Local Estimation of Copula Based Value-at-Risk

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Abstract
In this paper we propose the local maximum likelihood method for dynamically estimate copula parameters. We study the estimates statistical properties and derive the expression for their asymptotic variance in the case of Gaussian copulas. The local estimates are able to detect temporal changes in the strength of dependence among assets. These dynamics are combined with a GARCH type modeling of each individual asset to estimate the Value-at-Risk. The performance of the proposed estimates is investigated through Monte Carlo simulation experiments. In an application using real data, an out-of-sample test indicated that the new methodology may outperform the constant copula model when it comes to Value-at-Risk estimation.

Keywords: copulas; local maximum likelihood estimation; value-at-risk.

JEL codes: C13; C32; C52.

Resumo
Neste artigo, nós propomos o método de máxima verossimilhança local para estimar parâmetros de cópulas dinamicamente. Nós estudamos as propriedades de estimativas locais e derivamos a expressão para suas variâncias assintóticas para o caso de cópulas Gaussianas. As estimativas locais são usadas para detectar mudanças temporais na força de dependência entre ativos. Estas dinâmicas são então incorporadas na estimação do Valor em Risco (VaR) com um modelo GARCH para cada marginal. A performance das estimativas propostas é investigada por meio de experimentos de simulação de Monte Carlo. Em aplicações utilizando dados reais, os testes fora-da-amostra indicaram que a nova metodologia pode ser melhor que o modelo de cópula global quando se estima Valor em Risco.

Palavras-chave: cópulas; estimação por máxima verossimilhança local; valor em risco.
1. Introduction

In this paper we use local maximum likelihood estimation to assess temporal trends in copula based risk measures. We select the well known risk measure Value-at-Risk (VaR) to illustrate this new methodology.\(^1\)

When estimating the VaR of a portfolio, it is desirable to take into account the dependence structure among its components. In this case a multivariate distribution must be proposed and dynamically estimated. This multivariate distribution should be able to take care of the linear and non-linear forms of dependence existing among the assets, as well as it should allow for different marginal distributions. These requirements are not fulfilled by the multivariate normal distribution. Even though, the most commonly used technique for VaR estimation is based on the simplifying normal assumption.

All the pitfalls in the multivariate normality assumption are well documented in the literature. The use of univariate time-varying volatility models and copulas can solve part of these pitfalls. Nevertheless, in order to capture all dynamics found in the multivariate distribution of financial returns, the fit of marginal time-varying volatility is not enough, somehow it is also necessary the modeling of the temporal changes in the dependence structure. This can be accomplished using multivariate GARCH models (Engle (2000) and Tse and Tsui (2002)), or time-varying correlation coefficient (Cherubini et al., 2004), or time-varying copulas (Dias and Embrechts (2004), Mendes (2005) and Van Den Goorbergh et al. (2005)). One should note that besides the VaR estimation, the time-varying estimation of the multivariate distribution is the basis for many important financial applications, for example, portfolio selection, option pricing and asset pricing models.

Copulas are now very popular in finance (see Georges et al. (2001), Embrechts et al. (2003), Cherubini et al. (2004), Fermanian and Scaillet (2004), among others). Applications using dynamic copulas were proposed more recently. The extension of the standard definition of copula to the conditional case appeared first in a discussion paper of A. Patton in 2001, available at www.econ.ucsd.edu and published as Patton (2006). In this paper he models exchange rates assuming GARCH models for the margins and the Gaussian and the Joe-Clayton copulas for the dependence structure. The time variation the copula parameters is specified according to an auto-regressive equation similar to the GARCH model for conditional volatilities. The estimation is via maximum likelihood in two stages.

\(^1\)Loosely speaking, the VaR of a portfolio is the value large enough to cover its losses over a N-day holding period with a probability of \((1 - \alpha)\). The bottom line in the computation of the VaR is the estimation of the \((1 - \alpha)\)-quantile of the distribution \(F\) of the portfolio. Since \(F\) is unknown, it reduces to the estimation of \(F\). The historical or empirical VaR, for example, is computed using the empirical distribution function as an estimate of \(F\).
Fermanian and Scaillet (2004) introduced the concept of pseudo-copulas. They showed that the copula models defined in Patton (2006) and Rockinger and Jondeau (2001) are all pseudo-copulas. They proposed a nonparametric estimator of the conditional pseudo-copulas, derived its normal asymptotic distribution, and built up a goodness of fit test statistics.

Cherubini et al. (2004), Chapter 5, following ideas in Patton (2006), fitted GARCH(1,1) models to the margins and modeled the dependence structure with a time-varying correlation coefficient and a Gaussian copula. Dias and Embrechts (2004) applied univariate GARCH models and the t-copula model with time varying correlation to high frequency data. Mendes (2005) extended Rockinger and Jondeau (2001) model and assumed a parametric pseudo-copula conditional to the position of lag 1 past joint and lag 2 past joint observations in the unit square, combined with marginal FIGARCH estimation. Van Den Goorbergh et al. (2005) allowed the Kendall’s correlation coefficient to evolve through time according to current values of the conditional marginal variances. The basic idea was to model a stylized fact that the strength of dependence in stock markets usually rises with volatility. Except for Fermanian and Scaillet (2004), all the above cited works assume some parametric form for the copula parameter, which is then estimated by maximum likelihood.

In this paper we propose the use of a local maximum likelihood estimation procedure to capture the dynamics of the copula parameters in a non-parametric way. Our main objective is to assess how this different modeling strategy carries over the computation of risk measures. We show we are able to detect temporal trends in the parameters of the copula linking the assets composing the portfolio.

As in Patton (2006) estimation is carried on in two steps. In the first stage, maximum likelihood estimates of GARCH type models are obtained for the margins. In the second step we estimate the copula parameters and compare with two alternative models: The bivariate GARCH model (Gaussian and t-student) and a constant copula model.

With this aim, in Section 2 we provide a brief review of copula definitions and classical estimation. In Section 3, we review the definition and properties of local maximum likelihood estimators. We define the local maximum likelihood estimates for copula models, we argue that they are asymptotically normally distributed, and analytically find their asymptotic variance in the case of Gaussian copulas. In Section 4, we perform simulation experiments to investigate whether or not the local estimates are able to capture linear trends. In this Section, we also provide an illustrative example how the proposed method may be used in the computation of the conditional VaR. In Section 5, we apply the methodology to 10 equally weighted portfolios composed of international indexes. Having found temporal trends in the copula parameters, this information is incorporated in a parametric model which may then be used for predictions and to compute the conditional-local-estimated VaR. We also perform out-of-sample tests (Kupiec test and the loss function index) to evaluate the performance of the proposed estimation.
tion procedure. We compare with the results from the constant copula fit, and a bivariate GARCH fit. Finally, in Section 6, we provide some concluding remarks.

2. Copulas and Classical Estimation

For simplicity, we consider bivariate copulas, although the inference method proposed is intended and work for higher dimensions. Let $(X_1, X_2)$ be a continuous random variable (rv) in $\mathbb{R}^2$ with joint distribution function (cdf) $F$ and marginal cdfs $F_i$, $i = 1, 2$. Consider the probability integral transformation of $X_1$ and $X_2$ into uniformly distributed random variables (rvs) on $[0, 1]$ (denoted uniform$(0, 1)$), that is, $(U_1, U_2) = (F_1(X_1), F_2(X_2))$. The copula $C$ pertaining to $F$ is the joint cdf of $(U_1, U_2)$. Copulas are invariant under strictly increasing transformations of the marginal variables and play an important role when constructing multivariate distributions with given marginals.

For continuous random variables (Sklar’s theorem, Sklar 1959), there exists a unique 2-dimensional copula $C$ such that for all $(x_1, x_2) \in [-\infty, \infty]^2$:

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

(1)

To measure monotonic dependence, one may use the copula based Kendall’s $\tau$ correlation coefficient, given by

$$\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2)dC(u_1, u_2) - 1$$

Unlike the Pearson linear correlation coefficient $\rho$, Kendall’s $\tau$ does not depend upon the marginal distributions.

In order to measure upper tail dependence one may use the upper tail dependence coefficient defined as

$$\lambda_U = \lim_{\alpha \to 0^+} \lambda_U(\alpha) = \lim_{\alpha \to 0^+} Pr(X_1 > F_1^{-1}(1 - \alpha) \mid X_2 > F_2^{-1}(1 - \alpha))$$

provided the limit $\lambda_U \in [0, 1]$ exists, and where $F_i^{-1}$ is the generalized inverse, i.e., $F_i^{-1}(u_i) = \sup\{x_i \mid F_i(x_i) \leq u_i\}$, for $i = 1, 2$. The lower tail dependence coefficient $\lambda_L$ is defined in a similar way. Both the upper and the lower tail dependence coefficients may be expressed using the pertaining copula:

$$\lambda_U = \lim_{u \to 1^-} \frac{C(u, u)}{1 - u}$$
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where \( C(u_1, u_2) = Pr(U_1 > u_1, U_2 > u_2) \) and \( \lambda_L = \lim_{u \to 0+} \frac{C(u,u)}{u} \) if these limits exist. If \( \lambda_U = 0 (\lambda_L = 0) \), the two variables \( X_1 \) and \( X_2 \) are said to be asymptotically independent in the upper (lower) tail. Whenever \( \lambda_U \in (0, 1) \) (or \( \lambda_L \in (0, 1) \)), these measures quantify the amount of extremal dependence within the class of asymptotically dependent distributions.

Given a set of \( N \) independent and identically distributed (iid) bivariate observations from a bivariate distribution with copula \( C_\theta \), \( \theta \in \Theta \), both margins and copula parameters may be estimated using the IFM method (Joe, 1997). Usually, two versions of the IFM method are considered: the fully parametric and the semiparametric, see Genest and Rivest (1993), Shi and Louis (1995) and Chebrian et al. (2002). The fully parametric approach starts with the estimation of parametric marginal distributions. The transformed uniform \((0,1)\) data are used to maximize the copula likelihood function with respect to \( \theta \). The final results are very sensitive to the correct specification of all margins, but, as shown by Genest et al. (1995), and Shi and Louis (1995), the resulting estimator is consistent and asymptotically normally distributed. In the semiparametric method, the standardized data in the first step are obtained through their empirical cdfs. Although being preferred by many authors to avoid problems on the specification of the marginal cdfs (Frahm et al., 2004), this estimation procedure suffers from loss of efficiency, see Genest and Rivest (1993).

The behavior of the maximum likelihood estimators of copula parameters were investigated through simulations by Capéraà et al. (1997) in the case of the Gumbel or logistic model, by Genest (1987) in the case of the Frank family, and by Mendes et al. (2007) in the case of the Joe, Clayton, Galambos and Husler-Reiss copulas. Genest (1987) investigated the performance of four estimators considering samples of size 10 to 50, and found that, in the case of small samples, the method of moments estimator appears to have smaller mean squared error than the maximum likelihood estimator. In the next section, we propose to estimate copula parameters using the local maximum likelihood estimates.

3. Local Maximum Likelihood Estimation

In this section, we introduce the local maximum likelihood estimates for copula models. These estimates maximize a weighted log likelihood, where weights are provided by a kernel function (see Tibshirani and Hastie (1987)). Assuming that the copula parameter \( \theta \) is locally constant over time, the local maximum likelihood estimator \( \hat{\theta}(t) \) at time \( t \) satisfies

\[
\hat{\theta}(t) = \arg\max_{\theta} \sum_{i=1}^{N} K(t - T_i, h)\log(c(u_i, v_i | \theta(t)))
\]  

where \( T_i \) is a point process indicating the time of occurrence of \((u_i, v_i)\), and \( c(u_i, v_i | \theta(t)) \) is the copula density function given \( T_i = t \), and where \( \theta = \theta(t) \) is a smooth function of \( t \). The kernel function \( K_h(t, h) = K(t - T_i, h) \) is a posi-
tive symmetric function whose variability is governed by the temporal bandwidth $h > 0$. The kernel function $K$ applied to the time separation $t - T_i$ determines the weights. As such, at different times, the estimates can be viewed as a generalization of the idea of a weighted average. At the end of the series, where it is not possible to use a symmetric Kernel function, it is used a right truncated Kernel function.

In this paper, we take $K$ to be the a normal density function with zero mean and standard deviation $h$. In practice, the choice of $h$ is based on the combination of preliminary experimentation, expertise in the field of application, and on the visual assessment of the bias-variance trade-off, which can be done by experimenting several choices for $h$. The final estimates should have small (local) bias and still capture the dynamics in the dependence structure.

One appealing characteristic of this method is its simplicity: it can be applied to any continuous copula family and it easily extends to the vector parameter case, $\theta \in \mathbb{R}^k$ with $k > 1$.

The uncertainty in the local likelihood estimates can be quantified through bootstrap samples (see Davison and Ramesh (2000)). According to Davison and Hinkley (1997), under weak assumptions, if the number of bootstrap samples is sufficiently large, then the empirical variability within the bootstrapped estimates shall give a good approximation for the uncertainty of the local likelihood estimator. Nevertheless, due to the presence of bias in the estimates, the construction of bootstrap confidence intervals about parameter estimates becomes complicated. Deciccio and Romano (1988) proposed methods for constructing confidence intervals that account for bias in the estimates. Bowman and Azzalini (1997) call the attention to the fact that confidence intervals may be constructed without correcting for bias, although, in this case, the variability bands cover $E(\hat{\theta}(t))$ rather than $\hat{\theta}(t)$. The variability of the local estimates for copulas may be also assessed by means of their asymptotic distribution.

Hall and Tajvidi (2000), in the context of univariate series, derive the asymptotic Normal distribution of the local maximum likelihood estimates. By considering a stationary process $\{(W_i, Y_i, S_i), i \geq 1\}$ for describing the copula data $\{(U_i, V_i, T_i), i \geq 1\}$, where $S_i$ a point process analogous to the previously defined process $T_i$, it is straightforward to extend the results of Hall and Tajvidi (2000) to the copula case. In order to clarify details for the interested reader, we show in Appendix A how the Hall and Tajvidi (2000) theorem could be extended to copula models.

The formula for the variance in the case of the Gaussian copula may be obtained as an application of this extension. In this case, the dependence parameter $\theta$ is the linear correlation coefficient $\rho$, and the copula density $c(u, v | \rho)$ is given by:

\[
c(u, v | \rho) = \frac{1}{\sqrt{1 - \rho^2}} e^{\exp \left( \frac{\zeta_1^2 + \zeta_2^2}{2} + \frac{2 \rho \zeta_1 \zeta_2 - \zeta_1^2 - \zeta_2^2}{2(1 - \rho^2)} \right)}
\]
where $\zeta_1 = \Phi^{-1}(u)$ and $\zeta_2 = \Phi^{-1}(v)$. The variance of the local maximum
likelihood estimate is given by

$$
\left[ \kappa \rho^4 - 4\rho^2 - 1 \right]^{-1} \left[ \frac{1}{\rho^2 - 1}(1 - \rho^2) \right]^{-1}
$$

where $\kappa = 1/(2h\sqrt{\pi})$.

Details are given in Appendix A.

It is important to note that the variance of the estimators for all copula families
can also be assessed through the use of bootstrap sampling techniques. It is a simple
procedure, but can be time consuming or computationally intensive, depending
on the length of the time series.

4. Simulations

In subsection 4.1 we present eight simulation experiments carried out to assess
the performance of the local maximum likelihood estimates. We assume that the
copula dependence parameter follows the dynamic behavior shown by the corre-
lation coefficient in Figure 4 (first row). This figure shows the evolution through
time of the Gaussian copula dependence parameter fitted to two financial returns.
We clearly note three distinct periods. In the first and the last quarters there is a
positive slope. In the middle part we observe small variation around a constant.
The copula families considered are: Normal, Gumbel, Galambos, Husler Reiss,
Clayton (Kimeldorf Sampson), Joe, Frank and BB7 (see formulas in Appendix B).
Some of these copulas allow for tail dependence. All experiments were imple-
mented in the S-Plus statistical package.

In subsection 4.2, we show an illustrative example where the VaR of a portfolio
is evaluated based on both a globally and a locally estimated Gaussian copula
model.

4.1 Local estimates of piecewise linear trends

Suppose the random vector $(U, V)$ follows the general copula model

$$(U_t, V_t) \sim C_{\theta(t)}(\cdot, \cdot), \text{ where: } \theta(t) = \theta_0 + \beta t \text{ and } t = 1, \ldots, T$$

where $\beta$ can vary for each sub-period of time. As anticipated, we investigate the
performance of the local estimates when detecting trends in the data. In each ex-
periment, the range for the copula parameter $\theta_t$ was set in such a way that the
Kendall’s tau ($\tau$) varies from 0.00 to 0.50 in 500 days with 3 distinct periods of
linear trend. In the first quarter $\tau$ varies from 0.00 to 0.25. In the second and third
quarters, $\tau$ remains constant. Finally, in the last quarter, $\tau$ varies from 0.25 to 0.50.
Therefore, in
Experiment 1: For the Gaussian copula, in the first quarter $\theta_t \in \{0.0000, 0.0031, 0.0062, \ldots, 0.3827\}$. In the second and third quarters: $\theta_t = 0.3827$. In the final quarter: $\theta_t \in \{0.3827, 0.3853, 0.3879, \ldots, 0.7070\}$.

Experiment 2: For the Clayton copula, in the first quarter $\theta_t \in \{0.0000, 0.0054, 0.0107, \ldots, 0.6660\}$. In the second and third quarters: $\theta_t = 0.6660$. In the final quarter: $\theta_t \in \{0.6660, 0.6768, 0.6875, \ldots, 2.0000\}$.

Experiment 3: For the BB7 copula, the $\delta$ parameter was kept constant and equal to 1.1. Then, in the first quarter $\theta_t \in \{0.0000, 0.0031, 0.0062, \ldots, 0.3884\}$. In the second and third quarters: $\theta_t = 0.3884$. In the final quarter: $\theta_t \in \{0.3884, 0.3933, \ldots, 1.0000\}$. In this case, the dynamic of the lower tail dependence coefficient varies from 0.00 to 0.50 in 500 days.

In addition, we considered the Gumbel, Galambos, the Husler-Reiss, and the Joe and Frank copula families, totalizing 8 experiments. For each simulation it is obtained a path $(\hat{\theta}_1, \ldots, \hat{\theta}_{500})$ of the local maximum likelihood estimates. The number of simulations is 1000 for each copula family. At each time $t$, $t = 1, 2, \ldots, 500$, we compute the average absolute bias, and the standard error, over the 1000 local estimates. We then average these figures over the 500 time points and report in Table 1 these results.

Table 1

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Copula</th>
<th>Average Bias</th>
<th>Average Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Normal</td>
<td>0.0496</td>
<td>0.0661</td>
</tr>
<tr>
<td>2</td>
<td>Clayton</td>
<td>0.1396</td>
<td>0.1794</td>
</tr>
<tr>
<td>3</td>
<td>BB7</td>
<td>0.0935</td>
<td>0.1223</td>
</tr>
<tr>
<td>4</td>
<td>Gumbel</td>
<td>0.0693</td>
<td>0.0914</td>
</tr>
<tr>
<td>5</td>
<td>Galambos</td>
<td>0.1886</td>
<td>0.2241</td>
</tr>
<tr>
<td>6</td>
<td>Husler-Reiss</td>
<td>0.1047</td>
<td>0.1385</td>
</tr>
<tr>
<td>7</td>
<td>Joe</td>
<td>0.1218</td>
<td>0.1560</td>
</tr>
<tr>
<td>8</td>
<td>Frank</td>
<td>0.3095</td>
<td>0.4011</td>
</tr>
</tbody>
</table>

From the simulations experiments, we observe that for all copulas families the local estimation with the chosen bandwidth ($h = 25$) could correctly infer the linear trend imputed to the parameters according to equation (3). The estimated models are very close to the true ones, as confirmed by Figure 1.
4.2 Local estimates of Value-at-Risk

We now report the results from an illustrative example of the use of the local estimation method for computing the in-sample VaR of a hypothetical portfolio. With this aim, we used one of the simulated series of Experiment 1 in the previous subsection, where the parameter \( \theta \) varies linearly on the interval \([0.00, 0.3827]\) in 125 days (\( \beta = 0.0031 \) in equation (3)), remains constant at 0.3827 during 250 days, and again varies linearly on the interval \([0.3827, 0.7070]\) in 125 days (\( \beta = 0.0026 \) in equation (3)). For this bivariate series we estimate the global and the local in-sample VaR(1\%) measures of a short position of an equally weighted portfolio. The marginal distributions are chosen to be the t-Student with 6 and 4 degrees of freedom. The straight line in Figure 2 is the VaR(1\%) evaluated using the global maximum likelihood estimation procedure. Once, we computed in-sample VaR, this risk measure remains constant for the entire path when the global estimation procedure is used.

Figure 1
Plot of the true \( \rho_t \) for the Gaussian model (solid line) and the average of the local \( \hat{\rho}_t \) estimates (dashed line)
Figure 2
Plot of the true $\rho_t$ for the Gaussian model (solid line) and the average of the local $\hat{\rho}_t$ estimates (dashed line)

Plot of the VaR(1%) measures over time, based on local and global (straight line) maximum likelihood estimation for a short position in an equally weighted portfolio of two assets.

Through the above illustrative example, it is clear the difference between the globally and locally estimated VaRs in the case where the dependence strength changes over time.

5. Out-of-sample VaR

In this section we analyze a sample of 3130 daily log-returns from five stock market indexes from January 3rd, 1994, to December 30th, 2005. The indexes are: (i) S&P500 (US), (ii) FTSE (UK), (iii) IBOVESPA (BR), (iv) MERVAL (AR) and (v) IPC (MX). We compute the one-step-ahead (out-of-sample) VaR for the ten pairs of indexes. The results indicate that temporal changes in the copula parameters carry over the Value-at-Risk. Moreover, statistical tests indicate that better estimates may be obtained whenever a temporal trend is detected and incorporated in the $k$-steps-ahead VaR forecasts, $k \geq 1$. As in Patton (2006) and following a suggestion of a referee, we compare the new conditional-local-estimated-copula VaR estimates to the conditional-global-estimated-copula based VaR, and those based on the well known bivariate GARCH model.

To assess the performance of our methodology we separate the data in an estimation sample and a validation sample containing the last 250 observations. The validation sample is used to compare, via out-of-sample back-testing, the performances of VaR estimates.
For the copula-based VaR computation we first fit by maximum likelihood ARMA-GARCH(1,1) models to the log-returns series in the estimation sample. The choice of the conditional error distribution for each margin was based on the AIC criterion and resulted in: (i) S&P500, t-student ($\nu_{SP500} = 8$), (ii) FTSE, t-student ($\nu_{FTSE} = 13$), (iii) IBOVESPA, t-student ($\nu_{IBOVESP A} = 7$), (iv) MERVAL, t-student ($\nu_{MERVAL} = 5$) and (v) IPC, t-student ($\nu_{IPC} = 5$), where $\nu$ represents the number of degrees of freedom. These distributions are used to obtain the pseudo-uniform $(0, 1)$ values for copulas fitting. For example, Figure 3 shows at the left hand side the scatter plot of the log-returns from IBOVESPA and IPC, and at the right hand side the standardized data.

![Figure 3](image)

At the left hand side the scatter plot of the log-returns from IBOVESPA and IPC, and at the right hand side the standardized uniform $(0, 1)$ data.

To choose the best copula model for each pair we fit the Gaussian and the BB7 copulas to the standardized data. Another elliptical copula, the $t$-copula could have been fitted to the data. However, for the application that follows, which focus on the comparisons between the constant and the dynamic VaR models, the Gaussian copula does the job at a much lower computational cost. In addition, to capture lower and upper tail dependence we have used the BB7 copula. The BB7 copula has an advantage over the $t$-copula: it is more flexible since it allows for asymmetric behavior during crisis, as it can handle different upper and lower strength of dependence at extreme levels. The choice of the best copula model for each pair was based on the AIC criterion. For example, for the pair IBOVESPA-IPC, the best global fit was provided by the BB7 copula. The estimates were $\hat{\theta} = 1.27$ and $\hat{\delta} = 0.54$. The estimated path of the local constant estimates $\hat{\theta}_t$ and $\hat{\delta}_t$ is shown in Figure 4 (second row). We can observe trends indicating that the strength of dependence changes over time, calling for dynamic models.

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Let \( \hat{\theta} \) represent the Gaussian and BB7 varying copula parameters. In order to estimate the VaR for the remaining 250 time periods (for \( t = 2881, \ldots, 3130 \)), we implemented the following algorithm:

For The Copula Models:

(i) Add the observation at time \( t \) to the estimation sample.

(ii) Fit ARMA-GARCH\((1,1)\) models to each margin. Transform the residuals obtaining the uniform\((0, 1)\) data.

(iii) Compute the global and the local maximum likelihood estimates of the copula parameter.

(iv) Computing the one-step-ahead local VaR: using the most recent 42 local estimates (approx. 2 commercial months) compute the least squares estimate of the trend slope. Predict the one-step-ahead copula parameter value based on the local estimate plus trend. Simulate 1000 observations from the predicted copula (Gaussian or BB7).

(v) Computing the one-step-ahead global VaR: Assume as the one-step-ahead copula parameter value the last global estimate. Simulate 1000 observations from the predicted copula (Gaussian or BB7).

(vi) For both bivariate simulated data obtained in (iv) and (v) apply the inverse cumulative distribution function to the margins, according to the t-student conditional distributions used in (ii).

(viii) For each bivariate data form an equally weighted “portfolio”, and compute their long position VaR as the \( \alpha \)% quantile.

For The Bivariate GARCH Model:

(i) Add the observation at time \( t \) to the estimation sample.

(ii) Fit ARMA models to each margin, and obtain the residuals.

(iii) Fit by maximum likelihood a bivariate GARCH\((1,1)\) model to the residuals.

(iv) Computing the one-step-ahead VaR: Predict the one-step-ahead conditional correlation and variances. Simulate 1000 observations from the predicted model.

(v) Apply the inverse cumulative distribution function to the margins, according to the conditional distribution used in (ii).

(viii) Form an equally weighted “portfolio”, and compute their long position VaR as the \( \alpha \)% quantile.
For all ten portfolios, the three series of 250 VaR forecasts are then compared with the observed values in a out-of-sample test.

One should note that the dynamics in the VaR series come from the conditional heteroscedastic model as well as from the local copula model. With the aim to verify the accuracy of the risk measures, we apply the Kupiec LR test and the loss function index on the empirical failure rates. The Kupiec test (Kupiec, 1995) is based on binomial theory and tests the difference between the observed and expected number of VaR exceptions of the effective portfolio losses.

It is known that the VaR measure is based on a $1 - \alpha$ confidence level, so when we observe $N$ losses in excess of VaR out of $T$ observations, it is experienced a $N/T$ proportion of excessive losses. Considering the null hypothesis $H_0 : \alpha = 1\% \ (5\%)$, the Kupiec’s test says if $N/T$ is statistically significant to reject or not $H_0$. In other words, it helps to evaluate if the VaR is inadequate for that level of confidence. Following binomial distribution, the probability of observing $N$ failures out of $T$ observations is $\binom{T}{N}(1 - \alpha)^{T-N}\alpha^N$, so that the test of the null hypothesis is given by a likelihood ratio test statistic (details in Kupiec, 1995) distributed as $\chi^2_1$ under $H_0$. It is well known that the power of this test, that is the ability to reject a bad model, rises with $T$. In this paper, we are working with 250 observations. The realized VaR exceptions and the Kupiec test p-values are reported in Table 2.
Table 2
VaR back-testing analysis (globally, locally estimation of Copulas and bi-variate GARCH model). RE – Realized exceptions in % and (p-value) of the Kupiec test

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>α = 5%</th>
<th>α = 1%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Global RE% (p-value)</td>
<td>Local RE% (p-value)</td>
</tr>
<tr>
<td>IBOVESPA and IPC</td>
<td>3.6(0.29)</td>
<td>4.0(0.45)</td>
</tr>
<tr>
<td>IBOVESPA and Merval</td>
<td>6.0(0.48)</td>
<td>5.2(0.89)</td>
</tr>
<tr>
<td>IBOVESPA and S&amp;P500</td>
<td>4.8(0.88)</td>
<td>4.8(0.88)</td>
</tr>
<tr>
<td>IBOVESPA and FTSE</td>
<td>4.4(0.67)</td>
<td>5.6(0.67)</td>
</tr>
<tr>
<td>IPC and S&amp;P500</td>
<td>4.0(0.45)</td>
<td>5.2(0.89)</td>
</tr>
<tr>
<td>IPC and FTSE</td>
<td>3.6(0.29)</td>
<td>4.8(0.88)</td>
</tr>
<tr>
<td>IPC and Merval</td>
<td>5.2(0.89)</td>
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</tr>
<tr>
<td>Merval and S&amp;P500</td>
<td>6.0(0.48)</td>
<td>4.4(0.67)</td>
</tr>
<tr>
<td>Merval and FTSE</td>
<td>3.6(0.29)</td>
<td>3.6(0.29)</td>
</tr>
<tr>
<td>FTSE and S&amp;P500</td>
<td>2.8(0.08)</td>
<td>2.8(0.08)</td>
</tr>
</tbody>
</table>

The null hypothesis was rejected only for the pair IPC - S&P500 when the copula parameter was globally estimated for a confidence level of \( \alpha = 1\% \). In summary, the results from the Kupiec’s test of the local-estimation-VaR are quite similar to the ones presented by the global-estimation-VaR. However, when they are not equal, the local-estimation-VaR presents a better performance. Since the numbers of exceptions are small as well as the results of the Kupiec’s test are almost equal, we also compared the risk measures based in different estimation methods by means of a loss function index.

This procedure was first introduced by Lopez (1998). The general form of these loss functions is:

\[
B_t = \begin{cases} 
  f(L_t, VaR_t) & \text{if } L_t < VaR_t \\
  g(L_t, VaR_t) & \text{if } L_t \geq VaR_t
\end{cases}
\]

where \( f(x, y) \) and \( g(x, y) \) are arbitrary functions such that \( f(x, y) \geq g(x, y) \) for a given \( y \). The “score” of the complete sample is

\[
B = \sum_{t=1}^{250} B_t
\]

The more accurate the VaR estimates, the lower score \( B \). In this paper, we considered the loss function suggested by Blanco and Ihle (1999).
Local Estimation of Copula Based Value-at-Risk

\[ B_t = \begin{cases} 
\frac{L_t - \text{VaR}_t}{\text{VaR}_t} & \text{if } L_t < \text{VaR}_t \\
0 & \text{if } L_t \geq \text{VaR}_t
\end{cases} \]

where \( L_t \) is the return of the portfolio for the day \( t \). The results for this index are reported in Table 3. Based in the loss function considered above, it can be observed the better performance of the local-estimated VaRs compared to the global-estimated ones.

Table 3
Performance of the VaR for \( \alpha = 5\% (\alpha = 1\%) \) computed according to the global and local estimation of copulas and the bi-variate GARCH. Total score of the loss function index \( B \)

<table>
<thead>
<tr>
<th></th>
<th>Global</th>
<th>Local</th>
<th>Bivariate GARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>IBOVESPA and IPC</td>
<td>2.25(0.30)</td>
<td>1.58(0.40)</td>
<td>1.44(0.47)</td>
</tr>
<tr>
<td>IBOVESPA and Merval</td>
<td>3.63(0.18)</td>
<td>2.07(0.15)</td>
<td>1.53(0.00)</td>
</tr>
<tr>
<td>IBOVESPA and S&amp;P500</td>
<td>2.25(1.40)</td>
<td>2.15(0.95)</td>
<td>2.47(0.65)</td>
</tr>
<tr>
<td>IBOVESPA and FTSE</td>
<td>1.65(0.40)</td>
<td>1.65(0.47)</td>
<td>1.61(0.56)</td>
</tr>
<tr>
<td>IPC and S&amp;P500</td>
<td>3.03(0.00)</td>
<td>2.77(0.12)</td>
<td>2.80(0.11)</td>
</tr>
<tr>
<td>IPC and FTSE</td>
<td>1.76(0.25)</td>
<td>1.20(0.14)</td>
<td>1.35(0.23)</td>
</tr>
<tr>
<td>IPC and Merval</td>
<td>4.63(1.00)</td>
<td>3.93(0.60)</td>
<td>3.54(0.55)</td>
</tr>
<tr>
<td>Merval and S&amp;P500</td>
<td>5.20(2.32)</td>
<td>4.23(1.90)</td>
<td>4.47(0.93)</td>
</tr>
<tr>
<td>Merval and FTSE</td>
<td>3.08(1.90)</td>
<td>2.69(1.80)</td>
<td>2.60(1.71)</td>
</tr>
<tr>
<td>FTSE and S&amp;P500</td>
<td>0.94(0.46)</td>
<td>0.82(0.45)</td>
<td>0.86(0.40)</td>
</tr>
</tbody>
</table>

6. Concluding Remarks

In this paper, we apply the local maximum likelihood method to estimate copula parameters. The local estimates are used to detect temporal changes in the strength of dependence among assets. These dynamics are then incorporated in the estimation of the Value-at-Risk (VaR). Our modelling strategy allows for the margins to follow some GARCH type model while the copula dependence structure changes over time and is estimated locally. The performance of the proposed estimates is investigated through Monte Carlo simulation experiments and applications using real data. Our exposition was restricted to the bivariate case, but models can be easily implemented and run relatively fast in higher dimensions.

There are many open questions on time varying copulas, and there is room for both theoretical and computational developments. Many applications will naturally follow. In this paper we provided empirical evidence that the dependence structure among asset returns (given by its copula) may be best dealt by the use of local estimation methods in copula models.

Through simulation experiments and out-of-sample back tests, such as the Kupiec test and a loss function index, we showed based in real portfolios that the Value-at-Risk estimation may be improved by using a local estimation procedure. These findings can be used by investors selecting portfolio components or by fund managers improving the accuracy of their risk measures.
References


Appendix: Extension of Hall and Tajvidi (2000) Result to Copula Models

For simplicity, suppose $\theta \in \mathbb{R}$. The generalization to the $k$-dimensional parameter space is straightforward. Consider a stationary process $\{(W_i, Y_i, S_i), i \geq 1\}$ for describing the data $\{(U_i, V_i, T_i), i \geq 1\}$, where $S_i$ a point process analogous to the previously defined process $T_i$. Given an integer $\varphi \geq 1$, and a compact interval $\mathcal{I}$ which we take without loss of generality to have unit length, let $\{(U_i, V_i, T_i), 1 \leq i \leq N\}$ denote those values of $(W_j, Y_j, S_j)$ for which $1 \leq j \leq \varphi$ and $S_j \in \mathcal{I}$. Let $(U, V, T)$ and $(W, Y, S)$ be generic values of the processes $(U_i, V_i, T_i)$ and $(W_i, Y_i, S_i)$, respectively.

Let

$$g(\cdot, \cdot | \theta) = \log c(\cdot, \cdot | \theta)$$

where $c(\cdot, \cdot | \theta)$ is the copula density of $(W, Y)$ conditional on $S = t, t \in \mathcal{I}$ (equivalently, of $(U, V)$ conditional on $T = t$). Also define

$$\theta'(t) = d\theta(t)/dt$$

$$\theta''(t) = d^2\theta(t)/dt^2$$

and

$$g'(\cdot, \cdot | \theta) = \frac{\partial g(\cdot, \cdot | \theta)}{\partial \theta}$$

and

$$g''(\cdot, \cdot | \theta) = \frac{\partial^2 g(\cdot, \cdot | \theta)}{\partial \theta^2}$$

Following the standard steps in the derivation of asymptotic distribution of maximum likelihood estimates, we denote by $V, V = V(t)$, the reciprocal of $E[g'(((U, V)|\theta(T)) |T = t]$. Since $E[g'(((U, V)|\theta(T)) |T = t] = -E[g''((U, V)|\theta(T)) |T = t]$, we may write

$$V(t) = \frac{1}{E[g'(((U, V)|\theta(T)) |T = t]} = \frac{1}{-E[g''((U, V)|\theta(T)) |T = t]}$$

(A.1)

We put $\kappa_2 = \int x^2 K(x)dx, \kappa = \int K^2(x)dx$ and assume the following conditions: (a) $p \equiv P(S_i \in \mathcal{I}) > 0$, the distribution of $S_i$, conditional on $S_i \in \mathcal{I}$, is absolutely continuous with density $\xi$, say; (b) $\xi$ has a continuous derivative in a neighborhood of $t$ and satisfies $\xi(t) > 0$; (c) for all values of $\theta$ in a neighborhood of $\theta(t)$, $c(\cdot, \cdot | \theta)$ satisfies the regularity conditions of Lehmann (1983), which are sufficient for the Cramer-Rao lower bound to be attained in the same neighborhood; (d) $\theta(\cdot)$ has two continuous derivatives in a neighborhood of $t$; (e) $K$ is a
symmetric compactly supported probability density and (f) \( h = h(\varphi) \rightarrow 0 \) and \( \varphi \rightarrow \infty \) in such a manner that \( \varphi h \rightarrow \infty \). Then \( \varphi \lambda \), where \( \lambda \equiv p \xi \), equals the intensity of the point process \( \{ T_1, \ldots, T_\lambda \} \) on \( I \).

Assuming that conditions (a)-(f) hold, if \( t \) is an interior point of \( I \), suppressing the argument \( t \) of \( \lambda, \theta, \theta', \theta'' \) and \( V \) it follows that

\[
\hat{\theta} = \theta + \kappa h^2 \left[ \frac{1}{2} \theta'' + \frac{\lambda \theta'}{\lambda} \right] + (\varphi \lambda h)^{-1/2} Z + o_p[h^2 + (\varphi h)^{-1/2}] \tag{A.2}
\]

where \( Z \) is asymptotically normal, \( N(0, \kappa V) \).

As noted by Hall and Tajvidi (2000), the terms of size \( h^2 \) on the right-hand side of A.2 represent the dominant contributions to systematic error or bias. The terms of size \( (\varphi h)^{-1/2} \) denote the dominant contributions to stochastic error, or error about the mean. When \( h = \varphi^{-1/2} \), the systematic and stochastic error terms are of identical orders.\(^3\) In this case, an optimal size of bandwidth is obtained. The variance of \( Z \) is the maximum-information variance associated with the Cramer-Rao lower bound.

To derive the asymptotic variance of the local maximum likelihood estimates of the linear correlation coefficient \( \rho \) in the case of a Gaussian copula, we first note that \( \rho(t) \) satisfies the log-likelihood equation:

\[
\sum_{i=1}^{N} g'(U_i, V_i | \hat{\rho}) K_i(h) = 0
\]

By the mean value theorem,

\[
\sum_{i=1}^{N} g(t(U_i, V_i | \hat{\rho}) K_i(h)) - \sum_{i=1}^{N} g(t(U_i, V_i | \rho) K_i(h)) = \left[ \sum_{i=1}^{N} g''(U_i, V_i | \rho^*) K_i(h) \right] (\hat{\rho} - \rho)
\]

where \( \rho^* \) is a point between \( \hat{\rho} \) and \( \rho \). Using the log-likelihood equation we obtain

\[
\sqrt{N} (\hat{\rho} - \rho) = -\frac{N^{-1/2} \sum_{i=1}^{N} g'(U_i, V_i | \rho) K_i(h)}{N^{-1} \sum_{i=1}^{N} g''(U_i, V_i | \rho^*) K_i(h)} \tag{A.3}
\]

Let \( R_N = N^{-1} \sum_{i=1}^{N} [g''(U_i, V_i | \rho^*) K_i(h) - E[g''(U_i, V_i | \rho) K_i(h)]]/E[g''(U, V | \rho) K(h)] \), and note that \( E[g''(U, V | \rho) K(h)](1 - R_N) \) equals to

\[
E[g''(U, V | \rho) K(h)] - N^{-1} \sum_{i=1}^{N} [g''(U_i, V_i | \rho^*) K_i(h) - E[g''(U_i, V_i | \rho) K_i(h)]]
\]

\(^3\)If \( h = \varphi^{-1/2} \), then \( (\varphi h)^{-1/2} = \varphi^{-2/5} = h^2 \).
Then we can rewrite A.3 as

\[
\sqrt{N} \left( \hat{\rho} - \rho \right) = - \frac{N^{-1/2} \sum_{i=1}^{N} g'(U_i, V_i | \rho) K_i(h)}{E[g''(U, V | \rho)K(h)][1 - R_N]} \quad (A.4)
\]

Under the assumed regularity conditions and for large \( N \), \( N^{-1} \sum_{i=1}^{N} [g''(U_i, V_i | \rho) K_i(h)] \) behaves like \( N^{-1} \sum_{i=1}^{N} [g''(U_i, V_i | \rho) K_i(h)] \) and by the law of large numbers \( N^{-1} \sum_{i=1}^{N} [g''(U_i, V_i | \rho) K_i(h)] \longrightarrow E[g''(U, V | \rho)K(h)] \). Then \( R_N \) converges in probability to zero. Moreover, the Fisher information number based on a pair of \( rvs \) is given by

\[
I_1(\rho) = E[g'(U, V | \rho)K(h)]^2 = -E[g''(U, V | \rho)K(h)] \quad (A.5)
\]

and, as usually, it holds that

\[
E[g'(U, V | \rho)K(h)] = 0
\]

Applying the Central Limit Theorem and Slutsky’s Theorem (Bickel and Doksum, 1977) to the right hand side of (A.4), we conclude that \( \sqrt{N} (\hat{\rho} - \rho) \) tends to a normal random variable with variance given by \( 1/I_N(\rho) \).

In order to find the expression for the variance we need to find the expected values in equation (A.5).

\[
E \left[ g'(U, V | \rho)K(h) \right]^2 = E \left\{ \left[ \frac{\rho(1 - \rho^2)^2 + \zeta_1 \zeta_2 (1 + \rho^2) - \rho (\zeta_1^2 + \zeta_2^2)}{(1 - \rho^2)^2} \right] K(h) \right\}^2
\]

\[
= \kappa \frac{\rho^4 - 4\rho^2 - 1}{(\rho^2 - 1)(1 - \rho^2)}
\]

where \( \kappa = 1/(2h\sqrt{\pi}) \).
B. Appendix: Copula Models

Gaussian Copula. It is the copula pertaining to the multivariate normal distribution, a member of the large class of elliptical copulas. It is given by

\[ C^G_{\theta}(u, v) = \int_{-\infty}^{\phi^{-1}(u)} \int_{-\infty}^{\phi^{-1}(v)} \frac{1}{2\pi(1-\theta^2)^{1/2}} \exp \left( -\frac{s^2 - 2\theta st + t^2}{2(1-\theta^2)} \right) dsdt \]

where \( \theta \) is the linear correlation coefficient between the two random variables.

Gumbel copula: This is an Extreme Value copula as well as an Archimedean copula. It has the following form:

\[ C^G_{\theta}(u, v) = \exp \left\{ \left( -\log u \right)^\theta + \left( -\log v \right)^\theta \right\} \]

The coefficient of tail dependence is given by \( \lambda_U = 2 - 2^{1/\theta} \).

Galambos copula: It is an Extreme Value copula:

\[ C^G_{\text{Gal}}(u, v) = uv \exp \left[ \left( -\log u \right)^{-\theta} + \left( -\log v \right)^{-\theta} \right] \]

for \( \theta \geq 0 \). It has upper tail dependence given by \( \lambda_U = 2 - 2^{1/\theta} \).

Joe copula: The Joe copula is an Archimedean copula and has the form:

\[ C^J_{\theta}(u, v) = 1 - \left[ (1 - u)^\theta + (1 - v)^\theta - (1 - u)^\theta (1 - v)^\theta \right]^{1/\theta} \]

where \( \theta \geq 1 \). The upper tail dependence coefficient is given by \( \lambda_U = 2 - 2^{1/\theta} \).

Husler Reiss copula: The Husler and Reiss copula is an Extreme Value copula given by:

\[ C^HR_{\theta}(u, v) = \exp \left( -\hat{u}\Phi \left[ 1 + \frac{1}{2} \theta \log \left( \frac{u}{v} \right) \right] + \hat{v}\Phi \left[ 1 + \frac{1}{2} \theta \log \left( \frac{u}{v} \right) \right] \right) \]

where \( \hat{u} = -\log u, \hat{v} = -\log v, \theta \geq 0 \) and \( \Phi \) is the cdf of a standard normal. It has upper tail dependence is given by \( \lambda_U = 2 - 2\Phi(1/\theta) \).

Clayton copula. This is an Archimedean copula given by:

\[ C^C_{\theta}(u, v) = \left[ u^{-\theta} + v^{-\theta} - 1 \right]^{-1/\theta} \]

where \( \theta > 0 \). The tail dependence coefficient is given by \( \lambda_L = 2^{-1/\theta} \). The Kimeldorf and Sampson copula (KS) is another name of the Clayton copula.
**BB7 copula:** BB7 copula (Joe, 1997), an Archimedean copula, has the form of:

\[
C_{\theta,\delta}^{BB7}(u, v) = 1 - \left(1 - \left([1 - (1-u)^\delta]^{-\theta} + [1 - (1-v)^\delta]^{-\theta} - 1\right)^{-1/\theta}\right)^\delta \tag{B.1}
\]

where \(\theta \geq 0\) and \(\delta \geq 1\).

**Frank copula:** It is an Archimedean copula given by:

\[
C_{\theta}^{Fr}(u, v) = -\theta^{-1} \log \left(\frac{\eta - (1 - e^{-\theta u})(1 - e^{-\theta v})}{\eta}\right)
\]

where \(\theta > 0\) and \(\eta = 1 - e^{-\theta}\). For the Frank copula, \(\lambda_L = \lambda_U = 0\).