The Relation Between Expected Returns and Volatility in the Brazilian Stock Market

Ricardo R. G. Avelino**

Abstract

This paper applies the Markov switching regression model of stock returns with volatility feedback of Turner et al. (1989), suitably extended to incorporate endogenous regime shifts, as in Kim et al. (2008), to examine the intertemporal relationship between risk premium and volatility in the Brazilian stock market over the period 1995-2011. The results suggest that there is a positive relationship between risk premium and expected volatility once the volatility feedback effect is taken into account. By contrast, unanticipated increases in volatility have a negative impact on risk premium. This negative impact increases by 50% when I account for endogeneity of regime shifts.

Keywords: Risk Premium, Markov Switching, Endogenous Regime Shifts, Volatility Feedback.

JEL Codes: C13, C34, G10.
1. Introduction

From May to October 2008, the Ibovespa fell from 72.593 to 37.257 points, at a rate of 13.3% compounded monthly, after advancing more than sevenfold in the preceding 68 months. What explains this dramatic intertemporal variation in stock market prices and, in particular, the huge declines observed during short intervals of time in periods of crisis? Changes in the investors’ perception of risk, which may be proxied by movements in stock market volatility, are a possible answer. Unforeseen events such as a slowdown of the global economy that adversely impacts the demand for commodities, for instance, increase the riskiness from holding stocks of some major components of the Ibovespa to the extent that they dampen the expected value of the future flow of marginal product of existing capital and increase the variance of the firm’s marginal return on capital. If investors require a larger premium to hold stocks when the riskiness of capital investments increases, unanticipated increases in volatility will initially cause an immediate drop in stock prices and an associated increase in expected excess returns to induce them to hold the risky asset.

The intertemporal relation between risk premium, defined as the difference between the expected return on a market portfolio and the risk-free rate, and risk, measured by the volatility of the stock market, has been the subject of extensive research in the U.S. Pindyck (1984), for example, attributes the sharp reduction in stock prices in the U.S. during the 1970s to increases in market volatility and changing risk premia. Although it may seem that investors require a larger expected return from risky assets during times when volatility is higher, this is not a direct implication of theoretical asset pricing models.\footnote{Merton (1980) provides conditions under which the risk premium is approximately proportional to the \textit{ex ante} variance of the market return. Abel (1988) and Glosten et al. (1993), on the other hand, demonstrate that equilibrium does not imply a positive relation between the first two moments of returns.}

The empirical evidence is mixed. The results in French et al. (1987), Campbell and Hentschel (1992) and Kim et al. (2002), for instance, suggest the existence of a positive relationship between risk premium and stock market volatility. Campbell (1987), Glosten et al. (1993) and Whitelaw (1994), by contrast, document a negative relationship between expected excess returns and market volatility. One possible reason for the conflicting evidence is the failure of some of these studies to appropriately distinguish anticipated from unanticipated changes in volatility.

This paper employs the Markov switching regression model of stock returns with volatility feedback originally proposed by Turner et al. (1989), suitably extended to incorporate endogenous regime shifts, as in Kim et al. (2008), to examine the intertemporal relationship between risk premium and volatility in the Brazilian stock market. In this model, the two regimes are characterized by their variances and can be interpreted as a low and a high variance state.

During the process of trading between time periods \(t−1\) and \(t\), partial informa-
Information about the state at time $t$ is revealed to the agents, who use this information to update their prior probabilities of the states. Volatility feedback refers to the immediate decline in stock prices that follows an unanticipated increase in volatility if there is a positive relationship between risk premium and the anticipated level of volatility and if volatility is persistent. As pointed out earlier, this immediate decline in stock prices and associated increase in expected excess returns are necessary to induce agents to hold the risky asset when future expected market volatility rises. The incorporation of volatility feedback allows us to disentangle the impacts of anticipated and unanticipated changes in volatility. (See, for example, Turner et al. (1989) and Campbell and Hentschel (1992).)

There are basically four reasons for employing a Markov switching specification. First, it allows for heteroskedasticity and persistence in volatility, capturing two well-documented features of the distribution of stock returns: the pronounced peaks and heavy tails (see, e.g., Fama (1963) and Mandelbrot (1963)), which Turner et al. (1989) argue is a feature of unconditional normal observations subject to heteroskedasticity, and the predictability of the variance of stock returns (see, inter alia, Bollerslev et al. (1987); Engle et al. (1987), and Schwert (1989a,b)). Second, the evidence in Hamilton (1994) suggests that in a model that allows for both Markov switching and ARCH dynamics, most of the ARCH dynamics die out at the monthly return horizon. Third, as emphasized by Kim et al. (2002), the model avoids the complications inherent to the estimation of a nonlinear relationship between returns and regression errors in the QGARCH model or the statistical issues involved in the two-step procedure of French et al. (1987). Fourth, the Markov switching specification allows us to make different assumptions about the contents of the information set available to the agents and to make a distinction between what is known by the agents and by the econometrician at a specific point, thereby enabling us to test for the presence of volatility feedback.

In practice, the agents’ updated probability of the state, before the revelation of the final realized value of the excess return, is unobservable to the econometrician and replaced by the actual value of the state, leading to measurement error in the explanatory variables and rendering regime shifts endogenous. Kim et al. (2008) point out the substantial impact on the parameter estimates that results from the assumption of exogeneity when regime shifts are endogenous.

The model is applied to monthly Bovespa index returns in excess of the risk-free rate over the period 1995-2011 under different assumptions about the agents’ information set. The main results are as follows. When volatility feedback is ignored, the model identifies two distinct regimes, which are characterized by their volatilities and are very persistent. There is evidence that agents require a positive risk premium to hold the risky asset in the low variance state. However, the risk premium decreases as the anticipated level of risk increases, implying the existence of a negative relationship between expected excess returns and stock market volatility.
When volatility feedback is taken into account and the volatility feedback parameter is left unrestricted, this negative relationship between expected excess returns and stock market volatility disappears. In this case, the estimates suggest the existence of a positive, although not statistically significant, relation between risk premium and anticipated volatility. The estimates also provide evidence of a strong volatility feedback effect, indicating that agents are often surprised by unanticipated increases in volatility, and of endogeneity of regime shifts. Accounting for endogeneity leads to an increase of more than 50% in the volatility feedback parameter in absolute value.

The Markov switching regression model with volatility feedback can also be derived from the approximate log-linear present value framework of Campbell and Shiller (1988), as in Kim et al. (2002), which enables us to express the volatility feedback parameter in terms of the other parameters of the model. Imposing this restriction basically increases the precision of the estimate of the parameter associated with the relationship between risk premium and market volatility. The restriction cannot be rejected by a likelihood ratio (LR) test and leads to a positive and statistically significant relation between expected excess returns and the anticipated level of volatility. This relation is strengthened when endogeneity of regime shifts is taken into account. With endogenous regime shifts and the volatility feedback parameter restricted, as suggested by the LR tests, the unconditional estimate of the annualized risk premium is 4.05%.

An alternative explanation for the ex post negative relation between returns and volatility is the leverage effect, which posits that, when the price of the stock falls, the debt to equity ratio (financial leverage) of firms rises, making them riskier. This in turn tends to raise the contemporaneous and future volatility of returns. (See, e.g., Black (1976) and Christie (1982).) In this case, the direction of causality is reversed and we should expect to find lingering ARCH effects in the residuals of a model that incorporates only volatility feedback, as pointed out by Kim et al. (2002). Nevertheless, the ARCH-LM statistics of the different models in general do not display evidence of serial correlation.

Surprisingly, there is sparse evidence of the intertemporal relation between risk premium and volatility in the Brazilian stock market. To the best of my knowledge, the study of Tabak and Guerra (2007) is the only one that attempts to uncover the relationship between realized returns and current and future volatility using seemingly unrelated regressions, based on a sample of 25 stocks over the period 1990-2002. Their results suggest that stock returns and volatility are contemporaneously positively correlated, while there is a negative relationship between changes in volatility and returns. Nevertheless, the authors do not analyze the empirical relation between excess returns and volatility at the market level. Nor do they make a distinction between anticipated and unanticipated changes in market volatility.

The rest of the paper is structured as follows. Section 2 presents the Markov
switching regression model of stock returns and the alternative assumptions about
the information set available to agents. Section 3 discusses the maximum likelihood
estimation of the more general specification considered in the paper, which incor-
porates volatility feedback and endogeneity of regime shifts. Section 4 describes
the dataset used in this paper. Section 5 contains the empirical estimates of the
parameters of the different models and the results of formal statistical tests that
assess the adequacy of the alternative specifications. Finally, Section 6 concludes.

2. A Markov Switching Model of Stock Returns

Let \( r_t \) denote the continuously compounded excess return at time \( t \) with time-
varying expectation \( E[r_t|\Psi_{t-1}] \), where \( \Psi_{t-1} \) denotes the information available to
agents at \( t-1 \), and variance

\[
\sigma_{S_t}^2 = \begin{cases} 
\sigma_0^2 & \text{if } S_t = 0 \\
\sigma_1^2 & \text{if } S_t = 1 
\end{cases}
\]  

(1)

According to equation (1), the variance is a function of the index \( S_t \), which is
assumed to follow a first-order Markov process, with transition probabilities given
by

\[
P[S_t = 1|S_{t-1} = 1] = p
\]

\[
P[S_t = 0|S_{t-1} = 1] = 1 - p
\]

\[
P[S_t = 0|S_{t-1} = 0] = q
\]

\[
P[S_t = 1|S_{t-1} = 0] = 1 - q
\]

The two states \( S_t = 0 \) and \( S_t = 1 \) can be associated, respectively, with a low
volatility and a high volatility regime (\( \sigma_1 > \sigma_0 \)).

To complete the description of the model, it is necessary to specify the con-
ditional expectation of \( r_t \), which in turn depends on the information set \( \Psi_{t-1} \).
In the sequel, I make alternative assumptions about the agents’ information set,
following Kim et al. (2002).

2.1 Agents do not observe or do not react to information about the
volatility regime

The benchmark specification considered in this paper posits a constant risk
premium

\[ r_t = \mu_0 + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_{S_t}^2) \]
namely, it assumes that agents do not require additional return to compensate themselves for increased risk in the high variance state. In this case, the information set $\Psi_{t-1}$ collapses to the empty set.

2.2 Agents observe past returns and do not update their inferences about the state until the beginning of the next trading period

In this case, agents are unsure of the state and observe only past returns, i.e., $\Psi_{t-1} = \{r_{t-1}, r_{t-2}, \ldots\}$. Nevertheless, they know the underlying model that generates the states and the parameters governing their evolution. Since there are only two states, the prior distribution assigned by the agents to the states can be summarized by the probability of the high variance state, $P(S_t = 1|\Psi_{t-1})$, and the excess return can be expressed as a linear function of this probability

$$r_t = \mu_0 + \mu_1 P[S_t = 1|\Psi_{t-1}] + \varepsilon_t$$

As long as $\mu_1 > 0$, equation (2) implies that agents require an increase in the risk premium to invest in the risky asset at time $t$ when faced with an increase in the prior probability that the high variance state will prevail in that period. However, equation (2) does not allow us to distinguish the impact on the risk premium of unanticipated from anticipated changes in stock market volatility. The volatility feedback effect, incorporated into the discussion below, prevents us from confusing a negative relationship between realized returns and realized volatility with the underlying relationship between market volatility and the equity premium.

2.3 Agents observe past returns and revise their inferences about the state based on the information acquired during the process of trading

Although the true state is not known by the agents at the beginning of period $t$, it is plausible that it is partially revealed to them during period $t$ through the process of trading. If this is the case and stock market volatility is persistent and positively related to the equity premium, an exogenous increase in the level of market volatility (caused, for instance, by new information about future discounted expected returns) initially results in a drop of stock prices since investors require an increase in expected return in exchange for expected future volatility. The volatility feedback effect refers to this reduction in stock prices which results from an unanticipated increase in volatility. (See, e.g., Turner et al. (1989) and Campbell and Hentschel (1992).) It can be incorporated into the analysis by adding an additional term to equation (2).

$$r_t = \mu_0 + \mu_1 P[S_t = 1|\Psi_{t-1}] + \delta \left\{ P[S_t = 1|\Psi'_t] - P[S_t = 1|\Psi_{t-1}] \right\} + \varepsilon_t$$

The information set $\Psi'_t$ in expression (3) contains all the elements of $\Psi_{t-1} = \{r_{t-1}, r_{t-2}, \ldots\}$.
\{r_{t-1}, r_{t-2}, \ldots\} \text{ plus the partial information about the state acquired between } t - 1 \text{ and } t, \text{ before the final realized value of } r_t \text{ is revealed. Kim et al. (2002) show that equation (3) can also be derived from the approximate log-linear present value framework of Campbell and Shiller (1988) under the assumption that future expected returns are a linear function of the volatility of news. In this case, the following relation holds:}

\[\delta = -\frac{\mu_1}{1 - \zeta \lambda}, \quad \lambda \equiv p + q - 1\]

The parameter \(\zeta\) is the average of the ratio of the stock price and the sum of the stock price and the dividend, a number slightly smaller than one. Thus, if volatility is persistent, namely, \(\lambda > 0\), a negative volatility feedback effect is equivalent to a positive relationship between excess returns and market volatility.

As emphasized earlier, \(P\left[S_t = 1 | \Psi_t\right]\) is unobservable to the econometrician and it is often replaced by the true value of \(S_t\) when the model is estimated. This is the strategy adopted in this paper. Replacing \(P\left[S_t = 1 | \Psi_t\right]\) with the actual value of \(S_t\) leads to a measurement error in the explanatory variables, rendering \(S_t\) endogenous. The next section is devoted to the maximum likelihood estimation of the parameters under the assumption of endogeneity.

### 3. Maximum Likelihood Estimation of the Parameters

In this section, I discuss the maximum likelihood estimation of the parameters of the model given by equations (1) and (3). The estimation of the model is considered in detail in Kim et al. (2008) and is based on a straightforward generalization of the nonlinear filter proposed by Hamilton (1989), suitably extended to incorporate endogenous switching. Rewrite equation (3) as

\[r_t = x_t' \beta + \sigma S_t u_t\]

where \(x_t = [1 \quad P\left[S_t = 1 | \Psi_{t-1}\right] \quad S_t]'\) and \(\beta = [\mu_0 \quad \mu_1 - \delta \quad \delta]'\). The transition probabilities can be constrained to lie in the interval \([0, 1]\) by adopting a probit specification for \(S_t\):

\[S_t = \begin{cases} 1 & \text{if } \eta_t > a_{S_{t-1}} \\ 0 & \text{if } \eta_t \leq a_{S_{t-1}} \end{cases}\]

where \(a_1 = \Phi^{-1}(p)\), \(a_0 = \Phi^{-1}(1 - q)\) and \(\Phi\) denotes the cumulative distribution function of the standard normal.

The error terms \(u_t\) and \(\eta_t\) are assumed to follow a bivariate normal distribution with correlation coefficient \(\rho\), i.e.,

\[
\begin{bmatrix}
  u_t \\
  \eta_t
\end{bmatrix}
\sim i.i.d. N\left(
\begin{bmatrix}
  0 \\
  0
\end{bmatrix},
\begin{bmatrix}
  1 & \rho \\
  \rho & 1
\end{bmatrix}
\right)
\]
The filter consists of five steps and draws on the information contained in the observed series \( r_t \) to make probabilistic inference about the historical sequence of the states \( S_t \). It is iterated from \( t = 1 \) to \( T \) and can be initialized using the unconditional probabilities of the chain in the first iteration. The appendix provides the five equations that comprise the discrete-time filter. The filter requires a minor modification in the second step of the algorithm developed by Hamilton (1989), since the conditional density of \( r_t \), given the past returns and the current and past values of \( S_t \), is not simply a univariate normal. It includes two extra terms that emerge from the nonzero correlation between \( u_t \) and \( \eta_t \).

The likelihood of \( r_t \), conditional only on the past returns, is obtained in step 3, as a by-product of the filter,

\[
f(r_t | r_{t-1}, ..., r_1) = \sum_{s_t} \sum_{s_{t-1}} f(r_t, S_t = s_t, S_{t-1} = s_{t-1} | r_{t-1}, ..., r_1)
\]

and can be used to construct the sample log likelihood.

\[
L = \sum_{t=1}^{T} \log f(r_t | r_{t-1}, ..., r_1)
\]

This log likelihood function can be maximized numerically to find the maximum likelihood estimate of the parameter vector \( \theta = (\mu_0, \mu_1, \delta, \sigma_0, \sigma_1, p, q, \rho)^{\prime} \).

Kim et al. (2008) conducted Monte Carlo experiments and concluded that the maximum likelihood estimates of the parameters of the endogenous Markov switching model are very close to their true values when the model is correctly specified, even for relatively small sample sizes containing 200 observations. The maximum likelihood estimates assuming exogenous switching, by contrast, result in biased estimates when \( \rho \neq 0 \); moreover, the bias does not vanish as the sample size increases.

They also show that the Wald and LR tests can be used to test for endogenous switching and present evidence that the two tests have similar power. However, the LR test has approximately correct size, whereas the Wald test is somewhat oversized.

4. Description of the Data

The dataset used in this study consists of monthly returns from January 1995 to July 2011. The sample starts just after the Real plan in mid-1994 and excludes the hyperinflation period of the 1980s and early 1990s, since I am primarily interested in the impact of large discrete, but temporary, shifts in market volatility. Tabak and Guerra (2007), examining 25 Brazilian stocks over the period June 1990 – April 2002, find evidence of a structural break in 1994. Furthermore, the stock market is highly illiquid prior to 1995.
The Relation Between Expected Returns and Volatility in the Brazilian Stock Market

Figure 1
Evolution of Monthly Excess Returns

This figure shows the evolution of monthly excess returns over the period 1995-2011. The monthly excess return is computed as 100 times the difference between the continuously compounded Bovespa index return and the log of one plus the beginning-of-period Selic rate.

The monthly excess return series is the continuously compounded return on the Bovespa index, less the log return on the one-month risk-free rate. I use the beginning-of-period Selic rate as a proxy for the risk-free rate. Figure 1 illustrates the evolution of excess returns. The sample is not extremely long, but encompasses several episodes characterized by high volatile returns and by large movements of the index such as the Mexican crisis in 1995, the Russian crisis in 1998 and the recent global financial crisis which arose in 2007/2008 in the U.S.

Table 1 presents the summary statistics of monthly excess returns. Some features are noteworthy. The distribution of excess returns is extremely fat-tailed, which is often attributed to a tendency for new information to come infrequently in big lumps. (See Shiller (1981).) The statistics also show that excess returns are negatively skewed. They exceed 10% in absolute value only on four occasions: October 1997, August 1998, December 1998 and October 2008. All these movements correspond to a decline in stock prices. During the period under consideration, the mean of monthly excess returns equals -0.10%. The negative value is influenced by the high interest rates that prevailed in the first half of the sample under a fixed exchange rate regime. The volatility of excess returns is also substantially higher in the first subperiod, decreasing from 5.08% in the 1995-2002 subsample to 2.98% in the 2003-2011 subsample. Finally, we see that the first two autocorrelations of excess returns are very small for the entire sample, in spite of the moderate value.
This table presents the summary statistics of monthly continuously compounded Bovespa index returns in excess of the Selic rate over the period 1995-2011 as well as over the subperiods 1995-2002 and 2003-2011.

5. Empirical Results

This section presents the empirical estimates of the parameters under different assumptions about the information set $\Psi_t$ and discusses the results of formal statistical tests aimed at discriminating among the alternative specifications. I am particularly interested in how the incorporation of volatility feedback impacts the relationship between excess returns and stock market volatility and in the evidence of endogeneity of regime shifts.

5.1 Markov Switching Model with Constant Risk Premium

I start the discussion of the empirical results with the maximum likelihood estimates of the Markov switching regression model with constant risk premium, presented in the second column of Table 2 under the heading $\mu_1 = 0$. The estimate of $\mu_0$ is positive, although not statistically significant at conventional levels (the coefficient of 0.0038 has an associated $p$ value of 11.33%). This is surprising in view of the negative mean of monthly excess returns from 1995 to 2011. Turning now to the estimates of $\sigma_0$ and $\sigma_1$, notice that the model clearly identifies two distinct regimes. The standard deviation in the high variance state is roughly twice as big as in the low variance state. Moreover, we see that the parameters $\sigma_0 = 0.0255$ and $\sigma_1 = 0.0557$ are estimated very precisely.

The transition probabilities $p = 0.9590$ and $q = 0.9668$ also imply that the two regimes are very persistent. Once in the low (high) variance state, the system is expected to stay in the regime on average 30.1 (24.4) months. The parameter estimates of $p$ and $q$ imply that the unconditional probabilities of the low and high
Table 2
Estimates of the Markov Switching Model with State Dependent Volatility and without Volatility Feedback

<table>
<thead>
<tr>
<th></th>
<th>$\mu_1 = 0$</th>
<th>$\mu_1 \neq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0$</td>
<td>0.0038</td>
<td>0.0069</td>
</tr>
<tr>
<td></td>
<td>(0.0024)</td>
<td>(0.0026)</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>-0.0183</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0080)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>0.0255</td>
<td>0.0247</td>
</tr>
<tr>
<td></td>
<td>(0.0020)</td>
<td>(0.0018)</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.0557</td>
<td>0.0537</td>
</tr>
<tr>
<td></td>
<td>(0.0050)</td>
<td>(0.0044)</td>
</tr>
<tr>
<td>$p$</td>
<td>0.9590</td>
<td>0.9668</td>
</tr>
<tr>
<td></td>
<td>(0.0295)</td>
<td>(0.0220)</td>
</tr>
<tr>
<td>$q$</td>
<td>0.9668</td>
<td>0.9643</td>
</tr>
<tr>
<td></td>
<td>(0.0217)</td>
<td>(0.0212)</td>
</tr>
<tr>
<td>Log likelihood</td>
<td>367.6450</td>
<td>370.0464</td>
</tr>
<tr>
<td>ARCH-LM Statistic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K = 1$</td>
<td>0.5862</td>
<td>0.1384</td>
</tr>
<tr>
<td></td>
<td>(0.4439)</td>
<td>(0.7099)</td>
</tr>
<tr>
<td>$K = 3$</td>
<td>1.6031</td>
<td>0.7860</td>
</tr>
<tr>
<td></td>
<td>(0.6587)</td>
<td>(0.8528)</td>
</tr>
<tr>
<td>$K = 6$</td>
<td>4.1144</td>
<td>1.9108</td>
</tr>
<tr>
<td></td>
<td>(0.6612)</td>
<td>(0.9277)</td>
</tr>
<tr>
<td>$K = 12$</td>
<td>8.7091</td>
<td>5.0976</td>
</tr>
<tr>
<td></td>
<td>(0.7276)</td>
<td>(0.9546)</td>
</tr>
<tr>
<td>Observations</td>
<td>199</td>
<td></td>
</tr>
</tbody>
</table>

This table shows the maximum likelihood estimates of the parameters of the Markov switching regression model of stock returns with state dependent volatility and without volatility feedback. The estimates under the assumption of a constant and a time-varying risk premium are presented, respectively, under the headings $\mu_1 = 0$ and $\mu_1 \neq 0$. The standard errors are reported in parentheses below the coefficients. The bottom of the table shows the ARCH-LM statistics for a $k$-th order autoregression of squared standardized residuals ($k = 1, 3, 6$ and $12$) and the associated $p$ values in parentheses.
variance states are, respectively, 0.5526 and 0.4474. Hence, roughly half of the observations should fall into the low variance state and the remaining ones into the high variance state.

The bottom of the table shows the ARCH-LM statistics for the squared standardized residuals and the associated p values for lags 1, 3, 6 and 12. There is no evidence that the squared errors are serially correlated. The p values always exceed 40%.

5.2 Markov switching model with time-varying risk premium and uncertainty of the state

This subsection presents the empirical estimates of the parameters when the agents observe only past returns and forecast $S_t$, the value of the state in the next period, based on this information. These estimates are presented in the last column of Table 2. We observe that the estimates of the standard deviation of the low and high variance regimes as well as the estimates of the transition probabilities change marginally when $\mu_1$ is allowed to be different from zero. However, the estimate of $\mu_0$ increases almost 100% and becomes statistically significant, indicating that agents require a positive risk premium to hold the risky asset in the low variance state. The estimate of $\mu_1$ is also statistically different from zero, but negative, providing evidence that the risk premium decreases as the anticipated level of risk, summarized by $P(S_t = 1|\Psi_{t-1})$, increases. In other words, the model does not support the existence of a positive relationship between the risk premium and the variance of excess returns. This result contrasts with the findings of Turner et al. (1989), who report a positive value of $\mu_1$, statistically different from zero, when uncertainty is assumed.

A likelihood ratio test of the hypothesis that $\mu_1 = 0$ yields a statistic of 4.80. The corresponding p value, calculated from the $\chi^2$ distribution with one degree of freedom, is 2.84%, enabling us to reject the restricted model. Furthermore, the squared standardized residuals do not display any evidence of serial correlation, as can be inferred from the ARCH-LM statistics. The corresponding p values are always greater than 71%.

It is straightforward to compute from the output of the extended filter the probability that the economy is in a particular state at a specific time conditional on currently available information. The top of Figure 2 displays the filtered probabilities of the high variance regime and, at the bottom, the corresponding smoothed probabilities, that is, the probabilities of observing the state at a specific time conditional on the full history of $r_t$.

These probabilities can be used to infer the unobserved state and to indicate changes in the volatility regime. For instance, if we adopt the rule of associating $P(S_t = 1|r_T, \ldots, r_1) > 0.5$ with the high volatility state, we see that the high variance regime prevails throughout the first half of the sample, with the exception of the brief period from May 1996 to July 1997. The low variance state characterizes
The Relation Between Expected Returns and Volatility in the Brazilian Stock Market

Figure 2
Probability of the High Variance Regime – Model with Time-Varying Risk Premium and without Volatility Feedback

(a)

(b)

The top of the figure depicts the filtered probabilities \( P(S_t = 1|r_t, \ldots, r_1) \) for the period from Jan 1995 to Jul 2011, as inferred from the Markov switching model of excess returns with time-varying risk premium and without volatility feedback. The bottom of the figure shows the corresponding smoothed probabilities, that is, \( P(S_t = 1|r_T, \ldots, r_1) \).
the second half of the sample, in which the smoothed probability that \( S_t = 1 \) is always less than one half, apart from the period from July 2008 to June 2009, which followed the burst of the U.S. housing bubble and the subprime crisis. Since the mean of excess returns equals \(-0.4234\) from 1995 to 2002 and \(0.2098\) from 2003 to 2011, it is not surprising that the estimated value of \( \mu_1 \) is negative.

5.3 Markov switching model with time-varying risk premium, uncertainty of the state and volatility feedback

As noted earlier, the empirical negative relationship between risk premium and market volatility documented in the previous subsection may be due to the inability to take into account the so-called volatility feedback effect and to distinguish anticipated from unanticipated changes in stock market volatility. In what follows, I present the estimates of the parameters incorporating volatility feedback into the model, both estimating \( \delta \) along with the other parameters and imposing the restriction \( \delta = -\mu_1 / (1 - \zeta \lambda) \).

5.3.1 \( \delta \) unrestricted

The second column of Table 3 shows the estimates of the parameters when volatility feedback is allowed and changes in regime are assumed to be exogenous. The value of \( \delta \) is left unrestricted. Several comments are in order. The estimated value of \( \mu_0 \) is not statistically significant, unlike \( \mu_0 \) in Table 2 with a time-varying risk premium and uncertainty of the state, and it is substantially smaller than the values reported in Table 2. Moreover, the estimate of \( \mu_1 \) turns out to be positive, though not statistically different from zero, enabling us to reject the assumption of a negative relationship between risk premium and expected future volatility.

There is also an increase in the estimated value of \( \sigma_0 \), which jumps from approximately 0.0250 in Table 2 to 0.0306 in the second column of Table 3, suggesting that excess returns are more volatile in the low variance regime when volatility feedback is taken into account. The increase in the variance of excess returns that results from a higher estimate of \( \sigma_0 \) is offset by a reduction in the probability of staying in the high variance regime, which declines from approximately 0.96 to 0.79. This translates into an unconditional probability of the high variance state of only 0.1241. Finally, there is evidence of a strong volatility feedback effect, as indicated by the estimated value of \( \delta \) of \(-0.0817\), which is statistically significant at any reasonable level of significance.

The model with uncertainty about the state and in which agents do not react to unanticipated changes in volatility emerges as a special case of the model of this section under the constraint that \( \delta = 0 \). The LR test of this hypothesis yields a statistic of 5.30 with an associated \( p \) value of 2.13%, indicating that volatility feedback is an important feature of excess returns.

The last column of Table 3 takes account of the potential measurement error
Table 3
Estimates of the Markov Switching Model with State Dependent Volatility, Volatility Feedback and $\delta$ unrestricted

<table>
<thead>
<tr>
<th></th>
<th>$\Psi_{t-1} = {r_{t-1}, r_{t-2}, \cdots}$</th>
<th>$\Psi'_t = {S_t}$, $\rho = 0$</th>
<th>$\Psi'_t = {S_t}$, $\rho \neq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_0$</td>
<td>0.0004</td>
<td>0.0001</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0033)</td>
<td>(0.0031)</td>
<td></td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.0188</td>
<td>0.0158</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0216)</td>
<td>(0.0189)</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>-0.0817</td>
<td>-0.1293</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0159)</td>
<td>(0.0123)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>0.0306</td>
<td>0.0318</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0018)</td>
<td>(0.0019)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.0513</td>
<td>0.0512</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0091)</td>
<td>(0.0081)</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>0.7882</td>
<td>0.7705</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0815)</td>
<td>(0.0723)</td>
<td></td>
</tr>
<tr>
<td>$q$</td>
<td>0.9700</td>
<td>0.9579</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0142)</td>
<td>(0.0160)</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.7554</td>
<td>0.7554</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.1212)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>372.6980</td>
<td>375.7116</td>
<td></td>
</tr>
<tr>
<td>ARCH–LM Statistic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K = 1$</td>
<td>4.8008</td>
<td>0.9149</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0284)</td>
<td>(0.3388)</td>
<td></td>
</tr>
<tr>
<td>$K = 3$</td>
<td>8.4436</td>
<td>1.0540</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0377)</td>
<td>(0.7882)</td>
<td></td>
</tr>
<tr>
<td>$K = 6$</td>
<td>19.1328</td>
<td>8.6613</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0039)</td>
<td>(0.1935)</td>
<td></td>
</tr>
<tr>
<td>$K = 12$</td>
<td>31.0741</td>
<td>16.4642</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0019)</td>
<td>(0.1709)</td>
<td></td>
</tr>
</tbody>
</table>

This table shows the maximum likelihood estimates of the parameters of the Markov switching regression model of stock returns with time-varying risk premium, volatility feedback and the volatility feedback parameter $\delta$ unrestricted. The estimates under the assumptions of exogeneity and endogeneity are presented, respectively, under the headings $\rho = 0$ and $\rho \neq 0$. The standard errors are reported in parentheses below the coefficients. The bottom of the table shows the ARCH-LM statistics for a $k$-th order autoregression of squared standardized residuals ($k = 1, 3, 6$ and 12) and the associated $p$ values in parentheses.
problem that emerges when we replace $P \left[ S_t = 1 | \Psi_t \right]$ with the true value of $S_t$ in equation (3). In other words, it recognizes that regime shifts may be endogenous. We note that most of the estimates concur with the values reported in the second column. However, it is worth emphasizing that there is a marked change in the estimated value of $\delta$. The volatility feedback effect increases more than 50% in absolute value when we allow for endogeneity, changing from $-0.0817$ to $-0.1293$.

Turning now to the estimated value of $\rho$, we see that there is strong evidence of a positive correlation between regime shifts and excess returns. The likelihood ratio statistic of the hypothesis that $\rho$ equals zero is 6.03 and has a corresponding $p$ value of 1.41%, favoring the model with endogenous switching to the detriment of the model with exogenous regime shifts. Kim et al. (2008) also find evidence of endogenous regime shifts using a sample of monthly excess returns for a value-weighted portfolio of all NYSE listed stocks over the period 1952-1999.

5.3.2 $\delta$ restricted

Table 4 shows the estimates of the coefficients when $\delta$ is restricted. The value of $\zeta$ is set to 0.997, as in Kim et al. (2002). In the model with exogenous switching, there is a negligible change in the values of most parameters. The main impact is on the standard error of $\mu_1$, which declines from 0.0216 to 0.0084. As a result, the estimated value of $\mu_1$, which increases from 0.0188 to 0.0201, turns out to be statistically significant, supporting the existence of a positive relationship between the equity premium and stock market volatility and, hence, of a negative volatility feedback effect.

The last column of Table 4 presents the estimates of the coefficients under the assumption of endogenous switching. The restricted and unrestricted estimates of $\delta$ coincide under the assumption of endogeneity and are roughly 50% greater than the value reported in the second column of Table 4. There is also a substantial increase in the estimated value of $\mu_1$, which almost doubles, confirming that agents require a positive risk premium to demand the risky asset under a high volatility state and supporting once again the existence of a volatility feedback effect. Restricting $\delta$ has an impact on the probability of remaining in a high volatility regime. The parameter $p$ falls from 0.7852 to 0.7113. The expected duration of the high volatility state, conditional on being in that state, is reduced to only 3.5 months.

Imposing the restriction on the estimation also entails a slight reduction in the maximum values attained by the likelihood function. Focusing on the model with exogenous switching, the LR statistic of the hypothesis that $\delta = -\mu_1 / \left(1 - \zeta \lambda \right)$ is less than 0.01 and has an associated $p$ value of 94.83%, not permitting the rejection of the restriction. The same conclusion is achieved when we allow for a nonzero

\footnote{The estimates are not sensitive to the use of sensible alternative values of $\zeta$ such as 0.996 and 0.998.}
Table 4

Estimates of the Markov Switching Model with State Dependent Volatility, Volatility Feedback and $\delta$ restricted

<table>
<thead>
<tr>
<th></th>
<th>$\Psi_{t-1} = {r_{t-1}, r_{t-2}, \ldots}$</th>
<th>$\Psi_{t-1} = {r_{t-1}, r_{t-2}, \ldots}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Psi_t' = {S_t}, \rho = 0$</td>
<td>$\Psi_t' = {S_t}, \rho \neq 0$</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>0.0003</td>
<td>-0.0019</td>
</tr>
<tr>
<td></td>
<td>(0.0029)</td>
<td>(0.0030)</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.0201</td>
<td>0.0399</td>
</tr>
<tr>
<td></td>
<td>(0.0084)</td>
<td>(0.0149)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>-0.0815</td>
<td>-0.1293</td>
</tr>
<tr>
<td></td>
<td>(0.0142)</td>
<td>(0.0125)</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>0.0306</td>
<td>0.0321</td>
</tr>
<tr>
<td></td>
<td>(0.0017)</td>
<td>(0.0020)</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.0513</td>
<td>0.0502</td>
</tr>
<tr>
<td></td>
<td>(0.0089)</td>
<td>(0.0088)</td>
</tr>
<tr>
<td>$p$</td>
<td>0.7852</td>
<td>0.7113</td>
</tr>
<tr>
<td></td>
<td>(0.0703)</td>
<td>(0.0823)</td>
</tr>
<tr>
<td>$q$</td>
<td>0.9699</td>
<td>0.9583</td>
</tr>
<tr>
<td></td>
<td>(0.0144)</td>
<td>(0.0205)</td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td>0.6837</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.1604)</td>
</tr>
<tr>
<td>Log likelihood</td>
<td>372.6959</td>
<td>374.8472</td>
</tr>
<tr>
<td>ARCH–LM Statistic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K = 1$</td>
<td>4.5479</td>
<td>0.8796</td>
</tr>
<tr>
<td></td>
<td>(0.0330)</td>
<td>(0.3483)</td>
</tr>
<tr>
<td>$K = 3$</td>
<td>6.0469</td>
<td>0.9637</td>
</tr>
<tr>
<td></td>
<td>(0.1094)</td>
<td>(0.8100)</td>
</tr>
<tr>
<td>$K = 6$</td>
<td>13.6933</td>
<td>6.4296</td>
</tr>
<tr>
<td></td>
<td>(0.0333)</td>
<td>(0.3768)</td>
</tr>
<tr>
<td>$K = 12$</td>
<td>25.0433</td>
<td>14.5855</td>
</tr>
<tr>
<td></td>
<td>(0.0146)</td>
<td>(0.2649)</td>
</tr>
</tbody>
</table>

This table shows the maximum likelihood estimates of the parameters of the Markov switching regression model of stock returns with time-varying risk premium, volatility feedback and the volatility feedback parameter $\delta$ restricted. The estimates under the assumptions of exogeneity and endogeneity are presented, respectively, under the headings $\rho = 0$ and $\rho \neq 0$. The standard errors are reported in parentheses below the coefficients. The bottom of the table shows the ARCH-LM statistics for a $k$-th order autoregression of squared standardized residuals ($k = 1, 3, 6$ and 12) and the associated $p$ values in parentheses.
Figure 3
Probability of the High Variance Regime – Model with Volatility Feedback, Endogenous Switching and \( \delta \) Restricted

(a)

(b)

The top of the figure depicts the filtered probabilities \( P(S_t = 1|\{r_t, \ldots, r_1\}) \) for the period from Jan 1995 to Jul 2011, as inferred from the Markov switching model of excess returns with volatility feedback, endogenous switching and the volatility feedback parameter \( \delta \) restricted. The bottom of the figure shows the corresponding smoothed probabilities, \( P(S_t = 1|\{r_T, \ldots, r_1\}) \).
correlation between $u_t$ and $\eta_t$, in spite of the moderate reduction in the estimated value of $\rho$, which falls from 0.7554 to 0.6837. In this case, the observed value of the LR statistic is 1.73 and the associated $p$ value equals 18.86%. If the goal instead is to test the assumption of endogenous switching, the LR test yields a statistic of 4.30. The corresponding $p$ value of 3.81% does not support again the assumption of exogenous switching, reinforcing the conclusion that we arrived at by keeping $\delta$ unrestricted.

With endogenous switching, a move from the low to the high volatility regime raises the annualized risk premium from $-2.28\%$ to $47.88\%$. The unconditional probabilities of the states imply an estimate of the equity premium of $4.05\%$, higher than the sample average of excess returns of $-9.56\%$, because the volatility feedback model does not necessarily associate large negative returns with a decline in the risk premium.

The top of Figure 3 plots the filtered probabilities of the high variance regime for the model with volatility feedback, endogenous switching and $\delta$ restricted. The bottom of the figure gives the corresponding smoothed probabilities.

Employing again the criterion of associating $P(S_t = 1 | r_T, ..., r_1) > 0.5$ with the high volatility state, we see that the smoothed probabilities suggest that the high volatility regime is much less frequent than indicated by the model without volatility feedback. The high volatility state characterizes the first quarter of 1995, which coincided with the Mexican crisis, the period from August 1997 to December 1998, which encompasses the Asian and Russian crises, September 2001, marked by the terrorist attacks of September 11, the months preceding the presidential election of 2002 and the period from July 2008 to March 2009, which experienced a huge decline in the stock market following the most severe downturn in the global economy in the postwar period.

Table 5 summarizes the high variance periods in the stock market for the model with volatility feedback, endogenous switching and $\delta$ restricted and for the model with time-varying risk premium and without volatility feedback.

Table 5

<table>
<thead>
<tr>
<th>Model with time-varying risk premium and no volatility feedback</th>
<th>Model with volatility feedback, endogenous switching and $\delta$ restricted</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 1995 to January 1996</td>
<td>January to March 1995</td>
</tr>
<tr>
<td>May 1997 to December 2002</td>
<td>August 1997 to December 1998</td>
</tr>
<tr>
<td>April to November 2008</td>
<td>September 2001</td>
</tr>
<tr>
<td></td>
<td>June to September 2002</td>
</tr>
<tr>
<td></td>
<td>July 2008 to March 2009</td>
</tr>
</tbody>
</table>

This table presents the high variance episodes according to the Markov switching regression model with time-varying risk premium and without volatility feedback (first column) and to the Markov switching regression model with volatility feedback, endogenous switching and the volatility feedback parameter $\delta$ restricted (second column).
An alternative explanation for the \textit{ex post} negative relationship between returns and volatility is the leverage effect, whereby a drop in stock prices raises the financial leverage of firms, leading to a subsequent increase in volatility. As pointed out above, Kim et al. (2002) claim that, if the leverage hypothesis holds, the residuals from a Markov switching specification that allows only for discrete changes in volatility should display lingering ARCH effects. The bottom of Table 4 shows the ARCH-LM statistics for a $k$-th order autoregression of squared standardized residuals ($k = 1, 3, 6$ and 12). There is evidence of autocorrelation only under the assumption of exogeneity of regime shifts. However, this assumption is rejected by the LR test. Assuming endogeneity of regime shifts, it is not possible to reject the null hypothesis of no serial correlation. The smallest $p$ value equals 26.49%. I note in passing that we reach the same conclusion when the volatility feedback parameter is left unrestricted. This evidence is consistent with the results reported by Schwert (1989b) for the U.S., which suggest that changes in financial leverage can explain only a small proportion of the movements in stock volatility over time.

6. Conclusion

This paper applied the Markov switching regression model of equity returns of Turner et al. (1989) and Kim et al. (2008) to monthly Bovespa index excess returns over the period 1995 – 2011 to investigate the intertemporal relationship between risk premium and market volatility. I make a variety of assumptions about the information set available to agents.

When volatility feedback is ignored, the estimates suggest that the low and the high volatility regimes are very persistent and have similar unconditional probabilities, and provide evidence of a negative relationship between risk premium and volatility.

However, this negative relationship is not robust to the incorporation of volatility feedback into the model. When the volatility feedback parameter is estimated along with the other parameters of the model, there is evidence that regime shifts are endogenous and of a positive, although not statistically significant, relationship between risk premium and volatility.

The volatility feedback parameter can be expressed in terms of the other parameters of the model when the model is derived from the approximate log-linear present value framework of Campbell and Shiller (1988). Imposing this restriction on the estimation leads to a positive and statistically significant relationship between risk premium and volatility, suggesting that agents require an additional return to hold the risky asset when the anticipated level of risk increases. This restriction is not rejected by a likelihood ratio test.
The Relation Between Expected Returns and Volatility in the Brazilian Stock Market

References


A. Appendix

This appendix describes the nonlinear filter employed to compute the likelihood function of the model with endogenous switching. The filter uses as input

\[ P(S_{t-1} = s_{t-1}|r_{t-1}, ..., r_1) \]

Then it proceeds through the five steps below:

Step 1: Compute the joint density of \( S_t = s_t \) and \( S_{t-1} = s_{t-1} \), given the previous returns \( \{r_{t-1}, ..., r_1\} \):

\[
P(S_t = s_t, S_{t-1} = s_{t-1}|r_{t-1}, ..., r_1) = P(S_t = s_t|S_{t-1} = s_{t-1}) \frac{P(S_{t-1} = s_{t-1}|r_{t-1}, ..., r_1)}{P(S_{t-1} = s_{t-1}|r_{t-1}, ..., r_1)}
\]

Step 2: Obtain the joint density distribution of \( r_t \) and \( S_t = s_t, S_{t-1} = s_{t-1} \) as

\[
f(r_t, S_t = s_t, S_{t-1} = s_{t-1}|r_{t-1}, ..., r_1) = f(r_t|S_t = s_t, S_{t-1} = s_{t-1}, r_{t-1}, ..., r_1) \times P(S_t = s_t, S_{t-1} = s_{t-1}|r_{t-1}, ..., r_1)
\]

where

\[
f(r_t|S_t = 0, S_{t-1} = j, r_{t-1}, ..., r_1) = \phi \left( \frac{r_t - \hat{x}_t'\beta}{\sigma_0} \right) \Phi \left( \frac{a_j - \rho \left( r_t - \hat{x}_t'\beta \right) / \sigma_0}{\sqrt{1 - \rho^2}} \right) \times \frac{1}{\sigma_0 P(S_t = 0|S_{t-1} = j)}
\]

and

\[
f(r_t|S_t = 1, S_{t-1} = j, r_{t-1}, ..., r_1) = \phi \left( \frac{r_t - \hat{x}_t'\beta}{\sigma_1} \right) \Phi \left( \frac{-a_j + \rho \left( r_t - \hat{x}_t'\beta \right) / \sigma_1}{\sqrt{1 - \rho^2}} \right) \times \frac{1}{\sigma_1 P(S_t = 1|S_{t-1} = j)}
\]
Step 3: Calculate the likelihood function of the $t$-th observation as

$$P(r_t|r_{t-1},...,r_1) = \sum_{s_t} \sum_{s_{t-1}} f(r_t,S_t = s_t,S_{t-1} = s_{t-1}|r_{t-1},...,r_1)$$

Step 4: Divide the output of step 2 by the output of step 3 to get the joint probability of $S_t = s_t,S_{t-1} = s_{t-1}$, conditional on the history $\{r_s\}_{s=1}^t$,

$$P(S_t = s_t,S_{t-1} = s_{t-1}|r_1,...,r_1) = \frac{f(r_1,S_t = s_t,S_{t-1} = s_{t-1}|r_{t-1},...,r_1)}{P(r_t|r_{t-1},...,r_1)}$$

Step 5: Sum over $S_{t-1}$ to get the marginal probability of $S_t$, given $\{r_s\}_{s=1}^t$,

$$P(S_t = s_t|r_1,...,r_1) = \sum_{s_{t-1}} P(S_t = s_t,S_{t-1} = s_{t-1}|r_t,...,r_1)$$