A Multi-Period Mean-Variance Portfolio Selection Problem

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Abstract
In a recent paper, Li and Ng (2000) considered the multi-period mean variance optimization problem, with investing horizon $T$, for the case in which only the final variance $\text{Var}(V(T))$ or expected value of the portfolio $E(V(T))$ are considered in the optimization problem. In this paper we extend their results to the case in which the intermediate expected values $E(V(t))$ and variances $\text{Var}(V(t))$ for $t = 1, \ldots, T$ can also be taken into account in the optimization problem. The main advantage of this technique is that it is possible to control the intermediate behavior of the portfolio’s return or variance. An example illustrating this situation is presented.

1. Introduction
Mean-Variance portfolio selection is a classical financial problem introduced by Markowitz (1959) in which it is desired to reduce risks by diversifying assets allocation. The main goal is to maximize the expected return for a given level of risk or minimize the expected risk for a given level of expected return. Optimal portfolio selection is the most used and well known tool for economic allocation of capital (see Campbell et al. (1997), Elton and Gruber (1995), Jorion (1992), Steinbach (2001)). More recently it has been extended to include tracking error optimization (see Roll (1992), Rudolf et al. (1999)) and semivariance models.
Other objective functions can also be considered (Zenios, 1993), and a unifying approach to these methodologies can be found in Duarte Jr. (1999).

All previous papers consider single period optimization problems. But typically portfolio strategies are multi-period, since the investor can re-balance his position from time to time. Recently there have been a continuing effort in extending portfolio selection from the single period to the multi-period case. Using a stochastic linear quadratic theory developed in Chen et al. (1998), the continuous-time version of the Markowitz’s problem was studied in Zhou and Li (2000), with closed-form efficient policies derived, along with an explicit expression of the efficient frontier. In Zhou and Yin (2003) and Yin and Zhou (2004) the authors treated the continuous-time and discrete-time versions of the Markowitz’s mean-variance portfolio selection with regime switching, and derived the efficient portfolio and efficient frontier explicitly.

In Li and Ng (2000) the authors extended the mean-variance allocation problem for the discrete-time multi-period case, with final time $T$. As usual in these problems, the authors only considered the variance and expected value of the portfolio at the final time $T$. This could lead to too aggressive strategies since the intermediate variances and expected values are not taken into account in the performance criterion or constrains. The main goal of this paper is to generalize the results of Li and Ng (2000) for the case in which the intermediate variances and expected values of the portfolio are also considered in the performance criterion or constrains. First we consider a problem in which the performance criterion can be written as a linear combination of the expected values and variances for $t = 1, \ldots, T$. A solution for this problem is derived in Theorem 2, based on backward equations. From this solution, numerical procedures for a multi-period problem with performance criterion written as a linear combination of the mean values of the portfolio and restrictions on the variances, and performance criterion written as a linear combination of the variances of the portfolio and restrictions on the mean value, are presented. The proposed techniques are based on solving a set of equations so that, if a solution exists, then an optimal solution for the problem is derived. The main relevance of the technique presented in this paper is that it is possible to have a better control of the intermediate values of the variance and expected values of the portfolio, avoiding it to reach undesirable high and low values respectively. We consider only the case in which all the assets are risky. The case with one riskless asset can be easily deduced by considering one of the assets with null variance.

This paper follows the same approach and notation as in Li and Ng (2000). It is organized in the following way. The notation, basic results, and problem formulation that will be considered throughout the work are presented in Section 2. Section 3 presents the solution of an auxiliary problem. The main problems are analyzed and solved in Section 4. In Section 5 an example comparing the technique introduced here with the one in Li and Ng (2000) is presented. The
paper in concluded in Section 6 with some final comments.

2. Preliminaries

On a probabilistic space \((\Omega, \mathcal{F}, \mathbb{P})\) we shall consider a financial model in which there are \(n + 1\) risky assets represented by the random return vector \(\bar{S}(t) \in \mathbb{R}^{n+1}\). We denote by \(\mathcal{F}_t\) the \(\sigma\)-field generated by the random vectors \(\{\bar{S}(s); s = 0, \ldots, t\}\). It will be convenient to write

\[
\bar{S}(t) = \begin{pmatrix} S_0(t) \\ S(t) \end{pmatrix}, \quad S(t) = \begin{pmatrix} S_1(t) \\ \vdots \\ S_n(t) \end{pmatrix}
\]

A portfolio \(\bar{H}(t-1)\) will be a vector belonging to \(\mathbb{R}^{n+1}\) such that it is \(\mathcal{F}_{t-1}\)-measurable, \(t = 1, 2, \ldots, T\). We write

\[
\bar{S}(t) = \begin{pmatrix} H_0(t-1) \\ H(t-1) \end{pmatrix}, \quad H(t-1) = \begin{pmatrix} H_1(t-1) \\ \vdots \\ H_n(t-1) \end{pmatrix}
\]

and we have that \(H_i(t-1)\) represents the amount of asset \(i\) in the portfolio at time \(t\). Notice that \(\bar{H}(t-1)\) is chosen at the beginning of time \(t\), and thus depends of \(\bar{S}(s)\) from \(s = 0\) up to the closing values at time \(s = t-1\). Let \(V(t)\) represent the value of the portfolio at the end of time \(t\). It follows that

\[
V(t) = \bar{H}(t-1)' \bar{S}(t). \tag{1}
\]

To have a self-financing portfolio it must keep the same value at the beginning of period \(t + 1\), when the portfolio is re-balanced, and thus

\[
V(t) = \bar{H}(t)' \bar{S}(t). \tag{2}
\]

Let us define

\[
\mathcal{R}_i(t) = \frac{S_i(t+1)}{S_i(t)}, \quad \bar{\mathcal{R}}(t) = \begin{pmatrix} \mathcal{R}_0(t) \\ \mathcal{R}_1(t) \\ \vdots \\ \mathcal{R}_n(t) \end{pmatrix}, \quad \mathcal{R}(t) = \begin{pmatrix} \mathcal{R}_1(t) \\ \vdots \\ \mathcal{R}_n(t) \end{pmatrix}. \tag{3}
\]

Notice that \(\mathcal{R}_i(t)\) is \(\mathcal{F}_{t+1}\)-measurable. We have that \(\mathcal{R}(t)\) can be written as

\[
\bar{\mathcal{R}}(t) = \bar{\eta}(t) + \bar{Z}(t),
\]

where \(\bar{Z}(t)\) are null mean vectors and \(\bar{\eta}(t) \in \mathbb{R}^{n+1}\) represents the mean value of \(\mathcal{R}(t)\). We write
\[ \bar{Z}(t) = \begin{pmatrix} Z_0(t) \\ Z(t) \end{pmatrix}, \quad Z(t) = \begin{pmatrix} Z_1(t) \\ \vdots \\ Z_n(t) \end{pmatrix}, \]
\[ \bar{\eta}(t) = \begin{pmatrix} \eta_0(t) \\ \eta(t) \end{pmatrix}, \quad \eta(t) = \begin{pmatrix} \eta_1(t) \\ \vdots \\ \eta_n(t) \end{pmatrix}, \]
and make the following hypothesis:

**Hypothesis 1**: \( \{ \bar{Z}(t); t = 0, \ldots, T - 1 \} \) are independent random vectors.

Since \( \bar{S}(t) \) is a function of \( \{ \bar{Z}(s); s = 0, \ldots, t - 1 \} \), we have from Hypothesis 2 that \( \bar{Z}(t) \), and thus \( \bar{R}(t) \), is independent from the sigma field \( \mathcal{F}_t \).

**Hypothesis 2**: \( E(\bar{R}(t)\bar{R}(t)') > 0 \) for each \( t = 0, \ldots, T - 1 \).

Define now

\[ U_i(t) = H_i(t)S_i(t), \quad U(t) = \begin{pmatrix} U_1(t) \\ \vdots \\ U_n(t) \end{pmatrix}. \]

It follows that

\[ V(t) = U_0(t) + e'U(t) \]

where \( e \) is the vector formed by one in all components, and

\[ V(t + 1) = H_0(t)S_0(t + 1) + H(t)'S(t + 1) \]
\[ = H_0(t)S_0(t)R_0(t) + \sum_{i=1}^n H_i(t)S_i(t)R_i(t) \]
\[ = R_0(t)U_0(t) + \sum_{i=1}^n R_i(t)U_i(t) \]
\[ = R_0(t)V(t) + (R(t) - R_0(t)e)'U(t). \] (4)

Defining

\[ P(t) = R(t) - R_0(t)e = (e - I)R(t). \]

it follows from Hypothesis 2 that
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and thus 

\[
E(\mathcal{P}(t)\mathcal{P}(t)') > 0
\]

and thus \(E(\mathcal{P}(t)\mathcal{P}(t)') > 0\). Define

\[
\phi(t) = E(\mathcal{P}(t)\mathcal{P}(t)')
\]

\[
\varphi_0(t) = E(\mathcal{R}_0(t)^2)
\]

\[
\varphi(t) = E(\mathcal{R}_0(t)\mathcal{P}(t))
\]

\[
\chi(t) = E(\mathcal{P}(t)) = \eta(t) - \eta_0(t)e.
\]

From the Schur’s complement,

\[
\varphi^2(t) - \varphi(t)'\phi(t)^{-1}\varphi(t) > 0.
\]

Define

\[
A_2(t) = \varphi_0^2(t) - \varphi(t)'\phi(t)^{-1}\varphi(t) > 0
\]

\[
A_1(t) = \eta_0(t) - (\eta(t) - \eta_0(t)e)'\phi(t)^{-1}\varphi(t)
\]

\[
B(t) = (\eta(t) - \eta_0(t)e)'\phi(t)^{-1}(\eta(t) - \eta_0(t)e).
\]

Consider a set of positive numbers \(\alpha(t)\) for \(t = 1, \ldots, T\), \(\nu(t)\) for \(t \in I_\nu = \{\vartheta_1, \ldots, \vartheta_{c_\nu}\}, \vartheta_{c_\nu} \leq T\), and \(\sigma^2(t)\) for \(t \in I_{\sigma} = \{\zeta_1, \ldots, \zeta_{c_\sigma}\}, \zeta_{c_\sigma} \leq T\). The investor, with an initial wealth \(v_0\), looks for the best investing strategy \((\mathcal{U}_0(t), \mathcal{U}(t))\) \(t = 0, \ldots, T - 1\) such that:

1. The sum of the expected value of the portfolio \(E(V(t))\), weighted by the value \(\alpha(t)\) at time \(t\), is maximized, subject to the constraints that the variance of the portfolio \(\text{Var}(V(t))\) is less than or equal to a pre-specified upper bound \(\sigma^2(t)\) at each time \(t \in I_{\sigma}\).

2. The sum of the variance of the portfolio \(\text{Var}(V(t))\), weighted by the value \(\alpha(t) \geq 0\) at time \(t\), is minimized, subject to the constraints that the expected value of the portfolio \(E(V(t))\) is greater than or equal to a pre-specified lower bound \(\nu(t)\) at each time \(t \in I_{\nu}\).

There is no loss of generality in assuming that \(\alpha(T) > 0\), so that the final period is always considered in the optimization problem (if not so, we could redefine \(T\) so that it would coincide with the largest weight \(\alpha(t)\) strictly greater than zero). For simplicity we shall denote

\[
\sigma = \begin{pmatrix} \sigma(\zeta_1) \\ \vdots \\ \sigma(\zeta_{c_\sigma}) \end{pmatrix}, \nu = \begin{pmatrix} \nu(\vartheta_1) \\ \vdots \\ \nu(\vartheta_{c_\nu}) \end{pmatrix}, \alpha = \begin{pmatrix} \alpha(1) \\ \vdots \\ \alpha(T) \end{pmatrix}.
\]
We can mathematically formalize the above problems as follows:

**Problem $PE(\sigma)$:**

\[
\max \sum_{t=1}^{T} \alpha(t)E(V(t))
\]

subject to

\[
\begin{align*}
    \text{Var}(V(t)) &\leq \sigma^2(t), t \in I_{\sigma} \\
    V(t) &= R_0(t-1)V(t-1) + \mathcal{P}(t-1)'U(t-1) \\
    v_0 &= U_0(0) + e'U(0) \\
    t &= 1, \ldots, T.
\end{align*}
\]

**Problem $PV(\nu)$:**

\[
\min \sum_{t=1}^{T} \alpha(t)\text{Var}(V(t))
\]

subject to

\[
\begin{align*}
    E(V(t)) &\geq \nu(t), t \in I_{\nu} \\
    V(t) &= R_0(t-1)V(t-1) + \mathcal{P}(t-1)'U(t-1) \\
    v_0 &= U_0(0) + e'U(0) \\
    t &= 1, \ldots, T.
\end{align*}
\]

An alternative problem that will be considered is as follows. For a sequence of positive numbers $\ell(t)$ and $\rho(t)$, $t = 1, \ldots, T$, set

\[
\ell = \left( \begin{array}{c}
\ell(1) \\
\vdots \\
\ell(T)
\end{array} \right), \quad \rho = \left( \begin{array}{c}
\rho(1) \\
\vdots \\
\rho(T)
\end{array} \right).
\]

**Problem $PMV(\ell, \rho)$:**

\[
\max \sum_{t=1}^{T} \alpha(t) (\ell(t)E(V(t)) - \rho(t)\text{Var}(V(t)))
\]

subject to
\[ V(t) = R_0(t-1)V(t-1) + \mathcal{P}(t-1)U(t-1) \]
\[ v_0 = U_0(0) + e^tU(0) \]
\[ t = 1, \ldots, T \]

where \( \rho(t) \) represents the investor’s risk aversion coefficient, that is, the larger \( \rho(t) \) is, the bigger the investor’s risk aversion at time \( t \) is. The case in which \( \rho(t) = 0 \) means that the investor doesn’t care at all with the risk at time \( t \). Similar comments hold for \( \ell \).

To avoid unfeasible problems to \( PV(\nu) \) and \( PE(\sigma) \), we make the following assumptions:

**Hypothesis 3** For \( t \in I_\nu \),
\[ \prod_{s=1}^{t} \eta_{0}(s) \geq \frac{\nu(t)}{v_0}. \]

**Hypothesis 4** For \( t \in I_\sigma \),
\[ \left( \prod_{s=0}^{t-1} \rho_{0}^{2}(s) - \prod_{s=0}^{t-1} \eta_{0}^{2}(s) \right) \leq \frac{\sigma^{2}(t)}{v_0^2}. \]

From these hypothesis the following propositions are easily shown. For the proof, just make \( U(t) = 0 \) for all \( t \), so that 100% of the resources are applied in the risky asset 0, and verify that from hypothesis 3 and 4 the expected value and variance of the resulting portfolio will satisfy the restrictions of Problems \( PV(\nu) \) and \( PE(\sigma) \).

**Proposition 1** Under hypothesis 3, Problem \( PV(\nu) \) is always feasible.

**Proposition 2** Under hypothesis 4, Problem \( PE(\sigma) \) is always feasible.

3. **Solution of an Auxiliary Problem**

Define
\[ \lambda = \begin{pmatrix} \lambda(1) \\ \vdots \\ \lambda(T) \end{pmatrix}, \xi = \begin{pmatrix} \xi(1) \\ \vdots \\ \xi(T) \end{pmatrix}. \]

In order to solve the problems \( PE(\sigma), PV(\nu) \) and \( PMV(t, \rho) \), we shall consider the following auxiliary problem:
Problem **AUX (λ, ξ):**

\[
\min \sum_{t=1}^{T} \alpha(t) \left( E \left( \xi(t)V(t)^2 - \lambda(t)V(t) \right) \right)
\]

subject to

\[
\begin{align*}
V(t) &= R_0(t-1)V(t-1) + P(t-1)U(t-1) \\
v_0 &= U_0(0) + e'U(0) \\
t &= 1, \ldots, T
\end{align*}
\]

where \( \xi(t) \geq 0 \) for each \( t = 1, \ldots, T \). It is convenient to set \( \xi(0) = 0 \) and \( \alpha(0) = 0 \). We make the following backward recursive definitions:

\[
\begin{align*}
p(t) &= \alpha(t)\xi(t) + A_2(t)p(t+1) \\
q(t) &= -\alpha(t)\lambda(t) + A_1(t)q(t+1) \\
w(t) &= -\frac{q(t+1)^2}{4p(t+1)}B(t) + w(t+1)
\end{align*}
\]

for \( t = T-1, \ldots, 0 \), with

\[
\begin{align*}
p(T) &= \alpha(T)\xi(T) \\
q(T) &= -\alpha(T)\lambda(T) \\
w(T) &= 0.
\end{align*}
\]

Notice that \( A_2(t) > 0 \) and \( \alpha(t)\xi(t) \geq 0 \) for all \( t = 0, \ldots, T \). Since by hypothesis \( \alpha(T)\xi(T) > 0 \) we have that \( p(t) > 0 \) for all \( t = 0, \ldots, T \). Let us apply dynamic programming to solve problem **AUX (λ, ξ)**. Let us define the intermediate problems.

Problem **AUX (λ, ξ, τ, v_τ):**

\[
\min \sum_{t=\tau}^{T} \alpha(t) \left( E \left( \xi(t)V(t)^2 - \lambda(t)V(t) \right) \mid \mathcal{F}_\tau \right)
\]

subject to

\[
\begin{align*}
V(t) &= R_0(t-1)V(t-1) + P(t-1)U(t-1) \\
v_\tau &= U_0(\tau) + e'U(\tau) \\
t &= \tau, \ldots, T
\end{align*}
\]

and let \( J(\tau, v_\tau) \) be the value function of this problem.
The value function is given by
\[ J(\tau, v_{\tau}) = p(\tau)v_{\tau}^2 + q(\tau)v_{\tau} + w(\tau) \quad (5) \]
and the optimal strategy for \( t = \tau, \ldots, T - 1 \) is given by
\[ U(t) = -\phi(t)^{-1}\varphi(t)V(t) - \frac{q(t + 1)}{2p(t + 1)}\phi(t)^{-1}\chi(t). \quad (6) \]

**Proof** Let us show the result by induction on \( t \). For \( \tau = T \) we have that
\[ J(T, v_T) = \alpha(T)\xi(T)v_T^2 - \alpha(T)\lambda(T)v_T + 0 = p(T)v_T + q(T)v_T + w(T) \]
proving (7) for \( \tau = T \). Suppose (7) holds for \( \tau + 1 \). Then
\[ J(\tau, v_{\tau}) = \alpha(\tau)\xi(\tau)v_{\tau}^2 - \alpha(\tau)\lambda(\tau)v_{\tau} + \min_{u_{\tau}} \{ E(J(\tau + 1, V(\tau + 1))|F_{\tau}) \} \]
where
\[ V(\tau + 1) = R_0(\tau)v_{\tau} + P(\tau)'u_{\tau}. \]

We have that
\[ E(J(\tau + 1, V(\tau + 1))|F_{\tau}) = p(\tau + 1)E((R_0(\tau)v_{\tau} + P(\tau)'u_{\tau})^2|F_{\tau}) + q(\tau + 1)E((R_0(\tau)v_{\tau} + P(\tau)'u_{\tau})|F_{\tau}) + w(\tau + 1). \]

Notice now that
\[ E\left((R_0(\tau)v_{\tau} + P(\tau)'u_{\tau})^2 \mid F_{\tau}\right) = \varphi_0^2(\tau)v_{\tau}^2 + 2\varphi(\tau)u_{\tau}v_{\tau} + u_{\tau}'\phi(\tau)u_{\tau} \]
and
\[ E(R_0(\tau)v_{\tau} + P(\tau)'u_{\tau}|F_{\tau}) = \eta_0(\tau)v_{\tau} + \chi(\tau)u_{\tau}. \]

Thus
\[ J(\tau, v_{\tau}) = (\alpha(\tau)\xi(\tau) + p(\tau + 1)\varphi_0^2(\tau))v_{\tau}^2 + (q(\tau + 1)\eta_0(\tau) - \alpha(\tau)\lambda(\tau))v_{\tau} + w(\tau + 1) + \min_{u_{\tau}} f(u_{\tau}) \]
where the function $f(u_r)$ is given by

$$f(u_r) = 2p(\tau + 1)\varphi(\tau)'u_r + p(\tau + 1)u_r'\phi(\tau)u_r + q(\tau + 1)\chi(\tau)'u_r. \quad (7)$$

Taking the derivative and making equal to zero we obtain that the optimal strategy $u_r^*$ is given by

$$u_r^* = -\phi(\tau)^{-1}\varphi(\tau)v_r - \frac{q(\tau + 1)}{2p(\tau + 1)}\phi(\tau)^{-1}\chi(\tau)$$

which coincides with the optimal strategy given in (6). Replacing this in (7) leads to

$$f(u_r^*) = (2p(\tau + 1)\varphi(\tau)'v_r + q(\tau + 1)\chi(\tau)'u_r + p(\tau + 1)u_r'\phi(\tau)u_r$$

$$+ p(\tau + 1)\left(\varphi(\tau)v_r + \frac{q(\tau + 1)}{2p(\tau + 1)}\chi(\tau)\right) = 2q(\tau + 1)\varphi(\tau)'\phi(\tau)^{-1}\chi(\tau)\tau v_r$$

$$- \frac{q(\tau + 1)^2}{2p(\tau + 1)}\chi(\tau)'\phi(\tau)^{-1}\chi(\tau) + p(\tau + 1)\varphi(\tau)'\phi(\tau)^{-1}\varphi(\tau)v_r^2$$

$$+ q(\tau + 1)\varphi(\tau)'\phi(\tau)^{-1}\chi(\tau)v_r + \frac{q(\tau + 1)^2}{4p(\tau + 1)}\chi(\tau)'\phi(\tau)^{-1}\chi(\tau)$$

$$= -p(\tau + 1)\varphi(\tau)'\phi(\tau)^{-1}\varphi(\tau)v_r^2 - q(\tau + 1)\varphi(\tau)'\phi(\tau)^{-1}\chi(\tau)v_r$$

$$- \frac{q(\tau + 1)^2}{4p(\tau + 1)}\chi(\tau)'\phi(\tau)^{-1}\chi(\tau).$$

Thus,

$$J(\tau, v_r) = (\alpha(\tau)\xi(\tau) + p(\tau + 1)\left(\varphi_0^2(\tau) - \varphi(\tau)'\phi(\tau)^{-1}\varphi(\tau)\right)\tau v_r^2$$

$$+ (q(\tau + 1)\left(\eta(\tau) - \varphi(\tau)'\phi(\tau)^{-1}\chi(\tau)\right) - \alpha(\tau)\lambda(\tau)\right)\tau v_r$$

$$+ w(\tau + 1) - \frac{q(\tau + 1)^2}{4p(\tau + 1)}\chi(\tau)'\phi(\tau)^{-1}\chi(\tau)$$

$$= (\alpha(\tau)\xi(\tau) + A_2(\tau)p(\tau + 1))\tau v_r^2 + (A_1(\tau)q(\tau + 1) - \alpha(\tau)\lambda(\tau))\tau v_r + w(\tau)$$

$$v_r + w(\tau + 1) - \frac{q(\tau + 1)^2}{4p(\tau + 1)}\xi(\tau) = p(\tau)\tau v_r^2 + q(\tau)v_r + w(\tau)$$

proving the desired result.
4. Solution to the Problems

We represent the set of optimal solutions for the problems $AUX(\lambda, \xi)$, $PMV(\ell, \rho)$, $PE(\sigma)$ and $PV(\nu)$ by $\Pi(AUX(\lambda, \xi))$, $\Pi(PMV(\ell, \rho))$, $\Pi(PE(\sigma))$ and $\Pi(PV(\nu))$ respectively. We denote by $\{V_t(t)\}_{t=0}^T$ the value of the portfolio when we use an investing strategy $U = \{U(0), \ldots, U(T-1)\}$.

**Proposition 3** We have that

$$\Pi(PMV(\ell, \rho)) \subset \bigcup_{\lambda \in \mathbb{R}} \Pi(AUX(\lambda, \rho)).$$

**Proof** We shall show that if $U \in \Pi(PMV(\ell, \rho))$ then $U \in \Pi(AUX(\lambda, \rho))$ with

$$\lambda(t) = \ell(t) + 2\rho(t)E(V_{\lambda t}(t))$$

for $t = 1, \ldots, T$. Suppose by contradiction that $U \notin \Pi(AUX(\lambda, \rho))$, so that for some $U^* \in \Pi(AUX(\lambda, \rho))$,

$$\sum_{t=1}^T \alpha(t) \left( E(\rho(t)V_{U^*t}(t)^2 - \lambda(t)V_{U^*t}(t)) \right) < \sum_{t=1}^T \alpha(t) \left( E(\rho(t)V_{Ut}(t)^2 - \lambda(t)V_{Ut}(t)) \right)$$

and thus

$$\sum_{t=1}^T \alpha(t)(\rho(t)E(V_{U^*t}(t)^2 - V_{Ut}(t)^2) - \lambda(t)E(V_{U^*t}(t) - V_{Ut}(t))) < 0. \quad (9)$$

Recall now that for a function of the form

$$f(x, y) = ax - ay^2 - by$$

where $a \geq 0$, we have for $\bar{x}, \bar{y}$ fixed, that for all $x, y$

$$f(x, y) \leq f(\bar{x}, \bar{y}) + \nabla f(\bar{x}, \bar{y})' \left( x - \bar{x} \right)$$

where

$$\nabla f(\bar{x}, \bar{y}) = \left( a - \frac{2a\bar{y} + b}{2\bar{y}} \right).$$

Setting $a = \rho(t)$, $b = \ell(t)$, $\bar{x} = E(V_{U^*t}(t)^2)$, $\bar{y} = E(V_{Ut}(t))$ and $x = E(V_{U^*t}(t)^2)$, $y = E(V_{Ut}(t))$, so that
\[ f(x, y) = \rho(t)(E(V_{t\ell,\ell}\tau(t)^2) - E(V_{t\ell,\ell}\tau(t)^2)) - \ell(t)E(V_{t\ell,\ell}\tau(t)) \]
\[ = \rho(t)\text{Var}(V_{t\ell,\ell}\tau(t)) - \ell(t)E(V_{t\ell,\ell}\tau(t)) \]

and
\[ \nabla f(\bar{x}, \bar{y}) = \left( -2\rho(t)E(V_{t\ell,\ell}\tau(t)) + \ell(t) \right) = \left( \rho(t) - \lambda(t) \right) \]
we conclude from (10) that
\[ \rho(t)\text{Var}(V_{t\ell,\ell}\tau(t)) - \ell(t)E(V_{t\ell,\ell}\tau(t)) \leq \rho(t)\text{Var}(V_{t\ell,\ell}\tau(t)) - \ell(t)E(V_{t\ell,\ell}\tau(t)) \]
\[ + \rho(t) - \lambda(t) \left( E(V_{t\ell,\ell}\tau(t)^2) - E(V_{t\ell,\ell}\tau(t)^2) \right). \]

From (9)
\[ \sum_{t=1}^{T} \alpha(t) \left( \rho(t) - \lambda(t)E(V_{t\ell,\ell}\tau(t)^2) - E(V_{t\ell,\ell}\tau(t)^2)E(V_{t\ell,\ell})(t) - E(V_{t\ell,\ell})(t) \right) \]
\[ = \sum_{t=1}^{T} \alpha(t)(\rho(t)E(V_{t\ell,\ell}\tau(t)^2) - V_{t\ell,\ell}\tau(t)^2) - \lambda(t)E(V_{t\ell,\ell}\tau(t) - V_{t\ell,\ell})(t)) < 0 \]

and from (11)
\[ \sum_{t=1}^{T} \alpha(t)(\rho(t)\text{Var}(V_{t\ell,\ell}\tau(t)) - \ell(t)E(V_{t\ell,\ell}\tau(t))) \]
\[ \leq \sum_{t=1}^{T} \alpha(t)(\rho(t)\text{Var}(V_{t\ell,\ell}\tau(t)) - \ell(t)E(V_{t\ell,\ell}\tau(t))) + \]
\[ \sum_{t=1}^{T} \alpha(t) \left( \rho(t) - \lambda(t) \right) \left( E(V_{t\ell,\ell}\tau(t)^2) - E(V_{t\ell,\ell}\tau(t)^2) \right) \]
\[ < \sum_{t=1}^{T} \alpha(t)(\rho(t)\text{Var}(V_{t\ell,\ell}\tau(t)) - \ell(t)E(V_{t\ell,\ell}\tau(t))) \]

in contradiction with the fact that \( U \in \Pi(PMV(\ell, \rho)). \)

**Proposition 4** Suppose that \( \ell(t) = \bar{\ell} > 0, \) \( t = 1, \ldots, T \) and \( \rho(t) = 0 \) for \( t \notin I_\sigma. \)
If \( U \in \Pi(PMV(\ell, \rho)) \) with
\[ \sigma(t)^2 = \text{Var}(V_{t\ell,\ell}\tau(t)), \ t \in I_\sigma \]
then \( U \in \Pi(PE(\sigma)) \).

**Proof** Suppose by contradiction that \( U \notin \Pi(PE(\sigma)) \), so that for some \( U^* \in \Pi(PE(\sigma)) \),

\[
\text{Var}(V_{U^*}(t)) \leq \sigma(t)^2 = \text{Var}(V_U(t)), \quad t \in \sigma
\]

and

\[
\sum_{t=1}^{T} \alpha(t) E(V_{U^*}(t)) > \sum_{t=1}^{T} \alpha(t) E(V_U(t)).
\]

Thus

\[
\sum_{t=1}^{T} \alpha(t) \left( \rho(t) \text{Var}(V_{U^*}(t)) - \bar{\ell} E(V_{U^*}(t)) \right) < \sum_{t=1}^{T} \alpha(t) \left( \rho(t) \text{Var}(V_U(t)) - \bar{\ell} E(V_U(t)) \right)
\]

in contradiction with the fact that \( U \in \Pi(PMV(\ell, \rho)) \).

**Proposition 5** Suppose that \( \rho(t) = \bar{\rho} > 0 \) for \( t = 1, \ldots, T \) and \( \ell(t) = 0 \) for \( t \notin I_\nu \).

If \( U \in \Pi(PMV(\ell, \rho)) \) with

\[
\nu(t) = E(V_{U^*}(t)), \quad t \in I_\nu
\]

then \( U \in \Pi(PV(\nu)) \).

**Proof** Suppose by contradiction that \( U \notin \Pi(PV(\nu)) \), so that for some \( U^* \in \Pi(PV(\nu)) \),

\[
E(V_{U^*}(t)) \geq \nu(t) = E(V_U(t)), \quad t \in I_\nu
\]

and

\[
\sum_{t=1}^{T} \alpha(t) \text{Var}(V_{U^*}(t)) < \sum_{t=1}^{T} \alpha(t) \text{Var}(V_U(t)).
\]

Thus
\[
\sum_{t=1}^{T} \alpha(t) \left( \rho \text{Var}(V_{U^*}(t)) - \ell(t) \mathbb{E}(V_{U^*}(t)) \right)
\]
\[
< \sum_{t=1}^{T} \alpha(t) \left( \rho \text{Var}(V_{U}(t)) - \ell(t) \mathbb{E}(V_{U}(t)) \right)
\]
in contradiction with the fact that \( U \in \Pi(PMV(\ell, \rho)) \).

From the optimal strategy (6) we have that
\[
V(t+1) = R_0(t)V(t) + P(t)'U(t)
\]
\[
= R_0(t)V(t) - P(t)' \left( \phi(t)^{-1} \varphi(t)V(t) + \frac{q(t+1)}{2p(t+1)} \phi(t)^{-1} \chi(t) \right)
\]
\[
= (R_0(t) - P(t)' \phi(t)^{-1} \varphi(t)) V(t)
- \frac{q(t+1)}{2p(t+1)} P(t)' \phi(t)^{-1} \chi(t)
\]
and taking the expected value we obtain that
\[
E(V(t+1)) = (\eta_0(t) - \chi(t)' \phi(t)^{-1} \varphi(t)) E(V(t))
- \frac{q(t+1)}{2p(t+1)} \chi(t)' \phi(t)^{-1} \chi(t)
\]
\[
= A_1(t) E(V(t)) - \frac{q(t+1)}{2p(t+1)} B(t).
\]
where
\[
p(t) = \alpha(t) \rho(t) + A_2(t)p(t+1) \quad \text{(12)}
\]
\[
q(t) = -\alpha(t) \lambda(t) + A_1(t)q(t+1)
\]
for \( t = T - 1, \ldots, 0 \), with
\[
p(T) = \alpha(T) \rho(T) \quad \text{(13)}
\]
\[
q(T) = -\alpha(T) \lambda(T).
\]
From equation (8) we have that to obtain \( U \in \Pi(PMV(\ell, \rho)) \) we must have \( U \in \Pi(AUX(\lambda, \rho)) \) with \( \lambda \) such that
\[
\lambda(t) = \ell(t) + 2p(t)E(V_{U^*}(t)), \quad t = 1, \ldots, T. \quad \text{(14)}
\]
Notice from (12), (13) that \( p(t) \) doesn’t depend on \( \lambda(t) \) and can be obtained by recursion from equations (12), (13). In order to obtain \( q(t) \) such that (14) is satisfied, define recursively for \( t = T - 1, \ldots, 0 \)

\[
a(t) = -\alpha(t)\ell(t) + \frac{2p(t + 1)}{2p(t + 1) + b(t + 1)B(t)}A_1(t)a(t + 1)
\]

\[
b(t) = -2\alpha(t)\rho(t) + \frac{2p(t + 1)}{2p(t + 1) + b(t + 1)B(t)}A_1(t)^2b(t + 1)
\]

\[
c(t) = \frac{2p(t + 1)}{2p(t + 1) + b(t + 1)B(t)}A_1(t)
\]

\[
d(t) = -\frac{a(t + 1)}{2p(t + 1) + b(t + 1)B(t)}B(t)
\]

with

\[
a(T) = -\alpha(T)\ell(T)
\]

\[
b(T) = -2\alpha(T)\rho(T).
\]

Define also for \( t = 0, \ldots, T - 1 \)

\[
v(t + 1) = c(t)v(t) + d(t)
\]

with (recall that \( v_0 \) is the initial value of the portfolio)

\[
v(0) = v_0
\]

and

\[
z(t + 1) = A_2(t)z(t) + \frac{(a(t + 1) + b(t + 1)v(t + 1))^2}{4p(t + 1)^2}B(t)
\]

with

\[
z(0) = v_0^2.
\]

**Theorem 2** An optimal solution \( U \in \Pi(\text{PMV}(\ell, \rho)) \) is obtained from

\[
U(t) = -\phi(t)^{-1}\varphi(t)V_U(t) - \frac{q(t + 1)}{2p(t + 1)}\phi(t)^{-1}\chi(t)
\]

with \( p(t), t = T, \ldots, 0 \), given by equations (12), (13), and \( q(t) \) given by

\[
q(t) = a(t) + b(t)v(t)
\]
for \( t = T, \ldots, 0 \). For this strategy the expected value is given by

\[
E(V_U(t)) = v(t)
\]

and the variance is given by

\[
\text{Var}(V_U(t)) = z(t) - v(t)^2.
\]

Moreover \( q(t) \) satisfies

\[
q(t) = -\alpha(t)\lambda(t) + A_1(t)q(t + 1)
\]

with \( q(T) = -\alpha(T)\lambda(T) \) and

\[
\lambda(t) = \ell(t) + 2\rho(t)E(V_U(t)).
\]

**Proof** Let us show by induction on \( t = 0, \ldots, T \) that

\[
v(t) = E(V_U(t)).
\]

For \( t = 0 \) it is true from (16). Suppose it holds for \( t \). Then

\[
E(V_U(t + 1)) = A_1(t)E(V_U(t)) - \frac{q(t + 1)}{2p(t + 1)}B(t)
\]

\[
= A_1(t)v(t) - \frac{q(t + 1)}{2p(t + 1)}B(t).
\]

From (15) it follows that

\[
v(t + 1) = \frac{2p(t + 1)A_1(t)v(t) - a(t + 1)B(t)}{2p(t + 1) + b(t + 1)B(t)}
\]

\[
= 2p(t + 1)A_1(t)v(t) - a(t + 1)B(t)
\]

that is,

\[
v(t + 1) = A_1(t)v(t) - \frac{(a(t + 1) + b(t + 1)v(t + 1))B(t)}{2p(t + 1)}
\]

\[
= A_1(t)v(t) - \frac{q(t + 1)}{2p(t + 1)}B(t)
\]

and from (20) and (21) it follows that \( E(V_U(t + 1)) = v(t + 1) \). Let us show now that

\[
q(t) = -\alpha(t)\lambda(t) + A_1(t)q(t + 1), t = T - 1, \ldots, 0
\]

\[
q(T) = -\alpha(T)\lambda(T)
\]

\[
\lambda(t) = \ell(t) + 2\rho(t)E(V_U(t)).
\]
We have by definition that

\[
q(t) = a(t) + b(t)v(t) = -\alpha(t)(\ell(t) + 2\rho(t)v(t)) + \frac{2p(t+1)}{2p(t+1) + b(t+1)B(t)}A_1(t)(a(t+1) + b(t+1)A_1(t)v(t))
\]

and

\[
A_1(t)v(t) = \frac{2p(t+1) + b(t+1)B(t)}{2p(t+1)}v(t+1) + \frac{a(t+1)B(t)}{2p(t+1)}
\]

Replacing (23) into (22) leads to

\[
q(t) = -\alpha(t)\lambda(t) + A_1(t)(a(t+1) + b(t+1)v(t+1)) = -\alpha(t)\lambda(t) - A_1(t)q(t+1)
\]

yielding to the desired result. Finally notice that after some manipulations, we conclude that

\[
E(V_U(t+1)^2) = A_2(t)E(V_U(t)^2) + \frac{q(t+1)^2}{4p(t+1)^2}B(t)
\]

so that \(z(t) = E(V_U(t)^2)\), completing the proof of the Theorem.

To solve \(PV(\nu)\) we take \(\rho(t) = 1, t = 1, \ldots, T\), and \(\ell(t) = 0\) for \(t \notin I_\nu\) and therefore from Proposition 5 we have to find \(\ell(t) \geq 0\) such that for \(t \in I_\nu = \{\vartheta_1, \ldots, \vartheta_\nu\}, \vartheta_\nu \leq T\)

\[
\nu(t) = E(V_U(t)) = v(t).
\]

We have in this case \(\nu\) variables \(\ell(t)\) and \(t_\nu\) equations that need to be solved simultaneously.

In a similar way, to solve problem \(PE(\sigma)\) we take \(\ell(t) = 1, t = 1, \ldots, T\) and \(\rho(t) = 0\) for \(t \notin I_\sigma\) and therefore from Proposition 4 we have to find \(\rho(t) \geq 0\) such that for \(t \in I_\sigma = \{\zeta_1, \ldots, \zeta_\sigma\}, \zeta_\sigma \leq T\),

\[
\sigma(t)^2 = Var(V_U(t)) = z(t) - v(t)^2.
\]

Again in this case we have \(\tau_\sigma\) variables \(\rho(t)\) and \(\tau_\sigma\) equations that need to be solved simultaneously.

It is important to point out that there is no guarantee that these problems will have a solution. What we have is that if a solution exists, then an optimal solution for the problem \(PE(\sigma)\) or \(PV(\nu)\) is obtained from (19).
5. Example

As mentioned in Section 1, the main advantage of the technique presented in this paper is that it is possible to control the intermediate behavior of the portfolio’s return or variance for this multi-period problem. In this section we present an example of this situation, in which it is desirable to have the intermediate values of the variance bounded by some reference values.

We borrow the date from an example presented in Li and Ng (2000). Let us consider the case in which there are 3 risky assets and finite horizon time $T = 12$.

We assume that

$$E(Z(t)Z(t)') = \begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix},$$

$$\eta(t) = \begin{pmatrix} 1.162 \\ 1.246 \\ 1.228 \end{pmatrix}, \quad t = 0, \ldots, 12.$$

We consider two portfolios. Portfolio 1 is obtained from the multi-period optimization problem $P E(\sigma)$ as presented in Li and Ng (2000), that is, it is desired to maximize the expected value of the portfolio $E(V(12))$, with a restriction on the final variance $\text{Var}(V(12)) \leq 2$. Portfolio 2 is obtained from the multi-period optimization problem $P E(\sigma)$ as presented in Section 2. For the vector $\alpha$, we considered

$$\alpha = \begin{pmatrix} 0.001 \\ 0.0019 \\ 0.0035 \\ 0.0066 \\ 0.0123 \\ 0.0231 \\ 0.0433 \\ 0.0811 \\ 0.152 \\ 0.2848 \\ 0.5337 \\ 1.0 \end{pmatrix}.$$ 

The idea behind this choice is to gradually increase the weights on the expected value of the portfolio $E(V(t))$ in the performance criterion, given more importance to the latest values. For $I_\sigma$, we considered $I_\sigma = \{4, 6, 8, 10, 12\}$, with $\sigma(4) = 0.1382, \sigma(6) = 0.2965, \sigma(8) = 0.4343, \sigma(10) = 0.8517, \sigma(12) = 2.0$. The idea behind this choice is to reduce the value of the variance of the portfolio at the intermediate times $I_\sigma = \{4, 6, 8, 10, 12\}$, at a cost of a lower value for the expected return $E(V(t))$. Figure 1 presents the trajectory of the variances of Port-

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folios 1, 2 (star and solid line, respectively), and the variance of reference (plus signal). We can see from this figure that Portfolio 2 follows the reference as desired. The mean value of the portfolios are presented in figure 2. As expected, the price paid for having a lower variance is that the mean value of Portfolio 1 overcome the mean value of Portfolio 2.

Figure 1
Variance of the portfolios; Portfolio 1 in *, Portfolio 2 in −, and Variance of reference in +

Figure 2
Expected value of the portfolios; Portfolio 1 in *, Portfolio 2 in −
6. Conclusions

In this paper we have considered a multi-period mean variance optimization problem, with investing horizon $T$. First we consider a problem (Problem $PMV(\ell, \rho)$) in which the performance criterion can be written as a linear combination of the expected values $E(V(t))$ and variances $Var(V(T))$ for $t = 1, \ldots, T$. A solution for this problem is derived in Theorem 4, based on backward equations. With this solution, a numerical procedure for a problem with performance criterion written as a linear combination of the expected values $E(V(t))$ and restrictions on the variances $Var(V(t))$ for $t \in I_\sigma$ (Problem $PE(\sigma)$), and performance criterion written as a linear combination of the variances $Var(V(T))$ and restrictions on the expected values $E(V(t))$ for $t \in I_\nu$ (Problem $PV(\nu)$) are presented. To solve $PV(\nu)$ we have to find $\ell(t) \geq 0$ such that for $t \in I_\nu$, $\nu(t) = E(V(t)) = \nu(t)$, which leads to $i_\nu$ variables $\ell(t)$ and $i_\nu$ equations that need to be solved simultaneously. In a similar way, to solve problem $PE(\sigma)$ we have to find $\rho(t) \geq 0$ such that for $t \in I_\sigma$, $\sigma(t)^2 = Var(V(t))$, yielding to $i_\sigma$ variables $\rho(t)$ and $i_\sigma$ equations that need to be solved simultaneously. If a solution exists, then (19) is an optimal solution for the problem $PE(\sigma)$ or $PV(\nu)$.

The main relevance of the technique presented in this paper is that it is possible to control the intermediate behavior of the portfolio’s return or variance for this multi-period problem. An example of this situation is presented in Section 5. In this example we consider two portfolios. The first one is obtained from the multi-period problem as in Li and Ng (2000) with horizon time $T = 12$ for the case in which there is only a final restriction on the value of the variance ($Var(V(12)) \leq 2$). The second portfolio is obtained from Problem $PE(\sigma)$, with intermediate restrictions on $\sigma(t)$ at times 4, 6, 8, 10, 12. The trajectories of the variances of these two portfolios, as well as the reference variance, are presented in figure 1. It can be seen that, as desired, the variance of the second portfolio tracked the desired reference. We believe that this simple example illustrates the possible usefulness of the technique presented in this paper for real problems.

References


