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Inada Conditions Imply that Production Function must be Asymptotically Cobb-Douglas

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Abstract

We show that every twice-continuously differentiable and strictly concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ can be bracketed between two C.E.S. functions at each open interval. In particular, for the Inada conditions to hold, a production function must be asymptotically Cobb-Douglas.
JEL Classification: E13, E23
Key Words: Inada Condition, Production Function, Elasticity of Substitution

1 Introduction

The celebrated Inada conditions, that a (per-capita) production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ should satisfy

$$f(0) = 0,\ f'(0) = \infty,\ f'(\infty) = 0,\ \text{and}\ f(\infty) = \infty \tag{1}$$

on top of being strictly increasing ($f'(k) > 0$) and strictly concave ($f''(k) < 0$) for all $k \in \mathbb{R}_+$, are widely used in the applied literature. In 1963 Inada noticed that those conditions had been implicitly used by Usawa in his series of two-sector growth models, and that those conditions were sufficient to ensure existence of equilibria. In addition, those conditions are intuitively very plausible and easily justified. It is then not surprising that the assumptions (1) have not yet been subject to a more thorough investigation. In this note we show that they impose strong restrictions on the

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asymptotic behavior of the elasticity of substitution between capital and labor. In particular, for (1) to hold we must have that the production function is asymptotically Cobb-Douglas (that is, its elasticity of substitution is asymptotically equal to one), as k approaches either zero or infinity.

2 The Result

We assume that $f \in C^2(\mathbb{R}_+)$ is increasing and strictly concave. The elasticity of substitution between capital and labor is given by

$$\sigma(k) \equiv -\frac{f'(k)\left[f(k) - kf'(k)\right]}{kf(k)f''(k)} \geq 0, \tag{2}$$

and is assumed bounded and continuous.

In order to prove Proposition 1 below, we will make use of two Lemmas, which show that any production function can be approximated both from above and from below by suitable C.E.S. functions. Let $\sigma_\epsilon \equiv \sigma(0) - \epsilon$, $\sigma^\epsilon \equiv \sigma(0) + \epsilon$ and define

$$h(k) \equiv \frac{f(k) - kf'(k)}{f'(k)},\ C_k^\sigma \equiv \left(\frac{h(k) + k}{h(k) + k^{\frac{1}{\sigma}}}\right)^{\frac{\sigma}{\sigma-1}} \text{ and } \alpha_k^\sigma \equiv \frac{h(k)}{h(k) + k^{\frac{1}{\sigma}}}.$$

Lemma 1 *For every $\epsilon > 0$ there exists $k > 0$ such that*

$$\frac{f(k)}{C_k^{\sigma_\epsilon}}\left(1 - \alpha_k^{\sigma_\epsilon} + \alpha_k^{\sigma_\epsilon} x^{\frac{\sigma_\epsilon - 1}{\sigma_\epsilon}}\right)^{\frac{\sigma_\epsilon}{\sigma_\epsilon - 1}} \leq f(x) \leq \frac{f(k)}{C_k^{\sigma^\epsilon}}\left(1 - \alpha_k^{\sigma^\epsilon} + \alpha_k^{\sigma^\epsilon} x^{\frac{\sigma^\epsilon - 1}{\sigma^\epsilon}}\right)^{\frac{\sigma^\epsilon}{\sigma^\epsilon - 1}}$$

for all $x \in [0, k]$.

Proof. *Continuity and boundedness of $\sigma(k)$ assure that for any $\epsilon > 0$ there exists $k > 0$ such that*

$$\sigma_\epsilon \leq \sigma(x) \leq \sigma^\epsilon \text{ for all } x \in [0, k].$$

From the definition of h above we have $\frac{dh(x)}{h(x)} = \frac{1}{\sigma(x)}\frac{dx}{x}$. Hence

$$\frac{1}{\sigma^\epsilon}\frac{dx}{x} \leq \frac{dh(x)}{h(x)} \leq \frac{1}{\sigma_\epsilon}\frac{dx}{x} \text{ for all } x \in [0, k].$$

Integrating and substituting for h we get:

$$h(k)\left(\frac{x}{k}\right)^{\frac{1}{\sigma_\epsilon}}+x\leq\frac{f(x)}{f'(x)}\leq h(k)\left(\frac{x}{k}\right)^{\frac{1}{\sigma^\epsilon}}+x \text{ for all } x\in[0,k], \tag{3}$$

where the inequality follows from $h(k)>0$, which comes from f being concave. Integrating again:

$$f(k)\frac{\left(\frac{h(k)}{k^{\frac{1}{\sigma_\epsilon}}}+x^{\frac{\sigma_\epsilon-1}{\sigma_\epsilon}}\right)^{\frac{\sigma_\epsilon}{\sigma_\epsilon-1}}}{\left(\frac{h(k)}{k^{\frac{1}{\sigma_\epsilon}}}+k^{\frac{\sigma_\epsilon-1}{\sigma_\epsilon}}\right)^{\frac{\sigma_\epsilon}{\sigma_\epsilon-1}}}\leq f(x)\leq f(k)\frac{\left(\frac{h(k)}{k^{\frac{1}{\sigma^\epsilon}}}+x^{\frac{\sigma^\epsilon-1}{\sigma^\epsilon}}\right)^{\frac{\sigma^\epsilon}{\sigma^\epsilon-1}}}{\left(\frac{h(k)}{k^{\frac{1}{\sigma^\epsilon}}}+k^{\frac{\sigma^\epsilon-1}{\sigma^\epsilon}}\right)^{\frac{\sigma^\epsilon}{\sigma^\epsilon-1}}} \text{ for all } x\in[0,k].$$

Now use the definitions of C_k^σ and α_k^σ and we are done. ■

Lemma 2 *If $\sigma(\infty)$ is well defined, then for every $\epsilon>0$ there exists $k>0$ such that*

$$\frac{f(k)}{C_k^{\sigma_e}}\left(1-\alpha_k^{\sigma_e}+\alpha_k^{\sigma_e}x^{\frac{\sigma_e-1}{\sigma_e}}\right)^{\frac{\sigma_e}{\sigma_e-1}}\leq f(x)\leq\frac{f(k)}{C_k^{\sigma^\epsilon}}\left(1-\alpha_k^{\sigma^\epsilon}+\alpha_k^{\sigma^\epsilon}x^{\frac{\sigma^\epsilon-1}{\sigma^\epsilon}}\right)^{\frac{\sigma^\epsilon}{\sigma^\epsilon-1}}$$

for all $x\in[k,\infty)$, where now $\sigma_\epsilon\equiv\sigma(\infty)-\epsilon$, $\sigma^\epsilon\equiv\sigma(\infty)+\epsilon$.

Proof. *Just repeat the proof of Lemma 1 with proper adjustments.* ■

We are now in position to state

Proposition 1 *1) If $\sigma(0)<1$ then $f(0)=0$, $f'(0)<\infty$ and $\lim_{k\searrow0}\frac{kf'(k)}{f(k)}=1$;*

2) If $\sigma(0)>1$, then $f(0)>0$, $f'(0)=\infty$ and $\lim_{k\searrow0}\frac{kf'(k)}{f(k)}=0$;

3) If $\sigma(\infty)<1$, then $f(\infty)<\infty$, $f'(\infty)=0$, and $\lim_{k\nearrow\infty}\frac{kf'(k)}{f(k)}=0$;

4) If $\sigma(\infty)>1$, then $f(\infty)=\infty$, $f'(\infty)>0$, and $\lim_{k\nearrow\infty}\frac{kf'(k)}{f(k)}=1$.

Proof. *We can set ϵ in Lemmas 1 and 2 such that:*

1) $\sigma(0)<1\Rightarrow\sigma^\epsilon<1$;

2) $\sigma(0)>1\Rightarrow\sigma_\epsilon>1$;

3) $\sigma(\infty)<1\Rightarrow\sigma^\epsilon<1$;

4) $\sigma(\infty)>1\Rightarrow\sigma_\epsilon>1$.

Taking respectively the limit for $x\searrow0$ in Lemma 1 and for $x\nearrow\infty$ in Lemma 2 the results for $f(0)$ and $f(\infty)$ follow from the asymptotic properties of the C.E.S. function.

To derive the results for $\frac{kf'(k)}{f(k)}$ we divide (3) (and the equivalent expression from the proof of Lemma 2) by x and take the limit. Likewise, to get the results for $f'(0)$ and $f'(\infty)$ we divide the inequalities in Lemmas 1 and 2 by x, and then take the limit. ■

In particular, if we force $f(0) = 0$ and $f'(0) = \infty$, then $\sigma(0)$ must be equal to 1. Since the Cobb-Douglas functional form is the one that has $\sigma(k)$ constant and equal to 1, we conclude that the Inada conditions force the production function to be asymptotically Cobb-Douglas. Whenever evidence points out that the Cobb-Douglas functional form is not appropriate for some application, we are forced to give up the full force of the Inada conditions, and perhaps some of its implications.

References

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