Measuring Progress Toward Basic Opportunities for All

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Abstract

Universal access to basic opportunities is undoubtedly a central objective of development. In this study, we propose an alternative scalar measure. In addition to being sensitive to the number of available opportunities, this new measure is also concerned with how these opportunities are allocated, and is therefore known as an equity sensitive coverage rate. The proposed measure is increasing in the overall coverage rate and decreasing in deviations from the ideal of equal opportunity. It is obtained by deducting a penalty equal to the deviation from the ideal of equal opportunity from the overall coverage rate. We show that this equity sensitive measure has several suitable properties. In particular, we demonstrate that it is consistent with both first- and second-order stochastic dominance. As a consequence, this measure is Pareto-consistent and sensitive to improvements in the allocation of the available opportunities. According to this index, any increase in the number of opportunities that also leads to an increase in the average coverage rate among the most vulnerable groups would always represent a progress towards opportunities for all. In addition, we propose an estimation procedure for the index and its variance.

Keywords: Opportunity Measure, Coverage Rate, Opportunity Inequality.

JEL Codes: I30, I38.
1. Introduction

Although theories of distributive justice differ on countless aspects, all of them advocate that universal access to basic opportunities is a necessary step towards justice, fairness, and equity. They disagree on the extent to which universal access will suffice. Economic theory, backed by a wide range of empirical studies, has also emphasized that basic opportunities for all are central to fostering development and to reducing poverty and inequality.

As a consequence, it is not surprising that universal access to basic opportunities has always been a central issue for research in development economics and a major objective of public policy. The International Covenant on Economic, Social and Cultural Rights (ICESCR)\(^1\) is just one example of many international efforts devoted to promoting access to basic opportunities for all.\(^2\) Many of the Millennium Development Goals (MDGs) aim precisely at the universalization of access to selected basic opportunities.

To effectively promote a goal, invariably, one needs an indicator to measure progress. It is, therefore, quite surprising that very little effort has been devoted to constructing adequate scalar measures of the distance to universal access to basic opportunities, despite all the importance that pursuing this goal has for development.

Since universal access is attained if and only if the overall coverage rate (proportion of the population with access to a given opportunity) reaches 100\%, traditionally, the distance to universalization has been measured by the gap between the prevailing overall coverage rate and the ideal 100\%. For instance, the indicator used to monitor the Seventh Millennium Development Goal is the proportion of the population using an improved drinking water source.

Nevertheless, the use of the overall coverage rate to monitor progress towards universal access has a fundamental shortcoming: Absolute insensitivity to the allocation of opportunities.\(^3\) This is certainly not an issue when access is universal. In this case, the number of available opportunities is equal to or greater than the population size and everyone has access to an opportunity. However, when coverage is only partial, opportunities are scarce and a variety of alternative allocations are possible. Among these alternatives, fairness may vary considerably. Nonetheless, the overall coverage rate treats all of them as equals.

Consider, for instance, two societies (I and II) made of two ethnic groups (A and B) of the same population size. Assume that there are enough opportunities to cover only one half of the total population in each society. Hence, the overall

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\(^1\)Based on the Universal Declaration of Human Rights (UDHR).

\(^2\)Others include 1990 Jomtien World Declaration on Education for All and 1978 Alma-Ata Declaration on Health for All.

\(^3\)Having access to more than one opportunity brings no advantage. Since you can only go to one school or one health clinic. Assuming that all schools or health clinics are of the same quality, having access to more than one does not represent any advantage.
coverage rate is 50% in both cases. Also, assume the allocation of these scarce opportunities is quite distinct in the two societies. In Society I, all opportunities are allocated to ethnic group A and none to group B, whereas in Society II both ethnic groups receive an equal share. Hence, while in Society I the coverage rates for groups A and B are 100% and 0% respectively, in Society II the coverage rate is 50% for both groups. In sum, although both societies have the same overall coverage rate, they differ markedly in terms of the disparities in coverage by ethnicity. Measured by their overall coverage rates, the two societies would look alike. Nevertheless, any measure sensitive to group differences in access would consider Society II closer to the ideal of opportunity for all than Society I.

In this study, we assume that any proper indicator of the distance to the goal of opportunities for all must incorporate some sensitivity to the degree of fairness characterizing the allocation of the available opportunities. In this sense, the overall coverage rate is not a proper measure. Such measures should take into consideration both the number of opportunities available and the fairness characterizing their allocation among members of the population.

It is widely recognized that allocation matters when evaluating progress towards universal access and that overall coverage rates do not take this concern into consideration. Incidence analysis has been a common answer to this shortcoming. The idea behind incidence analysis is to replace one overall coverage rate with many group-specific coverage rates; one for each relevant socioeconomic or demographic group.

In our example, incidence analysis would require the estimation of coverage rates by ethnic group. Hence, it would clearly uncover the distinct nature of the progress achieved by the two societies. As this example well illustrates, incidence analysis always provides a view of how distant societies are from universal access. It does not, however, provide any summary scalar measure of this distance. The objective of this paper is to fill part of this gap.

Using the same principles guiding incidence analysis, we introduce a scalar measure which could be used to compute the distance to universalization. To emphasize its sensitivity to the allocation of opportunities, we refer to it as an equity-sensitive coverage rate.

The availability of a scalar measure is fundamental if one wants to introduce basic opportunities for all as a formal and quantitative goal for societies, governments and public policy, or to promote the proper consideration of universal access in standard impact evaluation studies. Such summary measures can also be essential to improve targeting, to measure poverty in terms of limited access to basic opportunities and to improve the effectiveness of social policy in promoting universal access.

Finally, it is worth mentioning that the absence of equity-sensitive scalar measures of the distance to universalization is not a consequence of any lack of scalar measures of inequality of opportunity. In fact, measuring inequality of opportu-
The task we undertake is not to measure inequality of opportunity per se, but to incorporate equity concerns in the evaluation of the distance to full coverage.

This paper is organized as follows: Section 2 introduces the equity-sensitive coverage rate, defined by the overall coverage rate and a penalty for improperly allocated opportunities. Section 3 examines six alternative interpretations of the measure that help to illustrate different features of the equity-sensitive coverage rate. Section 4 formally establishes six properties of the measure. While all these six properties are important, they might not be equally appealing. Probably, the two most appealing properties of the measure are

(i) its Pareto consistency and

(ii) its sensitivity to pro-vulnerable transformations.

Section 5 discusses the statistical procedures necessary for its estimation. Finally, Section 6 concludes, summarizing the main contribution of the measure and highlighting future areas of research.

2. Introducing an Equity-Sensitive Coverage Rate

The main objective of this section is to introduce the equity-sensitive coverage rate, $\theta$, defined by the overall coverage rate and a penalty for improperly allocated opportunities. The last subsection presents this measure, $\theta$. In order to build the measure, $\theta$, the first subsection briefly discusses the main building concepts of this framework: basic opportunities, circumstances, and the equality of opportunity principle. Similarly, the second subsection builds a measure of inequality of opportunity, essential to the equity-sensitive coverage rate.

2.1 Key Concepts: Basic opportunities, circumstances and the equality of opportunity principle

Let $\Omega$ be a society and $\omega$ a member of this society. $O$ is a basic opportunity and $I(\omega)$ is an indicator of whether member $\omega$ had access to $O$. Without any loss of generality, we assume that $I(\omega) = 1$ for all those who had access and $I(\omega) = 0$, otherwise.

What an opportunity is and what means to be basic are important and non-trivial questions. For now, however, it will suffice to say that a basic opportunity is whatever society $\Omega$ wants all its members to have. Access to food and literacy are natural examples.
Universal coverage is accomplished whenever \( \{ \omega : I(\omega) = 1 \} = \Omega \). In this case, the overall coverage rate, \( \mu = P[I = 1] \), will be equal to one. Universal coverage, \( \{ \omega : I(\omega) = 1 \} = \Omega \), implies full coverage, \( \mu = P[\omega : I(\omega) = 1] = 1 \). To simplify the exposition, we will treat universal and full coverage as equivalent.\(^6\)

Since universal coverage occurs only if \( \mu = 1 \), it may seem natural to measure progress towards the ideal of opportunities for all by movements of \( \mu \) towards one. However, when \( \mu \) is lower than one, the performance of societies with identical available opportunities could differ drastically according to how the available opportunities are allocated. Surely, when full coverage is reached, the allocation of opportunities becomes trivial and unique: each member has one opportunity. This fact, however, does not lead us necessarily to treat alternative allocations as irrelevant over the entire process, and as a consequence, to consider that progress could be comprehensively measured by advances in the overall coverage rate \( \mu \).

On the contrary, among societies with identical overall coverage rates, we consider as closer to the ideal of opportunities for all, those societies in which opportunities are allocated closer to the principle of equal opportunity. In other words, we assume that proper progress towards the goal of opportunity for all must take into consideration both the proportion of members being covered, \( \mu \) and the extent to which the available slots were allocated consistently with the equality of opportunity principle.

Even though the precise meaning of equal opportunity remains open to debate in many aspects, for our purposes, it will be enough to recognize that for every society certain personal, family and community characteristics should be unrelated to the access to all basic opportunities. For instance, school enrollment rates among boys and girls should be identical and the percentage of dwellings with adequate access to water and sanitation should be the same for all neighborhoods in a given city. Usually, these personal, family and community characteristics are referred to as circumstances. We follow this tradition.

Accordingly, for the purpose of this study, we assume that equal opportunity would prevail if and only if circumstances are unrelated to the access to basic opportunities. In this case, the specific coverage rates for all circumstance groups must all be the same, and therefore equal to the overall rate. Formally, let \( X(\omega) \) be the vector of all circumstances characterizing the situation of person \( \omega \). Then, equal opportunity will prevail whenever \( p(x) = \mu \), where \( p(x) \equiv P[I = 1|X = x] \). We refer to \( p(x) \) as the specific coverage rate of circumstance group \( x \). The group-specific coverage rate profile that includes all circumstance groups will be called \( p \).

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\(^6\)Universal coverage means that everyone has access, \( \{ \omega : I(\omega) = 1 \} = \Omega \), whereas full coverage simply means that \( \mu = P[\omega : I(\omega) = 1] = 1 \).
2.2 Measuring deviations from the equality of opportunity principle

Equal opportunity requires that all group-specific coverage rates should be identical, $p(x) = \mu$. For this principle to be fulfilled, opportunities ought to be allocated to each circumstance group proportional to its population size. More precisely, the number of opportunities allocated to a circumstance group should equal $100\mu\%$ of its size. We refer to this ideal share $\mu dF_X(x)$ as the group $x$ equitable share, as opposed to the group $x$ actual share, $p(x)dF_X(x)$.

Any opportunity allocated to a group above its equitable share is a violation of the equality of opportunity principle. Hence, it can be easily seen that

$$\pi = \int_{p(x) > \mu} (p(x) - \mu) dF_X(x)$$

measures the total number of opportunities whose allocation violates the equality of opportunity principle, expressed as a proportion of the total population. Since the number of opportunities required for full coverage is equal to the total population, $\pi$ also measures the total number of opportunities whose allocation violates the equality of opportunity principle expressed as a proportion of the total number of opportunities required for full coverage. To simplify the exposition, we refer to $\pi$ as the proportion of improperly allocated opportunities.

It is worth noting that $\pi$ is a proportion relative to all opportunities required for full coverage and not the fraction of available opportunities that were improperly allocated. To obtain the number of improperly allocated opportunities as a proportion of the total number of available opportunities we have to divide $\pi$ by $\mu$. This ratio is denoted by $\delta$. So,

$$\delta = \frac{\pi}{\mu}$$

To the extent that $\pi$ is a measure of absolute inequality, $\delta$ is a measure of the degree of relative inequality of opportunity.

2.3 Obtaining an equity-sensitive coverage rate

At any point in time, the distance to the ideal of opportunity for all comes from two sources. On the one hand, the number of opportunities available may not be sufficient to ensure access for all. On the other hand, some of the available opportunities may not have been allocated according to the equality of opportunity ideal, leading some circumstance groups to have opportunities in excess of their equitable share.

Accordingly, we propose to evaluate the current situation by the total number of opportunities allocated to circumstance groups up to their equitable share, expressed as a percentage of the total number of opportunities needed for full coverage. In this case, the concern for equity is introduced by deducting from the available opportunities those allocated to groups in excess of their equitable share.
and so violating the equality of opportunity principle. In other words, we evaluate the current situation by

$$\theta \equiv \mu - \pi$$

This measure is certainly an equity-sensitive coverage rate. The equity-sensitive coverage rate is defined by the overall coverage rate and the penalty for improperly allocated opportunities. It is, however, just one among many possible alternatives. In fact, the sensitivity to the inequality of opportunity could be introduced in many different ways. Nevertheless, as we are going to show, this formulation leads to a measure with clear interpretation, that has good properties and is easy to estimate. Most of this paper is devoted to establishing the properties of $\theta$ as an equity-sensitive coverage rate. We also propose a simple estimator for $\theta$ that is consistent, asymptotically normal and easy to compute. We further derive its asymptotic variance.

3. Interpretations for the Equity-Sensitive Measure $\theta$

The main objective of this section is to discuss six alternative interpretations of the equity-sensitive coverage rate. By discussing these complementary interpretations, we aim to illustrate more clearly different features of the measure, $\theta$. The interpretations of $\theta$ to be discussed are:

(i) $\theta$ as available opportunities discounted by improperly allocated ones;

(ii) $\theta$ as available opportunities discounted by the opportunity gap;

(iii) $\theta$ as available opportunities discounted by one half of the average distance between the actual and the ideal coverage profile;

(iv) $\theta$ as the weighted average of the average coverage rate among the vulnerable groups and the overall coverage rate, using the population shares of vulnerable and non-vulnerable groups as weights;

(v) $\theta$ as the weighted average of the circumstance group-specific coverage rate profiles, using the population weights altered to overweight the vulnerable group and underweight the non-vulnerable one; and

(vi) $\theta$ as the cumulative function of the group-specific coverage profile truncated at the overall mean.

In order to facilitate the interpretation of $\theta$, let $V$ denote the set of opportunity-vulnerable circumstance groups, in the sense that $V \equiv \{x : p(x) \leq \mu\}$. These are the groups that receive opportunities below their equitable share. We denote its complement by $V^c$ and refer to it as the set of non-vulnerable circumstance groups. Further, let $U = V \cup V^c$ denote the set of all circumstance groups.
3.1 Available opportunities discounted by improperly allocated ones

Since,
\[ \pi = \int_{V^c} (p(x) - \mu) dF_X(x) \]

It follows that \( \pi \) is the area between the actual circumstance group-specific coverage rate profile, \( p \), and the corresponding profile that would be observed if equal opportunity prevails – a horizontal line at \( \mu \) (see Figure 1). It is, therefore, the number of opportunities allocated in excess to the non-vulnerable circumstance groups expressed as a percentage of the total number of opportunities needed for full coverage.

Figure 1
Percentage of 16-year-olds who completed the 8th grade – Brazil, 2007

Source: Estimates based on the Brazilian Household Survey (PNAD) 2007.

From its definition, it follows that the equity-sensitive coverage rate, \( \theta \), is given by
\[
\theta = \int_U p(x) dF_X(x) - \pi = \int_V p(x) dF_X(x) + \int_{V^c} \mu dF_X(x)
\]
\[
= \int_U \min \{ p(x), \mu \} dF_X(x)
\]

Hence, it is equal to the area below both

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(i) the actual circumstance group-specific coverage rate profile and
(ii) the profile that would be observed if equality of opportunity prevails – a horizontal line at $\mu$ (See Figure 2).

If we say that an opportunity was properly allocated whenever it does not surpass the equitable share of the circumstance group to which it has been allocated, then we can say that is the proportion of all opportunities needed for full coverage that are available and were properly allocated.

Figure 2
Percentage of 16-year-olds who completed the 8th grade – Brazil, 2007

Source: Estimates based on the Brazilian Household Survey (PNAD) 2007.

3.2 Available opportunities discounted by the opportunity gap

Since

$$\mu = \int_V p(x)dF_X(x) + \int_{V^c} p(x)dF_X(x)$$

it follows that

$$\int_V (\mu - p(x)) dF_X(x) = \int_{V^c} (p(x) - \mu) dF_X(x)$$

Therefore,
\[ \pi = \int_V (\mu - p(x)) \, dF_X(x) \]

Hence, \( \pi \) has an additional interpretation. It is the total number of opportunities that all vulnerable groups need for their specific coverage rates to reach the overall average, expressed as a percentage of the total number of opportunities needed for full coverage. We refer to this proportion as the opportunity gap. It is equal to the area above the actual circumstance group-specific coverage rate profile, \( p \), and below the profile that would be observed if equality of opportunity prevails (see Figure 3).

**Figure 3**
Percentage of 16-year-olds who completed the 8th grade – Brazil, 2007

Source: Estimates based on the Brazilian Household Survey (PNAD) 2007.

Note that

\[ \delta = \frac{\pi}{\mu} = \frac{1}{\mu} \int_V (\mu - p(x)) \, dF_X(x) \]

has an expression identical to the traditional poverty gap, as long as the poverty line is set at the mean, \( \mu \).

Given this alternative expression for the penalty, \( \pi \), \( \theta \) can by expressed as

\[ \theta = \mu - \int_V (\mu - p(x)) \, dF_X(x) \]
or by the difference between the overall coverage rate and the opportunity gap (see Figure 4).

Figure 4
Percentage of 16-year-olds who completed the 8th grade – Brazil, 2007

Source: Estimates based on the Brazilian Household Survey (PNAD) 2007.

3.3 Available opportunities discounted by one half of the average distance between the actual and the ideal coverage profile

Since

\[ E[p(X) - \mu] = \int_V (\mu - p(x)) dF_X(x) + \int_{V^e} (p(x) - \mu) dF_X(x) \]

and

\[ \pi = \int_V (\mu - p(x)) dF_X(x) = \int_{V^e} (p(x) - \mu) dF_X(x) \]

it follows that

\[ \pi = \frac{1}{2} E[p(X) - \mu] \]

Therefore, the penalty \( \pi \) is equal to one half of the area between the actual circumstance specific coverage profile, \( p \), and the profile that would be observed if equality of opportunity prevails – a horizontal line at \( \mu \) (See Figure 5).
From these expressions, it follows that $\theta$ can alternatively be expressed by

$$\theta = \mu - \frac{1}{2} E |p(X) - \mu|$$

or by the difference between the area below the ideal circumstance specific coverage profile (horizontal line at $\mu$) and one half of the area between the actual and ideal circumstance group-specific coverage profile.

### 3.4 Truncated mean I

In this subsection, we discuss the equity-sensitive coverage rate as the weighted average of the average coverage rate among the vulnerable groups and the overall coverage rate, using the population shares in vulnerable and non-vulnerable groups as weights.

Let $t(X) = \min\{p(X), \mu\}$. As already shown

$$\theta = \int_U \min\{p(x), \mu\} \cdot dF_X(x)$$

Hence,

$$\theta = \int_U \min\{p(x), \mu\} \cdot dF_X(x) = \int_U t(x) \cdot dF_X(x) = E [t(X)]$$
Therefore, whereas the overall coverage rate is the mean of the circumstance group-specific coverage rates, \( \mu E[p(X)] \), the equity-sensitive coverage rate, \( \theta \), is the mean of the same circumstance group-specific coverage rates truncated at the mean, i.e., \( \theta = E[t(X)] \).

Alternatively, let \( \alpha = \int_{V^c} df_X(x) = P[p(X) > \mu] \) denote the proportion of the population in non-vulnerable groups. In this case, since

\[
\theta = \mu - \int_V (\mu - p(x))dF_X(x)
\]

It follows that

\[
\theta = \int_V p(x)dF_X(x) + \alpha \mu
\]

Since

\[
E[p(X)|V] = \frac{1}{1 - \alpha} \int_V p(x)dF_X(x)
\]

It follows that

\[
\mu = (1 - \alpha).E[p(X)|V] + \alpha.E[p(X)|V^c]
\]

and

\[
\theta = (1 - \alpha).E[p(X)|V] + \alpha \mu
\]

Hence, while the overall coverage, \( \mu \), is equal to the weighted average of the average coverage rates for the opportunity vulnerable and non-vulnerable groups, the equality of opportunity-sensitive coverage rate, \( \theta \), is equal to the weighted average of the average coverage rate among the opportunity-vulnerable groups and the overall coverage. In both cases, the weights are the population shares.

### 3.5 Weighted average of the circumstance group-specific profiles

Since,

\[
\mu = \int_{V \cup V^c} p(x)dF_X(x)
\]

It follows that

\[
\theta = \int_V p(x)dF_X(x) + \alpha \left( \int_{V \cup V^c} p(x)dF_X(x) \right)
\]

Therefore,
\[ \theta = (1 + \alpha) \int_V p(x) dF_X(x) + \alpha \int_{V^c} p(x) dF_X(x) \]

From this expression, it follows that

\[ \theta = (1 - \alpha^2) E[p(x)|V] + \alpha^2 E[p(X)|V^c] \]

while as usual

\[ \mu = (1 - \alpha) E[p(X)|V] + \alpha E[p(X)|V^2] \]

Hence, both the overall coverage rate, \( \mu \), and the equity-sensitive coverage rate, \( \theta \), are equal to the weighted average of the average coverage among the opportunity-vulnerable and non-vulnerable groups. The weights, however, are different. Since, \( 0 \leq \alpha \leq 1 \), it follows that \( \alpha^2 \leq \alpha \) and \( 1 - \alpha^2 \leq 1 - \alpha \). Therefore, in the average used to obtain \( \theta \), the weight given to the non-vulnerable group is below its population share and the weight given to the vulnerable group is above its population share. In sum, this expression reveals that \( \theta \) is a weighted average of all circumstance specific coverage rates. However, the weights are the population weights altered to overweight the vulnerable group and underweight the non-vulnerable one.

As we are going to show in the next section, two main properties of \( \theta \) follow straight from this expression. First, the index is Pareto consistent, in the sense that it will increase as the coverage rate of any circumstance group increases. Secondly, \( \theta \) is inequality-sensitive, in the sense that opportunities given to the most deprived groups would have an impact on \( \theta \) greater than if the same opportunities were given to the least vulnerable groups.

### 3.6 Truncated mean II

To further investigate the interpretation of \( \theta \), it is useful to construct a new random variable, \( R \). This variable associates each society member \( \omega \) with the specific coverage rate of the circumstance group to which he or she belongs, \( (\omega) \). More precisely, \( R(\omega) = p(X(\omega)) \). Hence, \( F_R(r) = P\{x : p(x) \leq r\} \) and, in particular, \( F_R(\mu) = 1 - \alpha \). Moreover, by a simple change in variables, it follows that

\[ \int_V p(x) dF_X(x) = \int_0^\mu r dF_R(r) \]

Since

\[ \int_0^\mu r dF_R(r) = \mu F_R(\mu) - \int_0^\mu F_R(r) dr = (1 - \alpha) \mu - \int_0^\mu F_R(r) dr \]

it also follows that
\[ \theta = \int V p(x)dF_X(x) + \alpha \mu = \mu - \int_0^\mu F_R(r)dr = \int_0^\mu (1 - F_R(r))dr \]

This expression allows an alternative interpretation for \( \theta \): the horizontal area between the cumulative function \( F_R(r) \) and the vertical line \( p = 1 \) when \( r \) varies from zero to \( \mu \) (see Figure 6). Moreover, this expression is central to establishing the relationship between first- and second-order dominance changes and improvements in \( \theta \).

Figure 6
Percentage of 16-year-olds who completed the 8th grade – Brazil, 2007

Since \( 0 \leq R_R(r) \leq 1 \) and

\[ \mu = \int_0^1 (1 - F_R(r)).dr \]

it follows that \( \mu \leq 1 \) and

\[ 0 \leq \theta = \int_0^\mu (1 - F_R(r)).dr \leq \int_0^1 (1 - F_R(r)).dr = \mu \leq 1 \]

Thus, while the overall coverage rate \( \mu \) is equal to the horizontal area between the cumulative function \( F_R(r) \) and the vertical line \( p = 1 \) with \( r \) varying from zero to one, the equity-sensitive coverage rate \( \theta \) is equal to the same area with the variation of \( r \), however, limited to \( \mu \) instead of one (see Figures 6 and 7).
4. Properties

In this section, we formally establish six properties of the equity-sensitive coverage rate. While all these six properties are important, they might not be equally appealing. Probably, the two most appealing properties of this measure are

(i) its Pareto consistency and

(ii) its sensitivity to pro-vulnerable transformations.

However, in order to establish the Pareto consistency of $\theta$, we need first to establish its consistency with the first-order stochastic dominance. Similarly, in order to establish a robust sensitivity to pro-vulnerable transformations of $\theta$, we need first to establish its consistency with the second-order stochastic dominance. The section is organized as follows: the first two properties establish attainable lower and upper bounds and specify its sensitivity to scale. The next two properties focus on the consistency with first-order stochastic dominance and its related Pareto consistency. The last two properties focus on the consistency with second-order stochastic dominance and its related sensitivity to pro-vulnerable transformations.

4.1 The range: $\mu^2 \leq \theta \leq \mu$

Since $\theta = \mu - \pi$ and $\pi = \frac{1}{2}E[p(X) - \mu] \leq 0$, it follows that $\theta \leq \mu$, with its upper bound, $\theta = \mu$, being attained only when $\pi = 0$. Since, $\pi = 0$ if and only if
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$p(X) = \mu$, almost surely, it follows that $\theta = \mu$ only when equality of opportunity prevails.

The equity-sensitive coverage rate $\theta$ also has a lower bound. In fact, $\mu^2 \leq \theta \leq \mu$. To show this result, first notice that

$$\int_0^\mu r \text{d}F_R(r) = \mu - \int_\mu^1 r \text{d}F_R(r) \geq \mu - \alpha$$

where we use the facts that $\mu = \int_0^1 r \text{d}F_R(r)$, $F_R(1) = 1$ and $\alpha = 1 - F_R(\mu)$. Hence,

$$\theta = \int_0^\mu r \text{d}F_R(r) + \alpha, \mu \geq \mu - \alpha(1 - \mu)$$

Moreover, since $F_R$ is a non-decreasing function, $1 - F_R(x) \geq 1 - F_R(\mu) = \alpha$ for all $x \leq \mu$, as a consequence,

$$\theta = \int_0^\mu (1 - F_R(r)) \text{d}r \geq \alpha \mu$$

Together, these two inequalities imply that

$$\theta \geq \max \{\mu - \alpha(1 - \mu), \alpha \mu\}$$

To complete the argument, consider the two logical possibilities:

(i) $\mu \leq \alpha$ and

(ii) $\mu > \alpha$.

If $\mu \geq \alpha$, then

$$\theta \geq \mu - \alpha(1 - \mu) \geq \mu - \mu(1 - \mu) = \mu^2$$

If $\mu > \alpha$, then

$$\theta \geq \alpha \mu \geq \mu^2$$

We have, therefore, established that it is always the case that $\theta \geq \mu^2$.

Just as the upper bound, this lower bound is also attainable. To verify its attainability, suppose opportunities are distributed among circumstance groups such that either $p(x) = 0$ or $p(x) = 1$. So, either everyone in a group has an opportunity or nobody has it. This is certainly the maximum deviation from the ideal of equal opportunity one may obtain. In this case,

$$V^c = \{x : p(x) = 1\}$$

so that
\[ \alpha = P[p(X) = 1] \]

and

\[ \mu = E[p(X)] = P[p(X) = 1] = \alpha \]

Hence,

\[ \pi = \int_{V^c} (p(x) - \mu) dF_X(x) = \int_{V^c} (1 - \mu) dF_X(x) = \alpha(1 - \mu) = \mu(1 - \mu) \]

and

\[ \theta = \mu - \pi = \mu - \mu(1 - \mu) = \mu^2 \]

Hence, the lower bound is attainable.

4.2 Sensitivity to scale

To investigate the impact of changes in scale, \( p_I, \mu_I, \) and \( \theta_I \) denote the initial group-specific coverage profile, the overall and the equity-sensitive coverage rates, respectively, and \( p_F, \mu_F \) and \( \theta_F \) denote the corresponding final coverage profiles and rates. Accordingly, \( \pi_I \) and \( \pi_F \) denote the initial and final penalties (total inequality of opportunity).

At least two types of balanced growth are worth considering: parallel and proportional. Under a parallel shift, \( p_F(x) = p_I(x) + \lambda, \) for all \( x. \) Hence, \( \mu_F = \mu_I + \lambda. \) Since, in this case, \( \mu_F - p_F(x) = \mu_I - p_I(x), \) it follows that \( \pi_F = \pi_I. \)

Therefore, \( \theta_F = \theta_I + \lambda. \) Under a proportional shift, \( p_F(x) = \lambda p_I(x). \) Hence, \( \mu_F = \lambda \mu_I. \) Since, in this case, \( \mu_F - p_F(x) = \lambda(\mu_I - p_I(x)), \) it follows that \( \pi_F = \lambda \pi_I, \) leading to \( \theta_F = \lambda \theta_I. \)

In sum, under both sources of balanced growth, the change in the equity-sensitive coverage rate mimics the corresponding change in the overall rate. When all group-specific coverage rates increase by 1 percentage point, both the overall, \( \mu, \) and the equity-sensitive coverage rate, \( \theta, \) would also increase by 1 percentage point. Likewise, when all group-specific coverage rates increase by 1%, both the overall, \( \mu, \) and the equity-sensitive coverage rate, \( \theta, \) would also increase by 1%.

4.3 First-order stochastic dominance

We say \( R_F \) strictly first-order stochastically dominates \( R_I \) whenever \( F_{R_F}(r) \leq F_{R_I}(r), \) for all \( r \geq 0, \) with \( F_{R_F}(r) < F_{R_I}(r), \) for at least some \( r. \) Notice that since

\[ \mu = \int_0^1 (1 - F_R(r)) dr = 1 - \int_0^1 F_R(r) dr \]
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strict first-order stochastic dominance implies that

\[ \mu_F - \mu_I = \int_0^1 (F_{R_1}(r) - F_{R_F}(r))dr > 0 \]

Moreover, since \( \phi = P[R_F < \mu_F] < 1 \), strict first-order stochastic dominance also implies that

\[ \int_{\mu_I}^{\mu_F} (1 - F_{R_F}(r))dr \geq (\mu_F - \mu_I)(1 - \phi) > 0 \]

Therefore, since

\[ \theta = \int_0^\mu (1 - F_R(r))dr \]

\[ \theta_F - \theta_I = \int_{\mu_I}^{\mu_F} (1 - F_{R_F}(r))dr + \int_0^{\mu_I} (F_{R_1}(r) - F_{R_F}(r))dr \]

and strict first-order dominance as already shown implies that

\[ \int_{\mu_I}^{\mu_F} (1 - F_{R_F}(r))dr > 0 \]

and

\[ \int_0^{\mu_I} (F_{R_1}(r) - F_{R_F}(r))dr \geq 0 \]

Therefore, it follows that strict first-order stochastic dominance yields \( \theta_F - \theta_I > 0 \).

In sum, if \( R_F \) strictly first-order stochastically dominates \( R_I \), then necessarily both the overall and the equity-sensitive coverage rates are going to be higher in the final situation: \( \mu_F > \mu_I \) and \( \theta_F > \theta_I \).

4.4 Sensitivity to Pareto improvements

In terms of coverage, we say that a Pareto improvement has occurred when none of the circumstance group-specific coverage rates decreases, increasing, however, for at least one group with a positive proportion of the population. More specifically, a Pareto improvement occurs when

\[ P_F(x) = p_I(x) + \lambda(x) \]

with \( \lambda(x) \geq 0 \) for all \( x \) and \( P[\lambda(X) > 0] > 0 \).

Next, we show that a Pareto improvement occurs only if \( R_F \) first-order dominates \( R_I \). As a consequence of the results from the previous section, any Pareto
improvement will lead necessarily to an increase in $\theta$. In this sense, we say $\theta$ is Pareto consistent.

To establish that Pareto improvements lead to first-order stochastic dominance, notice that since $\lambda(x) \geq 0$, $R_F = R_I + \lambda(X) \leq r$ implies $R_I \leq r$. As a consequence,

$$\{\omega : R_F(\omega) \leq r\} = \{\omega : R_I(\omega) + \lambda(X(\omega)) \leq r\} \subseteq \{\omega : R_I(\omega) \leq r\}$$

Therefore

$$F_{R_F}(r) = P[R_F \leq r] = P[R_I + \lambda(X) \leq r] \leq P[R_I \leq r] = F_{R_I}(r)$$

Furthermore, notice that $F_{R_F}(r) = F_{R_I}(r)$ for all $r$ would imply $E[R_F] = E[R_I]$, such that $E[\lambda(X)] = 0$. However, $P[\lambda(X) > 0]$ implies $E[\lambda(X)] > 0$. Hence, we must have $F_{R_F}(r) > F_{R_I}(r)$ for at least some $r$.

In sum, we have shown that $p_F(x) = p_I(x) + \lambda(X)$ with $\lambda(x) \geq 0$ and $P[\lambda(X) > 0]$ implies that $R_F$ strictly first-order stochastically dominates $R_I$.\textsuperscript{7} From the result of the previous section, it then follows that any Pareto improvement would lead to an increase in both overall, $\mu$, and equity-sensitive coverage rates, $\theta$, i.e., $\mu_F > \mu_I$ and $\theta_F > \theta_I$. Hence, despite being sensitive to the inequality among group-specific coverage rates, $\theta$ will increase after any Pareto improvement, even when the circumstance groups with the highest coverage rates are benefited. This result reveals that the sensitivity of $\theta$ to equity is bounded.

4.5 Second-order stochastic dominance

We say $R_F$ second-order stochastically dominates $R_I$, whenever

$$\int_0^a F_{R_F}(r)dr \leq \int_0^a F_{R_I}(r)dr$$

for all $a \geq 0$. Notice that since

$$\mu = \int_0^I (1 - F_R(r))dr = 1 - \int_0^I F_R(r)dr$$

second-order stochastic dominance implies $\mu_F \geq \mu_I$ and hence that

$$\int_{\mu_I}^{\mu_F} (1 - F_{R_F}(r))dr \geq 0$$

Moreover, since

$$\theta = \int_0^\mu (1 - F_R(r))dr$$

\textsuperscript{7}It can be shown that the opposite is also true. If $R_F$ first-order dominates $R_I$, then $p_F(x) \geq p_I(x)$ for all $x$ and $P[p_F(X) > p_I(X)] > 0$. 

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\[ \theta_F - \theta_I = \int_{\mu_F}^{\mu_I} (1 - F_{R_F}(r))dr + \left( \int_0^{\mu_I} F_{R_I}(r)dr - \int_0^{\mu_I} F_{R_F}(r)dr \right) \]

and second-order dominance, as already shown, implies that

\[ \int_{\mu_F}^{\mu_I} (1 - F_{R_F}(r))dr \geq 0 \]

and

\[ \int_0^{\mu_I} F_{R_I}(r)dr - \int_0^{\mu_I} F_{R_F}(r)dr \geq 0 \]

Therefore, it follows from second-order stochastic dominance that \( \theta_F - \theta_I > 0 \).

### 4.6 Sensitivity to pro-vulnerable transformations

We say that a pro-vulnerable transformation has occurred when

\[ p_F(x) = p_I(x) + \lambda(x) \]

with

\[ \Gamma(a) = \int_{V_F(a)} \lambda(x)dF_X(x) \geq 0 \]

for all \( a \geq 0 \), where \( V_F(a) = \{ x : p_F(x) \leq a \} \). Notice that \( V_F(1) = \Omega \). Hence, \( E[\lambda(X)] = \Gamma(1) \). Since \( \Gamma(1) \geq 0 \), it follows that \( E[\lambda(X)] \geq 0 \) and, as a consequence

\[ \mu_F E[p_F(X)] \geq E[p_I(X)] = \mu_I \]

Moreover, the condition \( \Gamma(a) \geq 0 \) implies that

\[ \int_{V_F(a)} p_F(x)dF_X(x) \geq \int_{V_F(a)} p_I(x)dF_X(x) \]

or,

\[ E[p_F(X)|V_F(a)] \geq E[p_I(X)|V_F(a)] \]

Hence, the transformation is pro-vulnerable to the extent that it increases the average coverage of the most vulnerable groups, with vulnerability being defined after the transformation.

Next, we show that a pro-vulnerable transformation leads to second-order stochastic dominance. Hence, as a consequence of the results obtained in the previous section, a pro-vulnerable transformation leads to an increase in \( \theta \). In other words, we show that \( \Gamma(a) \geq 0 \) for all \( a \geq 0 \) implies \( \theta_F \geq \theta_I \). To demonstrate this result, it suffices to show that \( \Gamma(a) \geq 0 \) for all \( a \geq 0 \) implies second-order dominance defined as
\[
\int_0^a R_{R_f}(r)dr \leq \int_0^a F_{R_i}(r)dr
\]

for all \( a \geq 0 \).

To obtain this result, note that

\[
\int_{V_{p}(a)} p_I(x)dF_X(x) = \int_{V_{p}(a)} p_I(x)dF_X(x) + \int_{V_{p}(a)} \lambda(x)dF_X(x)
\]

\[
= \int_{V_{p}(a)} p_I(x)dF_X(x) + \Gamma(a)
\]

Therefore,

\[
\int_{V_{p}(a)} p_F(x)dF_X(x) - \int_{V_{I}(a)} p_I(x)dF_X(x) = \int_{V_{p}(a)} p_I(x)dF_X(x)
\]

\[
- \int_{V_{I}(a)} (x)dF_X(x) + \Gamma(a)
\]

Next, observe that

\[
\int_{V_{p}(a)} p_I(x)dF_X(x) - \int_{V_{I}(a)} p_I(x)dF_X(x) \geq a(F_{R_f}(a) - F_{R_i}(a))
\]

Hence,

\[
\int_{V_{p}(a)} p_F(x)dF_X(x) - \int_{V_{I}(a)} p_I(x)dF_X(x) \geq a(F_{R_f}(a)) + \Gamma(a)
\]

Next, through a change in variables and integration by parts, note that

\[
\int_{V(a)} p(x)dF_X(x) = \int_0^a rdF_R(r) = aF_R(a) - \int_0^a F_R(r)dr
\]

Hence,

\[
\int_{V_{p}(a)} p_F(x)dF_X(x) - \int_{V_{I}(a)} p_I(x)dF_X = a(F_{R_f}(a) - F_{R_i}(a))
\]

\[
- \left( \int_0^a F_{R_f}(r)dr - \int_0^a F_{R_i}(r)dr \right)
\]

Therefore,
\[ a(R_F(a)) = F_R_1(a) + \Gamma(a) \leq a(F_{R_F}(a) - F_{R_I}(a)) \]

\[ = \left( \int_0^a F_{R_F}(r)dr - \int_0^a F_{R_I}(r)dr \right) \]

and, as a consequence,

\[ \int_0^a F_{R_F}(r)dr - \int_0^a F_{R_I}(r)dr \geq \Gamma(a) \]

Hence, \( \Gamma(a) \geq 0 \) for all \( a \geq 0 \) implies second-order dominance and, as a consequence of the results of the previous section, that \( \theta_F \geq \theta_I \). In sum, following a pro-vulnerable transformation, both the overall and equity-sensitive coverage rates, \( \mu \) and \( \theta \), would never decrease. They could, however, stay the same.

Finally, it is worth noticing that the equity-sensitive coverage rate \( \theta \) can always be expressed as a weighted average of the average coverage rate among the vulnerable groups and the overall average, i.e.,

\[ \theta = (1 - \alpha).E[p(X)|V] + \alpha.\mu \]

As a result, any transformation that increases the average coverage among the vulnerable groups without reducing the overall coverage will always increase the value of the equity-sensitive coverage rate \( \theta \).

5. Statistical Inference

Let us assume that one has access to a random sample of the population with information on whether person \( i \) had access to a given opportunity (\( I_i = 1 \) if person \( i \) had access and \( I_i = 0 \) otherwise) and a vector of variables indicating his/her circumstances, \( x_i = (x_{i1}, \ldots, x_{im}) \).

Given this information, one needs to follow three steps to estimate the equity-sensitive coverage rate

\[ \theta = \mu - \frac{1}{2}E[p(X) - \mu] \]

Before we proceed, however, it is worth noticing that since

\[ P[I = 1|X = x] = p(x) \]

and

\[ \mu = P[I = 1] = E[p(X)] \]

\( \theta \) can be rewritten as
\[ \theta = E[p(X)] - \frac{1}{2}E[p(X) - E[p(X)]] \]

which is actually the expression to be mimicked in order to estimate \( \theta \). This expression also indicates the central role played by the group-specific coverage profile, \( p \), in estimating \( \theta \).

These conditional probabilities could be estimated through a variety of parametric, nonparametric or semi-parametric procedures. In all cases, the three-step procedure described below would apply. One of the simplest alternatives would be to estimate a logistic model, linear in the parameters. In this case, the first step in estimating \( \theta \) would be to fit the logistic regression

\[ \ln \left( \frac{P[I = 1|X = (x_1, \ldots, x_m)]}{1 - P[I = 1|X = (x_1, \ldots, x_m)]} \right) = \sum_{k=1}^{m} x_k \beta_k \]

where \( x_k \) denotes the row vector of variables representing the \( k \)-dimension of circumstances, hence, \( x = (x_1, \ldots, x_m) \) and \( \beta, \ldots, \beta_m \) will be a corresponding column vector of parameters. From the estimation of this logistic regression one obtains estimates of the parameters \( \{\hat{\beta}_k\} \) to be denoted by \( \{\hat{\beta}_{k,n}\} \) and an estimate of their asymptotic variance matrix, \( \hat{V}_{\beta,n} \), where \( n \) denotes the sample size.

As a second step, given the estimated coefficients, one can obtain the predicted probability of access to the opportunity in consideration for each individual in the sample, i.e., for each individual \( i \) one should compute

\[ \hat{p}_{i,n} = \frac{\exp(x_i, \hat{\beta}_n)}{1 + \exp(x_i, \hat{\beta}_n)} \]

Next, as a third and final step one simply computes

\[ \hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \hat{p}_{i,n} \]

and

\[ \hat{\theta}_n = \hat{\mu}_n - \frac{1}{2n} \sum_{i=1}^{n} |\hat{p}_{i,n} - \hat{\mu}_n| \]

Under the assumptions that

(i) the regression has been correctly specified and

(ii) its coefficients have been consistently estimated, it follows that almost surely,

\[ \lim_{n \to \infty} \hat{\mu}_n = \mu \] and

\[ \lim_{n \to \infty} \hat{\theta}_n = \theta \]
\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} |\hat{p}_{i,n} - \hat{\mu}_n| \right) = \mathbb{E} |p(X) - \mu|
\]
also almost surely. Hence, \(\lim_{n \to \infty} \hat{\theta}_n = \theta\) almost surely and \(\hat{\theta}_n\) is a consistent estimator of \(\theta\).

To obtain the asymptotic variance of this estimator, notice that \(\hat{\theta}_n\) can also be expressed as

\[
\hat{\theta}_n = \hat{\mu}_n - \frac{1}{2n} \sum_{i=1}^{n} |\hat{p}_{i,n} - \hat{\mu}_n| = \mu_n - \frac{1}{2n} \left( \sum_{i \in L_n} (a_i \hat{\mu}_n - \hat{p}_{i,n}) + \sum_{i \in H_n} (\hat{p}_{i,n} - \hat{\mu}_n) \right)
\]
or alternatively as

\[
\hat{\theta}_n = \frac{1 + \hat{\alpha}_n}{n} \sum_{i \in L_n} \hat{p}_i + \frac{\hat{\alpha}_n}{n} \sum_{i \in H_n} \hat{p}_i
\]
where

\[
\hat{\alpha}_n = \frac{\#H_n}{n}
\]

\(L_n\) denotes the set of all individuals with predicted probability of access below average and \(H_n\) designates the complement. So, \(L_n = \{i : \hat{p}_{i,n} \leq \hat{\mu}_n\}\) and \(H_n = \{i : \hat{p}_{i,n} > \hat{\mu}_n\}\). Hence, considering that the sets \(L_n\) and \(H_n\) are not affected by marginal changes in the parameter \(\beta_n\), \(\hat{\theta}_n\) can be expressed as a linear function of \(\hat{p}_{i,n}\), and as a consequence

\[
\frac{\partial \hat{\theta}_n}{\partial \beta_n} = \frac{1 + \hat{\alpha}_n}{n} \sum_{i \in L_n} \frac{\partial \hat{p}_{i,n}}{\partial \beta_n} + \frac{\hat{\alpha}_n}{n} \sum_{i \in H_n} \frac{\partial \hat{p}_{i,n}}{\partial \beta_n}
\]
Moreover, since

\[
\ln \left( \frac{\hat{p}_{i,n}}{1 - \hat{p}_{i,n}} \right) = x_i, \hat{\beta}_n
\]

\[
\frac{\partial \hat{p}_{i,n}}{\partial \beta_n} = \hat{p}_{i,n}(1 - \hat{p}_{i,n}) x'_i
\]
So,

\[
\frac{\partial \hat{\theta}_n}{\partial \beta_n} = \frac{1 + \hat{\alpha}_n}{n} \sum_{i \in L_n} \hat{p}_{i,n}(1 - \hat{p}_{i,n}) x'_i + \frac{\hat{\alpha}_n}{n} \sum_{i \in H_n} \hat{p}_{i,n}(1 - \hat{p}_{i,n}) x'_i
\]
As a consequence, a consistent estimator of the asymptotic variance of \( \hat{\theta}, \hat{V}_{\theta,n} \) is given by

\[
\hat{V}_{\theta,n} = \frac{\partial \hat{\theta}_n}{\partial \beta_n'} \hat{V}_{\beta,n} \frac{\partial \hat{\theta}_n}{\partial \beta_n}
\]

where \( \hat{V}_{\beta,n} \) is a consistent estimator of the variance of \( \hat{\beta}_n \).

6. Limitations and Extensions

In this section, we discuss three limitations of the proposed index and develop some extensions based on these limitations. The limitations addressed are

(i) the lack of invariance of the index to the set of circumstances,
(ii) its insensitivity to a variety of increases and decreases in inequality, and
(iii) its lack of subgroup consistency.

6.1 \( \theta \) as an upper bound for the true Human Opportunity Index

The measure \( \theta \) is certainly an inequality-sensitive coverage rate. However, it is not invariant with respect to the set of circumstances \( (x) \). In fact, \( \theta \) is a function of the chosen set of circumstances. Hence, we should more properly refer to this index as \( \theta(x) \) instead of simply \( \theta \).

If there were an overall agreement on what the relevant circumstances are and if all of them were measured, this lack of invariance of the measure \( \theta \) would not constitute a drawback. Unfortunately, however, there is no consensus on the relevant set of circumstances and many are not often measured. As a result, the set of circumstances tends to vary across studies, sacrificing comparability.

Undeniably, the lack of invariance of \( \theta \) to the set of circumstances is a drawback. Fortunately, however, it has a monotonic property that mitigates, at least partially, the consequences of its lack of invariance. According to this property, the measure \( \theta \) never increases as the set of circumstances expands. In other words, it simply establishes that \( \theta(x) \geq \theta(x,z) \) for any additional set of circumstances \( z \). The objective of this subsection is to demonstrate this property.

First, recall that \( \theta(x) \) can be expressed as

\[
\theta(x) = E[p(x)] - \frac{1}{2}E[p(x) - E[p(x)]]
\]

where \( p(x) \) is the circumstance specific coverage rate. Hence,

\[
p(x) = P[D = 1|x] = E[D|x]
\]

where \( D \) is an indicator of access to a basic service or good. Similarly, for the expanded set of circumstances, we have that
\[ \theta(x, z) = E[p(x, z)] - \frac{1}{2} E[p(x, z) - E[p(x, z)]] \]

By the Law of Iterated Expectation, it follows that
\[
E[D|x] = E[ED[x, z]|x]
\]

In other words,
\[ p(x) = E[p(x, z)|x] \]

Hence, in particular, again by the Law of Iterated Expectation
\[
E[p(x)] = E[E[p(x, z)|x]] = E[p(x, z)] = \mu
\]

Then it follows that
\[ \theta(x) = \mu - \frac{1}{2} E[p(x) - \mu] \]

and
\[ \theta(x, z) = \mu - \frac{1}{2} E[p(x, z) - \mu] \]

As a consequence,
\[
2 (\theta(x) - \theta(x, z)) = E[p(x, z) - \mu] - E[E[p(x, z) - \mu|x]]
\]

Since
\[ p(x) - \mu = E[p(x, z) - \mu|x] \]

Finally, since \( g(a) = |a| \) is a convex function, it follows from Jensen’s Inequality that
\[
|E[p(x, z) - \mu|x]| \leq E[|p(x, z) - \mu||x|
\]

As a consequence
\[
E[E[p(x, z) - \mu|x]] \leq E[E[|p(x, z) - \mu||x]] = E[p(x, z) - \mu]
\]

and
\[
2 (\theta(x) - \theta(x, z)) = E[p(x, z) - \mu] - E[E[p(x, z) - \mu|x]] \geq 0
\]

Hence,
\[ \theta(x) \geq \theta(x, z) \]

as we would like to show. This establishes the fact that the observed HOI is an upper bound for the true HOI.
6.2 Strict sensitivity to inequality

We have shown that improvements in the distribution of opportunities (in the sense of second-order dominance) would never reduce \( \theta \). In this weak sense, \( \theta \) is a genuine inequality-sensitive coverage rate.

However, since \( \theta \) can always be expressed as

\[
\theta = E[p(x)|p(x) \leq \mu] \cdot P[p(x) \leq \mu] + (1 - P[p(x) \leq \mu]) \cdot \mu
\]

Not all strict reductions in inequality would lead to an increase in \( \theta \). And not all strict increases in inequality would lead to a decrease in \( \theta \). As a matter of fact, \( \theta \) is entirely insensitive to a variety of increases and decreases in inequality.

More specifically, \( \theta \) is completely insensitive to reallocation of opportunities among below-average groups, \( V \), i.e., circumstance groups belonging to \( V = \{x : p(x) \leq \mu\} \). Even though, reallocation of opportunities among groups in this set could unambiguously lead to increases or decreases in inequality, the measure \( \theta \) is insensitive to such reallocations.

Reallocations of this type would never lead to changes in

(i) \( E[p(x)|p(x) \leq \mu] \) (average among below-average groups),

(ii) \( \mu \) (overall average), and

(iii) \( P[p(x) \leq \mu] \) (proportion of the population in vulnerable groups).

These are the three immediate determinants of \( \theta \). If none of them changes, \( \theta \) will not change. By similar reasoning, it can also be shown that \( \theta \) is insensitive to reallocations among above-average opportunity groups, \( V^C = \{x : p(x) > \mu\} \).

In fact, \( \theta \) is only sensitive to changes that either change

(i) the overall coverage rate, \( \mu \), or lead to

(ii) reallocations of opportunities between below-average and above-average groups.

In this sense, \( \theta \) is similar to the poverty gap: an index that is insensitive to both (i) income transfers among the poor and (ii) income transfers among the non-poor.

In this sense, \( \theta \) is only weakly sensitive to inequality. When inequality declines, \( \theta \) would never increase, but it may not increase either. As a consequence, \( \theta \) may remain unchanged even when inequality has clearly increased.

As in the case of the poverty gap, this undesirable property is a consequence of \( \theta \) being an average of a concave function, which is not, however, strictly concave. In fact,

\[
\theta = E[\min\{p(x), \mu\}]
\]

and the function \( f(t) = \min\{t, \mu\} \) is concave, but not a strictly concave function.
As in the case of the poverty gap, a solution could be obtained by replacing it with a strictly concave increasing function. The most natural alternative is to use a logarithmic function. In this case,

$$\ln(\theta^*) = E[\ln(p(x))]$$

Hence,

$$\theta = \exp\{E[\ln(p(x))]\}$$

It is easy to see that $\theta^*$ is the geometric mean of the circumstance-specific coverage rates, $p(x)$. In this case, any proportional increases in all circumstance-specific coverage rates would increase $\theta^*$ by the same proportion. More specifically, if $p_1(x) = \lambda p(x)$, then $\theta_1^* = \lambda \theta^*$. Moreover, it follows from the strict concavity of the logarithmic function that if $p_1$ strictly second-order dominates $p$, then $\theta_1^* > \theta^*$. In the case of $\theta$, all one can show is that $\theta_1 \geq \theta$. So, it may be the case that $p_1$ strictly second-order dominates $p$ and $\theta_1 = \theta$. Nevertheless, it is certainly true that it would never be the case that $\theta_1 < \theta$.

6.3 Subgroup consistency

In principle, if opportunities improved for each and every region of a country, then it also should improve when measured for the whole country. $\theta$, however, does not have this property.

To exemplify, we consider a country with two regions and eight circumstance groups with identical population sizes. Table 1 presents the initial and final coverage rates for each of the eight groups. Circumstance groups 1 to 3 and 5 form region A, whereas the groups 4 and 6 to 8 make up region B.

From the initial to the final period, two changes occur. Firstly, the coverage rate of all groups increased proportionately by 1%. Secondly, opportunities are transferred from the most vulnerable groups 3 and 4 to the least vulnerable groups 5 and 6. These transfers unambiguously increased the degree of inequality of opportunity.

Therefore, from period 0 to 1, both overall coverage and inequality increased for the country as a whole, as well as within each region. The evolution of the opportunity index will depend, therefore, on how these two transformations are weighted.

In this case, $\theta$ displays an awkward property. Within each region, the scale effect dominates, leading to an improvement in the regional specific overall opportunity indices. This scale dominance occurs because in region A the increase in inequality was restricted to above-average groups, while in region B, it remained restricted to below-average groups.
<table>
<thead>
<tr>
<th>Circumstance group</th>
<th>Initial period</th>
<th>Final period</th>
<th>Change</th>
<th>Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1- Region A</td>
<td>0.100</td>
<td>0.101</td>
<td>0.001</td>
<td>1% increase</td>
</tr>
<tr>
<td>2- Region A</td>
<td>0.200</td>
<td>0.202</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>3- Region A</td>
<td>0.300</td>
<td>0.258</td>
<td>-0.042</td>
<td>1% increase and 4.5%points</td>
</tr>
<tr>
<td>4- Region B</td>
<td>0.300</td>
<td>0.258</td>
<td>-0.042</td>
<td>coverage loss</td>
</tr>
<tr>
<td>5- Region A</td>
<td>0.400</td>
<td>0.449</td>
<td>0.049</td>
<td>1% increase and 4.5%points</td>
</tr>
<tr>
<td>6- Region B</td>
<td>0.400</td>
<td>0.449</td>
<td>0.049</td>
<td>coverage gain</td>
</tr>
<tr>
<td>7- Region B</td>
<td>0.500</td>
<td>0.505</td>
<td>0.0057</td>
<td>1% increase</td>
</tr>
<tr>
<td>8- Region B</td>
<td>0.600</td>
<td>0.606</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Average region A</td>
<td>0.250</td>
<td>0.253</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>Average region B</td>
<td>0.450</td>
<td>0.455</td>
<td>0.005</td>
<td></td>
</tr>
<tr>
<td>Overall average</td>
<td>0.350</td>
<td>0.354</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>Proportion of the population in below-average groups: Region A</td>
<td>0.500</td>
<td>0.500</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Proportion of the population in below-average groups: Region B</td>
<td>0.500</td>
<td>0.500</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Proportion of the population in below-average groups</td>
<td>0.500</td>
<td>0.500</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>Average coverage for the below-average groups in region A</td>
<td>0.150</td>
<td>0.152</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>Average coverage for the below-average groups in region B</td>
<td>0.350</td>
<td>0.354</td>
<td>0.004</td>
<td></td>
</tr>
<tr>
<td>Average coverage for the below-average groups for the country</td>
<td>0.225</td>
<td>0.205</td>
<td>-0.020</td>
<td></td>
</tr>
<tr>
<td>O index for region A</td>
<td>0.200</td>
<td>0.202</td>
<td>0.002</td>
<td></td>
</tr>
<tr>
<td>O index for region B</td>
<td>0.400</td>
<td>0.404</td>
<td>0.004</td>
<td></td>
</tr>
<tr>
<td>O index for the country</td>
<td>0.288</td>
<td>0.279</td>
<td>-0.008</td>
<td></td>
</tr>
<tr>
<td>O* index for region A</td>
<td>0.311</td>
<td>0.309</td>
<td>-0.001</td>
<td></td>
</tr>
<tr>
<td>O* index for region B</td>
<td>0.221</td>
<td>0.220</td>
<td>-0.001</td>
<td></td>
</tr>
<tr>
<td>O* index for the country</td>
<td>0.436</td>
<td>0.434</td>
<td>-0.002</td>
<td></td>
</tr>
</tbody>
</table>
For the country as a whole, however, the increase in inequality mixed below-average and above-average groups and ended up dominating the increase in coverage. So the country’s overall opportunity index declined, even though

(i) the average coverage rate of each region increased,
(ii) the overall opportunity index for each region increased and
(iii) there were no transfers of opportunities or population between regions.

As in the previous section, the option for the geometric mean would also resolve the inconvenience. When the geometric mean is adopted, the country’s overall opportunity index is given by the geometric mean of the regional specific indices, i.e.

\[ \theta^* = \Pi_r (\theta^*_r)^{\alpha_r} \]

where \( \theta^*_r \) is the population weighted geometric mean of the circumstance group-specific coverage rates in region \( r \) and \( \alpha_r \) is the fraction of the population in region \( r \). As a consequence, the country’s index would always increase when all regional indices increase.

### 7. Summary and Final Remarks

Universal access to basic opportunities is undoubtedly a central objective of development. It has been the core of many world meetings and declarations, including the Millennium Development Goals. Despite all its importance, progress towards universality continues to be measured by the overall coverage rate: a measure entirely insensitive to how opportunities are allocated.

In this study, we propose an alternative scalar measure. Besides being sensitive to the number of available opportunities, this new measure also cares about how these opportunities are allocated. For this reason, it is called an equity-sensitive coverage rate. The proposed measure is increasing on the overall coverage rate and decreasing on deviations from the ideal of equal opportunity.

We consider that equality of opportunity prevails when the probability of access to an opportunity is unrelated to a selected set of personal, family, and community characteristics collectively referred to as circumstances. Hence, ideally, every circumstance group-specific coverage rate should equal the overall coverage. Accordingly, we measure the deviation from the ideal of equal opportunity by the average distance of each actual group-specific coverage rate from the overall coverage.

The proposed measure is obtained by deducting a penalty equal to the deviation from the ideal of equal opportunity from the overall coverage rate. We show that this measure has many simple and intuitive interpretations. For instance, it can be interpreted as the proportion of available opportunities needed to ensure
full coverage and which have been allocated according to the equal opportunity principle. It can also be interpreted as a weighted average of all circumstance group-specific coverage rates, where the weights are larger than the population share among the vulnerable groups and smaller than the population share among the non-vulnerable groups.

We also show that this equity-sensitive measure has several suitable properties. In particular, we demonstrate that it is consistent with both first- and second-order stochastic dominance. As a consequence, this measure is Pareto consistent and sensitive to improvements in the allocation of the available opportunities. According to this index, any increase in the number of opportunities that also leads to an increase in the average coverage rate among the most vulnerable groups would always represent a progress towards opportunities for all.

In this study, we also demonstrate how the proposed index and its asymptotic variance could be estimated.

We expect that equity-sensitive measures will help to properly introduce basic opportunities for all as a formal and quantitative goal for societies, governments, and public policy, and to promote the proper consideration of universal access in standard impact evaluation studies. Such equity-sensitive measure may also be helpful in improving targeting, in measuring poverty and in improving the effectiveness of social policy in promoting universal access to basic opportunities.

References


