

Recovering Risk-Neutral Densities from Brazilian Interest Rate Options

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Abstract

Building Risk-Neutral Density (RND) from options data is one useful way for extracting market expectations about a financial variable. For a sample of IDI (Brazilian Interbank Deposit Rate Index) options from 1998 to 2009, this paper estimates the option-implied Risk-Neutral Densities for the Brazilian short rate using three methods: Shimko, Mixture of Two Log-Normals and Generalized Beta of Second Kind. Our in-sample goodness-of-fit evaluation shows that the Mixture of Log-Normals method provides better fitting to option's data than the other two methods. The shape of log-normal distributions seems to fit well to the mean-reversal dynamics of Brazilian interest rates. We have also calculated the RND implied Skewness, showing how it could have provided market early-warning signals of the monetary policy outcomes in 2002 and 2003. Overall, Risk-Neutral Densities implied on IDI options showed to be a useful tool for extracting market expectations about future outcomes of the monetary policy.

Keywords: risk-neutral density; interest rate options; generalized beta; mixture of log-normals.

JEL codes: C13; C16; E47; E52; G12; G13; G17.

Resumo

A construção de densidades neutras ao risco a partir de dados de opções é uma forma de extrair as expectativas do mercado de uma variável financeira. Este artigo usa uma amostra de opções de IDI de 1998 até 2009 e estima a densidade neutral ao risco implícita em opções para a taxa de juros de curto prazo brasileira, usando três métodos: Shimko, Mistura de duas Log-Normais e Distribuição Beta generalizada do tipo dois. A avaliação dentro da amostra mostra que a Mistura de Normais possui uma melhor aderência aos dados de opções do que os outros métodos no período de análise. O formato da distribuição log-normal parece aderir bem à dinâmica das taxas de juros brasileiras. Também foi calculada a assimetria implícita nas distribuições neutras ao risco, e foi mostrado como ela poderia ter fornecido sinais dos futuros movimentos das taxas de juros em 2002 e 2003. De uma forma geral, as densidades neutras ao risco mostraram-se uma ferramenta útil para extração de expectativas de mercado sobre futuros desenvolvimentos da política monetária.

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1. Introduction

Many techniques have been applied in order to extract market expectations. Building Risk-Neutral Density (RND) from options data is one of them. This information may be useful for financial stability analysis. Supervisory institutions can assess monetary policy impacts on expectations by inferring whether the market is attributing a high probability of a significant change on financial variables, such as interest rate or exchange rate. On the other way, market expectations on financial variables may influence monetary policy decisions. Using option-implied RND, one can calculate, for example, the probability that interest rate will stay inside a specific range of values.

Other areas of research that require densities forecasts may also benefit from Risk-neutral densities. This is the case of strategic asset allocation and market risk models.

Risk-neutral densities may also be used to price illiquid options, where we cannot find a market price. In this way, we would have a price for the illiquid option that would be in a non-arbitrage condition with the other options.

This article aims to evaluate methods for extracting the Risk-Neutral Densities implied in the main Brazilian interest rate option: the IDI (Interbank Deposit Rate Index) option. As far as we know, this is the first paper to recover RND for Brazilian interest rate options. Applications of RND for the Brazilian markets use basically currency options.¹

The IDI option has some special features that make its pricing different from other fixed income options. In fact, the IDI option is similar to an Asian option on the geometric average of the one-day interbank interest rate (CDI), between the trade date and the expiration of the option. Therefore, option-pricing formulas from traditional models must be adapted to be used with these options.

In order to build RND's, we apply three methods: Shimko (1993) methodology, the fitting of Mixture of two Lognormals, and fitting of a Generalized Beta distribution of second kind (GB2). After the estimation of 1,879 RND's for each method, we have assessed the in-sample goodness-of-fit of each method with option data. The Mixture of Log-Normals provided the best in-sample fitting. We have also calculated the implied Skewness using the Mixture of Log-Normals, and analyzed its behavior in two specific periods of our sample.

The paper is organized as follows: Section 2 revises the Risk Neutral Density literature; Section 3 shows the IDI option main characteristics; on Section 4 our methodology is described; Section 5 gives an overview our dataset; Section 6 presents the results and finally section 7 concludes the paper.

¹Campa *et al.* (1999), Castro (2002) and Chang & Tabak (2007) are some of the papers that have extracted RND's from Brazilian Real exchange rate options. Micu (2005) compares six methods for extracting RND of 12 emerging markets currencies against the US Dollar, including the Brazilian real.

2. Risk-Neutral Density (RND)

Black's (1976) method, assume that the future price's follows a geometric Brownian motion and that the volatility is constant, i.e., options on the same asset should provide the same implied volatility values. However, these are too strong assumptions. In practice, implied volatilities vary along strike prices and expiration dates. One stylized shape of the Strike x Implied Volatility figure is the so-called "smile": options too in-the-money or too out-of-the-money usually results in higher implied volatilities compared to at-the-money options. Depending on the underlying asset, other shapes occur also, such as a "sneer" or "smirk". This constitutes evidence against the Black's method that would produce a flat line shape.

Moreover, for interest rate options, the underlying asset price doesn't follow a geometric Brownian motion. The nominal interest rate can be negative. Therefore, as a consequence of the non-adherence of the Black's method, many other methods and models came out in order to incorporate the underlying asset price dynamics, such as the Vasicek (1977), Heath *et al.* (1992) and many others. Nevertheless, many practitioners still use the Black's model for interest rate options, given its simplicity. This the case of the Brazilian market of IDI options, subject of this paper.

Once we have a set of option prices for a specific time to maturity, we can recover the risk-neutral probability distribution (Ross, 1976). Breeden and Litzenberger approach (1978) gives an exact formula² for recover the risk-neutral density:

$$\frac{\sigma^2 C(K)}{\sigma^2 K} \Big|_{S=X} = e^{-rT} pdf(S) \quad (1)$$

where $C(K)$ is the option price as a function of the strike price K , r is the continuous interest rate of the underlying asset, T is the time to expiration and $pdf(S)$ is the risk-neutral density function as a function of the underlying asset price S .

There are many methods for recovering this risk-neutral density function pdf or the risk-neutral cumulative distribution CDF embedded in option prices. Jackwerth (1999) reviews this literature, and classify these methods into parametric and non-parametric.

Assuming that a CDF is defined by a limited set of parameters, parametric methods just provide ways in order to estimate them. Jackwerth (1999) divides these methods into three groups: i) expansion methods, ii) generalized distribution methods and iii) mixture methods. The expansion methods add a sequence of correction terms in order to obtain a better-fitting distribution. For instance, Jondeau & Rockinger (2001) use Gram-Charlier expansion to extract RND. Generalized

²The non-arbitrage conditions $C'(K) < 0$ and $C''(K) > 0$ of equation (1) are satisfied since we have ensured that: i) RND's with negative probability densities were not considered and ii) the areas under the RND curves are less than or equal to one. As we will see later, we have also excluded prices that violate the non-arbitrage condition: $Optionprice < SpotIDI - Strike * exp(-DIFut * t)$.

distribution methods use distributions with some additional parameters in order to obtain a better fit of the RND. This is the case of the Generalized Beta of Second Kind that we use in our paper which has been used in many articles, as for instance in Liu *et al.* (2007). Mixture methods create new distributions from combinations of well-known simple distributions like the normal. The most common combination is the Mixture of Two Lognormals, which we use in our paper, and have been used by, for instance, by Coutant *et al.* (2001).

Non-parametric methods consist of fitting CDF's to observed data by means of more general functions. These methods are divided into three groups: i) kernel methods, ii) maximum-entropy methods and iii) curve fitting methods. Kernel methods use regressions without specifying the parametric form of the function (for example, see Ait-Sahalia & Lo (1998)). Maximum-entropy methods fit the CDF by minimizing some specific form of loss function, as in Buchen & Kelly (1996). Curve-fitting methods try to fit some very flexible curve. This is the case of Shimko (1993) that proposed a curve-fitting method to the smile by fitting a quadratic polynomial.

In our paper, we use three methods for recovering the risk-neutral distribution (RND): i) the approach of Shimko (1993) ii) the fitting of Mixture of two Lognormals (M2N), and iii) fitting of a Generalized Beta distribution of second kind (GB2). We will describe these methods on section 4.

3. IDI Option Characteristics

The IDI option is the main interest rate option traded in Brazil. The average daily value of contracts traded at the Brazilian Exchange BM&F-Bovespa in the period from June 2nd, 2003 to April 23rd, 2009 was US\$ 8.86 millions, with 36 trades each day on average.

The IDI option is of European style, and mature in the first business day of the corresponding month of expiration. The underlying asset of this option contract is an index called IDI (interbank deposit index). It is calculated according to the recursive formula:

$$IDI_t = IDI_{t-1}(1 + i_{t-1}) \quad (2)$$

where i_{t-1} is the Average Rate of One-Day Interbank Deposit Certificate (CDI) converted to percentage per day at time $t - 1$.

Since 1997, the CDI has been expressed in annual interest rates, and then is converted to daily rates in the following way:

$$i_s = (1 + i_{s\text{annum}})^{1/252} - 1 \quad (3)$$

Thus, the IDI_t index reflects the daily accumulation of the average rates of the CDI, so that from time t to T , we have:

$$IDI_T = IDI_t \prod_{s=t}^{T-1} (1 + i_s) \quad (4)$$

Hence, the IDI call option payoff at expiration date T is:

$$C_T = \text{Max}[IDI_T - K; 0] \quad (5)$$

where K is the Strike index.

The IDI index was initially set at 100.000 points on January 2, 1997. Then, it was reset to 100.000 several times. One has to be careful to use the right index series when pricing and trading the option.

The IDI option has some special features that make its pricing different from other fixed income options. In fact, the IDI option is similar to an Asian option on the geometric average of the one-day interbank interest rate (CDI), between the trade date and the expiration of the option. Therefore, traditional option-pricing formulas for bonds must be adapted to be used with these options. For instance, Vieira Neto & Pereira (2000) used the short-term rate model of Vasicek (1977) model to obtain a closed-form formula to the IDI options, while Fajardo & Ornelas (2003) used the CIR model (Cox *et al.*, 1985). Barbedo *et al.* (2010) implemented the HJM model (Heath *et al.*, 1992) for ID options, which generalizes both the Vasicek and CIR models.

Although these models use sophisticated diffusion processes for the short-term interest rate or the term structure, most Brazilian traders use a simple modified Black (1976) model for pricing IDI options. They use the future price of the IDI Index as the forward price, and then price an option over a forward price using the Black model. The IDI index's future price is calculated using the Spot IDI and the term structure of the CDI curve, which is derived from Interbank Deposit Future contracts (DI Future)³

4. Methodology

4.1 Shimko:1993

Shimko (1993) suggested a non-parametric implementation of the Breeden and Litzenberger approach (1978). It consists of implied volatility computation and the interpolation of the volatilities' curve against the strike price. This approach comprises the following steps:

- 1) Implied volatility computation by Black's formula on future price of IDI index;

³More information about the DI Future contracts can be found at http://www.cmegroup-bmfbovespa.com/pages/eng/documents/DI_Futures_flyer/_122608.pdf and the fixed-to-CDI Swap contract. In our paper, we calculate the future price of IDI converting all interest rates to continuously compounded rates.

- 2) Volatility smile fitting by a parabolic function of the strikes;
- 3) CDF and pdf calculations using formulas derived by Breeden and Litzenberger approach (1978).

We assume the volatility smile effect as a function of strike price X_t and estimate the following equation by ordinary least squares (OLS):

$$\sigma_t(X_t) = A_0 + A_1 \cdot X_t + A_2 \cdot (X_t)^2 \quad (6)$$

Note that we need at least three data of options per day for identifying a parabola uniquely. The annual implied volatility function $\nu_t(X_t)$ is given by:

$$\nu_t(X_t) = \sigma_t(X_t) \cdot \sqrt{\tau} \quad (7)$$

Taking the first and second derivatives on X_t , we obtain:

$$\nu'_t = (A_1 + 2A_2 \cdot X) \cdot \sqrt{\tau} \quad (8)$$

$$\nu''_t = 2A_2 \sqrt{\tau} \quad (9)$$

In order to obtain the CDF and the pdf, we substitute (6), (8) e (9) in the following formulas (10) e (11), calculated by Breeden & Litzenberger (1978):

$$CDF(X_t)|_{S=X} = 1 + F \cdot n(d_1) \cdot \nu' - N(d_2) \quad (10)$$

$$pdf(X_t)|_{S=X} = F \cdot n(d_1) \cdot [\nu'' - d_1 \cdot d_{1X}] - n(d_2) \cdot d_{2X} \quad (11)$$

where F is the future price of the IDI index (see equation (12) below). $N(d_i)$ is the Gaussian normal distribution function,

$$d_1 = \frac{\ln(F/X) + (\sigma^2(X_t)/2) \cdot \tau}{\sigma(X_t) \cdot \sqrt{\tau}}$$

$$d_2 = d_1 - \sigma(X_t) \cdot \sqrt{\tau}$$

$$d_{1X} = \nu' - \frac{1}{X\nu} - d_1 \cdot \frac{\nu'}{\nu}$$

$$d_{2X} = -\frac{1}{\nu} \left(\frac{1}{X} + d_1 \cdot \nu' \right), \text{ and}$$

$$n(d_i) = \frac{\exp(-d_i^2/2)}{\sqrt{2\pi}}$$

Through the paper the future price F of the IDI index is calculated as follows:

$$F = IDI_0 \exp(rT) \quad (12)$$

where IDI_0 is the spot IDI index and r is the interest rate of the DI future contract with the same expiration date of the option, or the interest rate interpolated from the fixed-to-CDI Swap contract curve.

4.2 Mixture of two log-normals (M2N)

Following Melick & Thomas (1997) and Liu *et al.* (2007) we use a Mixture of Log-Normals to model the Risk-Neutral Densities. More specifically, we model the future price of the IDI index using a mixture of two lognormals densities:

$$\begin{aligned} pdf_{MLN}(x|w, F_1, \sigma_1, F_2, \sigma_2) &= w * pdf_{LN}(x|F_1, \sigma_1) \\ &+ (1 - w) * pdf_{LN}(x|F_2, \sigma_2) \end{aligned} \quad (13)$$

with

$$pdf_{ln}(x|F, \sigma) = \frac{1}{x\sigma\sqrt{2\pi T}} \exp\left(-0.5 \left[\frac{\log(x) - \log(F) + 0.5\sigma^2 T}{\sigma\sqrt{T}}\right]^2\right)$$

We use the DI future contract interest rate r to reduce the number of free parameters of the distribution. We do that by making the expectation of the distribution equal to the IDI Future price:

$$F = IDI_0 e^{rT} = wF_1 + (1 - w)F_2 \quad (14)$$

Therefore, we have five overall parameters, but only four free parameters. It is more flexible than a single lognormal, and can represent asymmetric and bimodal shapes. The parameters F_1 and F_2 are the expectation of the two distributions of the mixture, and make possible asymmetry and bimodality. The sigma parameters determine volatility and allow fat tails.

The price of an European call option is the weighted average of two Black (1976) call option formulas $C_b(F, T, K, rs)$:

$$\begin{aligned} C(F_1, \sigma_1, F_2, \sigma_2, w, K, r, T) &= wC_B(F_1, \sigma_1, K, r, T) \\ &+ (1 - w)C_B(F_2, \sigma_2, K, r, T) \end{aligned} \quad (15)$$

One advantage of the M2N is that we have a closed-formula for the moments, so that we do not need to calculate them numerically as in the Shimko's case. The Skewness and Kurtosis can be calculated from the following formulas:

$$\begin{aligned} E[IDI_T^n] &= wF_1^n \exp(0.5(n^2 - n)\sigma_1^2 T) \\ &+ (1 - w)F_2^n \exp(0.5(n^2 - n)\sigma_2^2 T) \end{aligned} \quad (16)$$

As this is an extension of the Black model, the M2N also admit negative nominal interest rates. This is obviously a flaw of the model since the estimate pdf will associate a positive probability for negative interest rates. However, as Brazilian interest rates mean is very high, this problem is minimized.

The parameters estimation of the M2N was done using an adaptation of the algorithm of Jondeau and Rockinger⁴ for the IDI option characteristics and data. This algorithm estimates parameters by minimizing of the squared errors of the theoretical and actual option prices. However, this condition may bring a bias because out-of-the-money options have lower dollar prices and thus will have lower squared errors (SE) than in-the-money options. Then this method would have a bias of bringing better fitting for in-the-money options. For this reason, we have also estimated parameters minimizing the absolute percentage error (APE), as defined by:

$$APE = |TP - AP|/AP \quad (17)$$

where TP is the theoretical price using the estimated parameters and AP is the actual traded price.

4.3 Generalized beta function of second kind (GB2)

The Generalized Beta distribution of the second kind (GB2) is often used to model Income distribution (See McDonald (1984)) and was first used to model asset's prices in Bookstaber & McDonanld (1987). It has been used also to extract pdf from option prices. For instance, Dutta & Babbel (2005) used the GB2 to extract risk-neutral densities from US Dollar interest rate options.

The GB2 has the following density function:

$$pdf_{GB2}(x|a, b, p, q) = \frac{ax^{ap-1}}{b^{ap}B(p, q)[1 + (x/b)^a]^{p+q}} \quad (18)$$

where B is the Beta function.

It is often assumed that all parameters are positive and that $pdf = 0$ if $x < 0$. Note that this ensures that the index is not negative, but interest rates may still be negative.

As in the case of the Mixture of Normals, We use the DI future contract interest rate r to reduce the number of free parameters of the distribution. Therefore, we have four overall parameters, but only three free parameters. We do that by setting the expectation of the distribution equal to the IDI Future price:

$$F = IDI_0 \exp(rT) = \frac{bB(p + 1/a, q - 1/a)}{B(p, q)} \quad (19)$$

⁴The original algorithm of Jondeau and Rockinger is available at the website <http://www.hec.unil.ch/MatlabCodes/rnd.html>. Among the changes we have done in the algorithm, we use formula (14) to reduce the number of parameters.

The cumulative distribution function of the GB2 (CDF_{GB2}) is defined as:

$$CDF_{GB2}(x|a, b, p, q) = I(y|p, q) \quad (20)$$

where $y = \left(1 + (x/b)^{-a}\right)^{-1}$ and I is the incomplete Beta Function.

The price of a European call option assuming that the future IDI follows a GB2 is:

$$\begin{aligned} C(a, b, p, q, K, r, T) &= IDI_0 [1 - I(y|p + 1/a, q - 1/a)] \\ &\quad - K \exp(-rT) [1 - I(y|p, q)] \end{aligned}$$

with $y = (1 + (K/b)^{-a})^{-1}$

This distribution has a flexible shape also. We have also a formula for the moments, when they exists.⁵

Again parameters estimation of the GB2 was done using an adaptation of the algorithm of Jondeau and Rockinger.⁶ Among the changes we have done in the algorithm, we use formula (19) to reduce the number of parameters. for the IDI option characteristics and data. We have also estimated using both minimization of Squared Errors (SE) and Absolute Percentage Errors (APE).

5. Dataset

Our dataset consists of 66,489 daily average call option prices during the period from January 15th 1998 to March 26th 2009. The IDI call option contracts were much more traded than the put ones so that we decided to use only the IDI call option data. Moreover, the sample was also filtered by number of observations per RND⁷ (higher than or equal to 4) and time-to-maturity (higher than 5 business days). We have also excluded prices that violate the non-arbitrage condition,⁸ as well as RND with negative probability densities in the Shimko method. After all exclusions the sample was reduced 11,099 daily average option prices and 1,879 risk-neutral densities, through 1,456 days. Therefore, we have built RND with 5.9 options on average. It is worth to mention that in the beginning of the period (1998-2001) we were not able to build many RND's due to our exclusions criteria, especially the minimum number of options per RND. This period was characterized by low liquidity, and this may affect results.

Besides the IDI Options data, we have collected also data from interest rates of futures contract of Average Rate of One-Day Interbank Deposit (DI Futures) that

⁵The n^{th} moment does not exist when $n < aq$.

⁶The original algorithm of Jondeau and Rockinger is available at the website <http://www.hec.unil.ch/MatlabCodes/rnd.html>.

⁷A set of options can be used to estimate a Risk Neutral Density when they have the same traded date and the same expiration date.

⁸ $Optionprice < SpotIDI - Strike * \exp(-DIFut * t)$

expires at the same date as of the respective call option, and from the IDI index itself.

Note that we may have problems with the lack of synchronism between the traded time of the option and the DI Futures. This may include some noise in our risk-neutral densities.

There are some structural breaks along the CDI time series. Since the beginning of the IDI call option negotiations (on July 18, 1997), the Brazilian economy has been pervaded by various shocks. Particularly, the period until 2002 was too turbulent mainly due to the Asian in 1997, Russian in 1998 and Brazilian in 1999 crises, as we can see in CDI Figure 1:

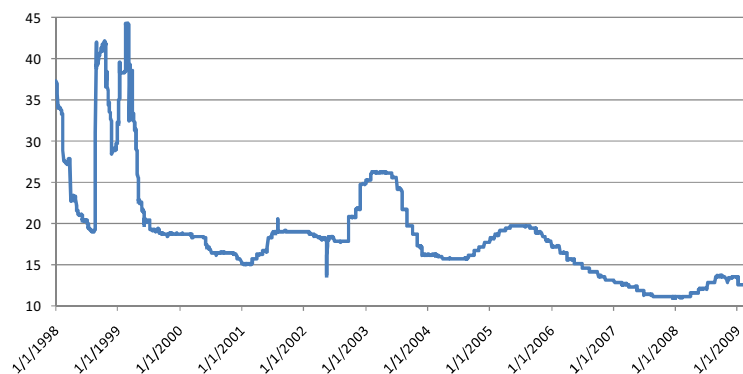


Figure 1
CDI-CETIP annual interest rate

6. Results

We have extracted the risk-neutral densities using the three methods – Shimko, M2N and GB2 – for all combination of trade date and maturity that satisfy the conditions described on section 5. We can draw some qualitative conclusions. First, Shimko's method seems to be more flexible to fit option data, but it comes out with very weird shapes outside the range of strikes, since this region is actually an extrapolation of the 2nd degree curve fitted. Therefore, when the strike range is very narrow, Shimko's approach turns out to be inadequate.

Mixture of Normals has a special feature that is to naturally represent bimodality. This is especially relevant for short-term maturities, when the market is divided into two outcomes of the COPOM⁹ meeting.

⁹COPOM is the committee responsible for deciding the target for the short-term interest rate in Brazil. It is composed by the board of directors from the Central Bank of Brazil. COPOM stands for "Monetary Policy Committee".

When running the fitting procedure, we have to bound the parameters of the GB2 and M2N in order to avoid very strange shapes. Allowing parameters very small or very large often produce shapes not very friendly.

6.1 Goodness of fit

We assess the goodness-of-fit of the distributions to the option data using the absolute percentage error (APE). We believe that the assessment using APE is more intuitive than squared errors (SE), since it is an error as percentage of the call price.

The percentiles of APE are shown on table 1. For the M2N and GB2 we provide errors using two algorithms: one minimizing the APE and other minimizing SE. Obviously the algorithm minimizing APE has better results in terms of APE.

As we can see in Table 1, the M2N method provided the best fitting to option's data for the period of our analysis. Shimko's method provided good results as well. GB2 distribution had the worst performance, with very high errors for the highest percentiles.

The magnitude of the errors for the best method - the M2N minimizing APE - was quite good. The median of 0.48% means a pricing error acceptable if we compare with a bid-ask spread.

Table 1
Absolute percentage error

Method	Percentiles				
	20	40	50	60	80
Shimko	0.22%	0.89%	1.48%	2.45%	7.10%
GB2 (Min APE)	0.07%	1.29%	2.46%	4.48%	15.67%
M2N (Min APE)	0.00%	0.11%	0.48%	1.06%	4.78%
GB2 (Min SE)	0.64%	1.84%	3.07%	4.83%	21.74%
M2N (Min SE)	0.22%	0.78%	1.20%	1.95%	6.94%
GB2 (Min APE)	0.22%	0.89%	1.48%	2.45%	7.10%

We may also analyze errors conditional on moneyness and through time. Table 2 shows that out-of-the money options had better fitting than in-the-money, except for the M2N using minimum APE, which has a very low error. We see that GB2 performs very poorly with out-of-the-money options. For in-the-money options, Shimko was the best performer.

Regarding the time period, in all cases the 2003-2005 had the best fit. The M2N using minimum APE was the best performer in all periods considered.

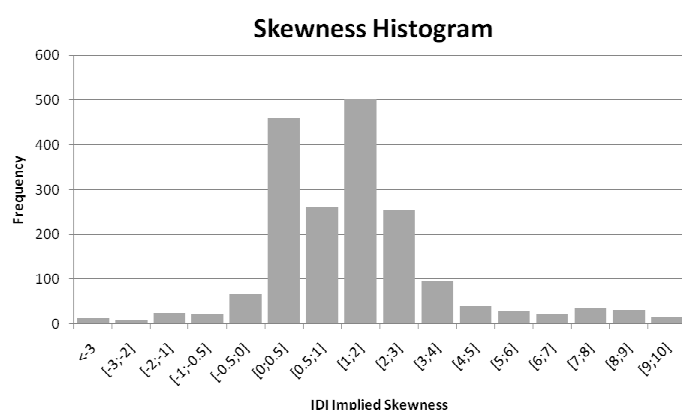
Table 2

Absolute percentage error by moneyness and period

Method	Median by Moneyness			Median by Period		
	< -1%	[-1%;1%]	>1%	1998-2002	2003-2005	2006-2009
Shimko	0.29%	1.89%	3.45%	1.71%	0.99%	1.70%
GB2 (min APE)	1.09%	3.06%	14.02%	4.19%	1.40%	3.09%
M2N (min APE)	0.76%	0.43%	0.00%	0.37%	0.27%	0.89%
GB2 (min SE)	1.05%	3.59%	37.92%	4.39%	1.86%	3.45%
M2N (min SE)	0.74%	1.35%	4.34%	1.24%	0.81%	2.03%

6.2 Implied skewness

In this section, we analyze the skewness implied in the RND's. We have chosen the Mixture of Log-Normals for this analysis, since this was the best fitting method. The implied skewness was calculated using formula (16). Note that this skewness is for the IDI index, and not for the interest rate itself. Anyway, a positive skewness indicates more fear of a sharp rise in CDI rate than a decline. Figure 2 shows the distribution of the implied skewness for all RND in the sample. We see that skewness are positive for most of the sample, indicating that investors are frequently worried about sharp rises in interest rates in Brazil.

**Figure 2**

Histogram of the IDI implied skewness

We analyze the implied skewness near changes of the CDI general trend. For instance, figure 2 shows the implied skewness of the call options with expiration in October 2003. It shows also the continuously compounded Spot CDI and the DI Futures converted to interest rates with expiration on October 2003. In June 2003, a downward trend in the CDI interest rate has begun, following a monetary policy easing performed by the Central Bank. A long-term picture of this trend can be better seen on figure 1.

Figure 3 shows strong negative skewness at the end of May. Note that this negative skewness incorporates market expectations of a strong decline of the interest rates, which actually happened latter. In order to analyze this process in more detail, Figure 4 shows the Implied Risk-Neutral Densities extracted using M2N in three specific dates,¹⁰ all of them using options with expiration in October 2003. On March 28th, we had an almost symmetric distribution with center around 24%. On May 27th, the market incorporated a higher possibility of a drop on interest rates, with a negative skewed distribution. Note that this day there is fatter left tail. On June 23rd, the distribution was again symmetric, centered in a lower level, under 22%. Note also that the dispersion in June was much lower, meaning that the market has reached a consensus regarding the following steps of the monetary policy.

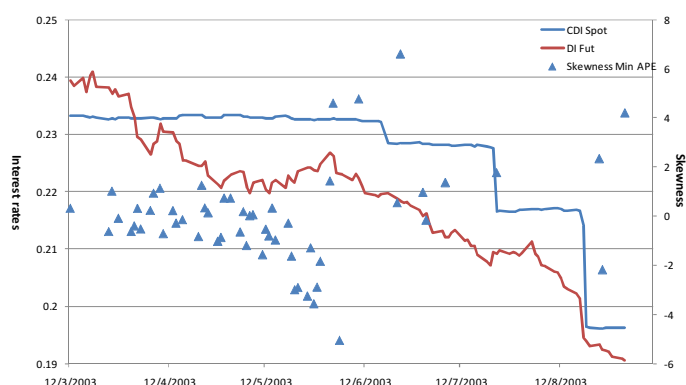


Figure 3
Spot CDI, future of DI and IDI implied skewness

Another interesting period to be analyzed is the Brazilian Presidential elections campaign of 2002. The possibility of a victory of the left-wing Labor Party candidate Luiz Inácio Lula da Silva leads to a rise on the future interest rate expectations. Figure 4 shows the implied Skewness of RND with expiration in January 2003, the continuously compounded SELIC target, and the continuously compounded DI Future with for the expiration on January 2003.

¹⁰Note that these RND's refer to the expected average interest rate from this date until the expiration of this option. As these RND's have different time to maturity, they are not directly comparable.

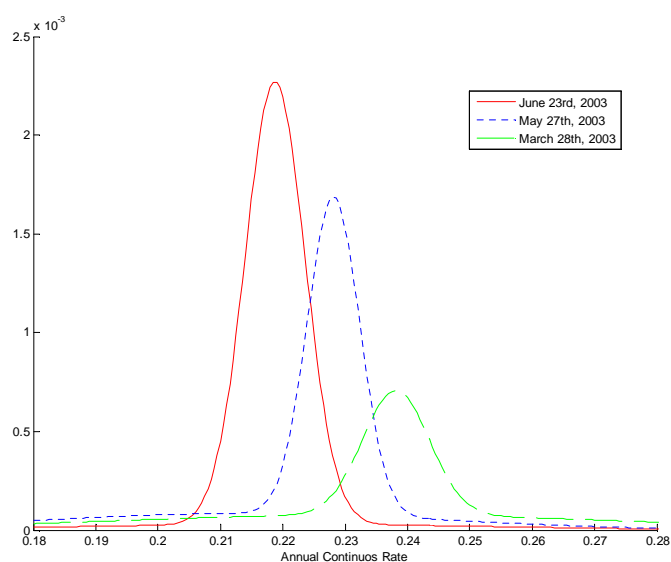


Figure 4
Risk-neutral densities for specific dates

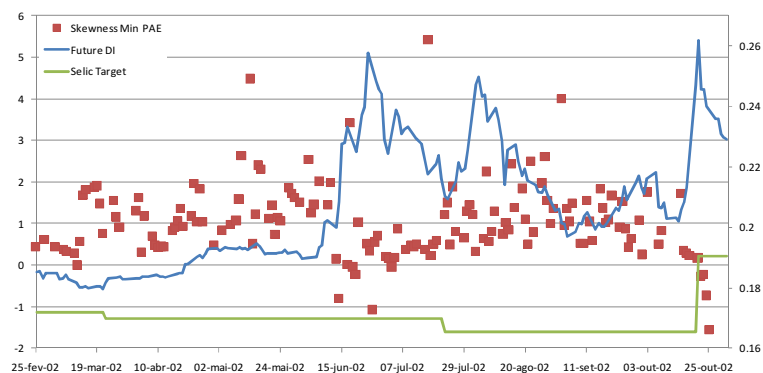


Figure 5
Interest rates and IDI implied skewness

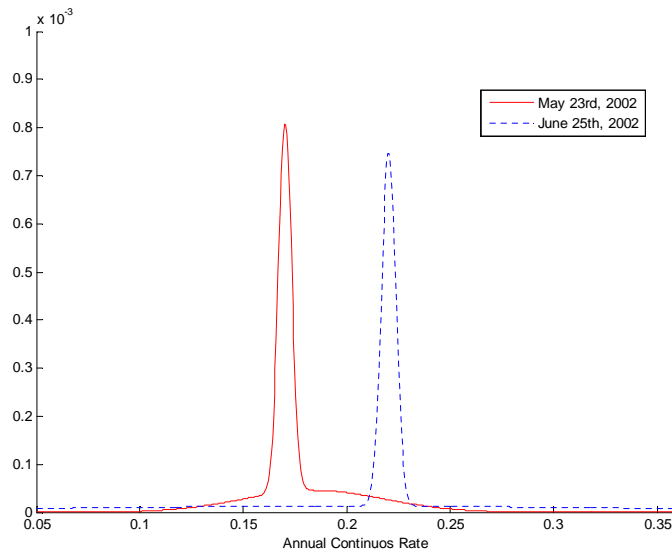


Figure 6
Risk-neutral densities for specific dates

It is interesting to note that DI Future has increased despite the Selic Target rate cut in the COPOM meeting of July 17th and downward bias of the meeting of June 19th. We also see a strong positive implied Skewness, along the year of 2002, until October, month of the elections. Note that this positive Skewness incorporates market expectations of a strong rise of the interest rates. Thus, no matter the effort of the Central Bank in fostering lower interest rates, the market was forecasting a strong possibility of a spike in the target interest rate that actually happened in the meeting of October 14th. The overshooting of the exchange rate caused by capital outflows was indicating a higher inflation in the future and this forced the Central Bank to raise the target rate. After the spike of the target and future rates, the skewness lowered, indicating less risk of a rise over the DI Future rate. This episode was a typical case where the derivative market was indicating a trend different from those intended by the Central Bank.

We may also analyze this episode looking to Risk-Neutral Densities of specific dates. For instance, figure 6 shows RND in May 23rd and June 25th, two days after the meeting where COPOM decides for downward bias in the target rate. We see that in May, the mean interest rate expectation was lower, but skewness was higher (1.15 against 0.35 in May). So, market increased expectation of the mean rate, but reduced the probability of a spike.

Figure 7 shows another example in September and October 16th, two days after the meeting where COPOM decides to raise interest rates in 300 basis points.

We see that in September, the mean interest rate expectation was lower, but skewness was higher (1.84 in September against 0.35 in October). So, market increased expectation of the mean rate given the rise in the target rate, but reduced the probability of another spike.

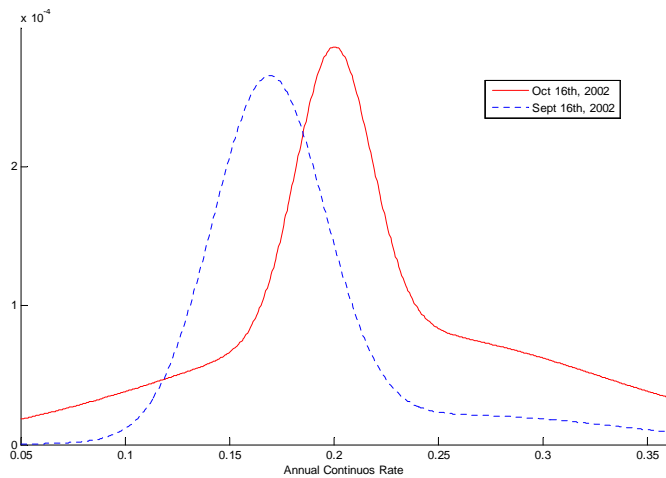


Figure 7
Risk-neutral densities for specific dates

7. Conclusion

We have estimated the IDI option-implied Risk-Neutral Densities for the Brazilian short rate CDI using three methods: Shimko, Mixture of Two Log-Normals and Generalized Beta of Second Kind. The in-sample goodness-of-fit evaluation showed that the Mixture of Log-Normals method provided better fitting to option's data than the other two methods during the period of our analysis.

We have also calculated the implied Skewness and showed how it can provide early-warnings of market expectations for monetary policy developments in 2002 and 2003. Overall, the Risk-Neutral Densities implied on IDI options seems to be a useful tool for extracting expectations the market about future outcomes of the monetary policy. It may also be used to support monetary policy decisions, since it can express market expectations with details.

In order to use RND in practice, one must consider some limitations. First, when prices are taken from a very volatile day, time synchronization of prices may be a problem. Another potential problem is the over fitting of data, since we are using distributions with 3 and 4 free parameters, and on average six options per day. Such situation may add instability to the estimation through time.

In future research, other RND extraction methods may be tested for the IDI

options, such as Kernel, Maximum-entropy or Expansion methods. Also, we may combine interest rate and currency option-implied Risk-Neutral densities to forecast economic developments, and to support decisions in the areas of monetary policy, asset allocation and general business strategies.

One can also perform an out-of-sample evaluation of the RND forecast ability. With the available dataset it is not yet possible to have a non-overlapping series with enough data to perform such analysis. Another interesting analysis that can be done should we have enough data is to estimate the risk aversion from options and real world data. This risk aversion parameter can be used to transform risk-neutral into real-world densities.

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