\gamma}\Delta t. \tag{13}$$

3.2 Maximum likelihood estimation of stochastic volatility models

Following Hamilton [38], the E-GARCH models can be estimated by maximum likelihood. Given $u_t = \frac{dW_t}{\sigma_t}$, with zero mean and unit variance, Nelson [54] proposes using the generalized error distribution:

$$f(u_t) = \frac{\kappa \cdot \exp\left[-(1/2)\cdot|u_t/\zeta|^{\kappa}\right]}{\zeta \cdot 2^{[(\kappa+1)/\kappa]} \cdot \Gamma(1/\kappa)}, \tag{14}$$

where $\Gamma(\cdot)$ is the gamma function, ζ is a constant given by

$$\zeta = \left\{\frac{2^{(-2/\kappa)} \cdot \Gamma(1/\kappa)}{\Gamma(3/\kappa)}\right\}^{1/2},$$

and κ is a positive parameter governing the thickness of the tails. For $\kappa = 2$ the constant $\zeta = 1$ and (14) is just the standard normal density.

Due to computational constraints, we proceed slightly different. As shown in Bollerslev & Wooldridge [14], the maximization of the Gaussian log-likelihood

[4]Theoretically, by applying an Ito-Taylor expansion, up to the order judged significant important, the continuous-specification can be reasonably approximated. However, because we are working at daily frequency, $\Delta t = (1/252)$ is quite small and a more precise discretization does not result in significant different estimates from the ones get by use of the less precise above formulas. Thus, we chose to work with simple (12) and (13). See Kloeden & Platen [42] for an explanation of the Ito-Taylor approximation.

can provide consistent estimates even when the distribution of u_t is non-Gaussian. Thus, we estimate [5]:

$$\log\left(\sigma_t^2\right) = \omega + \beta \log\left(\sigma_{t-1}^2\right) + \phi \left| \frac{dW_{t-1}}{\sigma_{t-1}} \right| + \eta \frac{dW_{t-1}}{\sigma_{t-1}}, \tag{15}$$

instead of (9), under assumed normally distributed u_t's. As our u_t are non-normal, we should use the Bollerslev-Wooldridge robust [6] standard error.

4 The data

This study covers the period from January, 1st 1990 to December, 29th 2000, which is relatively homogeneous in terms of macroeconomic conditions and operating procedures by the U.S. monetary authorities. According to Meulendyke [53] and Thornton [66], from October 1989 on, the Federal Open Market Committee has adopted the procedure to discontinuously change the Federal Fund Rate Target by multiples of 25 basis points (0.25% or 0.0025). The daily interest rate data used is from the $H.15$ Release by the Board of Governors (available at the Federal Reserve Bank of Saint Louis' web site: www.stls.frb.org/fred). The published data are bond-equivalent yields (r_{BEY}) or discount rate (r_D). They were transformed to continuously-compounded yield-to-maturity by respectively: $r_{YTM} = \frac{1}{M}\ln(1 + r_{BEY} * \frac{M}{100})$ and $r_{YTM} = -\frac{1}{M}\ln(1 - r_D * \frac{M}{100})$, where M is time-to-maturity in years. The five rates to be used are: the Federal Funds Rate (r) (the U.S. overnight rate), the Federal Fund Rate Target (r^0),

[5] Indeed, this procedure is the one used by the E-Views program.
[6] Robust to mispecification of the density function.

3-Month Treasury Bill Rates (3-m), 6-Month Treasury Bill Rates (6-m), 1-Year Treasury Bill Rates (1-y). T-bills are secondary market rates on Treasury securities; the Daily Federal Funds Rate is the effective federal funds rate published by the Federal Reserve Bank of New York.

Tables 1.a and 1.b present some descriptive statistics for the levels and differences of the series respectively. In Table 1.a, we see that r has fluctuated in a broader range than the other rates. We cannot reject the hypothesis of nonstationarity of any series at 10% significance level in Table 1.a, however we can reject the hypothesis of nonstationarity of differences of the five series at 1% significance level in Table 1.b. The changes in r are significantly more volatile than the other series. Table 1.b also presents the autocorrelations of the differences. Given the number of observations ($T = 2753$), autocorrelations smaller than 0.0381 in absolute value ($\pm 2/\sqrt{T}$) are not significantly different from zero at (approximately) the 5% significance level. Some values shown in Table 1.b are significant, particularly the first two negative autocorrelations of the Federal Fund Rate.

5 Empirical Results

5.1 The failure of the standard specifications

We estimate (4) and the nested processes (7), (6) and (5) by GMM for the Federal Fund Rate from January, 1st 1990 to December, 29th 2000, as described in subsection 3.1. The estimates and the $t-statistics$ (between brackets) are presented in Table 2, together with Hansen's overidentifying restriction statistic,

$\chi^2 - Test$, and the respective $p - value$ (between brackets). Like Chan et al. [19], to test the processes' forecast power for the interest rate changes and the squared interest rate changes, we calculate the proportion of the total ex-post yield changes and ex-post squared yield changes that can be explained by the conditional expected-yield changes and conditional volatility respectively. The two measures are named R_1^2 for the mean and R_2^2 for the variance:

$$R_1^2 = 1 - \frac{\sum \left[\Delta r_i - \mu\left(r_i, X_i\right) \Delta t\right]^2}{\sum \left[\Delta r_i - \overline{\Delta r}\right]^2},$$

$$R_2^2 = 1 - \frac{\sum \left[(\Delta r_i)^2 - \left(\sigma_i\left(r_i\right)\right)^2 \Delta t\right]^2}{\sum \left[(\Delta r_i)^2 - \overline{(\Delta r)^2}\right]^2}.$$

We refer to these as the coefficients of determination. Note that because the conditional volatility function may not include an intercept, the R_2^2 can be negative for models other than Vas.

The results from Table 2 are very different from the obtained by Chan et al. [19] in their Table III, on page 1218 [7]. First, the pair (α_0, α_1) does not vary very much among the models: the difference between our bigger and our smaller estimate is around 25%, while in Chan et al. this differences is bigger than 100%. Second, our $\widehat{\gamma} = 0.6942$ ($t - statistic = 1.95$) is smaller and less significant than the $\widehat{\gamma} = 1.4999$ ($t - statistic = 5.95$) got by them. If their high $\widehat{\gamma}$ was the reason for the better performance of the BS model relative to the CIR

[7] Although the different data frequency, sample period, specific yield chosen and observation frequency limits quantitative comparisons between our study and Chan et al., the qualitative comparisons that follow are still possible. Chan et al. studies monthly one-month yields for Treasury Bills from June 1964 to December 1989 (307 observations). To support our point we have reestimated Table III by Chan et al. for the period from January 1990 to December 2000 using same series as them, which is presented below under the label of Table 3.

and Vas in the $\chi^2 - Test$, our value is the reason for the better performance of the CIR in the same test. Because our $\widehat{\gamma}$ is less extreme, we can not reject any model by the $\chi^2 - Test$ at 5% contrary to their rejection of CIR and Vas. The obtained coefficients of determination for the mean $R_1^{2\prime}s$ are low, like Chan **et al.** However, the most important conclusion from Table 2 is that the value of $\widehat{\gamma}$ is not an important feature in differentiating interest rate models: the R_2^2 is smaller than 1% for all models [8]. The low $R_j^{2\prime}s$ make clear the low power of these models in forecasting the interest rate changes and the volatility changes.

The second part of Table 2, under the heading of residuals, presents some descriptive statistics for the standardized residuals of the first estimated moment, that means $\widehat{dW} = \frac{dr - \widehat{\mu}(r)dt}{\sigma_1 r^{\gamma}\sqrt{dt}}$. The results are not in accordance with the hypothesized. If the dW were iid (standard) normals, we would have: $E_t\left[\widehat{dW}^2\right] = 1$; $E\left[\left(\widehat{dW}\right)^3\right] = 0$; $E\left[\left(\widehat{dW}\right)^4\right] = 3$ and $E\left[\widehat{dW}_t\widehat{dW}_{t+j}\right] = 0 \; \forall \; j > 0$. But this is far from what is got from the sample moments presented. Except for the standard deviation, which is close to 1, all models residuals have skewness and huge kurtosis, with the Jarque-Bera statistics rejecting normality, as well as significant autocorrelation [9]. The first and second order autocorrelations are significant for all models and the Ljung-Box $Q - statistics$ (not reported) reject the null hypothesis of no autocorrelation up to order k for all series and

[8] On pp.1210, before-last paragraph, Chan **et al.** claim: "...Using one-month Treasury bill yields, we find that the value of γ is the most important feature differentiating interest rate models. In particular we show that models which allow $\gamma > 1$ capture the dynamics of the short-term interest rate better than those which require $\gamma < 1$. This is because the volatility of the process is highly sensitive to the level of r ; the unconstrained estimate of γ is 1.5. We also show that the models differ significantly in their ability to capture the volatility of the short-term interest rate..."

[9] Given the number of observations (T) is 2753, autocorrelations smallers than 0.0381 in absolute value ($\pm 2/\sqrt{T}$) are not significantly different from zero at (approximately) 5% significance level.

k 6 120. If we take the above models as true, the significant negative low order autocorrelation may be interpreted as overreaction of the Federal Fund Rate market: the consequences of contemporaneous innovations are exaggerated in the present forecast of the following value, thus sponsoring an opposite sign innovation next instant.

The difference between our estimates for the daily Federal Fund Rate from January, 1st 1990 to December, 29th 2000 and Chan et al. estimates for the monthly one-month T-bill yields from January 1964-December 1989 may be due to the different frequency or to the different yield modeled. To make clear this is not the case, Table 3 below replicates Chan et al.'s Table III for the period January 1990 to December 2000 (coincident with ours) using the one-month U.S Treasury yield. The results support our points. First, the pair (α_0, α_1) does not vary much among models. It is surprising that these estimates, computed with only 131 observations, have shown more significance than the ones obtained by Chan et al. with 306 observations. Our second finding of a γ smaller than one has also shown up for the one-month Treasury yield. Lastly, the value of $\widehat{\gamma}$ is not an important feature in differentiating interest rate models.

The explanation for such different estimates for the same monthly one-month Treasury Bill series deserves more analysis and is not the goal of this paper. However, we attempt an explanation without testing it: the differences are not due to one structural break in the series between the periods January 1964-December 1989 and January 1990- December 2000, but due to the somewhat different regimes that were included in the first 25 years. From January 1964-December 1989, inflation, real interest rates and monetary pol-

icy operating procedures have experimented different standards, as widely reported. Notwithstanding, the estimated processes assume constant parameters and,worse, constant mean reversion. The lack of flexibility to deal with such regime switching may be the cause of Chan **et al.**'s different results.

As the drawbacks in level volatility models like (4) have been shown in papers like Brenner **et al.** [13], we have estimated (10) to $\gamma = 1,\ 1/2,\ 0$ and $\widehat{\gamma}$ (estimate from the level volatility equation) by Maximum Likelihood for the Federal Fund Rate from January, 1st 1990 to December, 29th 2000, as described in subsection 3.2. The estimates and the $t-statistics$ (between brackets) are presented in Table 4. The results from Table 4 are not qualitatively different from those obtained in Table 2 for the common parameters and other statistics. The pair (α_0, α_1) is a bit smaller and less significant, while the first two autocorrelations are negative and significant. The high degree of significance got for the E-GARCH parameters of equation (9) is interesting. Confronted with the lower degree of significance of the estimated γ in Table 2, this may be pointing the better fit of the stochastic level volatility models relative to the level volatility models. However, because the still low $R_1^{2\prime}s$ we postpone this issue to next section.

5.2 The reversion to the target improvement

Since it has become common practice among monetary authorities to set a target for the short-term interest rate, and to enforce it as a policy measure, isn't it reasonable to include the target as a relevant variable in the forecast of the rate level changes? Figure 1.a plots the Federal Fund Rate Target,

and Figure 1.b plots the Federal Fund Rate and the Federal Fund Rate Target together, for the period from January,1st 1990 to December, 29th 2000, showing the Rate follows the Target closely.

Because Figure 1.b presents evidence that the rate target r^0 is a relevant instrument in the rate level process r , we estimate (11) under the level volatility and the E-GARCH stochastic level volatility assumptions.

Table 5 below presents the GMM estimates of level volatility versions of (11) using as instruments not only a constant and the contemporaneous rate r_t, but r_t^0 as well.

The variable long-run mean model estimates show interesting results. Except for the γ, the significance of other coefficient estimates have improved in general. Again, the (α_0, α_1) do not vary much among the models. However, the high estimates of α_1 point the impressive speed of $r's$ reversion to r^0. The $\widehat{\gamma} = 0.2734$ $(t - statistic = 1.30)$ of the unrestricted model has become even smaller and less significant than in Table 2. This lower $\widehat{\gamma}$ is the reason why the modified $VVas$ version $\chi^2 - Test$ statistic has improved and the VBS's one has worsened (when compared to Table 2). The $\chi^2 - Test$ statistic now rejects the modified VBS process, indicating such a high sensitivity of the volatility to the level of interest rate is unreasonable. The $R_1^{2\prime}s$ close to 0,35 mean that the drift term $\left[\alpha_0 + \alpha_1 \left(r_t^0 - r_t\right)\right] dt$ forecasts approximately 35% of the interest rate level changes. Like Table 2, the γ is not an important feature in differentiating interest rate models, with almost zero $R_2^{2\prime}s$. The $R_2^{2\prime}s$ have the standard interpretation of the coefficient of determination, which measures the proportion of the variation squared yield changes explained by the model conditional

volatility. The low values got just mean that the attempts to model a variable volatility did no better than the constant volatility of the $VVas$ process. Thus, the target inclusion considerably improves the mean-reversion models, and results that constant volatility models like $VVas$ perform as well as the variable volatility ones.

The residuals part of Table 5 presents some descriptive statistics for the standardized residuals of the first estimated moment: $\widehat{dW} = \frac{dr - \widehat{\mu}(r)dt}{\sigma_1 r^{\gamma} \sqrt{dt}}$. It shows that, once we take the target into account, the low order autocorrelations are not significantly negative, solving the apparent anomaly shown in Table 2. Although the Jarque-Bera still rejects normality, the modified drift does not imply it any more [10].

Table 6 below presents the Maximum Likelihood estimates of E-GARCH stochastic level volatility versions of (11). Compared to the estimates for (10) in Table 4, the significance of the estimates $(\widehat{\alpha}_0, \widehat{\alpha}_1)$ has improved. Like in the Table 5 for the level volatility versions of (11), the high estimates of α_1 points the impressive speed of $r's$ reversion to r^0, and the $R_1^{2\prime}s$ mean the drift term $\left[\alpha_0 + \alpha_1 \left(r_t^0 - r_t\right)\right] dt$ forecasts approximately 34% of the interest rate level changes. The residuals in Table 6 also show that the low order autocorrelations are not significant negative if we take the target into account.

Different from Table 5, the $R_2^{2\prime}s$ in Table 6 now present sensible and significant values, showing the superiority of the **stochastic volatility** model upon the **level volatility** ones on explaining the volatility. Between 14% and 24% of

[10] This statement does not mean the problem related to the huge kurtosis is solved. Just that normality of $\widehat{dW}$ is not hypothesized.

the variation of the squared yield changes are explained by the model conditional volatility. The $VSCIR$ is the model with highest R_2^2, meaning that the **stochastic level volatility** models do better than the **stochastic volatility** models (represented by the $VSVAS$).

6 A Preliminary Analysis of the Federal Fund Rate Target

The influence of the Federal Fund Rate Target in the Federal Fund Rate process raises two questions. First, it seems sensible to wonder what is the process followed by the Target and to try to estimate the Rate and the Target process jointly. This is the subject of section 6.1. Second, because the fast reversion of the Federal Fund Rate to its target means that r is almost a noise around r^0, one may wonder whether the Federal Fund Rate Target is a more efficient short-term factor than the Rate itself. This is the subject of section 6.2.

6.1 A model for the Federal Fund Rate Target and the joint-estimation of the Federal Fund Rate and its target

Figure 1.a shows that the Federal Fund Rate Target process (r^0) is a pure discontinuous process, what suggests dr^0 should be parametrized as a discrete jumps process.

Since the Federal Open Market Committee does not chose policy goals and routes on a daily basis, but only after accumulating a reasonable amount of information, and given medium to long-term nominal interest rates (r^1) are

widely accepted leading indicators that incorporate future expected inflation, we take these longer-term rates as possible factors sponsoring target moves and parametrize dr^0 to be:

$$dr_t^0 = dJ_t \tag{16}$$

where:

$$dJ_t = \begin{cases} \delta \frac{(\theta + r_t^1 - r_t^0)}{|\theta + r_t^1 - r_t^0|} & \textit{with probability } \lambda \left|\theta + r_t^1 - r_t^0\right| dt \\ 0 & \textit{with probability } 1 - \lambda \left|\theta + r_t^1 - r_t^0\right| dt \end{cases},$$

and r^1 is the longer-term rate proxing for expected future inflation.

Because dJ is not zero mean, it needs reformulation to continue having the interpretation of unexpected variation and we shall rewrite dr^0 as:

$$dr_t^0 = \delta\lambda \left(\theta + r_t^1 - r_t^0\right) dt + \left[dJ_t - \delta\lambda \left(\theta + r_t^1 - r_t^0\right) dt\right]; \tag{17}$$

what provides a non-standard foundation for the mean-reverting drift: it is the result of a non-zero mean discontinuous error.

Conditional on the information available at t, the above process is Poisson and presents the following two discrete time first moments:

$$E_t \left[\Delta r_t^0\right] = \delta\lambda \left(\theta + r_t^1 - r_t^0\right) \Delta t \tag{18}$$

$$E_t \left[\left(\Delta r_t^0 - E_t \left[\Delta r_t^0\right]\right)^2\right] = \delta^2\lambda \left|\theta + r_t^1 - r_t^0\right| \Delta t. \tag{19}$$

The Federal Open Market Committee procedure to move the rate by mul-

tiples of 25 basis points, since November 1989 [11], allows to restrict $\delta = .0025$ [12]. The choice of the r^1 can be based on best fit to data.

We have estimated (18) and (19) for r^1 $=\{r$, 3-m, 6-m, 1-y, 2-y, 3-y, 4-y, 5-$y\}$[13], by GMM, with a constant, r_t^0 and r_t^1 as instruments, for the period between 01/01/90 and 12/29/2000. Table 7.a shows the unrestricted model, while Table 7.b incorporates the restriction $\delta = .0025$.

Table 7.a estimates for the unrestricted δ shows that, except for the r proxy, we cannot reject the proposed model. The model performance increases with the maturity of the yield chosen to proxy for inflation, and the target jump size estimate $\widehat{\delta}$, around 0.0035 for the sensible proxies, is reasonably close to 0.0025. The λ is quite different among models, however this is not crucial, given the intensity of the jumps is given by $\lambda \left|\theta + r_t^1 - r_t^0\right| \Delta t$, dependent upon the other parameters and on the specific r^1.

Table 7.b estimates for the restricted $\delta = 0.0025$ shows that the model with better performance is the one for $r^1 = 6 - m$. However, its superiority is not big enough to decide for it[14].

To close, we have estimated the joint processes for the Federal Fund Rate and Target, with the 1-year T-bill as the r^1. The results of the joint processes presented in Table 8 are similar to the obtained for the individual processes (presented separately in Tables 5 and 7.a.). The CIR and the Vas versions

[11] See Meulendyke [53] and Thornton [66] for the empirical evidence.

[12] In this context, changes of 50 or 75 basis points are taken as the simultaneous realization of two or three jumps.

[13] Although r is is not a longer rate, it seemd sensible to perform this estimation as well, to check whether the results were coerent with the predicted. That mean, the longer-rates should perform better on explaining the target moves.

[14] With regard to monetary policy theory, a horizon of six month seems to short, given the lags involved in the transmission mechanisms. See Batini & Haldane [?]

confirm their superiority in modeling the data for the Federal Fund Rate. The modified drift is able to explain a significant part of the total variation of the rate level. Reinforcing the results from Table 5, the insignificant $R_2^{2\prime}s$ point that the level volatility functions don't do a good job in explaining the ex-post volatility changes than the constant Vasicek model.

6.2 The Federal Fund Rate Target seems the short-term factor

The fast reversion of the Federal Fund Rate to its target means that r is almost a noise around r^0, and raises the question whether the Federal Fund Rate Target is a more efficient short-term factor than the Rate. Table 9 presents the correlation matrix of the studied rates and of the difference between the Federal Fund Rate and the Federal Fund Rate Target, suggesting this may be the case. Indeed, the Target is slightly more correlated with the other maturity rates than the Rate itself, and the difference between the Rate and the Target is low correlated with the other rates, these meaning the Target drives out the Rate as a term structure factor.

7 Conclusion

We have shown that the univariate constant-mean-reversion processes like Vasicek [68], Cox, Ingersoll & Ross [24] and Brennan & Schwartz [12] present problems when taken to Federal Fund Rate data (the U.S. overnight rate). We have also shown that Chan et al. [19] suggestion to leave the exponent γ in the volatility function to adjust freely neither does result in a $\gamma > 1$, nor shows that

γ is important in differentiating interest rate models. Alternatively, we have shown that modeling the mean reverting process as reverting to the rate target results in an improved fit. The resulting process forecasts a considerable proportion of the ex post variation of the rate change. Furthermore, the proposed mean parametrization can be combined with the stochastic level volatility to explain part of the ex post volatility changes.

Figures and Tables

Table 1.a
Daily Levels of US Government Bond Yields from 01/01/1990 to 29/12/2000

Rate	F	r^0	3-m	6-m	1-y
Mean	0.0527	0.0522	0.0497	0.0511	0.0527
Maximum	0.1039	0.0825	0.0807	0.0820	0.0835
Minimum	0.0258	0.0300	0.0262	0.0277	0.0289
Std. Dev.	0.0138	0.0135	0.0122	0.0120	0.0117
Skewness	0.2438	0.1848	0.3107	0.2277	0.1869
Kurtosis	3.0847	3.02	3.1478	3.0538	3.0008
Jarque-Bera	28	16	47	24	16
Probability	0.00	0.00	0.00	0.00	0.00
ADF Test Statistic	-2.13	-2.07	-2.41	-2.43	-2.51
H_0: Nonstationarity			Reject at 10% (critical value=-2.58)		

The descriptive statistics reported in Table 1.a are the daily levels for the Federal Fund Rate (r), the Federal Fund Rate Target (r^0), the 3-month T-Bill (3-m), the 6-month T-Bill (6-m), the 1-year T-Bill (1-y), from the H.15 Release by the Board of Governors. Each series statistics were computed using the 2754 observations from January, 1st, 1990 to December, 29th, 2000.

Table 1.b
Daily Differences of US Government Bond Yields
from 01/01/1990 to 29/12/2000

Rate	dF	dr^0	d3-m	d6-m	d1-y
Maximum	0.0283	0.0075	0.0049	0.0024	0.0039
Minimum	-0.0270	-0.0050	-0.0049	-0.0040	-0.0037
Std. Dev.	0.0029	0.0004	0.0005	0.0004	0.0005
Skewness	0.8086	1.6051	-0.5752	-0.6018	-0.1897
Kurtosis	24.6688	137.22	18.8659	10.7033	9.2622
Jarque-Bera	54160	2067512	29027	6973	4515
Probability	0.00	0.00	0.00	0.00	0.00
Autocorr ρ_1	-0.36	0.00	0.08	0.07	0.09
ρ_2	-0.10	0.00	-0.08	-0.04	-0.01
ρ_3	-0.01	0.00	-0.03	-0.02	-0.02
ρ_4	0.00	0.03	0.00	0.02	-0.04
ρ_5	-0.01	0.00	0.07	0.05	0.01
ρ_{10}	0.09	0.03	0.07	0.07	0.05
ρ_{20}	-0.02	0.06	0.03	0.03	0.05
ρ_{60}	0.01	0.06	0.03	0.03	0.01
ρ_{90}	0.02	0.00	-0.02	-0.02	-0.01
ρ_{120}	0.00	0.00	0.06	0.01	-0.02
ADF Test Statistic	-13.75	-8.63	-9.20	-8.69	-8.42
H_0: Nonstationarity			Reject at 1% (critical value=-3.44)		

The descriptive statistics reported in Table 1.b are for the daily differences of the series ($dy_t=y_{t+1}-y_t$): the Federal Fund Rate (r), the Federal Fund Rate Target (r^o), the 3-month T-Bill (3-m), the 6-month T-Bill (6-m), the 1-year T-Bill (1-y), from the H.15 Release by the Board of Governors. Each series statistics were computed using the 2753 observations from January, 1st, 1990 to December, 28th, 2000.

Table 2
GMM estimates of alternative standard models for the daily Federal Fund Rate

Model	CKLS	BS	CIR	Vas
α_0	0.3108	0.3323	0.2968	0.2621
(t-statistic)	(5.98)	(7.09)	(6.71)	(6.27)
α_1	-5.9505	-6.3590	-5.6855	-5.0299
(t-statistic)	(-5.95)	(-7.04)	(-6.64)	(-6.19)
σ	0.35	0.87	0.20	0.04
(t-statistic)	(0.94)	(17.33)	(17.75)	(17.66)
γ	0.6942	1	0.5	0
(t-statistic)	(1.95)			
χ^2 Test	0	0.8114	0.2892	3.0519
(p-value)	(1.00)	(0.37)	(0.59)	(0.08)
R_1^2	0.0123	0.0123	0.0123	0.012
R_2^2	0.0068	0.0084	0.005	0
Residuals				
Std. Dev.	1.03	1.07	1.03	1.10
Skewness	1.75	2.18	1.51	-0.93
Kurtosis	21.78	25.55	21.17	24.57
Jarque-Bera	41853	60523	38933	53776
Probability	0.00	0.00	0.00	0.00
Autocorr. ρ_1	-0.35	-0.34	-0.35	-0.35
ρ_2	-0.06	-0.04	-0.07	-0.09
ρ_3	-0.02	-0.02	-0.01	0.00
ρ_4	0.01	0.01	0.01	0.01
ρ_5	0.01	0.01	0.01	-0.01
ADF Test Statistic	-5.07	-5.08	-5.16	-5.69
H_0: Nonstationarity		Reject at 1% (critical value=-3.44)		

The R_j^2 statistics are computed as the proportion of the total variation of the actual yield change (j=1) and their volatility (squared yield changes) (j=2) explained by the respective predictive values for each model. The χ^2-Tests evaluate overidentified restrictions imposed by alternative models on the unrestricted model. The daily Federal Fund Rate series for the period from January, 1st, 1990 to December, 29th, 2000 (2754 observations) is in annualized continuously-compounding rate.

Table 3
GMM estimates of alternative standard models for the monthly one-month Treasury Bill

Model	CKLS	BS	CIR	Vas
α_0	0.0294	0.0302	0.0297	0.0283
(t-statistic)	(2.48)	(2.55)	(2.50)	(2.39)
α_1	-0.6531	-0.7148	-0.6683	-0.6042
(t-statistic)	(-2.57)	(-2.83)	(-2.65)	(-2.40)
σ	0.04	0.22	0.05	0.01
(t-statistic)	(1.15)	(12.29)	(13.11)	(12.71)
γ	0.3858	1	0.5	0
(t-statistic)	(1.37)			
χ^2 Test	0	4.1397	0.1637	1.6941
(p-value)	(1.00)	(0.04)	(0.69)	(0.19)
R_1^2	0.0417	0.0385	0.0416	0.0405
R_2^2	0.013	0.0001	0.0111	0

The R_j^2 statistics are computed as the proportion of the total variation of the actual yield change (j=1) and their volatility (squared yield changes) (j=2) explained by the respective predictive values for each model. The χ^2-Tests evaluate overidentified restrictions imposed by alternative models on the unrestricted model. The monthly one-month U.S. Treasury bill yield for the period from January 1990 to December 2000 (132 observations) is in annualized continuously-compounding rate.

Table 4
Maximum Likelihood estimates of alternative E-GARCH models for the daily Federal Fund Rate

Model	CKLS	BS	CIR	Vas
α_0	0.2411	0.3439	0.2172	0.1512
(t-statistic)	(2.60)	(3.78)	(2.44)	(1.76)
α_1	-3.8254	-5.9018	-3.3577	-2.0265
(t-statistic)	(-2.21)	(-3.67)	(-2.00)	(-1.24)
γ	0.6942	1	0.5	0
(t-statistic)				
ω	-5.2660	-3.6421	-6.3311	-8.7243
(t-statistic)	(-9.24)	(-8.48)	(-10.14)	(-10.26)
β	0.4133	0.5038	0.3715	0.3287
(t-statistic)	(5.84)	(7.34)	(5.41)	(4.63)
ϕ	0.9363	0.9590	0.9254	0.9105
(t-statistic)	(9.36)	(10.28)	(9.06)	(8.64)
η	-0.1683	-0.2356	-0.1348	-0.0623
(t-statistic)	(-2.38)	(-3.60)	(-1.86)	(-0.82)
R_1^2	0.0075	0.0100	0.0066	0.0029
R_2^2	-5.7295	-3.0629	-7.5497	-19.5809
Residuals				
Std. Dev.	1.00	1.00	1.00	1.00
Skewness	2.36	2.28	2.34	2.32
Kurtosis	18.98	18.58	18.87	19.82
Jarque-Bera	31855	30203	31420	34936
Probability	0.00	0.00	0.00	0.00
Autocorr. ρ_1	-0.22	-0.21	-0.21	-0.21
ρ_2	-0.03	-0.01	-0.04	-0.07
ρ_3	-0.02	-0.01	-0.02	-0.02
ρ_4	-0.04	-0.03	-0.05	-0.05
ρ_5	0.04	0.06	0.04	0.03
ADF Test Statistic	-9.01	-7.36	-9.32	-9.94
H_0: Nonstationarity		Reject at 1% (critical value=-3.44)		

The R_j^2 statistics are computed as the proportion of the total variation of the actual yield change (j=1) and their volatility (squared yield changes) (j=2) explained by the respective predictive values for each model. The χ^2-Tests evaluate overidentified restrictions imposed by alternative models on the unrestricted model. The daily Federal Fund Rate series for the period from January, 1st, 1990 to December, 29th, 2000 (2754 observations) is in annualized continuously-compounding rate.

Table 5
GMM estimates of alternative standard models for the Federal Fund Rate

Volatility	CKLS	BS	CIR	Vas
α_0	0.0795	0.0705	0.0781	0.0793
(t-statistic)	(6.72)	(6.21)	(6.66)	(6.72)
α_1	179.2208	173.8845	178.3019	179.3418
(t-statistic)	(23.62)	(23.22)	(23.63)	(23.62)
σ	0.08	0.70	0.17	0.03
(t-statistic)	(1.52)	(16.81)	(18.31)	(18.47)
γ	0.2748	1	0.5	0
(t-statistic)	(1.30)			
χ^2 Test	0.6528	11.3556	2.0201	2.3106
(p-value)	(0.72)	(0.01)	(0.57)	(0.51)
R_1^2	0.3479	0.3478	0.348	0.3479
R_2^2	-0.0036	0.0040	-0.0005	0
Residuals				
Std. Dev.	1.02	1.10	1.01	1.07
Skewness	3.27	3.69	3.27	3.39
Kurtosis	31.56	34.48	29.86	36.15
Jarque-Bera	98452	119936	87642	131343
Probability	0.00	0.00	0.00	0.00
Autocorr. ρ_1	0.00	-0.04	-0.01	0.01
ρ_2	0.02	0.03	0.02	0.01
ρ_3	0.03	0.00	0.02	0.03
ρ_4	0.01	-0.01	0.01	0.02
ρ_5	-0.01	-0.01	-0.01	0.00
ADF Test Statistic	-8.15	-8.68	-8.29	-8.01
H_0: Nonstationarity	Reject at 1% (critical value=-3.44)			

The R_j^2 statistics are computed as the proportion of the total variation of the actual yield change (j=1) and their volatility (squared yield changes) (j=2) explained by the respective predictive values for each model. The χ^2-Tests evaluate overidentified restrictions imposed by alternative models on the unrestricted model. The daily Federal Fund Rate and daily Federal Fund Rate Target series for the period from January, 1st, 1990 to December, 29th, 2000 (2754 observations) are in annualized continuous-compounding rate.

Table 6
Maximum likelihood estimates of alternative E-GARCH models for the daily Federal Fund Rate

Model	VSCKLS	VSBS	VSCIR	VSVas
α_0	0.0305	0.0624	0.0383	0.0271
(t-statistic)	(2.33)	(4.12)	(2.84)	(2.14)
α_1	167.6011	174.0413	178.6638	163.6139
(t-statistic)	(18.15)	(19.07)	(17.69)	(16.82)
γ	0.2748	1	0.5	0
(t-statistic)				
ω	-5.2410	-0.4524	-4.3864	-6.0656
(t-statistic)	(-3.66)	(-3.21)	(-4.21)	(-3.28)
β	0.5416	0.9559	0.5673	0.5356
(t-statistic)	(4.10)	(56.64)	(5.18)	(3.61)
ϕ	0.6671	0.3239	0.6752	0.6668
(t-statistic)	(4.82)	(3.38)	(4.92)	(4.82)
η	-0.2649	-0.1225	-0.2754	-0.2488
(t-statistic)	(-2.29)	(-1.63)	(-2.39)	(-2.17)
R_1^2	0.3432	0.3474	0.3445	0.3419
R_2^2	0.2094	0.1411	0.2443	0.1654
Residuals				
Std. Dev.	1.00	1.00	1.00	1.00
Skewness	2.85	3.21	2.92	2.81
Kurtosis	22.52	26.18	22.34	23.42
Jarque-Bera	47400	66336	46820	51441
Probability	0.00	0.00	0.00	0.00
Autocorr. ρ_1	-0.02	-0.03	0.00	-0.03
ρ_2	0.03	0.04	0.04	0.02
ρ_3	0.01	0.01	0.02	0.02
ρ_4	-0.01	-0.01	-0.01	0.00
ρ_5	0.02	0.01	0.02	0.02
ADF Test Statistic	-8.47	-8.33	-8.45	-8.43
H_0: Nonstationarity	Reject at 1% (critical value=-3.44)			

The R_j^2 statistics are computed as the proportion of the total variation of the actual yield change (j=1) and their volatility (squared yield changes) (j=2) explained by the respective predictive values for each model. The χ^2-Tests evaluate overidentified restrictions imposed by alternative models on the unrestricted model. The daily Federal Fund Rate and daily Federal Fund Rate Target series for the period from January, 1st, 1990 to December, 29th, 2000 (2754 observations) are in annualized continuous-compounding rate.

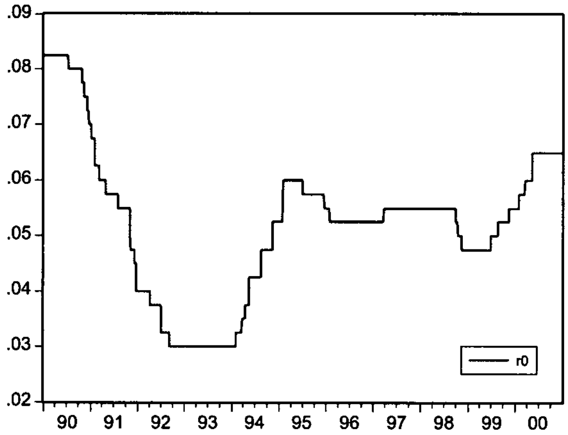

Figure 1.a

Daily Federal Fund Rate Target level from Jan. 1st, 1990 to Dec. 29th, 2000

This figure shows evolution of the Federal Fund Rate Target level. The rate is continuously compounded, annualized, and expressed in decimal number.

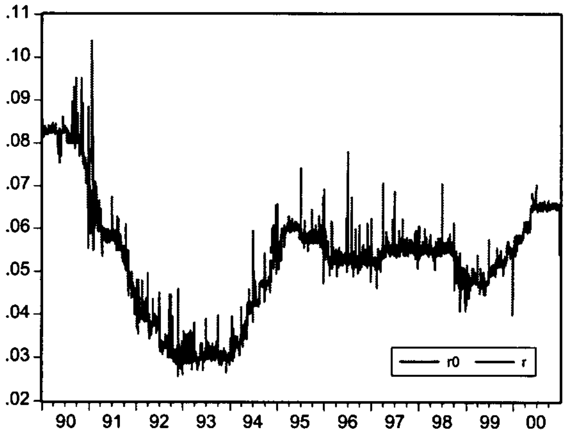

Figure 1.b

Daily Federal Fund Rate and Target levels from Jan. 1st, 1990 to Dec. 29th, 2000

This figure shows evolution of the Federal Fund Rate and Target levels. The rate is continuously compounded, annualized, and expressed in decimal number.

Table 7.a
GMM estimates for the unrestricted target process with alternative rates

r^0	ffr	3-m	6-m	1-y
δ	0.0074	0.0035	0.0032	0.0037
(t-statistic)	(3.24)	(6.59)	(7.42)	(6.38)
λ	382.42	987.11	919.59	544.84
(t-statistic)	(1.47)	(3.09)	(3.48)	(3.14)
θ	-0.0010	0.0017	0.0001	-0.0017
(t-statistic)	(-1.34)	(3.81)	(0.31)	(-2.41)
χ^2 Test	6.6181	4.5560	4.6906	3.4091
(p-value)	(0.0851)	(0.2074)	(0.1959)	(0.3327)

The χ^2-Tests evaluate overidentified restrictions.The daily Federal Fund Rate Target series (r^0) for the period from January, 1st, 1990 to December, 29th, 2000 (2754 observations) is in annualized continuously-compounding rate.

Table 7.b
GMM estimates for the restricted target process with alternative rates

r^0	r	3-m	6-m	1-y
λ	1355.42	1427.18	1193.22	807.33
(t-statistic)	(2.14)	(4.48)	(4.79)	(4.69)
θ	0.0000	0.0019	0.0004	-0.0014
(t-statistic)	(0.06)	(4.78)	(0.99)	(-2.06)
χ^2 Test	21.1109	8.6195	7.4168	8.0749
(p-value)	(0.0001)	(0.0713)	(0.1154)	(0.0889)

The χ^2-Tests evaluate overidentified restrictions.The daily Federal Fund Rate Target series (r^0) for the period from January, 1st, 1990 to December, 29th, 2000 (2754 observations) is in annualized continuously-compounding rate.

Table 8
GMM joint-estimates of the Federal Fund Rate and the Federal Fund Rate Target, with the 1-year yield

Volatility	VCKLS	VBS	VCIR	VVas
λ	685.8477	719.1360	706.4771	662.3925
(t-statistic)	(4.57)	(4.78)	(4.71)	(4.45)
θ	-0.0016	-0.0017	-0.0017	-0.0015
(t-statistic)	(-2.10)	(-2.30)	(-2.30)	(-1.89)
α_0	0.0730	0.0632	0.0714	0.0726
(t-statistic)	(6.47)	(5.79)	(6.35)	(6.48)
α_1	181.9661	178.4123	181.5851	181.5985
(t-statistic)	(26.09)	(25.85)	(26.10)	(26.01)
σ	0.07	0.69	0.16	0.03
(t-statistic)	(1.54)	(16.41)	(18.28)	(18.63)
γ	0.2481	1	0.5	0
(t-statistic)	(1.20)			
χ^2 Test	14.2043	26.2757	15.6684	15.4493
(p-value)	(0.16)	(0.01)	(0.15)	(0.16)
R_1^2	0.3475	0.3473	0.3475	0.3475
R_2^2	-0.005	0.003	-0.0014	0
Residuals				
Maximum	13.65	15.18	12.07	15.79
Minimum	-6.65	-6.40	-6.06	-7.69
Std. Dev.	1.05	1.13	1.03	1.10
Skewness	3.27	3.69	3.27	3.39
Kurtosis	31.91	34.37	29.84	36.21
Jarque-Bera	100784	119120	87541	131752
Probability	0.00	0.00	0.00	0.00
Autocorr. ρ_1	0.01	-0.03	0.00	0.02
ρ_2	0.02	0.04	0.03	0.02
ρ_3	0.03	0.01	0.02	0.03
ρ_4	0.01	0.00	0.01	0.02
ρ_5	-0.01	-0.01	-0.01	0.00
ADF Test Statistic	-8.13	-8.64	-8.28	-8.01
H_0: Nonstationarity		Reject at 1% (critical value=-3.44)		

The R_j^2 statistics are computed as the proportion of the total variation of the actual Federal Fund Rate change (j=1) and their volatility (squared yield changes) (j=2) explained by the respective predictive values for each model. The χ^2-Tests evaluate overidentified restrictions imposed by alternative models on the unrestricted model. The daily F. F. Rate, F. F. Rate Target and 1-year yield series for the period from January, 1st, 1990 to December, 29th, 2000 (2754 observations) are in annualized continuously-compounding rate.

Table 9
Correlation Matrix of the Daily Levels of US Yields from 01/01/1990 to 29/12/2000

	r	r^0	$(r-r^0)$	3-m	6-m	1-y
r	1	0.9837	0.1821	0.9719	0.9583	0.9257
r^0		1	0.0025	0.9853	0.9708	0.9363
$(r-r^0)$			1	0.0170	0.0206	0.0286
3-m				1	0.9935	0.9726
6-m					1	0.9904
1-y						1

Table 9 presents the estimated correlation matrix of the studied daily rates: the Federal Fund Rate (r), the Federal Fund Rate Target (r^o), the 3-month T-Bill (3-m), the 6-month T-Bill (6-m), the 1-year T-Bill (1-y), and the difference between the Federal Fund Rate and the Federal Fund Rate Target. The yields from January, 1st, 1990 to December, 29th, 2000 (2754 observations) are in annualized continuously-compounding rate.

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