STABILITY OF THE EDGEWORTH PROCESS WITH FIRMS BUT NO PRODUCTION

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ABSTRACT

In this paper an Edgeworth Process for economies with firms is defined, and its stability is proved. The process constrains trade to occur if and only if there is, at the given prices, a trade that increases the sum of utilities and profits without decreasing the utility of a household or the profits of a firm. A proof of convergence of prices to an equilibrium price vector is provided.

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1. INTRODUCTION

In the literature related to the stability of general equilibrium two kinds of restrictions have alternatively been imposed on non-tâtonnement trading processes, in order to obtain results. If the restrictions of the Hahn Process are satisfied, then every agent's post-trade excess demand for a commodity agrees in sign (if not zero) with the aggregate excess demand for that commodity. On the other hand, the Edgeworth Process constrains trade to occur if and only if there exists some trade that satisfies the budget constraints of households, determines a feasible allocation, and is Pareto-improving.

Besides the original works showing the stability of the Hahn Process for exchange economies (see [11] and [13]) a series of stability results in a Hahn Process framework were obtained by Franklin Fisher. The introduction of firms (but no production out of equilibrium) was effected in [5], and that of quantity of constraints in [8]. Also, production and consumption were allowed in [6, 7, and 9], in models that contained assumptions closely related to the Hahn Process restrictions.

The literature on the Edgeworth Process has been much less abundant, and the results obtained less general. In the original works of Uzawa [23, 24] and also in [1, 12, and 17], proofs of stability of the Edgeworth Process for exchange economies were obtained. Of these proofs, that in [1] is the most satisfactory. A discrete version of the process with demand correspondences instead of demand functions was shown to converge to the set of Pareto optimal

allocations in [3], with the use of a vector-valued Lyapounov function. In [21] price changes are determined by "short-run demands", which are the directions in which small trades provide the largest utility gains to households. This version of the process is also stable, as shown in [21]. Finally, in [10] the methods and results of [21] are extended to monetary exchange economies. But no further extensions appear to be available.

The main purpose of this paper is to begin to extend the Edgeworth Process in the direction of realism, as has been done for the Hahn Process. We define an Edgeworth Process for an economy with firms but no production and prove its stability. We also give a satisfactory proof of the convergence of prices to their equilibrium values. In previous works in which, as in the present one, demands are determined by utility (or profit) maximization [1, 12, 17, 23, and 24], such a proof is either absent or incorrect. The possibility of extensions will be discussed and some of the limitations of the process pointed out. It will be argued that the present form of the Edgeworth Process is not likely to produce sensible stability results in a world of money-mediated exchange like the one in Arrow and Hahn [1, p.340]. The question of the compatibility of the Edgeworth Process with models in which consumption and production take place out of equilibrium will also be discussed.

By an economy with firms but no production (and, by the way, households but no consumption) we mean an economy in which production and consumption occur only after equilibrium is reached. Households and firms exchange commitments to buy and sell commodities at equilibrium. When equilibrium is reached production and consumption take place, and the commitments are carried out. A more detailed description of such an economy is given in Section 2.

Although the economy modeled here retains many of the essential characteristics of the one modeled for the Hahn Process in [5], there are some differences between them. The most important of these is the absence of a separate treatment for money and of financial restrictions on exchange in the present work. It will be argued in Section 5 that in the Edgeworth Process assumption is

incompatible with money-mediated exchange. That is, the Edgeworth Process is essentially a barter process. Accordingly, there is no distinction between active and target excess demands, and we assume all realized profits of firms to be distributed at each point of time. This assumption as regards profit-distribution is made only for convenience, because under a rule that would guarantee asymptotic distribution of all realized profits our results would still hold. Moreover, without an explicit treatment of money this is a sensible assumption: if firms do not face financial constraints they need only hold money for production purposes. The reader can consider the results of this paper as concerning only this class of economies, or he can assume away the difficulties associated with a more detailed treatment of money.

The structure of the paper is as follows. In Section 2 we define the model, and in Section 3 the dynamic process is specified. The form of the Edgeworth Process proposed in this paper consists in assuming that transactions will occur at a moment of time if and only if there is some strictly Pareto superior (in a sense that will be specified below) allocation that is budget-feasible for all agents, and that when transactions take place the utilities of households and the profits of firms do not decrease, with at least one of these increasing strictly².

In Section 4 some preliminary results and the main theorem are proved. The method of proof owes much to [2] and to [24]. The

² We employ for shortness the expression "Pareto superior" to allocate a that is being compared to allocation b if:

- PS1) The utility of each household at a is not less than its utility at b.
- PS2) No firm has smaller profits at a than at b.

If also PS3 is satisfied, we say that a is strictly Pareto superior to b.

- PS3) Either some household strictly prefers a to b or some firm has strictly higher profits at a than at b.

discussion of limitations and possible extensions is presented in Section 5.

2. THE MODEL

As in [5], an economic process with firms but no production (or, indeed, consumption) before equilibrium is reached is modeled. Households and firms exchange commitments to buy and sell commodities, including labor. When equilibrium is reached, production and consumption take place and these commitments are carried out.

There are N commodities, H households, and F firms. The first commodity is the numeraire, and its price is assumed to be fixed at unity.

Each household has its preferences representable by a continuously differentiable, monotonic, and strictly quasiconcave utility function, U_h , defined on the nonnegative orthant of R^N .³ Also, there is no satiation consumption for household h .

At time t , household h holds a physical amount of commodity i and commitments to deliver and/or receive that commodity. As the distinction between these two components of its holdings of commodity i is irrelevant for our purposes, we consider only the "actual" stock of commodity i with household h in what follows. This is, of

³ We allow for the existence of a set of commodities whose consumption has no effect on U_h . The property of strict quasiconcavity applies to the restriction of U_h to the subspace of R^N that corresponds to the complement of that set, i.e., to the set of commodities whose consumption affects U_h in a non-trivial way.

course, the sum of the two types of holdings, and is denoted by $\bar{x}_{hi}$, or $\bar{x}_{hi}(t)$. For these, and other variables in the model, the notation will be frequently simplified by omitting explicit reference to their dependence on time.

The desired stock of commodity i by household h is denoted $x_{hi}(t)$, or simply x_{hi} . Desired stocks are determined by utility maximization at prices $p = (1, p_2, \dots, p_N)'$, where the prime denotes transposition, given a budget constraint

$$p'x_h = w_h \quad (1)$$

where $x_h = (x_{h1}, x_{h2}, \dots, x_{hN})'$ and the "virtual" wealth w_h of household h is given by

$$w_h = p'\bar{x}_h + s_h - d_h \quad (2)$$

and $\bar{x}_h = (\bar{x}_{h1}, \bar{x}_{h2}, \dots, \bar{x}_{hN})'$. The household is assumed to demand a zero amount of those commodities that enter trivially its utility function.

In equation (2), s_h is the share of household h in the desired earnings of firms, whence the name "virtual" wealth. And d_h is the share of household h in the actual earnings of firms. The meaning of the distinction between actual and desired earnings will be made clear below.

Each firm is assumed to have a production possibility set of the form

$$Y^f = \{v_f = (v_{f1}, v_{f2}, \dots, v_{fN})' \in R^N \mid \phi_f(v_f) \leq 0\}$$

where $0 \in Y^f$, and $\phi_f : R^N \rightarrow R$ is continuously differentiable. It

is also assumed that

$$\frac{\partial \phi_f}{\partial v_i} \geq 0 \quad (i=1, 2, \dots, N)$$

with the inequality holding strictly for $i=1$, and that ϕ_f is a strictly convex function of those variables from which it is not independent⁴. When $v_{fi} \geq 0$ we say that v_{fi} is an output of firm f , otherwise we say that v_{fi} is an input.

The actual commitment to deliver commodity i by firm f is denoted $\bar{v}_{fi}$ and the corresponding desired commitment is v_{fi} . In order to understand the maximizing behavior that generates the desired commitments of firms we need some further definitions. First, we define $v_f = (v_{f1}, v_{f2}, \dots, v_{fN})'$, $\bar{v}_f = (\bar{v}_{f1}, \bar{v}_{f2}, \dots, \bar{v}_{fN})'$, $g_f = v_f - \bar{v}_f$, and $\bar{v} = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_F)'$.

Now, the realized profits of firm f are denoted by $\bar{\pi}_f$, and given by

$$\bar{\pi}_f(t) = \int_0^t p(\tau)' \dot{\bar{v}}_f(\tau) d\tau + \bar{\pi}_f(0) \quad (3)$$

and its desired profits π_f are the optimal value of the problem

$$\text{Max } p'(v - \bar{v}_f) + \bar{\pi}_f$$

$$\text{subject to } v \in Y^f$$

where the maximization takes place over those commodities from which ϕ_f is not independent, and it is assumed that the firm

⁴ Note that this rules out constant returns to scale.

wants to end up with zero commitments of the remaining commodities. We also assume that for each positive price vector p this problem has a solution. By strict convexity this solution is unique, and we denote it by $v_f(p)$. Then we clearly have

$$\pi_f = p' (v_f(p) - \bar{v}_f) + \bar{\pi}_f. \quad (4)$$

For simplicity, we assume that firms are owned only by households, and not by other firms. All realized earnings of firms are distributed to households at each moment of time. The distribution is made according to the shares of firms that are held by households. The share of firm f that is held by household h is denoted by b_{hf} . Of course, we have

$$\sum_{h=1}^H b_{hf} = 1 \quad (f=1, 2, \dots, F). \quad (5)$$

Since there is no uncertainty and everyone has the same price expectations, namely that prices will not change, there is no market for these shares. The variables s_h and d_h of equation (2) satisfy

$$\begin{aligned} s_h &= \sum_{f=1}^F b_{hf} \pi_f & (h=1, 2, \dots, H) \\ d_h &= \sum_{f=1}^F b_{hf} \bar{\pi}_f & (h=1, 2, \dots, H). \end{aligned} \quad (6)$$

⁵ We refer the reader to [4] for a discussion of equations (3) and (4), and of other aspects of the basic model.

Taking (4) into account we see that

$$s_h - d_h = \sum_{f=1}^F b_{hf} (p' (v_f(p) - \bar{v}_f)) \quad (h=1, 2, \dots, H) \quad (7)$$

and thus demand functions $x_h = x_h(p, \bar{x}_h, \bar{v})$ can be defined, x_h being the utility maximizing consumption of household h at prices p and wealth w_h given by $\bar{x}_h$ and $\bar{v}$ substituted in (7) and (2). The fact that virtual wealths are used to generate demands constitutes a weakness of the model, since it implies that households always know the desired profits of firms.

Before proceeding, it is convenient to introduce some additional notation

$$\begin{aligned} v &= \begin{vmatrix} v_1 \\ v_2 \\ \vdots \\ v_F \end{vmatrix} & x &= \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_H \end{vmatrix} & \bar{x} &= \begin{vmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_H \end{vmatrix} & \bar{v} &= \begin{vmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_F \end{vmatrix} \\ x_i &= \sum_{h=1}^H x_{hi} & & & & & (i=1, 2, \dots, N) \\ \bar{x}_i &= \sum_{h=1}^H \bar{x}_{hi} & & & & & (i=1, 2, \dots, N) \\ x &= \sum_{h=1}^H x_h & & & & & \\ \bar{x} &= \sum_{h=1}^H \bar{x}_h & & & & & (8) \\ v_i &= \sum_{f=1}^F v_{fi} & & & & & (i=1, 2, \dots, N) \\ \bar{v}_i &= \sum_{f=1}^F \bar{v}_{fi} & & & & & (i=1, 2, \dots, N) \\ v &= \sum_{f=1}^F v_f & & & & & \\ \bar{v} &= \sum_{f=1}^F \bar{v}_f & & & & & \\ z_i &= x_i - \bar{x}_i & & & & & (i=1, 2, \dots, N) \\ G_i &= v_i - \bar{v}_i & & & & & (i=1, 2, \dots, N) \end{aligned}$$

$$Z = X - \bar{X}$$

$$G = V - \bar{V}$$

The fact that the economy is closed is expressed by

$$\bar{X}(t) - \bar{X}(0) = \bar{V}(t) - \bar{V}(0) \quad \forall t \geq 0. \quad (9)$$

Walras' Law can be deduced from (1), (2), (4), (5), (6) and (8), and has the form

$$p'(Z-G) = 0. \quad (10)$$

An allocation $(x, v) \in \bar{R}_+^{HN} \times R^{FN}$ ⁶ is said to be feasible if $v_f \in Y^f$ for all f , $1 \leq f \leq F$, and

$$\sum_{h=1}^H x_h - \sum_{f=1}^F v_f = \bar{X}(0) - \bar{V}(0).$$

The set S of feasible allocations is clearly convex.

A feasible allocation $(x, v) \in S$ is Pareto optimal if there is no feasible allocation $(\tilde{x}, \tilde{v}) \in S$ such that

$$U_h(\tilde{x}_h) \geq U_h(x_h) \quad (h=1, 2, \dots, H)$$

with at least one inequality holding strictly.

⁶ In general, $R_+^n = \{x = x_1, x_2, \dots, x_n\} \in R^n \mid x_i \geq 0, i=1, 2, \dots, n\}.$

A vector $(p, \bar{x}, \bar{v}) \in \bar{R}_+^{(H+1)N} \times R^{FN}$ is said to be a competitive equilibrium if $(\bar{x}, \bar{v}) \in S$ and

$$x_h(p, \bar{x}_h, \bar{v}) \leq \bar{x}_h \quad (h=1, 2, \dots, H)$$

$$v_f(p) \geq \bar{v}_f \quad (f=1, 2, \dots, F).$$

It can be shown that if $(p, \bar{x}, \bar{v})$ is a competitive equilibrium then $(\bar{x}, \bar{v})$ is a Pareto optimal allocation. It is worthwhile noticing that the equilibria of the process are autarchic. That is, there is no trade at equilibrium. For a discussion of the appropriateness of this equilibrium concept for non-tâtonnement models without production and consumption outside of equilibrium, see [18].

To each $(p, x, v) \in \bar{R}_+^{(H+1)N} \times R^{FN}$ we associate a set $B(p, x, v)$ consisting of all $(\tilde{x}, \tilde{v}) \in S$ such that $(\tilde{x}, \tilde{v})$ is Pareto superior to (x, v) and $\tilde{x}_h$ lies on the budget set of household h when its virtual wealth w_h is computed from (2) after substituting $\tilde{v}_f$ for $v_f(p)$ in (7). The set $B(p, x, v)$ is thus the set of all $(\tilde{x}, \tilde{v}) \in S$ that satisfy

$$p' \tilde{x}_h \leq p' x_h + \sum_{f=1}^F b_{hf} (p' (\tilde{v}_f - v_f)) \quad (h=1, 2, \dots, H)$$

$$p' \tilde{v}_f \geq p' v_f \quad (f=1, 2, \dots, F)$$

$$U_n(\tilde{x}_h) \geq U_n(x_h) \quad (h=1, 2, \dots, H)$$

and is clearly convex.

Finally, the functions $\psi_1^h: R_+^{2N} \rightarrow R$ and $\theta_1^f: R_+^N \times R^N \rightarrow R$ are defined. Their usefulness will be clear below.

$$\psi_i^h(p, x) = \frac{\partial U_h(x) / \partial x_i}{\partial U_h(x) / \partial x_1} \cdot \frac{1}{p_i} - 1$$

$$\theta_i^f(p, v) = \frac{\partial \phi_f(v) / \partial v_i}{\partial \phi_f(v) / \partial v_1} \cdot \frac{1}{p_i} - 1$$

$$\psi^h = (\psi_2^h, \psi_3^h, \dots, \psi_N^h)$$

$$\theta^f = (\theta_2^f, \theta_3^f, \dots, \theta_N^f).$$

3. THE DYNAMIC PROCESS

A verbal description of the workings of the process is given in what follows, and then the assumptions that characterize it are listed and discussed.

At each moment of time firms choose profit-maximizing inputs and outputs, considering the given prices. This determines the "virtual" profits that are to be taken into account by households, together with prices and actual stocks, in the determination of their demands. Agents are thus unaware of disequilibrium. The fact that households know the virtual profits of the firms of which they own shares is, of course, a strong informational requirement. Price changes are a linear sign-preserving function of excess demands.

As to the dynamic behavior of stocks, an Edgeworth Process for exchange economies is a process in which if there are strictly

Pareto improving trades available⁷, then households will be able to instantaneously find some such trade and perform the corresponding transactions, but otherwise stocks remain constant⁸. In the present model, when looking for Pareto improving trades, households do not face fixed budget constraints, but rather receive under each trade the wealth that corresponds to the profits that firms earn under that trade. The problem of compatibility of actions involved in finding a Pareto improving trade is thus more complex than it is for the case of an exchange economy. Note, however, that when transactions do not involve firms we have returned to an exchange economy case, so if households can find a strictly Pareto improving trade among themselves, then there is no increase in complexity. More important, it has been proved by Madden [16] that, in an exchange economy, if all households have positive endowments, whenever there is a Pareto improving trade there is also one such trade involving two agents. Since we assume the positivity of endowments of households and firms do not face non-negativity constraints, Madden's result applies to our model, and the informational problem is not as serious as it seems at first sight.

We assume that the time paths of p , $\bar{x}$, and $\bar{v}$ are given by a process satisfying the following assumptions:

Assumption I:

$$\dot{p}_i(t) = Z_i(t) - G_i(t) \quad (i=2,3,\dots,N)$$

$$\dot{p}_1(t) = 0$$

$$\dot{\bar{x}}(t) = g(p(t), \bar{x}(t), \bar{v}(t))$$

$$\dot{\bar{v}}(t) = k(p(t), \bar{x}(t), \bar{v}(t))$$

⁷ By a (strictly) Pareto improving trade we mean a trade in which the post-trade allocation $(\bar{x}, \bar{v})$ is (strictly) Pareto superior to the pre-trade allocation (x, v) .

⁸ On the realism of such an assumption, see [15].

where g and k are defined and satisfy a Lipschitz condition on $R_+^{(H+1)N} \times R^{FN}$. If a trajectory begins at $(p, \bar{x}, \bar{v}) \in R_+^{(H+1)N} \times R^{FN}$, then its closure lies on that set.⁹

This insures the existence and uniqueness of solutions to this system of differential equations, and their continuous dependence on initial conditions, as long as the solutions remain in $R_+^{(H+1)N} \times R^{FN}$.

Assumption II:

$$\bar{v}_f(t) \in Y^f \quad \forall t \geq 0 \quad (f=1, 2, \dots, F).$$

This says that production feasibility is maintained throughout the process. There is no similar assumption in [5], but there the stability result obtained implies that feasibility is asymptotically attained. Here, however, the assumption is essential. For if firms are allowed to make unfeasible commitments, this may determine that an allocation $(\bar{x}(t), \bar{v}(t)) \in S$ is reached such that $(\bar{x}(t), \bar{v}(t))$ is strictly Pareto superior to some Pareto optimal allocation. In this case, the fact that all trades must be Pareto improving would imply that the set of Pareto optima could never be approached.

Assumption III:

For any $(p, \bar{x}, \bar{v}) \in R_+^{(H+1)N} \times R^{FN}$ the motion $(p(t), \bar{x}(t), \bar{v}(t))$ having $(p, \bar{x}, \bar{v})$ as initial conditions remains in a bounded set.¹⁰

⁹ Suitable conditions on the utility functions could insure that prices and actual stocks remain positive, but we prefer to assume it directly.

¹⁰ Assumption III can be derived from more basic assumptions. See [4].

Assumption IV:

$$\dot{U}_h(t) = \frac{d}{dt} U_h(\bar{x}_h(t)) \geq 0 \quad t \geq 0$$

$$\dot{\pi}_f(t) \geq 0 \quad \forall t \geq 0$$

Assumption V:

For all $t \geq 0$,

$$0 = \dot{U}_h(t) = \dot{\pi}_f(t) \quad (h=1,2,\dots,H; f=1,2,\dots,F)$$

if and only if there exists no $(\bar{x}, \bar{v}) \in B(p, \bar{x}, \bar{v})$ such that $(\bar{x}, \bar{v})$ is strictly Pareto superior to $(\bar{x}, \bar{v})$, in which case there is no trade, that is,

$$0 = \dot{\bar{x}}(t) \quad 0 = \dot{\bar{v}}(t).$$

Assumption VI:

$$\dot{\bar{x}}(t) - \dot{\bar{v}}(t) = 0 \quad \forall t \geq 0.$$

Assumption VII:

For all $h \in \{1,2,\dots,H\}$, if there exists an increasing sequence $\{t_n\} \rightarrow \infty$ such that $\dot{\bar{x}}_h(t_n) \neq 0$ for all n , then there exists an M such that $U_h(\bar{x}_h(t_n)) > U_h(\bar{x}_h(t_{n-1}))$ for all $n \geq M$.

Assumptions IV and V are a natural extension of the Edgeworth Process to an economy with firms but no production. Assumption IV states that both the utilities of households and the profits of firms are non-decreasing in time. Assumption V states that at least one of these utilities or profits will increase at time t , unless

there is no $(\bar{x}, \bar{v}) \neq (\bar{x}, \bar{v}), (\bar{x}, \bar{v}) \in B(p, \bar{x}, \bar{v})$, in which case also there is no trade. Notice that if such an allocation $(\bar{x}, \bar{v})$ exists it gives a feasible direction $(\bar{x} - \bar{x}, \bar{v} - \bar{v})$ along which small movements are strictly Pareto improving. This direction is "feasible" because, for small $\lambda > 0$, $(\bar{x}, \bar{v}) + \lambda(\bar{x} - \bar{x}, \bar{v} - \bar{v})$ is a feasible allocation in which households do not violate their budget constraints.

Assumption VII states that a household will not continue trading forever unless such trading eventually leads to an increase in its utility. It is easy to see that as long as a Pareto improving trade exists, there exists a Pareto improving trade that increases the utility of every household and the profits of every firm. Assumption VII is therefore not necessarily incompatible with Assumption I-VI. That is, the cases ruled out by Assumption VII are never compelled to occur by the logic of the other assumptions. Assumption VII is necessary to rule out a form of perverse behavior of agents in which households, while receiving increasing amounts of dividends from firms, perform transactions that have the effect of keeping their utilities constant. In other words, we exclude the possibility that $\dot{\bar{x}}_h(t_h) \neq 0 \forall n$ while $U_h(\bar{x}_h(t_n)) = U_h(\bar{x}_h(t_1)) \forall n$, in which case, from Assumption V, the profits of firms and therefore their dividends should be increasing with n ¹¹.

¹¹ The following discrete time example shows that Assumption VII is necessary. Consider an economy with H households, one firm, and three commodities. The utility function of household 1 is $U_1(x_1, x_2, x_3) = (x_1, x_2, x_3)^{1/3}$. The firm's production function need not be specified, except for the fact that it can transform one unit of good 2 into one unit of good 3, and vice-versa. We assume that the firm is owned by household 1. Also, in the story that follows, households 2, 3, ..., H never trade. Given the Sonnenschein-Debreu [4, 20] result on excess demand functions, we can assume that

$$\bar{x}_1 = (1, 2, 1), p = (1, 2, 1) + Z = (1, -1, 1)$$

$$\bar{x}_1 = (1, 1, 2), p = (1, 1, 2) + Z = (1, 1, -1)$$

The discrete form of our price adjustment process is $\Delta p_i = Z_i, i = 2, 3$. Now, suppose that $\bar{x}_1(0) = (1, 1, 2)$ and $p(0) = (1, 2, 1)$, and that, in period zero, the household sells one unit of commodity 3, buys one unit of commodity 2, and receives the resulting profits from the firm. Then $\bar{x}_1(1) = (1, 2, 1)$, and, according to our assumption on excess demands and the discrete price adjustment rule, $p(1) = (1, 1, 2)$. Now, in period 1, household 1 reverses the transactions that it made in period zero, and receives the resulting profits. Then, again according to our assumptions, prices return to their initial values. That is, $\bar{x}_1(2) = \bar{x}_1(0)$, $p(2) = p(0)$. A cycle is thus formed in which households have constant utility but there is trade *ad infinitum*. Notice that the realized profits of the firm increase without bounds, but so do the losses of household 1, due to adverse price changes.

4. PROOF OF STABILITY

In this section the global stability of the process given by I-VII is proved. In other words, it is shown that, for any $(p(0), \bar{x}(0), \bar{v}(0)) \in R_+^{(H+1)N} \times R^{FN}$, the corresponding solution to the system of differential equations given in I converges to some competitive equilibrium. Before this is done, some preliminary results are needed.

Lemma 1: If $(p, x, v) \in R_+^{(H+1)N} \times R^{FN}$, then $B(p, x, v) = \{(x, v)\}$ only if there exist non-negative $\xi_1, \xi_2, \dots, \xi_H, \gamma_1, \gamma_2, \dots, \gamma_F$ and a vector $u = (u_2, u_3, \dots, u_N) \in R^{N-1}$ such that

$$\psi^h(p, x_h) = \xi_h u \quad (h=1, 2, \dots, H) \quad (11)$$

$$\theta^f(p, v_f) = \gamma_f u \quad (f=1, 2, \dots, F)$$

Proof: The proof is a simple extension of the proof of Lemma 2 in [21] to the case of an economy with firms, and is therefore omitted.

Lemma 2: If a solution to the process I-VII with initial conditions $(p(0), \bar{x}(0), \bar{v}(0)) \in R_+^{(H+1)N} \times R^{FN}$ satisfies $\bar{x}(t) = \bar{x}(0)$ and $\bar{v}(t) = \bar{v}(0)$ for all $t \geq 0$ then $(\bar{x}(0), \bar{v}(0))$ is a Pareto optimal allocation.

Proof: It follows from Assumption V and from strict convexity that, for all $t \geq 0$,

$$B(p(t), \bar{x}(t), \bar{v}(t)) = \{(\bar{x}(0), \bar{v}(0))\}.$$

Adapting the argument of Lemma 3 in [22], it can be shown that, unless $(\bar{x}(0), \bar{v}(0))$ is Pareto Optimal, there exists some continuous $\omega: \mathbb{R}_+ \rightarrow \mathbb{R}$ and scalars k_i , $i = 2, 3, \dots, N$, such that

$$\dot{p}_i(t) = k_i \omega(t) \quad (i = 2, 3, \dots, N). \quad (12)$$

Then $\omega(t) \neq 0$ for all $t \geq 0$. For suppose that $\omega(t_0) = 0$ for some $t_0 \geq 0$. Then $\dot{p}_i(t_0) = 0$, $2 \leq i \leq N$ and excess demands are zero at time t . This means that $(p(t_0), x(t_0), v(t_0))$ is a competitive equilibrium¹² for the Arrow-Debreu economy where household h has utility function U_h and initial endowments $\bar{x}_h(t_0) - \sum_{f=1}^F b_{hf} \bar{v}_f(t_0)$, and firm f has production possibility set Y^f . Of course, in such an economy a household may have negative initial endowments of some commodities, but its wealth at prices $p(t_0)$ is always non-negative. Then $(x(t_0), v(t_0))$ is a Pareto optimal allocation¹³ for this economy. It is easily checked that the set of feasible allocations for this economy is precisely S , whence $(x(t_0), v(t_0))$ is also a Pareto optimal allocation for the economy considered in the statement of the Lemma. Now, from $B(p(t_0), \bar{x}(t_0), \bar{v}(t_0)) = \{(\bar{x}(t_0), \bar{v}(t_0))\}$ we conclude that $(\bar{x}(t_0), \bar{v}(t_0))$ is also a Pareto optimal allocation for the economy considered in the Lemma. By assumption, $(\bar{x}(t_0), \bar{v}(t_0)) = (\bar{x}(0), \bar{v}(0))$.

It is clear from (12) that if $\omega(t) \neq 0$ for all $t \geq 0$, then each price varies monotonically in time. Assumption III then insures the convergence of prices to some p^* , and thus $(p^*, \bar{x}(0), \bar{v}(0))$ is a rest point of the process¹⁴. This being the case, excess demands

¹² Here the expression "competitive equilibrium" has the usual meaning, and not the one given to it elsewhere in this paper. That is, here "competitive equilibrium" means a not necessarily autarchic equilibrium of an Arrow-Debreu economy.

¹³ The remark of footnote (12) applies here to the expression "Pareto optimal allocation" and in the next sentence to "feasible allocations". That is, the same concepts have here what their usual meanings are in Arrow-Debreu theory.

¹⁴ See 19, where it is shown that a limit towards which a motion converges is necessarily a rest point.

are zero at $(p^*, \bar{x}(0), \bar{v}(0))$. By the reasoning in the previous paragraph we can then conclude that $(\bar{x}(0), \bar{v}(0))$ is a Pareto optimal allocation. The proof of the Lemma is complete.

Lemma 3: If $(p(t), \bar{x}(t), \bar{v}(t))$ is a solution to the process I-VII with initial conditions $(p(0), \bar{x}(0), \bar{v}(0)) \in R_+^{(H+1)N} \times R^{FN}$ where $(\bar{x}(0), \bar{v}(0))$ is a Pareto optimal allocation, then for all $t \geq 0$, $\bar{x}(t) = \bar{x}(0)$ and $\bar{v}(t) = \bar{v}(0)$. Moreover, the limit

$$p^* = \lim_{t \rightarrow \infty} p(t)$$

exists and is the unique equilibrium price vector associated to $(\bar{x}(0), \bar{v}(0))$.

Proof: We adapt an argument in [2] to the case of an economy with firms. That $\bar{x}(t) = \bar{x}(0)$ and $\bar{v}(t) = \bar{v}(0)$ for all $t \geq 0$ follows from Assumption V. It remains to prove the statement about prices.

Consider the function

$$V \equiv \frac{1}{2} \sum_{j=1}^N (p_j - p_j^*)^2$$

where $p^* = (1, p_2^*, \dots, p_N^*)'$ is the unique¹⁵ equilibrium price vector associated to $(\bar{x}(0), \bar{v}(0))$. We show that $\dot{V} < 0$ except when $p = p^*$. Then V is a Lyapounov function, and the Lemma follows from Assumption III.

¹⁵ Uniqueness of p^* follows from $\bar{x}(0) > 0$ and the differentiability of utility functions.

We have,

$$\dot{V} = \sum_{j=1}^N (p_j - p_j^*) (z_j - G_j) = - \sum_{j=1}^N p_j^* (z_j - G_j) \quad (13)$$

where the last equality follows from Walras' Law (10). If we define

$$z_h(t) = x_h(t) - \bar{x}_h(0)$$

$$g_f(t) = v_f(t) - \bar{v}_f(0)$$

we can rewrite (13) as

$$\dot{V} = - \left(\sum_{h=1}^H p^{*'} z_h - \sum_{f=1}^F p^{*'} g_f \right). \quad (14)$$

By the definition of a competitive equilibrium, and since $p^* > 0$,

$$z_h^* \equiv x_h(p^*, \bar{x}_h(0), \bar{v}(0)) - \bar{x}_h(0) = 0$$

$$g_f^* \equiv v_f(p^*) - \bar{v}_f(0) = 0.$$

It follows from the differentiability of the U_h 's and ϕ_f 's and from the strict convexity of preference relations and production possibility sets that $U_h(x_h(p(t), \bar{x}_h(0), \bar{v}(0))) > U_h(\bar{x}_h(0))$ unless $p = p^*$. Then, by revealed preference, $p^{*'} z_h(t) > 0$ for all h , $1 \leq h \leq H$ and all $t \geq 0$. This means that the expression in brackets in (14) is positive unless $p = p^*$ and V is indeed a Lyapounov function.

Theorem: If $(p(t), \bar{x}(t), \bar{v}(t))$ is a solution to the process I-VII with initial conditions $(p(0), \bar{x}(0), \bar{v}(0)) \in R_+^{(H+1)N} \times R^{FN}$ then $(\bar{x}(t), \bar{v}(t))$ converges to a Pareto optimal allocation $(\bar{x}^*, \bar{v}^*)$. Furthermore,

$$\lim_{t \rightarrow \infty} p(t) = p^* \quad (15)$$

exists and is the unique equilibrium price vector associated to $(\bar{x}^*, \bar{v}^*)$.

Proof: For any $(p, \bar{x}, \bar{v}) \in R_+^{(H+1)N} \times R^{FN}$ let $\omega(p, \bar{x}, \bar{v})$ be the ω -limit set of the motion associated to $(p, \bar{x}, \bar{v})$. In other words, $\omega(p, \bar{x}, \bar{v})$ is the set of all limit points of the solution having $(p, \bar{x}, \bar{v})$ as initial conditions. Given an arbitrary $(p, \bar{x}, \bar{v}) \in R_+^{(H+1)N} \times R^{FN}$, the set $\omega(p, \bar{x}, \bar{v})$ is nonempty, compact, and connected. This follows from Assumption III. Also, $\omega(p, \bar{x}, \bar{v})$ is an invariant¹⁶ set (see Theorem 3.7 in [22]).

Let $(p^*, \bar{x}^*, \bar{v}^*) \in \omega(p(0), \bar{x}(0), \bar{v}(0))$ and let $(p^*(t), \bar{x}^*(t), \bar{v}^*(t))$ be the motion associated to $(p^*, \bar{x}^*, \bar{v}^*)$. Since $\omega(p(0), \bar{x}(0), \bar{v}(0))$ is invariant, $(p^*(t_0), \bar{x}^*(t_0), \bar{v}^*(t_0)) \in \omega(p(0), \bar{x}(0), \bar{v}(0))$ for all $t_0 \geq 0$. Now Assumptions III and IV imply that each U_h converges to some limit, say, α_h , when $t \rightarrow \infty$. It follows that, for all $t \geq 0$,

$$U_h(\bar{x}_h^*(t)) = \alpha_h \quad (h=1, 2, \dots, H).$$

Given this, Assumption VII allows us to conclude that there exists some $T \geq 0$ such that $\bar{x}_h^*(t) = \bar{x}_h^*(T)$ for all $t \geq T$ and all h , $1 \leq h \leq H$. Suppose that

¹⁶ See [19]. A set I is invariant if any motion starting from a point in I remains in I .

$$\lim_{t \rightarrow \infty} \bar{\pi}_f^*(t) = +\infty \quad (16)$$

for some f , $1 \leq f \leq F$.

Given Assumption III, there exists and $M > 0$ such that

$$|p^* \bar{X}^*| < M \quad \forall t \geq 0.$$

Define $R^* \equiv p^* \bar{X}^*$. Then,

$$\dot{R}^* = p^* \dot{\bar{X}}^* + \dot{p}^* \bar{X}^* = \dot{D}^* + \dot{p}^* \bar{X}^*,$$

where

$$D \equiv \sum_{h=1}^H d_h.$$

Since $|\int_0^t \dot{R}^* d\tau| < 2M$ for all $t \geq 0$,

$$\int_0^t \dot{D}^* d\tau > nM \Rightarrow \int_0^t \dot{p}^* \bar{X}^* d\tau < -(n-2)M. \quad (17).$$

From (16),

$$\lim_{t \rightarrow \infty} \sum_{f=1}^F \bar{\pi}_f^*(t) = \lim_{t \rightarrow \infty} D^* = D^*(0) + \int_0^{\infty} \dot{D}^* d\tau = +\infty,$$

and hence (17) implies that

$$\int_0^{\infty} \dot{p}^* \bar{X}^* d\tau = -\infty.$$

Then, for some i , $2 \leq i \leq N$,

$$\int_T^\infty \dot{p}^{*i} \bar{x}^{*i} d\tau = -\infty$$

and since $\bar{x}^*(\tau) = \bar{x}^*(T)$ for $\tau \geq T$, $p^{*i}(t)$ becomes negative for large t . This contradicts assumption I, and therefore (16) cannot hold. It follows that there are scalars β_f such that

$$\lim_{t \rightarrow \infty} \bar{w}_f^*(t) = \beta_f \quad (f=1, 2, \dots, F).$$

Since $\omega(p(0), \bar{x}(0), \bar{v}(0))$ is compact and invariant, and

$$\omega(p^*, \bar{x}^*, \bar{v}^*) \subset \text{cl}\{(p^*(t), \bar{x}^*(t), \bar{v}^*(t)) \mid t \geq 0\},$$

we have $\omega(p^*, \bar{x}^*, \bar{v}^*) \subset \omega(p(0), \bar{x}(0), \bar{v}(0))$.

Now, consider $(p^{**}, \bar{x}^{**}, \bar{v}^{**}) \in \omega(p^*, \bar{x}^*, \bar{v}^*)$, and the associated solution $(p^{**}(t), \bar{x}^{**}(t), \bar{v}^{**}(t))$. Reasoning as above, we conclude that

$$U_h(\bar{x}_h^{**}(t)) = \alpha_h \quad (h=1, 2, \dots, H)$$

and that realized profits are also constant along this solution. By Lemma 2 and Assumption V, $(\bar{x}^{**}, \bar{v}^{**})$ is a Pareto optimal allocation. Then any limit point of $(\bar{x}(t), \bar{v}(t))$ is also a Pareto optimal allocation, since at such a point, as at $(\bar{x}^{**}, \bar{v}^{**})$, the utility of household h is α_h . It follows from the strict quasiconcavity of utility functions that $(\bar{x}^*, \bar{v}^*)$ is the unique limit point of $(\bar{x}(t), \bar{v}(t))$. By Lemma 2,

$$p^{**} = \lim_{t \rightarrow \infty} p^*(t)$$

is the unique equilibrium price vector associated to $(\bar{x}^*, \bar{v}^*)$. It remains to be proved that

$$p^{**} = \lim_{t \rightarrow \infty} p(t). \quad (18)$$

We use the fact that p^{**} is a limit point of $p(t)$ to prove the convergence of $p(t)$ to p^{**} . Let V be defined as in the proof of Lemma 3, but with p^{**} replacing p^* . The idea is that, for any $\varepsilon > 0$, if $(\bar{x}, \bar{v})$ is close enough to $(\bar{x}^*, \bar{v}^*)$ and $V \geq \varepsilon$, then $\dot{V} < 0$. Suppose that this is the case. Clearly, if t is large enough, and $p(t)$ lies in the neighborhood of p^{**} where $V < \varepsilon$, then p will never leave that neighborhood. But p enters that neighborhood for arbitrarily large t , and the result follows. We now make this reasoning rigorous.

Choose $\varepsilon > 0$, and assume that there is no $\delta > 0$ such that $V(p) \geq \varepsilon$ and $\|(\bar{x} - \bar{x}^*, \bar{v} - \bar{v}^*)\| < \delta$ imply $p^{**'}z(p, \bar{x}, \bar{v}) > 0$. Then there exists a sequence $(p_n, \bar{x}_n, \bar{v}_n)$, $n \in \mathbb{Z}_+$, such that $V(p_n) \geq \varepsilon$ and $p^{**'}z(p_n, \bar{x}_n, \bar{v}_n) \leq 0$ for all $n \in \mathbb{Z}_+$, and

$$\lim_{n \rightarrow \infty} (\bar{x}_n, \bar{v}_n) = (\bar{x}^*, \bar{v}^*).$$

Since prices remain in a bounded set and excess demand functions are continuous, there exists $\bar{p} \neq p^{**}$ such that

$$p^{**'}z(\bar{p}, \bar{x}^*, \bar{v}^*) \leq 0. \quad (19)$$

Reasoning as in the proof of Lemma 3, we conclude that (19) cannot hold, and therefore we have a contradiction. Then for all $\varepsilon > 0$ there exists some $\delta > 0$ such that $V(p) \geq \varepsilon$ and $\|(\bar{x} - \bar{x}^*, \bar{v} - \bar{v}^*)\| < \delta$ imply $p^{**'}z(p, \bar{x}, \bar{v}) > 0$, and thus $\dot{V} < 0$.

It follows that if $V(p(t)) < \varepsilon$ and for small enough $\delta > 0$,

$$\|(\bar{x}(\tau), \bar{v}(\tau)) - (\bar{x}^*, \bar{v}^*)\| < \varepsilon \quad \forall \tau \geq t,$$

then $V(p(\tau)) < \epsilon$ for all $\tau \geq t$. Since $(\bar{x}(t), \bar{v}(t))$ converges to $(\bar{x}^*, \bar{v}^*)$ and p^{**} is a limit point of $p(t)$, we conclude that

$$\lim_{t \rightarrow \infty} p(t) = p^*.$$

5. CONCLUSION

In this paper we define an Edgeworth Process for an economy with firms but no production, and prove its stability. Since the Edgeworth Process had been previously defined and proved to be stable only in exchange economies, this represents a step in the direction of greater realism. Also, we present a proof of convergence of prices to their equilibrium values. Such a proof could not be found (or was incorrect) in the previous literature related to the present form of the Edgeworth Process, in which demands are determined by myopic optimization at the given prices.

We now discuss the limitations of the model and its possible extensions.

There are two limitations on the class of models that are compatible with the Edgeworth Process assumption. First, these models cannot allow speculation to occur. To see this, consider for simplicity an exchange economy. Suppose that speculation is allowed. One cannot impose the condition that utilities are increasing in time because household h might want to exchange its actual stock $\bar{x}_h(t)$ for $\bar{x}_h(t')$, $t' > t$, with $U_h(\bar{x}_h(t)) > U_h(\bar{x}_h(t'))$ in the hope that this would bring speculative gains that would be reaped before or at equilibrium. If one tries to avoid this problem by constraining target utilities to be increasing in time, another kind of problem is found: prices may differ from what they were expected to be, and this implies the possibility of losses.

The second limitation is related to the introduction of money in the model. When a household sells a commodity, and receives money in exchange, its actual utility decreases in many cases, even if money enters the utility function. It is an essential feature of money-mediated exchange that some agents receive a commodity that they do not desire for its own sake, viz. money, with the intention of performing further trades in which that commodity is used to acquire other desired commodities. The behavior of such agents may be compared to that of speculators: they are accepting a temporary loss in (actual) utility, in order to obtain a gain later on. The discussion of the previous paragraph then applies to the present case, and we can conclude that the Edgeworth Process assumption is not compatible with money-mediated exchange.

Many of the characteristics of the model we present are related to these two limitations of the Edgeworth Process. Since speculation is not allowed, we assumed that at each moment of time agents believe that the economy will be in equilibrium at the given prices, after trade. This makes more reasonable the assumption that households and firms determine their demands by optimization of their objective functions (utilities or profits) at current prices, and, in the case of households, given their current virtual wealths. Therefore, given our assumption on expectations, the form of behavior we model follows from the assumption we made on the amount of information available to agents. Also, the reason why money is not explicitly introduced should now be clear, and without money in the model it is not unnatural to assume that all realized profits are distributed at each moment of time.

There is, however, one characteristic of the model that was not related to the above mentioned limitations. We did not allow production and consumption to occur outside of equilibrium. Our previous arguments do not imply that such an extension of the present model would not give a stable process, as long as speculative behavior is not permitted. This remains, however, an open question.

Another possible extension is to introduce transaction constraints, as done in [18] for the Hahn Process.

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