Risk Aversion or Model Uncertainty?  
An Empirical Cross-Sectional Analysis  
Across Countries

PEDRO ENGEL*  
CAIO ALMEIDA*  
JOÃO PAULO VALENTE**

Abstract
By analyzing a panel of macro data including both Emerging Markets (EM) and Advanced Economies (AE), we identify that an acceptable level of model uncertainty helps to explain the equity premium existing in all these markets. Model uncertainty aversion is in general higher for EMs than for AEs. In addition, the degree of cross-sectional heterogeneity across countries’ estimates of model uncertainty aversion is smaller than the corresponding heterogeneity of the risk aversion estimates in a traditional CRRA preference. We also compute separate costs of model risk and uncertainty for these economies in terms of present consumption, and conclude that the most significant effects come from uncertainty.

Keywords: Risk aversion, model uncertainty, equity premium puzzle, detection error probability  
JEL Codes: C5, C13, C14

1 Introduction
There is no doubt that one of the main open questions in the financial literature is the equity premium puzzle introduced by Mehra and Prescott (1985). The combination of high historical average stock returns and low risk-free interest rates demands a very high coefficient of relative risk aversion (RRA) to match the historical consumption growth process of the United States. This debate is not a singularity of the US economy. Campbell (2003) expands the puzzle to other economies by documenting high values of implied RRA coefficients for...
different developed economies. He identifies a high degree of variability among the RRA coefficients of these countries, making the puzzle under the traditional CCAPM model even more challenging.\(^1\)

Based on the framework proposed by Anderson, Hansen, and Sargent (2003) and Hansen and Sargent (2008), Barillas, Hansen, and Sargent (2009) reinterpret the RRA coefficient of Tallarini (2000) into a parameter that controls for model uncertainty aversion. Their interpretation for the agents’ preferences brings new insights to the problem. The idea is that when agents choose their consumption path, they are uncertain about possible future states and fear this uncertainty by considering worst case scenarios.

Barillas et al. (2009) calibrate the model for the US and find that plausible levels of model uncertainty can be equivalent to high levels of risk aversion when considering agents’ choices and effects on asset prices. Okubo (2015) expands this analysis to developed economies. His work suggests that there is also considerable variability in model uncertainty aversion levels among developed countries.

Since so far the analysis was focused on the US economy and little was done to evaluate the validity of the model within a broad empirical perspective, we build a new data set and expand the original analysis to a large group of countries including both Advanced Economies (AE) and Emerging Markets (EM). The idea is to test if the model is suitable to explain economic behavior, in particular the equity premium, for a broad set of economies or if there is a feature that makes the model suitable only for some particular set of countries. Moreover, we would like to know if the model is able to capture the expected higher level of uncertainty present in Emerging Markets. The main results are summarized below.

We find that the model uncertainty needed to explain the data for most countries remains on the reasonable bound. Additionally, we also find that the levels of model uncertainty present high variability among countries. This dispersion was also present in the estimates of the coefficient of RRA of Campbell (2003) and the model uncertainty aversion parameters of Okubo (2015). However, we show that the variability of model uncertainty parameters is much lower than of the RRA parameters when we restrict attention to detection error probabilities.

---

\(^1\) For a detailed revision of the literature on the Equity Premium Puzzle see Mehra (2003) and the references therein. For alternative asset pricing models that aim at solving the Equity Premium Puzzle, see Bansal and Yaron (2004), Barro (2006), and Campbell and Cochrane (1999) among others.
Barillas et al. (2009) also used this framework to reinterpret the large welfare gains eliminating risk found by Tallarini (2000). As Lucas (1987), they show that the elimination of risk only provides a small welfare benefit. According to them, a significant share of the gains found by Tallarini (2000) comes from reducing model uncertainty.\(^2\) Relying on this framework, we show that this result is also valid for other countries. Most of the welfare gains come from eliminating model uncertainty. In some cases, the benefits are more than 30 times larger than that of only reducing risk.

An apparent counterintuitive result we find from the data is that some Emerging Market countries like Brasil, Mexico, and India bear less risk and model uncertainty in absolute terms than other Advanced Economies like the US, France, and Denmark when considering the full sample. Those results can be attributed to the different time slot of data available for these economies. While most of the Advanced Economies have a long time series data available, Emerging Markets have a restricted sample that starts from 1999. When we restrict the analyses to an homogeneous dataset, all starting from 1999, the results are partially reversed.

The only unexpected result comes from Brazil. We would expect a high level of uncertainty for Brazil but observe the contrary. These results are in line with recent studies that suggest no equity premium in countries like Brazil.\(^3\) One possible explanation for the observed small level of uncertainty in Brazil is that while the interest rate is relatively high, the high volatility in the stock market makes it difficult to evaluate the equity premium when observing only a small time series data. Other possibility is that there are some possible characteristic present in the Brazilian economy that makes the equity premium small despite all uncertainty present. These characteristics are not captured by the model and further investigation is required.

Finally, we do a robustness exercise using an alternative method for calibrating the discount factor of EM and find similar results.

The rest of the paper is organized as follows. Section 2 develops the base model in detail for the interested reader. Section 3 describes the data. In

\(^2\) The welfare gain is computed by considering risk as the exogenous volatility of the consumption path. To compare gains from eliminating risk and model uncertainty we need to evaluate the agent preferences on both the exogenous stochastic consumption path and its deterministic trajectory with the same mean but zero variance. To evaluate the gains from eliminating risk alone, we must do the same procedure, but now setting uncertainty aversion to zero, which can be done by letting \(\theta \to \infty\).

\(^3\) See Cysne (2006) and Schor and Bonomo (2002).
section 4 we explain the details of the exercises and show the main results. Section 5 concludes.

2 The model

In this section, we introduce the model by describing the agents, characterized by their objective function and their constraints. The model is dynamic in the sense that the agent’s decision depends on history. The idea is that there is a transition equation that can be derived from equilibrium. This transition equation depends on parameter preferences. Since the preference is well defined by its parameters, we can compute a counter-factual with a different parameter of interest. Using this procedure, we can compute how much an agent would be willing to give up on consumption to avoid undesired uncertainties.

2.1 The transition equation

Many econometric papers have already shown that it is difficult to refuse the hypothesis that aggregate log consumption follows a geometric random walk. As used in Tallarini (2000), and later on Barillas et al. (2009), we use one of the following consumption plans:

1. geometric random walk:

   \[ c_t = c_0 + t\mu + \sigma\varepsilon_t + \varepsilon_{t-1} + \cdots + \varepsilon_1, \quad t \geq 1. \]

2. geometric trend stationary:

   \[ c_t = \rho^t c_0 + t\mu + \sigma\varepsilon_t + \rho\varepsilon_{t-1} + \cdots + \rho^{t-1}\varepsilon_1, \quad t \geq 1 \]

   where \( \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0,1) \) and \( c_t = \log C_t \).

It is not difficult to see that all these consumption plans are particular cases of the more general multivariate formulation

\[ x_{t+1} = Ax_t + B\epsilon_{t+1}, \]
\[ c_t = Hx_t, \]

where \( \epsilon_{t+1} \overset{iid}{\sim} \mathcal{N}(\mu, \Sigma) \) with dimension \( m \times 1 \); \( x_t \) is an \( n \times 1 \) state vector; and the eigenvalues of \( A \) are bounded in modulus by \( 1/\sqrt{\beta} \).
Note that this representation implies that the time $t$ element of the consumption plan can be expressed as the following function of $x_0$ and the history of shocks:

$$c_t = H(B\epsilon_t + AB\epsilon_{t-1} + \cdots + A^{t-1}B\epsilon_1) + HA^t x_0.$$  

We let $C(A,B,H;x_0)$ denote the set of consumption plans with the broad representation above.

As we can see, the transition path described lacks a vector of control variables, say $u_t$. In the absence of a control, we should understand this consumption path as an equilibrium consumption path. To clarify this idea, suppose the agent solves an optimization problem where he must choose a sequence of controls $\{u_t\}$ in order to maximize a discounted utility that is a function of the consumption path. Suppose also that the transition equation faced by the consumer can be written as $x_{t+1} = \hat{A}x_t + \hat{A}u_t + B\epsilon_{t+1}$, where $x_t$ is the state vector, $u_t$ is the control vector, and $\epsilon_{t+1}$ is the random vector of shocks.

If the optimal control for the agent can be written as $u_t = -Fx_t$, the equilibrium path for consumption would be $x_{t+1} = \hat{A}x_t + \hat{A}u_t + B\epsilon_{t+1} = (\hat{A} - \hat{A}F)x_t + B\epsilon_{t+1} = Ax_t + B\epsilon_{t+1}$, where $A = \hat{A} - \hat{A}F$.

From here on we will consider that the representative agent is optimizing for his consumption path in the sense presented above. The only layer of optimization will be carried out by nature. The idea is that in order to incorporate model uncertainty we consider an agent that is averse to ambiguity in the sense that his planning problem is distorted by a malevolent agent played by nature. The only role of nature is to distort the distribution of future consumption to the worst possible scenario.

### 2.2 The agent preferences

To incorporate the possibility of mistakes in the model, we consider here an agent with multiplier preferences. An agent is said to have multiplier preferences if his preference ordering over $C$, the set of consumption plans whose time $t$ elements $c_t$ are measurable functions of $(\epsilon^t, x_0)$, is described by

$$\min_{\{g_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \mathbb{E}[\beta^t G_t \{c_t + \beta \theta \mathbb{E}[g_{t+1} \log g_{t+1} | \epsilon^t, x_0]\} | x_0],$$

s.t.

$$G_{t+1} = g_{t+1} G_t, \quad \mathbb{E}[g_{t+1} | \epsilon^t, x_0] = 1, \quad g_{t+1} \geq 0, \quad G_0 = 1,$$
where $g_{t+1}$ is a positive measurable function of the history $X_t = \{(\varepsilon^t, x_0)\}; c_t = \log C_t; \varepsilon_{t+1} \stackrel{iid}{\sim} N(\mu, \sigma^2 \varepsilon)$; and $\varepsilon^t = (\varepsilon_t, \ldots, \varepsilon_1)$. We restrict attention only to the subset $C(A, B, H; x_0)$ of $C$ described by equation (1). Since this formula seems complicated at first sight, it is usually helpful to break it in parts. Think about the total “welfare”, $W$, of the agent as the sum of expected discounted (instantaneous) utilities $U_t$ that the agent obtain from consuming the bundle $c_t$ at time $t$; that is, consider

$$W = \sum_t U_t,$$

$$U_t = \beta^t \{E[G_t c_t \mid x_0] + \beta \theta E[G_t \log g_{t+1} \mid \varepsilon^t, x_0 \mid x_0]\}.$$

Defining the distorted expectation of a random variable $X_t$ as $\tilde{E}[X_t] = E[G_t X_t]$, we have that $U_t$ is the discounted sum of two terms:

$$\tilde{E}[c_t \mid x_0], \quad \text{and} \quad \theta \tilde{E}[\beta \log g_{t+1} \mid \varepsilon^t, x_0 \mid x_0].$$

The first term is just the expected utility of consumption under the worst case scenario. The second term is the expected value of discounted conditional entropy times a parameter $\theta$.

The conditional entropy is the function $E[g_{t+1} \log g_{t+1} \mid \varepsilon^t, x_0]$, which depends on the history of shocks $\varepsilon^t = (\varepsilon_t, \ldots, \varepsilon_1)$ and the initial condition $x_0$. The parameter $\theta$ measures the degree of concern of the agent with respect to model uncertainty. It can be understood as a parameter that constraints the choice set of the distorting sequence $g_{t+1}$ chosen by the nature. Note that if $\theta = \infty$ then $g_{t+1} = 1$ for all $t$, and $U_t = E[c_t]$; that is, the agent has no concern for model specification and his preference ordering is given by the usual representation,

$$W(x_0) = \sum_{t=0}^{\infty} \beta^t E[c_t \mid x_0].$$

The entropy is only one possible way to restrict the distribution choices considered by the agent. A more detailed description of entropy and its properties is given below. But first we explain the nature control variable $g_{t+1}$ and its space restrictions.
2.2.1 The agent constraints

To better understand the meaning of $g_{t+1}$ and of the constraints

$$G_{t+1} = g_{t+1}G_t, \quad E[g_{t+1} \mid \varepsilon^t, x_0] = 1, \quad g_{t+1} \geq 0, \quad G_0 = 1,$$

it is useful to think of $g_{t+1}$ as the ratio of two densities that represents a likelihood ratio. Let $f_{t+1}$ be the expected density for next period realization of the exogenous shock $\varepsilon_{t+1}$, and $\tilde{f}_{t+1}$ be its worst case distribution conditional on date $t$ information. With these assumptions, we can write $g_{t+1} = \frac{\tilde{f}_{t+1}}{f_{t+1}}$. Since $\tilde{f}_{t+1}$ and $f_{t+1}$ are both density functions, we must have $g_{t+1} \geq 0$ and $E[g_{t+1} \mid \varepsilon^t, x_0] = 1$ for all $t$, where the expectation is taken under the expected density $f_{t+1}$.

Now, since the density function of $\varepsilon_{t+1}$ is a measurable function of the history $X_t = \{(\varepsilon^t, x_0)\}$, so must be $g_{t+1}$. It is easy to see that $E[g_{t+1} \log g_{t+1} \mid \varepsilon^t, x_0] = 0$ only if $g_{t+1} = 1$ which means $\tilde{f}_{t+1} = f_{t+1}$, so that the conditional distribution is exactly the expected by the agent.

Now, to take into account the dynamic nature of the problem, it is common to factor a joint density $F_{t+1}$ over an $X_{t+1}$-measurable vector as $F_{t+1} = f_{t+1}F_t$, where $f_{t+1}$ is a one step ahead density conditioned on $X_t$. Following Hansen and Sargent, we factor a random variable $G_{t+1}$, forming

$$g_{t+1} = \begin{cases} \frac{G_{t+1}}{G_t} & \text{if } G_t > 0, \\ 1 & \text{if } G_t = 0. \end{cases}$$

Then $G_{t+1} = g_{t+1}G_t$ and $G_t = G_0 \prod_{j=1}^{t} g_j$. The random variable $G_0$ is set to unity so that $\{G_t : t \geq 0\}$ is a martingale. By construction, $G_t$ is a function of $\varepsilon^t$ and $x_0$ and $E[G_{t+1} \mid \varepsilon^t, x_0] = G_t$. We then have $E[G_t \mid x_0] = G_0 = 1$.

With all of this said, we can understand these restrictions as technical conditions to ensure that nature choices are perturbations that are in fact changes of measure. In particular, the optimal choice of nature will be the worst possible measure for the agent. The worst measure is computed by restricting the choices of nature using entropy as a discrepancy measure. We left in the appendix a brief description of entropy and its properties that makes it useful as a discrepancy measure.
2.3 The value function

The value function associated with the multiplier agent preferences solves the following Bellman equation:

\[
GW(x) = \min_{g \geq 0, Eg = 1} G \left( c + \beta \int [g(\varepsilon)W(Ax + B\varepsilon) + \theta g(\varepsilon)\log g(\varepsilon)]\pi(\varepsilon)d\varepsilon \right).
\]

The Bellman equation is a way to transform the sequential problem into a problem in which the agent only needs to solve for a 2 period time schedule. The idea is that solving for the entire sequence is equivalent to choose the best choice for today and the best choice for tomorrow supposing that the agent is optimizing for the remainder.

Dividing by \(G\) gives

\[
W(x) = c + \min_{g \geq 0} \beta \int [g(\varepsilon)W(Ax + B\varepsilon) + \theta g(\varepsilon)\log g(\varepsilon)]\pi(\varepsilon)d\varepsilon \quad \text{s.t.} \quad Eg = 1.
\]

(3)

By solving this problem, we can express \(W(x)\) as the sum of two components, the first of which is the expected discounted value of log consumption under the worst case scenario, while the second is \(\theta\) times discounted entropy:

\[
W(x) = J(x) + \theta N(x),
\]

(4)

where

\[
J(x) = c + \beta \int [\hat{g}(\varepsilon)J(Ax + B\varepsilon)]\pi(\varepsilon)d\varepsilon
\]

(5)

and

\[
N(x) = \beta \int [\hat{g}(\varepsilon)\log \hat{g}(\varepsilon) + \hat{g}(\varepsilon)N(Ax + B\varepsilon)]\pi(\varepsilon)d\varepsilon.
\]

(6)

Here \(J(x_t) = \hat{E}_t \sum_{j=0}^{\infty} \beta^j c_{t+j}\) is the expected discounted log consumption under the worst case joint density and

\[
G_tN(x_t) = G_t\beta E \left[ \sum_{j=0}^{\infty} \beta^j \frac{G_{t+j}}{G_t} E[g_{t+j+1} \log g_{t+j+1} \mid \epsilon^t, x_0] \right]
\]

is continuation entropy.

Substituting the minimizer into the above equation gives the risk-sensitive
risk aversion or model uncertainty?
An empirical cross-sectional analysis across countries

The recursion of Hansen and Sargent:

\[ W(x) = c - \beta \theta \log E \left[ \exp \left( \frac{-W(Ax + B\epsilon)}{\theta} \right) \right]. \] (7)

This same recursion is derived in the appendix from an Epstein and Zin preference relation. The conclusion is that identical consumption plans can be achieved with a different perspective over the preference parameters. While here \( \theta \) is related to the degree of ambiguity aversion of the agent, in the Epstein–Zin context, the parameter is related to risk aversion.

2.4 The minimizing martingale increment

The minimizing martingale increment is the optimal choice of nature and is given by

\[ \hat{g}_{t+1} = \left( \frac{\exp\{-W(Ax_t + B\epsilon_{t+1})/\theta\}}{E_t[\exp\{-W(Ax_t + B\epsilon_{t+1})/\theta\}]} \right). \] (8)

This is an exponential change of measure. As it is, we can see that it is not necessary to impose the condition that \( g_{t+1} > 0 \) since this restriction is naturally satisfied.

For the random walk model, we can show that

\[ \hat{g}_{t+1} \propto \exp\left\{ -\sigma \epsilon_{t+1} \right\} \]

and the worst-case density for the innovation \( \epsilon \) is

\[ \hat{\pi}(\epsilon_{t+1}) \propto \exp\left\{ -\frac{1}{2} \left( \epsilon_{t+1} + \frac{\sigma \epsilon}{(1 - \beta)\theta} \right)^2 \right\}, \]

which is the density function of a normal random variable with mean \( -\frac{\sigma \epsilon}{(1 - \beta)\theta} \), that is,

\[ \hat{\pi}(\epsilon_{t+1}) \sim \mathcal{N}(w(\theta), 1) \] (9)

where

\[ w(\theta) = -\frac{\sigma \epsilon}{(1 - \beta)\theta}. \] (10)

It means that the distorted distribution is still normal, but with a lower mean. How much lower the distorted mean is compared to the prior distribution depends on the level of concern of the agent with respect to model uncertainty, here represented by the parameter \( \theta \).
2.5 The discounted entropy

When the conditional densities for $\epsilon_{t+1}$ under the approximating and worst case models are $\pi \sim \mathcal{N}(0, 1)$ and $\hat{\pi} \sim \mathcal{N}(w(\theta), 1)$, respectively, the conditional entropy is $E_t \hat{g}_{t+1} \log \hat{g}_{t+1} = \int (\log \hat{\pi}(\epsilon) - \log \pi(\epsilon)) \hat{\pi}(\epsilon) d\epsilon = \frac{1}{2} w'(\theta) w(\theta)$. It then follows that discounted entropy becomes

$$
\beta E \frac{1}{\sum_{t=0}^{\infty} \beta^t \hat{g}_t E [\hat{g}_{t+1} | \epsilon^t, x_0] | x_0] } = \eta = \frac{\beta}{2(1-\beta)} w'(\theta) w(\theta).
$$

This formula allows us to compute explicitly how distant the worst case measure is from the prior as a function of the ambiguity aversion parameter $\theta$. It is clear that the greater the $\theta$, the less ambiguity averse the agent is and the closer the distorted measure is to the prior.

2.6 The value function for random walk log consumption

Using the formula for $w(\theta)$ from the random walk model tells us that discounted entropy is

$$
N(x) = \frac{\beta}{2(1-\beta)} \frac{\sigma_x^2}{(1-\beta)^2 \theta^2}.
$$

We can then compute the value function for the agent to be

$$
W(x_t) = \frac{\beta}{(1-\beta)^2} \left( \mu - \frac{\sigma_x^2}{2(1-\beta)\theta} \right) + \frac{1}{1-\beta} c_t,
$$

where the value function for the consumption process is

$$
J(x_t) = \frac{\beta}{(1-\beta)^2} \left( \mu - \frac{\sigma_x^2}{(1-\beta)\theta} \right) + \frac{1}{1-\beta} c_t,
$$

so that $W(x_t) = J(x_t) + \theta N(x_t)$.

We can interpret $J(x_t)$ as the value function for an uncertainty constraint agent where discounted entropy is bounded by $\eta$. To do so, we need to align $\theta$ and $\eta$ in a specific fashion (see Hansen & Sargent, 2008, p.159).

We shall use these value functions to construct compensating variations in the initial condition for log consumption $c_0$ in an elimination of model uncertainty experiment to be described bellow.
3 The data

Our study focuses on the largest advanced and emerging economies of the world. In this way, our data set is compounded by eight developed countries\textsuperscript{4} and ten emerging market economies.\textsuperscript{5} We do not include a few countries such as Argentina, China, and Saudi Arabia because of the lack of data availability for these economies. We use quarterly data from the beginning of 1970 to the end of 2015. As pointed out by Campbell (2003), there is no significant dispersion among the coefficients of relative risk aversion among developed countries when using annual data. In this way, given our interest in characterizing this variation, we opted to use quarterly data.

We rely on the International Financial Statistics of the International Monetary Fund (IFS-IMF) and the data provided by the Morgan Stanley Capital International (MSCI), obtained using Bloomberg. We use the following series: stock returns, consumer price index (CPI), short-term interest rate, total consumption, and GDP deflator. All data is in local currency and each countries’ data set is composed by the earliest to the latest available data within our predetermined date range.

Following Campbell (2003, appendix), we use the monthly MSCI National Price and Gross Return Indexes to build the quarterly stock market data. In this way, our stock series consider not only the stock price increases but also the returns of dividends accumulated within the quarter. It is important to mention that these indexes are representative, but not comprehensive of the whole equity market of the economy. Additionally, this representativeness can vary substantially among countries. These observations should be considered when analyzing the results. The stock indexes are deflated using the CPI, which is obtained from the IFS-IMF. The gross real return on stocks are defined as

\[ 1 + r_{e,t} = \frac{1 + R_{e,t}}{1 + \pi_t}, \]  

where \( r_{e,t} \) is the real return on stocks at time \( t \), \( R_{e,t} \) is the nominal return on stocks, and \( \pi_t \) is the inflation rate.

The short-term interest rate is also obtained from IFS-IMF database, except for Germany, France, Italy, and Turkey which the source is Bloomberg. Besides Italy, for which we use the 3-month Treasury bills, we adopt the money market

\textsuperscript{4} Australia, Canada, Germany, France, Great Britain, Italy, Japan, and United States.

\textsuperscript{5} Brazil, Chile, Colombia, India, Indonesia, Korea, Mexico, Russia, South Africa, and Turkey.
interest rate as a proxy for the risk-free short-term rate.

The source of the total consumption, GDP deflator, and population data is also the IFS-IMF. Since only the United States provides a reliable and extensive time series of household expenditure on non-durables and services, we decided to adopt total consumption as the proxy for household consumption for all countries. As pointed out by Mankiw (1985), Ogaki and Reinhart (1998b), Ogaki and Reinhart (1998a), Yogo (2006), and Pakos (2011), the treatment of durables can affect the estimates of the coefficient of intertemporal elasticity of substitution and the RRA. We need to consider this possible effect when evaluating the results since the estimation of consumption volatility is affected. Additionally, to keep the same methodology for all countries, we deflate consumption using GDP deflator. The real consumption per capita series were seasonally adjusted using the X-13ARIMA-SEATS Seasonal Adjustment Program from the US Census Bureau. Then, log consumption, $c_t$, in the model is defined as the log of the seasonally adjusted consumption per capita. Finally, because of the time convention sensitiveness of the correlation between real consumption growth and stock returns showed by Campbell (2003), we define consumption growth for the quarter as the next quarter’s consumption divided by this quarter’s consumption.

Table 1 describes the data of the 18 countries in our data set. The range of the sample of each country is described in the last two columns. Only three countries’ samples cover the whole time span of 1970 to 2015: Australia, Japan, and United States. On the other hand, India presents the shortest sample, with only 11 years.

Note that even though the Great Recession (2007–2009), when most stock markets suffered a sharp negative hit, is a representative period of most samples, all stock markets present real average return greater than the risk-free rate. In this way, even the countries that present a small sample display positive equity premium.

4 Results

4.1 Risk and Model Uncertainty Aversion

Table 1 also shows the estimated mean and variance of the consumption processes. They were obtained using maximum likelihood (ML) estimation and assuming that the consumption growth follows the random walk model. The second
Table 1. Description of the Full Data Set
This table reports an overview of the full data sample. The second and third columns show the results of ML estimation of the real consumption growth processes assuming a random walk model. Stocks and ST Rate are the quarterly average real return of stocks and short-term interest rate of each country, respectively. The last two columns show the initial and last date of the sample of each country.

<table>
<thead>
<tr>
<th>Country</th>
<th>μ</th>
<th>σ</th>
<th>Stocks (%)</th>
<th>ST Rate (%)</th>
<th>Min Date</th>
<th>Max Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>AU</td>
<td>0.005</td>
<td>0.007</td>
<td>1.447</td>
<td>0.613</td>
<td>1970-06-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>BR</td>
<td>0.004</td>
<td>0.010</td>
<td>2.413</td>
<td>2.025</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>CA</td>
<td>0.004</td>
<td>0.005</td>
<td>1.699</td>
<td>0.629</td>
<td>1975-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>CL</td>
<td>0.007</td>
<td>0.011</td>
<td>1.883</td>
<td>0.226</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>CO</td>
<td>0.011</td>
<td>0.032</td>
<td>4.442</td>
<td>0.539</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>DE</td>
<td>0.003</td>
<td>0.005</td>
<td>2.187</td>
<td>0.227</td>
<td>1995-06-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>FR</td>
<td>0.002</td>
<td>0.004</td>
<td>2.012</td>
<td>0.252</td>
<td>1995-06-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>GB</td>
<td>0.005</td>
<td>0.006</td>
<td>1.491</td>
<td>0.667</td>
<td>1988-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>ID</td>
<td>0.005</td>
<td>0.016</td>
<td>3.536</td>
<td>0.440</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>IN</td>
<td>0.016</td>
<td>0.014</td>
<td>0.971</td>
<td>0.201</td>
<td>2005-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>IT</td>
<td>0.001</td>
<td>0.006</td>
<td>1.461</td>
<td>0.591</td>
<td>1995-06-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>JP</td>
<td>0.006</td>
<td>0.034</td>
<td>2.311</td>
<td>0.250</td>
<td>1970-06-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>KR</td>
<td>0.008</td>
<td>0.008</td>
<td>2.708</td>
<td>0.245</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>MX</td>
<td>0.003</td>
<td>0.036</td>
<td>2.850</td>
<td>0.937</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>RU</td>
<td>0.011</td>
<td>0.014</td>
<td>3.270</td>
<td>-1.313</td>
<td>1999-03-01</td>
<td>2014-12-01</td>
</tr>
<tr>
<td>TR</td>
<td>0.006</td>
<td>0.029</td>
<td>3.162</td>
<td>1.512</td>
<td>2002-09-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>US</td>
<td>0.005</td>
<td>0.005</td>
<td>1.833</td>
<td>0.367</td>
<td>1970-06-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>ZA</td>
<td>0.003</td>
<td>0.008</td>
<td>2.751</td>
<td>0.615</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
</tbody>
</table>

and third columns of Table 1 show the results of the ML estimation for the 18 countries of our sample. India presents the highest quarterly average consumption growth (μ), around 1.6%, in our estimation. On the other hand, France presents the lowest μ, 0.2%. Once again, it is important to highlight the differences between sample periods among the countries. While India’s sample begins in 2005, the French times series are ten years longer, starting from 1995. Mexican, Japanese, and Colombian consumption processes are the most volatile in our sample. Their σ’s are higher than 0.03, more than double the average σ of the sample, 0.014.

After estimating the consumption processes and calculating equity return and the risk-free rate, we proceed to the computation of the model’s parameters γ, $\theta = \frac{1}{(1-\gamma)(1-\beta)}$ (see Appendix), and $p(\theta^* - 1)$. We first focus on how to obtain γ.
that attain the Hansen–Jagannathan bounds:

\[ \sigma(m) \geq \sigma^*(m), \]  

(15)

where

\[ \sigma(m) = \beta \exp \left( -\mu + \frac{\sigma}{2} (2\gamma - 1) \right) \left[ \exp \left( \sigma^2 \gamma^2 \right) - 1 \right]^{0.5}, \]  

(16)

and

\[ \sigma^*(m) = (1 - E[m]E[R])'\Sigma^{-1} (1 - E[m]E[R]), \]  

(17)

where \( E[m] = 1/R_f \), \( R_f \) is the real risk-free rate, \( R \) is the real stock return, and \( \Sigma = var(R) \). The detailed derivation is in the Appendix.

To find \( \gamma \) we first need to calibrate the discount factor, \( \beta \). Following Okubo (2015) and Garcia-Cicco, Pancrazi, and Uribe (2010), we set the same \( \beta \) for all AE and also a common \( \beta \) for all EM. Then, for the eight advanced economies in our sample, we follow Okubo (2015) and set \( \beta_{AE} = 0.995 \). On the other hand, for the EM, we calculate the common \( \beta \).

To calibrate \( \beta_{EM} \) we use the steady-state relation \( \beta_{EM} = 1/\bar{R}_{EM}^* \), where \( \bar{R}_{EM}^* \) is the average risk-free short-term rate of these economies. We define the country risk as the spread of the US dollar denominated five-year bond of the country to the same maturity US bond. After finding the risk estimate, we subtract it from the short-term interest rate to find the risk-free measure. However, a setback of this approach is the lack of US dollar denominated bonds. Only three countries present a sample longer than 15 years: Brazil, Colombia, and Mexico. Then, using these three time series, we found \( \beta_{EM} = 0.995 \). Note that using this approach we find the same value as \( \beta_{AE} \). However, this is not surprising since we expect arbitrage to eliminate the interest rate differences among countries after discounting for risk. In the robustness section, we use an alternative method for calibrating \( \beta_{EM} \).

After computing \( \beta \) and \( \gamma \), we proceed to the computation of \( \theta \) and \( p(\theta^{*-1}) \). We know that an agent with uncertainty aversion \( \theta \) has utility (value function)

---

6 If we calibrate \( \beta_{AE} \) using its steady-state relation with the risk-free short-term rate of the Advanced Economies since 1970, we find \( \beta_{AE} = 0.996 \), a similar number to the one used in the literature. Then, the results with this new calibration would be the same. Nevertheless, these results are available upon request.
represented by the recursion

\[ W_t = c_t - \beta \theta \log E \left[ \exp \left( -\frac{W_{t+1}}{\theta} \right) \right]. \]  \hspace{1cm} (18)

This is the equivalent recursion of a risk-averse investor (see Appendix). It means that there is an equivalent observation between risk averse agents and uncertainty averse agents. The difference between both is due to interpretation of the agent preference.

In the case of the uncertainty averse agent, the recursiveness arises from the worst case measure considered by the agent. Suppose that in the case of perfect foresight the consumption trajectory is given by, say Model A:

\[ c_{t+1} = c_t + \mu + \sigma \epsilon_{t+1}, \quad \epsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1). \]  \hspace{1cm} (19)

The consumption path that arises from the worst case scenario, say Model B, considered by the agents is

\[ c_{t+1} = c_t + (\mu + \sigma \omega) + \sigma \epsilon_{t+1}, \quad \epsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}(0, 1), \quad \omega = -\frac{\sigma}{\theta(1 - \beta)}, \]  \hspace{1cm} (20)

that is, the agent considers a path with smaller conditional mean, where the size of the decreased mean is determined by the degree of uncertainty of the agent. The question that must be answered is: what is a reasonable level of uncertainty aversion? To answer this question we follow Hansen and Sargent (2008) who propose the use of detection error probabilities, \( p(\theta^* - 1) \).

The idea of detection error probabilities is the following. The agent considers the baseline Model A, and the alternative Model B with lower conditional mean, \( \omega(\theta) \).

When looking at the consumption history, he compares the likelihood of Model A, \( L_A \), with the likelihood of Model B, \( L_B \).

Note that the likelihood of Model B depends on the level of uncertainty aversion \( \theta^{-1} \), where bigger uncertainty implies lower conditional mean for the worst case consumption scenario.

Considering a priori distribution of both consumption paths equally, we can link the uncertainty parameter level \( \theta \) with the probability of incurring a mistake that is, the probability of considering Model A, when Model B is the correct model and vice versa.
To compute the detection error probability we only need to calculate the probability of $L_A > L_B$ when the data is generated by Model B and the probability of $L_B > L_A$ when the Model A is the correct one. The detection error probability is then

$$p(\theta^*-1) = 0.5p\left(\ln\left(\frac{L_B}{L_A}\right) > 0 \bigg| \text{Model A is correct}\right) + 0.5p\left(\ln\left(\frac{L_A}{L_B}\right) > 0 \bigg| \text{Model B is correct}\right).$$

It is not difficult to show that

$$p(\theta^*-1) = \Phi\left(-\frac{\sqrt{T}}{\bar{\sigma}} \frac{\sigma}{\theta^*(1-\beta)}\right). \quad (21)$$

The derivation of the specific formulas for detection error probabilities due to Okubo (2015) is provided in the Appendix. This result gives a nice relation between the probability of committing mistakes and the uncertainty level parameter $\theta^{-1}$. It confirms the intuition that a larger degree of model uncertainty aversion $\theta^{-1}$ implies in a lower probability of committing mistakes $p(\theta^{-1})$, that is, $p(\theta^{-1})$ is a decreasing function. This result is expected since a higher $\theta^{-1}$ implies a smaller conditional mean for the worst case consumption process B, which in turn makes it easier to distinguish it from the approximating Model A.

The remaining issue is the level of uncertainty that is acceptable when we consider the possibility of choosing the wrong model. It is clear that no level of uncertainty should make the agent consider a model that is wrong with probability greater than half since this is the probability of committing mistakes when there are no model misspecification concerns ($\theta^{-1} = 0$). Hansen and Sargent (2008) argue that a bound for $p(\theta^*-1)$ between 0.15 and 0.20 implies a reasonable concern of committing mistakes. Note that this implies an upper bound for misspecification concerns that can be considered reasonable to have $\bar{\theta}^{-1} = \Phi^{-1}(0.15) \times \left(-2(1-\beta)/\sigma\sqrt{T}\right)$.

Table 2 reports $\gamma$, $\theta^*-1$, and $p(\theta^*-1)$ estimated for each country. The first column reports the risk aversion coefficient value that attains the Hansen–Jagannathan bound. The standard deviation of $\gamma$ among the countries in our sample is 10.89, while the mean is 15.64. This result is similar to Campbell (2003) and Okubo (2015), which also found significant RRA parameter variation...
Table 2. Risk Aversion and Detection Error Probability (Full Sample)

This table shows the results of the risk aversion parameter $\gamma$, the penalty parameter $\theta^{*-1}$, and the detection error probability $p(\theta^{*-1})$ of the calibration assuming that the consumption growth process follows a random walk model. The parameter $\gamma$ is chosen to satisfy the minimum of the Hansen–Jagannathan bounds: $\sigma(m) \geq \sigma^{*(m)}$.

The discount factor for all countries is set to $\beta = 0.995$.

<table>
<thead>
<tr>
<th>Country</th>
<th>$\gamma$</th>
<th>$\theta^{*-1}$</th>
<th>$p(\theta^{*-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AU</td>
<td>9.378</td>
<td>0.042</td>
<td>0.337</td>
</tr>
<tr>
<td>BR</td>
<td>2.909</td>
<td>0.010</td>
<td>0.468</td>
</tr>
<tr>
<td>CA</td>
<td>19.566</td>
<td>0.093</td>
<td>0.260</td>
</tr>
<tr>
<td>CL</td>
<td>16.353</td>
<td>0.077</td>
<td>0.239</td>
</tr>
<tr>
<td>CO</td>
<td>9.024</td>
<td>0.040</td>
<td>0.145</td>
</tr>
<tr>
<td>DE</td>
<td>31.215</td>
<td>0.151</td>
<td>0.257</td>
</tr>
<tr>
<td>FR</td>
<td>35.427</td>
<td>0.172</td>
<td>0.251</td>
</tr>
<tr>
<td>GB</td>
<td>15.944</td>
<td>0.075</td>
<td>0.323</td>
</tr>
<tr>
<td>ID</td>
<td>11.616</td>
<td>0.053</td>
<td>0.245</td>
</tr>
<tr>
<td>IN</td>
<td>4.308</td>
<td>0.017</td>
<td>0.446</td>
</tr>
<tr>
<td>IT</td>
<td>11.079</td>
<td>0.050</td>
<td>0.389</td>
</tr>
<tr>
<td>JP</td>
<td>5.183</td>
<td>0.021</td>
<td>0.169</td>
</tr>
<tr>
<td>KR</td>
<td>21.284</td>
<td>0.101</td>
<td>0.259</td>
</tr>
<tr>
<td>MX</td>
<td>4.848</td>
<td>0.019</td>
<td>0.285</td>
</tr>
<tr>
<td>RU</td>
<td>14.277</td>
<td>0.066</td>
<td>0.238</td>
</tr>
<tr>
<td>TR</td>
<td>3.966</td>
<td>0.015</td>
<td>0.380</td>
</tr>
<tr>
<td>US</td>
<td>34.383</td>
<td>0.167</td>
<td>0.124</td>
</tr>
<tr>
<td>ZA</td>
<td>29.989</td>
<td>0.145</td>
<td>0.182</td>
</tr>
</tbody>
</table>

Across countries, but using a sample including only advanced economies. Looking at individual countries, Brazil presents the lowest $\gamma$, 2.99, while France and the US have the highest, 35.13 and 34.64, respectively. We can also see in this column that the three highest $\gamma$ values are from advanced economies. On the other hand, the four lowest numbers are from emerging markets: Brazil, Turkey, India, and Mexico. These findings suggest not only a significant variation in risk aversion across countries but also between the two country groups. On average, AE’s investors are more risk averse than the ones based on emerging markets.

Given the values of $\gamma$, we find the inverse of the penalty parameter, $\theta^{*-1}$, and the detection error probabilities, $p(\theta^{*-1})$. The results are reported in the second and third columns of Table 2. Note that the estimates of $p(\theta^{*-1})$ also present significant variation across the countries, but the variability is considerably lower. Table 3 shows the coefficients of variation of $\gamma$, $\theta^{*-1}$, and $p(\theta^{*-1})$. When we consider all countries, the coefficient of variation (CV) of $\gamma$, 0.70, is double the
Table 3. Risk Aversion and Detection Error Probability (Full Sample)

This table shows the coefficient of variation of the risk aversion parameters $\gamma$, the penalty parameters $\theta^{*}-1$, and the detection error probabilities $p(\theta^{*}-1)$ of the calibration assuming that the consumption growth process follows a random walk model. The parameter $\gamma$ is chosen to satisfy the minimum of the Hansen–Jagannathan bounds: $\sigma(m) \geq \sigma^{*}(m)$. The discount factor for all countries is set to $\beta = 0.995$.

<table>
<thead>
<tr>
<th>Group</th>
<th>$\gamma$</th>
<th>$\theta^{*}-1$</th>
<th>$p(\theta^{*}-1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advanced economies</td>
<td>0.589</td>
<td>0.620</td>
<td>0.331</td>
</tr>
<tr>
<td>Emerging markets</td>
<td>0.744</td>
<td>0.812</td>
<td>0.374</td>
</tr>
<tr>
<td>All countries</td>
<td>0.698</td>
<td>0.746</td>
<td>0.351</td>
</tr>
</tbody>
</table>

The coefficient of $p(\theta^{*}-1)$, 0.35. The lower variation of detection error probabilities indicates that agents consider similar probabilities of committing mistakes when choosing the optimal allocation. This result stands even if we look within the country groups. The CV of $\gamma$ is 0.59 among AE and the coefficient of $p(\theta^{*}-1)$ is only 0.33. When we look to EM, the 2:1 ratio also stands, 0.74 for $\gamma$ and 0.38 for the detection error probability.

It is also important to compare the plausibility of the estimated parameters since it helps us to have a perception of the model suitability. On the first column of Table 2, we can see that only Brazil, India, Mexico, and Turkey present $\gamma$ estimates under 5.0, a common threshold in the literature. Even if we consider a looser bound of 10.0, the number of countries inside this limit would not be considerable. Only seven countries of eighteen in our sample would satisfy this bound, corroborating the equity risk premium puzzle to international data.

On the other hand, when we follow Barillas et al. (2009) and use model uncertainty to reinterpret the high level of $\gamma$, we find very different results. Note on Figure 1 that only Colombia and the US present a detection error probability below the 0.15 bound proposed by Hansen and Sargent (2008). Except for these two economies, model misspecification explains the equity premium for all countries in our sample. This is a stark contrast to the results of the specification without model uncertainty and indicates the better suitability of this model to our sample. Additionally, when using the full sample of all countries, we cannot identify any pattern difference between AE and EM, indicating that the model is equivalent for both country groups.
Figure 1. Detection error probabilities estimates using the full sample ($\beta_{AE} = \beta_{EM} = 0.995$).

4.2 Welfare Costs

After estimating the risk parameters for the countries and analyzing its feasibility, we study the welfare gains of eliminating the “traditional” risk and model uncertainty. Following Barillas et al. (2009), we describe how market prices of uncertainty extracted from data contain information about how much the representative consumer would be willing to pay to eliminate model uncertainty. We use as a point of comparison the certainty equivalent plan

$$c_{t+1} - c_t = \mu + \frac{1}{2}\sigma^2 \epsilon_t.$$  \hspace{1cm} (22)

We seek an adjustment to initial consumption that renders a representative consumer indifferent between the certainty equivalent plan and the original risky consumption plan. For the same initial conditions, the certainty equivalent path of consumption $\exp c_{t+1}$ has the same mean as the original plan $c_{t+1} - c_t = \mu + \sigma \epsilon_{t+1}$, but its conditional variance has been reduced to zero.

Recall the formula for the value function of the representative agent facing a random walk process for log consumption, specified by

$$U(c_0) = \frac{\beta}{(1-\beta)^2} \left( \mu - \frac{\sigma^2}{2(1-\beta)} \theta \right) + \frac{1}{1-\beta} c_0.$$ \hspace{1cm} (23)

We seek a proportional decrease in the certainty equivalent trajectory that leaves
$U$ equal to its value under the risky process. Let $c_0^I$ denote the initialization of the certainty equivalent trajectory for an agent. Evidently, it satisfies equation

$$
\frac{\beta}{(1-\beta)^2} \left( \mu + \frac{\sigma^2}{2} \right) + \frac{1}{1-\beta} c_0^I = \frac{\beta}{(1-\beta)^2} \left( \mu - \frac{\sigma^2}{2(1-\beta)\theta} \right) + \frac{1}{1-\beta} c_0.
$$  (24)

The left side is the value under the certainty equivalent plan, while the right side is the value under the original risky plan starting from $c_0$. Solving for $c_0 - c_0^I$ gives

$$
c_0 - c_0^I = \frac{\beta}{(1-\beta)} \left( \frac{\sigma^2}{2} + \frac{\sigma^2}{2(1-\beta)\theta} \right) = \frac{\beta \sigma^2}{2(1-\beta)} \left( 1 + \frac{1}{(1-\beta)\theta} \right).
$$  (25)

Note that this is exactly how much of today consumption the agent would be willing to give up to eliminate all risk and uncertainty from future consumption. As in Barillas et al. (2009), we now consider an agent who does not fear model uncertainty, so that $\theta = \infty$. We ask how much adjustment in the initial condition of a certainty equivalent path a $\theta = +\infty$ type of consumer would require. The compensating variation for the elimination of risk alone must satisfy

$$
\frac{\beta}{(1-\beta)^2} \left( \mu + \frac{\sigma^2}{2} \right) + \frac{1}{1-\beta} c_0 = \frac{\beta}{(1-\beta)^2} \mu + \frac{1}{1-\beta} c_0^I(r).
$$  (26)

In constructing the right side, we have set $\theta = \infty$ and replaced $c_0$ with $c_0^I(r)$. Solving the above equation for $c_0 - c_0^I(r)$ gives

$$
c_0 - c_0^I(r) = \frac{\beta \sigma^2}{2(1-\beta)}.
$$  (27)

Evidently, the part of the compensation that is accounted for by aversion to model uncertainty is

$$
c_0^I(r) - c_0^I = \frac{\beta \sigma^2}{2(1-\beta)} \left( \frac{1}{(1-\beta)\theta} \right) = \frac{\beta \sigma^2}{2(1-\beta)^2 \theta}.
$$  (28)

Here, it is important to notice that the results might be affected by the difference of the sample periods of the countries. A period of higher global uncertainty might have different weights in the sample of the countries and affect the results. For example, while Australian data starts in 1970 and ends in 2015,
the Brazilian sample has only 17 years, from 1999 to 2015. It is clear that the
Great Recession is more relevant to the Brazilian sample than to the Australian
one. As a consequence, a significant amount of the differences between the
countries’ results might be caused by the different sample periods, which is much
larger for most AE countries in our sample. In this way, we now use the same
sample period for all countries. We set the sample period from the beginning of
1999 to the end of 2015 and exclude all countries whose data does not fit in this
period.

Table 4 gives a brief description of this new data set. Note that now our
sample has fewer countries than before. India, Russia, and Turkey were excluded
because they do not have available data for the whole period. We can also
observe that only the sample of AE countries changed when comparing with the
full sample. All EM on Table 4 had their data beginning in 1999 and ending
in 2015 in the full sample. Then, we will only have changes in the results of
AE. Comparing Tables 4 and 1 we can notice a changing pattern in the sample
of the AE. For almost all countries, both µ and σ estimated using the shorter
sample are smaller than the previous ones. The biggest decrease in µ was in
the Japanese estimate, in line with their strong GDP growth deceleration in
last decades. We can also see a changing pattern of the stock returns. Except
for Australia and Canada, whose stock markets benefited from the commodity
boom in the last 15 years, all average stock returns decreased. Finally, all AE
short term rates are also smaller in this new sample than in the previous one.

We replicate the exercise of the preceding section, assuming \( \beta_{AE} = \beta_{EM} = 0.995 \) and that the consumption growth process follows a random walk model.
Table 5 show the results of the estimation of \( \gamma \) and \( p(\theta^{*}-1) \) that attain the
Hansen–Jagannathan bounds. The new \( \gamma \)'s computed using the 1999–2015
sample are shown in the second column of the Table 5. The most significant
changes were in the estimates for Australia, Great Britain, Italy, and Japan.
Australian and Japanese estimates, which were lower than ten using the full
sample, increased to well above this level. On the other hand, the estimates of
Great Britain fell from 15.9 to 5.2 and of Italy from 11.1 to 0.5. However, the
most significant result of our exercise is the general increase in the estimates of
\( p(\theta^{*}-1) \). Except for Australia, which the value did not change, all new detection
error probabilities estimates of the AE grew, indicating a lower degree of model
uncertainty in this period. The estimates for Japan and USA, which were
under 0.20 for the full sample, surpassed 0.30 when using the 1999–2015 sample.
**Table 4. Description of the homogeneous sample data set**

This table reports an overview of the 1999–2015 sample. The second and third columns show the results of ML estimation of the real consumption growth processes assuming a random walk model. Stocks and ST Rate are the quarterly average real return of stocks and short-term interest rate of each country, respectively. The last two columns show the initial and last date of the sample of each country.

<table>
<thead>
<tr>
<th>Country</th>
<th>μ</th>
<th>σ</th>
<th>Stocks (%)</th>
<th>ST Rate (%)</th>
<th>Min Date</th>
<th>Max Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>AU</td>
<td>0.003</td>
<td>0.006</td>
<td>1.786</td>
<td>0.457</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>BR</td>
<td>0.004</td>
<td>0.010</td>
<td>2.413</td>
<td>2.025</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>CA</td>
<td>0.003</td>
<td>0.004</td>
<td>1.903</td>
<td>0.130</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>CL</td>
<td>0.007</td>
<td>0.011</td>
<td>1.883</td>
<td>0.226</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>CO</td>
<td>0.011</td>
<td>0.032</td>
<td>4.442</td>
<td>0.539</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>DE</td>
<td>0.003</td>
<td>0.005</td>
<td>1.743</td>
<td>0.144</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>FR</td>
<td>0.002</td>
<td>0.004</td>
<td>1.339</td>
<td>0.139</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>GB</td>
<td>0.003</td>
<td>0.006</td>
<td>0.545</td>
<td>0.258</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>ID</td>
<td>0.005</td>
<td>0.016</td>
<td>3.536</td>
<td>0.440</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>IT</td>
<td>0.000</td>
<td>0.006</td>
<td>0.390</td>
<td>0.428</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>JP</td>
<td>0.002</td>
<td>0.008</td>
<td>1.063</td>
<td>0.021</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>KR</td>
<td>0.008</td>
<td>0.008</td>
<td>2.708</td>
<td>0.245</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>MX</td>
<td>0.003</td>
<td>0.036</td>
<td>2.850</td>
<td>0.937</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>US</td>
<td>0.003</td>
<td>0.004</td>
<td>0.835</td>
<td>0.027</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
<tr>
<td>ZA</td>
<td>0.003</td>
<td>0.008</td>
<td>2.751</td>
<td>0.615</td>
<td>1999-03-01</td>
<td>2015-12-01</td>
</tr>
</tbody>
</table>

Using the homogeneous sample, only the Colombian \(p(\theta^* - 1)\) that attain the HJ bounds is under the lower limit of 0.15 suggested by Hansen and Sargent (2008). Besides most countries’ \(p(\theta^* - 1)\) remaining on the reasonable bound as in the full sample exercise, the variability of the parameter is still lower when using the homogeneous sample. The coefficient of variation between countries’ estimates is more than two times bigger for \(\gamma\) than for \(p(\theta^* - 1)\) (Table 6).

Figure 2 presents the estimates of \(p(\theta^* - 1)\) for all countries. Looking at Figure 2 it is easy to see that the \(p(\theta^* - 1)\) estimates for AE are larger than 0.25 and greater than the figures of EM (except when comparing Canada and Mexico, which have the same values and for Brazil which has the second largest value). Additionally, except for Brazil, the estimates of EM are lower than 0.30 and Colombia and South Africa are the only countries with detection error probabilities under 0.20. Then, it seems that the robust approach assuming a random walk model for the consumption growth process indicates a higher level of uncertainty aversion for EM during the 1999–2015 period than for advanced economies.
Table 5. Risk Aversion and Detection Error Probability (Homogeneous Sample)
This table shows the results of the risk aversion parameter $\gamma$, the penalty parameter $\theta^{*-1}$, and the detection error probability $p(\theta^{*-1})$ of the calibration assuming that the consumption growth process follows a random walk model. The parameter $\gamma$ is chosen to satisfy the minimum of the Hansen-Jagannathan bounds: $\sigma(m) \geq \sigma^*(m)$. The discount factor for all countries is set to $\beta = 0.995$.

<table>
<thead>
<tr>
<th>Country</th>
<th>$\gamma$</th>
<th>$\theta^{*-1}$</th>
<th>$p(\theta^{*-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AU</td>
<td>17.945</td>
<td>0.085</td>
<td>0.331</td>
</tr>
<tr>
<td>BR</td>
<td>2.909</td>
<td>0.010</td>
<td>0.468</td>
</tr>
<tr>
<td>CA</td>
<td>34.782</td>
<td>0.169</td>
<td>0.271</td>
</tr>
<tr>
<td>CL</td>
<td>16.353</td>
<td>0.077</td>
<td>0.239</td>
</tr>
<tr>
<td>CO</td>
<td>9.024</td>
<td>0.040</td>
<td>0.145</td>
</tr>
<tr>
<td>DE</td>
<td>23.498</td>
<td>0.112</td>
<td>0.327</td>
</tr>
<tr>
<td>FR</td>
<td>25.899</td>
<td>0.124</td>
<td>0.345</td>
</tr>
<tr>
<td>GB</td>
<td>5.362</td>
<td>0.022</td>
<td>0.460</td>
</tr>
<tr>
<td>ID</td>
<td>11.616</td>
<td>0.053</td>
<td>0.245</td>
</tr>
<tr>
<td>IT</td>
<td>0.525</td>
<td>-0.002</td>
<td>0.504</td>
</tr>
<tr>
<td>JP</td>
<td>14.064</td>
<td>0.065</td>
<td>0.337</td>
</tr>
<tr>
<td>KR</td>
<td>21.284</td>
<td>0.101</td>
<td>0.259</td>
</tr>
<tr>
<td>MX</td>
<td>4.848</td>
<td>0.019</td>
<td>0.285</td>
</tr>
<tr>
<td>US</td>
<td>24.132</td>
<td>0.116</td>
<td>0.345</td>
</tr>
<tr>
<td>ZA</td>
<td>29.989</td>
<td>0.145</td>
<td>0.182</td>
</tr>
</tbody>
</table>

Table 6. Risk Aversion and Detection Error Probability (Full Sample)
This table shows the coefficient of variation of the risk aversion parameters $\gamma$, the penalty parameters $\theta^{*-1}$, and the detection error probabilities $p(\theta^{*-1})$ of the calibration assuming that the consumption growth process follows a random walk model. The parameter $\gamma$ is chosen to satisfy the minimum of the Hansen–Jagannathan bounds: $\sigma(m) \geq \sigma^*(m)$. The discount factor for all countries is set to $\beta = 0.995$.

<table>
<thead>
<tr>
<th>Group</th>
<th>$\gamma$</th>
<th>$\theta^{*-1}$</th>
<th>$p(\theta^{*-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Advanced economies</td>
<td>0.618</td>
<td>0.653</td>
<td>0.211</td>
</tr>
<tr>
<td>Emerging markets</td>
<td>0.699</td>
<td>0.753</td>
<td>0.396</td>
</tr>
<tr>
<td>All countries</td>
<td>0.645</td>
<td>0.688</td>
<td>0.323</td>
</tr>
</tbody>
</table>
Figure 2. Detection error probabilities estimates using the homogeneous sample ($\beta_{AE} = \beta_{EM} = 0.995$).

Figure 3 compares the potential gains of eliminating risk and model uncertainty for all countries between 1999 and 2015. It is evident when looking at the figure that the welfare gains from reducing uncertainty are much greater in developing countries. All AE are concentrated on the lower left of the graph, while the EM are spread upward and toward the right in the graph. The only exception is Brazil, which we should remember that has a high $\theta^*$ parameter, way above the 0.20 bound. The AE’ households would be willing to reduce their average consumption growth by a value between 0 and 10% in order to eliminate model uncertainty. For the EM, these figures are considerably higher. This result is not surprising since EM are usually considered more unstable economically and politically than AE. It is also noteworthy that even considering only emerging economies, the results of Colombia and Mexico stand out.

We could also ask if the EM has proportionately more model uncertainty than AE. Figure 4 shows the ratio between the welfare gains from eliminating model uncertainty and the gains from just removing risk. Looking at the figure, we notice that, on average, AE bear more model uncertainty proportionately than EM. From the eight AE countries in the sample, six have a ratio greater than 10, while for EM we have four out of seven.
Figure 3. Welfare gains from eliminating risk and model uncertainty between 1999 and 2015 ($\beta_{AE} = \beta_{EM} = 0.995$ and $p(\theta^{-1})$ estimated).

Figure 4. Ratio between welfare gains from eliminating only model uncertainty to gains from eliminating only risk between 1999 and 2015 ($\beta_{AE} = \beta_{EM} = 0.995$ and $p(\theta^{-1})$ estimated).
4.3 Robustness

In this section, we repeat previous exercises but using a different calibration for $\beta_{EM}$. Instead of using the risk-free short-term rate of Brazil, Colombia, and Mexico to calibrate $\beta_{EM}$, we use the average short-term rate of these countries without discounting for risk.\footnote{If we use the average short-term rate of all EM, then $\beta_{EM} = 0.994$, which will not significantly change the results from the previous section.}

Then, the value of $\beta_{EM}$ reduces from 0.995 to 0.989. Since the estimates for the detection error probabilities are almost not affected by the value of $\beta$ (Table 7), we discuss only the results of the welfare analysis.

Table 7. Risk Aversion and Detection Error Probability using $\beta_{EM} = 0.989$ (Homogeneous Sample)

<table>
<thead>
<tr>
<th>Country</th>
<th>$\gamma$</th>
<th>$\theta^{*-1}$</th>
<th>$p(\theta^{*-1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AU</td>
<td>17.945</td>
<td>0.085</td>
<td>0.331</td>
</tr>
<tr>
<td>BR</td>
<td>2.926</td>
<td>0.021</td>
<td>0.468</td>
</tr>
<tr>
<td>CA</td>
<td>34.782</td>
<td>0.169</td>
<td>0.271</td>
</tr>
<tr>
<td>CL</td>
<td>16.450</td>
<td>0.170</td>
<td>0.237</td>
</tr>
<tr>
<td>CO</td>
<td>9.076</td>
<td>0.089</td>
<td>0.143</td>
</tr>
<tr>
<td>DE</td>
<td>23.498</td>
<td>0.112</td>
<td>0.327</td>
</tr>
<tr>
<td>FR</td>
<td>25.899</td>
<td>0.124</td>
<td>0.345</td>
</tr>
<tr>
<td>GB</td>
<td>5.362</td>
<td>0.022</td>
<td>0.460</td>
</tr>
<tr>
<td>ID</td>
<td>11.685</td>
<td>0.118</td>
<td>0.244</td>
</tr>
<tr>
<td>IT</td>
<td>0.525</td>
<td>-0.002</td>
<td>0.504</td>
</tr>
<tr>
<td>JP</td>
<td>14.064</td>
<td>0.065</td>
<td>0.337</td>
</tr>
<tr>
<td>KR</td>
<td>21.411</td>
<td>0.225</td>
<td>0.258</td>
</tr>
<tr>
<td>MX</td>
<td>4.877</td>
<td>0.043</td>
<td>0.284</td>
</tr>
<tr>
<td>US</td>
<td>24.132</td>
<td>0.116</td>
<td>0.345</td>
</tr>
<tr>
<td>ZA</td>
<td>30.166</td>
<td>0.321</td>
<td>0.181</td>
</tr>
</tbody>
</table>

Figure 5 shows the welfare gains from eliminating risk and model uncertainty for all countries using the homogeneous sample. As expected, the lower $\beta_{EM}$ reduces both risk and model uncertainty elimination gains for all EM. However, emerging economies still present more gains than AE in general. The only exceptions are Brazil and Korea. In this way, even using a lower $\beta_{EM}$, the results
Figure 5. Welfare gains from eliminating risk and model uncertainty using $\beta_{EM} = 0.989$ (full sample and $p(\theta^{-1})$ estimated).

that welfare gains from reducing uncertainty are much greater in developing countries still holds.

5 Conclusion

We evaluate Barillas et al. (2009) model of ambiguity averse agents in light of a broader dataset by including developing economies and a longer time-span and show that including model uncertainty aversion also helps to explain the equity premium for countries other than the US. Additionally, when considering the same sample period for all countries, we find the fear for model misspecification higher for EM than for AE which is in line of what would be expected. This result is expected since EM usually have more political instability and are more susceptible to economic shocks. Brazil stands out from the other EM, since it possesses a low level of model uncertainty aversion, which can be checked by looking at its 0.468 detection error probability. This value is very close to the 0.5 achieved for a perfect foresight economy. This result corroborates the analysis of a very low (or none) equity premium observed in Brazil financial markets for this sample period. One possible explanation for this discrepancy is that since the stock market is more volatile when compared to Advanced Economies, the short time span of data available is not able to capture the long run equity premium present in the economy. Other possibility is that there
are unidentified factors that make the equity premium low in Brazil. Further investigation is required to understand the low equity premium in Brazil for the time span analyzed here. We also were able to identify model uncertainty as the most important factor of welfare loss when comparing to risk effects alone. This result suggests that a policy to prevent uncertainty may be considered to improve welfare with greater possibility of welfare gains for EM.

References


Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics, 121*(3), 823–866. [http://dx.doi.org/10.1162/qjec.121.3.823](http://dx.doi.org/10.1162/qjec.121.3.823)


Risk Aversion or Model Uncertainty?
An Empirical Cross-Sectional Analysis Across Countries

10.2307/1913656
Mankiw, N. G. (1985). Consumer durables and the real interest rate. The
dx.doi.org/10.3386/w9512
ideas.repec.org/a/eee/moneco/v15y1985i2p145-161.html
Ogaki, M., & Reinhart, C. M. (1998a). Intertemporal substitution and durable
Ogaki, M., & Reinhart, C. M. (1998b). Measuring intertemporal substitution:
The role of durable goods. Journal of Political Economy, 106(5), 1078–
1098.
Okubo, M. (2015, May). Model uncertainty and international differences in
risk aversion (Working paper No. 1329). Tsukuba, Japan: University of
pdf/1329.pdf
when both income and substitution effects are present: The role of durable
from https://books.google.com.br/books?id=HeL1E-wJqm4C
Monetary Economics, 45(3), 507–532. http://dx.doi.org/http://dx.doi
.org/10.1016/S0304-3932(00)00012-X
Appendix A
A.1 Computing Welfare Costs
Lucas, Reis and other authors compute the welfare cost of consumption fluctuations in a different fashion. They compute an annual cost instead of computing a present value of costs as we do here. In their computation they define the costs of fluctuations as the scalar $\lambda$ that solves the equation
\[ E \left[ \sum_{t=0}^{\infty} e^{-\beta t} u(C_t(1 + \lambda)) \right] = \sum_{t=0}^{\infty} e^{-\beta t} u(\bar{C}_t), \]
for $u(C_t) = \ln(C_t)$ it is not difficult to show that
\[ \ln(1 + \lambda) = 0.5 \left(1 - e^{-\beta}\right) \sum_{t=0}^{\infty} e^{-\beta t} \text{Var}(c_t) \]
Furthermore, if log consumption follows as stationary AR(1) process, $\text{Var}(c_t) = \sigma^2(1 - \eta^2)/(1 - \eta^2)$ for $t \geq 1$. Evaluating the sum on the right hand side of the above equation we get
\[ \ln(1 + \lambda) = \frac{0.5\sigma^2}{e^{\beta} - \eta^2} \]
A.2 Kreps–Porteo–Epstein–Zin preference representation
We use preferences that can be described by a recursive non-expected utility function à la Kreps and Porteus (1978),
\[ V_t = W(C_t, \mu(V_{t+1})), \]
where $W$ is an aggregator function.

The idea of this kind of representation is that the agent’s preference is a function of today’s consumption and future consumption. The future consumption is uncertain but we may be able to compute an equivalent quantity that the agent would be willing to consume for sure next period in exchange for all possible future consumption. This quantity $\mu$ is a certainty equivalent function
\[ \mu(V_{t+1}) = f^{-1}(E_t f(V_{t+1})), \]
where $f$ is a function that determines attitudes toward atemporal risk. It determines how much an individual would be willing to trade off expected future costs.
consumption to have a certain quantity of consumption for sure. Note that if $f$ is increasing and concave, then by Jensen inequality we have

$$E[f(X)] \leq f(E[X]).$$

And by monotonicity, for any $\mu$ such that

$$f(\mu(X)) = E[f(X)] \leq f(E[X]) \rightarrow \mu(X) \leq E[X]$$

Epstein–Zin use a constant relative risk aversion (CRRA) function to represent individual attitude towards risk

$$f(z) = z^{1-\gamma},$$

$$f(z) = \log(z), \quad \text{if } \gamma = 1,$$

and $\gamma$ is the coefficient of relative risk aversion.

Following Epstein and Zin (1991) it is common to use the CES aggregator $W$:

$$W(C_t, \mu) = \left(1 - \beta\right)C_t^{1-\eta} + \beta \mu^{1-\eta} \right]^{\frac{1}{1-\eta}},$$

$$\lim_{\eta \to 1} W(C_t, \mu) = C_t^{1-\beta} \mu^\beta.$$

Tallarini used a power certainty equivalent function to get the following recursive utility under uncertainty

$$V_t = C_t^{1-\beta} \left[ \left( E_t \left[ V_{t+1}^{1-\gamma}\right] \right)^{\frac{1}{1-\gamma}} \right]^\beta.$$

Taking logs gives

$$\log V_t = (1 - \beta)c_t + \frac{\beta}{1-\gamma} \log E_t \left[ V_{t+1}^{1-\gamma}\right],$$

where $c_t = \log C_t$ or

$$\frac{\log V_t}{(1 - \beta)} = c_t + \frac{\beta}{(1-\gamma)(1-\beta)} \log E_t \left[ V_{t+1}^{1-\gamma}\right].$$

Define $U_t = \frac{\log V_t}{(1 - \beta)}$ and $\theta = \frac{-1}{(1-\gamma)(1-\beta)}$, then

$$U_t = c_t - \beta \theta \log E_t \left[ \exp \left( \frac{-U_{t+1}}{\theta} \right) \right].$$
This is the risk-sensitive recursion of Hansen and Sargent (1995). In the special case that $\gamma = 1$ ($\theta = +\infty$) the recursion becomes the standard discounted expected utility recursion

$$U_t = c_t + \beta E_t U_{t+1}.$$ 

This relation only implies that the total utility of the agent can be expressed as the current utility plus a discounted expected future utility, where $\beta$ is the intertemporal discounting parameter.

The recursion implies the following Bellman equation for the random walk case

$$U(c) = c - \beta \theta \log \mathbb{E}_t \left[ \exp \left( -\frac{U(c + \mu + \sigma \epsilon)}{\theta} \right) \right].$$ 

The value function that solves this equation is

$$U(c) = \frac{\beta}{(1 - \beta)^2} \left( \mu - \frac{\sigma^2}{2\theta(1 - \beta)} \right) + \frac{1}{1 - \beta} c.$$ 

We model the agents’ preferences in a way that we can interpret $\theta$ as a parameter that measures concern for model specification. To do so, we must describe how the agents evaluate consumption in an environment of model uncertainty. There are two approaches that are equivalent under some conditions. The first uses multiplier preferences and the other uses constraint preferences.

### A.3 The meaning of entropy

Now, to understand a little bit about the penalizing factor, notice that $h(x) = x \log x$ is a convex function for values of $x \geq 0$, so that $h(y) - h(x) \geq h'(x)(y - x)$. Then we have

$$h(G_t) - h(1) = G_t \log G_t \geq G_t - 1 = h'(1)(G_t - 1),$$ 

so that

$$\mathbb{E}[G_t \log G_t \mid x_0] \geq 0.$$ 

Again, we only have $\mathbb{E}[G_t \log G_t \mid x_0] = 0$ if $G_t = 1$ in which case there is no probability distortion associated with time $t$ shock distribution. The factorization $G_t = \prod_{j=1}^t g_j$ implies the following decomposition of entropy

$$\mathbb{E}[G_t \log G_t \mid x_0] = \sum_{j=0}^{t-1} \mathbb{E} \left[ G_j \mathbb{E} \left[ g_{j+1} \log g_{j+1} \mid \varepsilon_j, x_0 \right] \mid x_0 \right].$$
We can then compute the discounted entropy over an infinite horizon as
\[
(1 - \beta) \sum_{j=0}^{\infty} \beta^j E[G_j \log G_j \mid x_0] = \sum_{j=0}^{\infty} \beta^j E[G_j E[g_{j+1} \log g_{j+1} \mid \varepsilon^j, x_0] \mid x_0].
\]
This term multiplied by the parameter $\theta$ is exactly the second term of the summation formula that describes agent preference and summarizes all uncertainty considered by the agent. The parameter $\theta$ represents how much weight the agent gives to all the uncertainty present in the economy.

A.4 Hansen–Jagannathan Bounds

We want to find $m$ such that $E[mR] = 1_N$, where $1_N \in \mathbb{R}^N$ is a vector of ones and $R$ is a vector of returns of the risky assets. We claim that any
\[
m = E[m] + (1_N - E[m]E[R])' \Sigma^{-1} (R - E[R]) + \varepsilon
\]
where $\Sigma = \text{Cov}(R, R')$ with $\varepsilon$ such that $E[\varepsilon] = 0$ and $E[\varepsilon R] = 0$ do the job. To see this, first note that
\[
E[mR]'E[m]E[R]' = E[m(R - E[R])']
\]
substituting $m$ from the claim we get
\[
E[m(R - E[R])'] = (1_N - E[m]E[R])'
\]
We conclude that $E[mR] = 1_N$. Now since $\sigma^2(\varepsilon) \geq 0$ we have
\[
\sigma^2(m) = E[(m - E[m])^2] = E[(1_N - E[m]E[R])' \Sigma^{-1} (R - E[R]) + \varepsilon]^2] \geq E[(1_N - E[m]E[R])' \Sigma^{-1} (R - E[R])(R - E[R])' \Sigma^{-1} (1_N - E[m]E[R])] = (1_N - E[m]E[R])' \Sigma^{-1} (1_N - E[m]E[R]),
\]
where, if exists a risk free asset, $E[m] = 1/R^f$.

A.5 Formulas for Detection Error Probabilities

Consider the following AR(1) process:
\[
c_{t+1} = c_t + \mu + \sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \text{iid} \mathcal{N}(0, 1).
\]
The average log-likelihood function for a sample of $t = 1, \ldots, T$ takes the form

$$\ln L = \frac{1}{T} \ln f(c_1) + \frac{1}{T} \sum_{t=2}^{T} \ln f(c_t | c_{t-1}).$$

Under the baseline Model A we have $\bar{\mu} = \mu$ and

$$\ln L_A = -\frac{1}{2} \ln 2\pi - \frac{1}{2} \ln \sigma^2 - \frac{1}{2T} \sigma^2 (c_1 - \mu)^2 - \frac{1}{T} \sum_{t=2}^{T} \frac{1}{2\sigma^2} (c_t - c_{t-1} - \mu)^2,$$

and under the worst case Model B, $\bar{\mu} = \mu + \sigma w$, and the log-likelihood is given by

$$\ln L_B = -\frac{1}{2} \ln 2\pi - \frac{1}{2} \ln \sigma^2 - \frac{1}{2T} \sigma^2 (c_1 - \mu - \sigma w)^2$$

$$- \frac{1}{T} \sum_{t=2}^{T} \frac{1}{2\sigma^2} (c_t - c_{t-1} - \mu - \sigma w)^2.$$

Thus, we obtain the log-likelihood ratio

$$\ln \left( \frac{L_A}{L_B} \right) = -\frac{1}{T} \left[ \frac{1}{2\sigma^2} (c_1 - \mu)^2 + \sum_{t=2}^{T} \frac{1}{2\sigma^2} (c_t - c_{t-1} - \mu)^2 \right]$$

$$+ \frac{1}{T} \left[ \frac{1}{2\sigma^2} (c_1 - \mu - \sigma w)^2 + \sum_{t=2}^{T} \frac{1}{2\sigma^2} (c_t - c_{t-1} - \mu - \sigma w)^2 \right].$$

To calculate the detection error probability under Model A, we only need to substitute $c_1 - \mu = \sigma \varepsilon_1$ and $c_t - c_{t-1} - \mu = \sigma \varepsilon_t$ for $t = 2, \ldots, T$ which yields

$$\ln \left( \frac{L_A}{L_B} \right) \mid \text{Model A is correct} = \frac{1}{T} \sum_{t=1}^{T} \left[ -\frac{1}{2\sigma^2} \varepsilon_t^2 + \frac{1}{2\sigma^2} (\varepsilon_t - w)^2 \right]$$

$$= \frac{1}{T} \sum_{t=1}^{T} (-w \varepsilon_t) + \frac{1}{2} w^2.$$
Therefore, the detection error probability under Model A is

\[
p\left( \ln \left( \frac{L_A}{L_B} \right) < 0 \mid \text{Model A is correct} \right)
= p \left( \frac{1}{T} \sum_{t=1}^{T} (-w\varepsilon_t) + \frac{1}{2} w^2 < 0 \right)
= p \left( \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\sigma}{\theta(1-\beta)} \right) \varepsilon_t < \frac{1}{2} \left( \frac{\sigma}{\theta(1-\beta)} \right)^2 \right)
= p \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t < -\frac{\sqrt{T}}{2} \frac{\sigma}{\theta(1-\beta)} \right)
= \Phi \left( -\frac{\sqrt{T}}{2} \frac{\sigma}{\theta(1-\beta)} \right),
\]

where \( \Phi \) is the Normal cumulative distribution function.

On the other hand, if we consider Model B as the correct one, we have \( c_1 - \mu = \sigma w + \sigma \varepsilon_1 \) and \( c_t - c_{t-1} - \mu = \sigma \varepsilon_t \) for \( t = 2, \ldots, T \) which yields

\[
\ln \left( \frac{L_A}{L_B} \right) \mid \text{Model B is correct} = \frac{1}{T} \sum_{t=1}^{T} \left[ -\frac{1}{2} (\varepsilon_t + w)^2 + \frac{1}{2} \varepsilon_t^2 \right]
= \frac{1}{T} \sum_{t=1}^{T} (-w\varepsilon_t) - \frac{1}{2} w^2
\]

Therefore, the detection error probability under Model B is

\[
p\left( \ln \left( \frac{L_A}{L_B} \right) > 0 \mid \text{Model B is correct} \right)
= p \left( \frac{1}{T} \sum_{t=1}^{T} (-w\varepsilon_t) - \frac{1}{2} w^2 > 0 \right)
= p \left( \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\sigma}{\theta(1-\beta)} \right) \varepsilon_t > \frac{1}{2} \left( \frac{\sigma}{\theta(1-\beta)} \right)^2 \right)
= p \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t > \frac{\sqrt{T}}{2} \frac{\sigma}{\theta(1-\beta)} \right)
= 1 - \Phi \left( \frac{\sqrt{T}}{2} \frac{\sigma}{\theta(1-\beta)} \right).
\]