A TWO-SECTOR INTERTEMPORAL OPTIMIZING MODEL OF CAPITAL ACCUMULATION AND EXTERNAL INDEBTEDNESS

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ABSTRACT

An intertemporal optimizing model for a small open economy is developed to study the dynamic interaction between capital accumulation and external indebtedness and the steady-state relation between the size of the capital stock and the level of external indebtedness. The analysis shows that in a context of growth, persistent current-account imbalances may result as the outcome of optimal behavior on the part of intertemporal maximizing agents. However, it is shown that by incurring external debt to partially finance its growth effort, the economy places itself on a constrained growth path that ultimately affects the optimal level of the steady-state capital stock.

* This paper contains results from my Ph.D. dissertation at the University of Chicago (1984). I would like to thank Professors Michael Mussaw and Jose Scheinkman for their guidance and encouragement. Financial support from the Conselho Nacional de Pesquisa e Desenvolvimento-CNPq (Brazil) is gratefully acknowledged.
1. INTRODUCTION

A proper analysis of current-account behavior requires an intertemporal optimizing model. Recently, a branch of the international economics literature (Dornbusch 1983, Obstfeld 1981, Sachs 1981c, Svensson and Razin 1981, among others), has discussed the issue of optimal consumption choices in an open-economy intertemporal setting. In this literature, the primary objective is to characterize the optimal paths of consumption and external indebtedness and to show how these paths respond to various shocks (internal, external, temporary or permanent, anticipated or unanticipated) and manifest themselves through changes in the current account of the balance of payments. An important implication from this literature is that a zero current-account position (external balance) is not in general a valid policy target since household's welfare is improved by the possibility of running current-account deficits or surpluses in response to exogenous shocks. A limitation of this literature, however, is the absence of an analysis of investment in physical capital. This limitation is important because shifts in investment opportunities over time should give rise to strong current-account effects (see Sachs 1981a). The omission of optimal investment plans is often justified on grounds of increased complexity and reduced tractability of the intertemporal model.

The objective of this paper is to develop an analytically tractable intertemporal model that allows study of the dynamic interaction between capital accumulation and external indebtedness, as well as the consumption-smoothing motives that influence the time path of external borrowing. The analysis is carried out in the context of a two-sector optimal growth model. A crucial feature of these models pertains to the steady-state configuration of
the economy. The standard result from earlier one-sector and two-sector models in a small open-economy set-up (see Hamada 1966, Bardhan 1965, Bazdarich 1976), is a steady-state equilibrium characterized by a unique level of the capital stock, implying independence of the optimal steady-state capital stock and the level of foreign indebtedness. This independence arises because the steady-state capital stock in that framework is exclusively a function of technological factors; more specifically it is that capital stock which equates its rate of return to the world interest rate. External indebtedness in that framework determines only the fraction of the capital stock which is not owned by domestic residents. This result contrasts with those of the present paper. Here, the steady-state equilibrium is characterized by an explicit relation between the optimal capital stock and the level of external indebtedness, and no longer by a unique level of the capital stock. This result enables us to stress the importance of initial conditions with respect to the state variables of the system for the long-run equilibrium of the economy. Also, contrary to earlier two-sector optimal-growth models, the analysis here distinguishes between a tradeable and a nontradeable goods sector. This distinction enables us to emphasize the constrained nature of the economy's optimal-growth path as it finances part of its growth effort through external debt and therefore commits itself to a long-run output composition that is consistent with the required amount of external debt servicing.

The paper is organized as follows. Section II develops the model and sets up the intertemporal optimizing problem. Section III describes the steady-state configuration of the system. Section IV discusses the dynamics and stability properties of the model and Section V concludes with a brief summary and discussion.

2. THE MODEL

Consider a small open economy consisting of identical, immortal households that produces and consumes an internationally tra-
deable good and a nontradeable good. This economy has access to an international market where it can lend or borrow tradeables directly at a constant interest rate $r$. At time $t = 0$, the state of the economy is characterized by a predetermined capital stock $K(0)$, and external debt $D(0)$, given by its past history.

2.1 Technology

Let gross output (denoted by superscript $g$) of tradeables ($x$) and nontradeables ($z$) be produced according to a neoclassical concave production function:

$$X^g = F(K_x), \quad F' > 0, \quad F'' < 0 \tag{1}$$

$$Z^g = G(K_z), \quad G' > 0, \quad G'' < 0 \tag{2}$$

where $K_x$ and $K_z$ are the quantities of capital used in the respective industry. At each point in time capital is fully employed, $K(t) = K_x(t) + K_z(t)$. For the purposes of this paper, it is implicitly assumed that labor employed in each industry is specific to that industry and fixed forever in supply.¹

Capital accumulation is determined by:

$$K(t) = I(t) - \delta K(t) \tag{3}$$

which states that net capital accumulation equals investment ($I$) minus depreciation, where $\delta$ is the exogenously determined rate of

¹ Extension of the analysis to incorporate the assumptions of the standard two-sector model of technology can be easily implemented (see Nunes 1984).
depreciation. It is assumed that the traded good \((X)\) is the only investment good available in the economy. The net output of tradeables is then determined by:

\[
x = x^g - \psi(I)
\]

where \(\psi(I)\) defines an investment cost function with the following properties:

\[
\psi'(I) > 1; \quad \psi''(I) > 0.
\]

This means that the process of transforming traded goods into real capital is costly and also that, a rising marginal cost of gross capital formation is imposed (in terms of units of traded goods, output used in the process). The introduction of the adjustment cost function \(\psi(I)\) into this model is essential to obtain a sensible path of capital accumulation. It eliminates the possibility of "jumps" of the stock of capital at any time period \(t\). Also, the benefit from treating one of the two consumption goods as the investment good is that it enables us to deal with an economy in which there are two distinct consumption goods, as well as an investment good, without dealing explicitly with a three-sector model. This is an important feature which differentiates the present model from those of the earlier literature on two-sector open-economy growth models which distinguished only between a single consumption good and an investment good (see Fisher and Frenkel 1974 and Bazdarich 1978).

The above specification of productive technology enables us to obtain the transformation function:

\[
z^g = z = T(X,K,I) = G[K-F^{-1}(X+\psi(I)]
\]

which indicates the maximum amount of nontraded goods \((z)\) that can
be produced as a function of the rate of investment (I) and the capital stock (K) available to the economy. Using the notation $T_i$ and $T_{ij}$ to denote first and second partial derivatives of $T$, we can summarize the properties of this transformation function as follows:\(^2\)

a) $T_X < 0$, the marginal cost of $X$ in terms of $Z$ is positive
\[(-T_X > 0)\]

b) $T_K > 0$, the marginal product of capital in terms of $Z$ is positive

c) $T_I = T_X \psi'(I) < 0$, the marginal cost of investment in terms of $Z$ is positive
\[(-T_I > 0)\]

d) $T_{XX} < 0$, the marginal cost of $X$ is an increasing function of $X$
\[(-T_{XX} > 0)\]

e) $T_{XX} > 0$, the marginal cost of $X$ is a decreasing function of $K$
\[(-T_{XX} < 0)\]

f) $T_{XI} = T_{XX} \psi'(I) < 0$, the marginal cost of $X$ in terms of $Z$ is an increasing function of $I$
\[(-T_{XI} > 0)\]

g) $T_{KK} < 0$, the marginal product of capital is a decreasing function of $K$

h) $T_{II} = [T_{XX} \psi'(I)]^2 + T_{XII} \psi''(I) < 0$, the marginal cost of investment in terms of $Z$, is an increasing function of $I$
\[(-T_{II} > 0)\]

i) $T_{IK} = T_{XX} \psi'(I) > 0$, the marginal cost of investment is a decreasing function of $K$
\[(-T_{IK} < 0)\]

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\(^2\) See section 1 of the Appendix for a complete derivation of the transformation function ($T$) and its properties.
2.2 External Sector

Let $M(t)$ denote the country's net imports of traded goods at time $t$ from the rest of the world. That is, when $M > 0$, there is excess of domestic demand for traded goods and when $M < 0$, there is excess supply of domestic production of tradeables.

Let $D(t)$ denote the debt of the home country to the rest of the world measured in terms of tradeables. That is, when $D > 0$, the country is a net debtor and when $D < 0$, it is a net creditor vis-à-vis the rest of the world.

The current account (in this model identical to the balance of payments) measures the rate at which this economy is paying off its outstanding indebtedness:

$$\frac{D(t)}{D(0)} = M(t) + rD(t)$$

(6)

where $r$ is the exogenously determined world real interest rate in terms of traded goods. The intertemporal budget constraint requires that:

$$\lim_{t \to \infty} e^{-rt} D(t) = 0.$$  

(7)

Thus, debt cannot grow boundlessly through borrowing that finances debt service indefinitely.

From (6) and (7) it follows that external debt at time $t$ is given by:

$$D(t) = D(0)e^{rt} + \int_{0}^{t} M(S)e^{r(t-S)} dS.$$  

(8)
That is, the stock of external debt at time $t$ must equal the sum of the compounded values of the initial net claims of the rest of the world on the economy and the country's net imports between time zero and $t$.

2.3 The Intertemporal Maximization Problem

The relevant objective for a social planner is to maximize the discounted sum of instantaneous utilities:

$$
\int_{0}^{\infty} U(x,z)e^{-\rho t} \, dt \quad U'(.) > 0, \quad U''(.) < 0.
$$

(9)

Here utility is additively separable and specifically we assume it to have functional form:

$$
U = \log (x^\alpha z^{1-\alpha}) \quad 0 < \alpha < 1.
$$

The rate of time preference ($\rho$) is assumed to be constant and equal to the world interest rate $r$. This assumption is convenient since it assures the existence of a steady-state equilibrium and makes the dynamics of the problem analytically tractable.

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3 Lower case letters define consumption of each good.

4 Nunes (1984) shows that neither the steady-state nor the stability properties of the system depend on the form of the utility function. However, in order to fully characterize the dynamics of the system a more restrictive class of utility functions must be adopted. One such class is that of additively separable utility functions. Sachs (1981c) and Obstfeld (1981) also adopt this formulation.
The optimization problem then consists of choosing the path of the control variables \( \{X(t), M(t), I(t)\} \) so as to maximize the utility functional (9), subject to the constraints implied by the capital accumulation process (3), the current account (6), the given initial values of the state variables \( K(0), D(0) \), and the condition that at each time \( t \):

\[
z(t) = Z(t) \quad (10)
\]

\[
x(t) = X(t) + M(t). \quad (11)
\]

That is, consumption of nontraded goods must equal total output of nontradeables and total consumption of traded goods must equal the sum of the net output of domestically produced tradeables and net imports from the rest of the world.

Substituting (10) and (11) into (9), and applying the Pontryagin Maximum Principle to determine a solution to this problem, we define the current value Hamiltonian:

\[
H[X, M, I; K, D, \lambda, \theta] = \alpha \log (X+M) + (1-\alpha) \log T(X, K, I) +
\]

\[
+ \lambda[I]
\]

where \( \lambda \) and \( \theta \) are costate variables denoting respectively the shadow price of capital and the shadow price of external debt. That is, they measure the marginal contribution of the corresponding state variables to the utility functional at time \( t \). Assuming an interior solution, the optimum path of the economy must satisfy the following necessary conditions (see Arrow 1968).

\[
\frac{3H}{3X} = 0 \implies \frac{2}{X} \left( -\frac{\alpha}{1-\alpha} \right) = -T_X(X, K, I) \quad (13)
\]
\[ \frac{H}{M} = 0 \implies x = -\frac{\alpha}{\theta} \quad (14) \]

\[ \frac{\partial H}{\partial I} = 0 \implies (1-\alpha) \frac{T_X(X,K,I)}{T(X,K,I)} = \lambda \quad (15) \]

\[ \frac{\partial H}{\partial \lambda} = I - \delta K = \dot{K} \quad (16) \]

\[ \lambda = [r \begin{bmatrix} T_K(X,K,I) \\ \frac{T_I(X,K,I)}{T_I(X,K,I)} \end{bmatrix}] \lambda \quad (17) \]

\[ \frac{\partial H}{\partial \theta} = M + rD = \dot{\theta} \quad (18) \]

\[ \dot{\theta} = (r - r)\theta = 0 \implies \theta = \theta(0), \text{ constant.} \quad (19) \]

Equation (13) is a restatement of the static equilibrium condition under which the marginal rate of substitution in consumption must at each \( t \) equal the marginal rate of transformation.

Equation (14) defines a consumption profile of traded goods that is constant through time\(^5\), since by (19), \( \theta \), the shadow price of debt (\( \theta \) is negative), in this model identical to the shadow price of traded goods, will be a constant along an optimal path.

Equation (15) after using (13) and (14) can be rewritten as:

\[ \frac{T_I}{T_X} = -\frac{\lambda}{\theta} \quad (15') \]

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\( ^5 \) This result obviously depends on the specific functional form of utility. See Nunes (1984) for a different result under a non-logarithmic formulation of the Cobb-Douglas utility function.
That is, the marginal cost of investment in terms of traded goods equals the shadow price of a unit of capital in terms of traded goods.

As shown in the appendix:

\[ T_I = - \frac{G'}{F'} \psi'(I) \text{ and } T_X = - \frac{G'}{F'}. \]

Substituting these results into (15') above we obtain:

\[ I = \psi^{-1}(- \frac{\lambda}{\delta}) \]

(20)

which defines an investment function that depends only on the shadow price of capital in terms of traded goods. This expression will provide an important result when we study the stability properties of the system.

Equations (16) and (18) restate the capital accumulation process and the current account, which are the differential equations that govern the evolution of the state variables \( K \) and \( D \).

Equation (17) describes the motion law of the shadow price of capital. It can be rewritten as:

\[ (r + \delta) = - \frac{T_K}{T_I} + \frac{\lambda}{\lambda} \]

which makes clear the arbitrage condition it implies. Namely, it will always be true that the rate of return to investment \((-T_K/T_I)\) plus the relative capital gain \((\lambda/\lambda)\) equals the opportunity cost of investment \((r + \delta)\). So if, for instance, we observe that the rate of return to investment is lower than \((r + \delta)\), it must be the case that the value of \( K \) is rising and thus \( \dot{\lambda} > 0 \), which leads one to expect that the capital stock should be falling and thus \( \dot{K} < 0 \).
This relation between the motion of $K$ and $\lambda$ will be proved when we study the dynamic properties of the systems below.

Finally, we need to impose one transversality condition necessary for optimality:

$$\lim_{t \to \infty} e^{-rt} \lambda(t)K(t) = 0. \tag{21}$$

3. THE STEADY STATE

In this section we want to investigate the steady-state relationship between the size of the economy's capital stock and its level of external indebtedness. We proceed by first evaluating the system of equations (13) - (19), that characterize the necessary conditions for optimality, at a stationary point ($\dot{K} = \dot{D} = \dot{\lambda} = \dot{\theta} = 0$). Making use of conditions (10)

$$T(X,K,\delta K) \left( \frac{\alpha}{1-\alpha} \right) + T_X(X,K,\delta K) = 0 \tag{13'}$$

\[\text{We note that all other transversality conditions are already satisfied. Namely:}\]

$$\lim_{t \to \infty} \lambda(t) \geq 0 \text{ by (15)}$$

$$\lim_{t \to \infty} \theta(t) \leq 0 \text{ by (14)}$$

$$\lim_{t \to \infty} e^{-rt} \theta(t)D(t) \text{ by (7)}.$$
Thus, the steady-state equilibrium is described by a system of two equations in three endogenous variables X, K and D. It is therefore indeterminate. This indeterminacy, however, is not a result of an incomplete specification of the model. Recall that the necessary conditions for optimality showed that one of the transition equations, namely (19), implied that the costate variable was a constant throughout an optimal path (θ = 0). Therefore one of the characteristic roots of the system of four differential equations that govern the dynamics of the model is necessarily equal to zero. This fact implies the indeterminacy of the steady-state equilibrium.

Given our interest in studying the relation between K(∞) and D(∞), we proceed by parameterizing the system (13') - (17') in D(∞), thus eliminating one variable. We can therefore state that for all D(∞), there exists X(∞), K(∞) and hence Z(∞), such that the system (13') - (17') is solved. Therefore, there is a stationary solution to the Hamiltonian function and by its concavity with respect to state variables, an optimal steady state (see Arrow and Kurz 1970, II.5 - Sufficiency Theorem).

Equation (13') yields a schedule on the (K,X) plane along which the marginal rate of substitution equals the marginal rate of transformation for a given level of external debt in the steady-state. It fixes the transformation curve for a given D(∞) and we therefore call this schedule \( T = 0 \). Equation (17'), on the other hand, yields a schedule on the (K,X) plane along which the shadow price of capital is stationary, and we call it \( \lambda = 0 \).

Totally differentiating (13'), we obtain:

\[
\frac{dx}{dK} \bigg|_{T = 0} > 0. \quad (22)
\]

See section 2 of the Appendix for a derivation of (22) and (23).
Thus, \((13')\) yields an upward-sloping schedule in the \((K, x)\) plane, and this should be obvious by inspection of equation \((13')\). There, holding \(D\) constant, as \(K\) increases, given that \(T_K > 0\) and \(T_{XK} > 0\), the only way to restore the equality between marginal rate of substitution and marginal rate of transformation is for \(x\) to increase.

Totally differentiating \((17')\), we obtain:

\[
\frac{dx}{dK} \bigg|_{\lambda = 0} < 0.
\]  

(23)

Thus, the \(\dot{\lambda} = 0\) schedule is downward-sloping. This should be so, since if we depart from a point on the \((K, x)\) plane at which \(\dot{\lambda} = 0\) by increasing \(x\) while holding \(K\) constant, that is, by reallocating some of the capital stock away from production of nontradeables \((Z)\) into production of tradeables \((I)\), the marginal product of capital in the traded goods sector will fall, and to restore the long-run equilibrium condition implied by \((17')\), the capital stock of the economy must decline. Figure 1 depicts these results.

Figure 1

Long-run equilibrium for a given level of external debt
We note that the \( \dot{\lambda} = 0 \) schedule \((17')\) does not depend on the parameter \( D \), while the \( \ddot{T} = 0 \) schedule \((13')\) obviously does. To determine how the latter shifts when \( D(\infty) \) changes, we use \((13')\) to evaluate the partial derivative of \( X \) with respect to \( D \), \( dK \) being set equal to zero:

\[
\frac{\partial X}{\partial D} \bigg|_{T=0} = -\frac{(\frac{\alpha}{1-\alpha}) \frac{rT}{(X - rD)^2}}{\left(\frac{\alpha}{1-\alpha}\right) \left[ \frac{(X - rD)T X - T}{(X - rD)^2} \right] + TXX} > 0. \tag{24}
\]

Thus, the \( \ddot{T} = 0 \) schedule in Figure 1 shifts upward when the steady-state level of external debt increases. That is, the higher a country's net debtor position (the higher its trade balance surplus), the lower will be the economy's capital stock and the larger its domestic production of traded goods in the steady state. Conversely, the higher a country's net creditor position (the higher its trade balance deficit), the larger will be the economy's capital stock, and the lower its domestic production of traded goods in the steady state.

Therefore, we have shown that in the steady state the capital stock of a particular country will be different for different levels of foreign indebtedness. We can represent this relation graphically under the assumption that \( \lambda \), the shadow price of capital, is being projected on the \((K,D)\) plane. Thus, for different long-run levels of external debt we will trace the steady-state equilibrium locus along the \( \dot{\lambda} = 0 \) schedule in the \((K,D)\) plane and obtain a relation as shown in Figure 2.
The steady-state relation between the capital stock and the level of external indebtedness.

\[ K(\infty) = \frac{I}{\delta} \]

Net Debtor

Net Creditor

\[ -D(\infty) = \frac{M}{r} \quad D(\infty) = -\frac{M}{r} \]

The nonuniqueness of the steady-state equilibrium and the particular negative relation it entails between the size of the capital stock and the level of external indebtedness can be rationalized for a net debtor country by noting that the long-run trade balance surplus it must generate in order to transfer the real resources it owes to the rest of the world imposes a constraint on how the capital stock will be allocated between the two productive sectors. To make this point, suppose we observe an economy that reaches its steady-state equilibrium with a certain capital stock, allocated across the two productive sectors so as to satisfy desired total consumption of tradeables and nontradeables, and also faces a zero external debt position (external balance). We now let this economy acquire some debt and ask: Can the previous level of the capital stock be sustained at the now relatively higher level of external indebtedness? The answer is clearly no, since this economy has now to service its external debt and in the present context this can only be done through transfers of traded goods to
the rest of the world (nontradeables are by definition nontransferrable). Therefore the higher external debt determines that a higher fraction of the economy's capital stock be allocated to production of tradeables. This leads to a fall in the marginal product of capital in the traded goods sector, and, as inspection of equation (17') dictates, the overall capital stock of the economy must fall in order to restore long-run equilibrium.

Therefore, the trade balance surplus a net debtor country has to generate imposes a shift in the productive structure of the economy, namely, the higher its net debtor position, the more "tradeables intensive" in production it should become. This movement requires a relative price change (an increase in the relative price of tradeables), which also induces a change in the pattern of domestic consumption. Specifically, the ratio of nontraded to traded goods in consumption should be higher, as can readily be inferred from the equality between the marginal rate of substitution and the marginal rate of transformation implied by equation (13').

Summarizing the above discussion we can establish a few propositions regarding steady-state comparisons between different countries. Namely, we should expect countries with relatively higher external debt to:

1) enjoy a relatively lower level of overall consumption (negative income effect from higher debt),
2) produce relatively less nontraded goods (substitution or resource allocation effect from higher debt), and
3) have a lower relative price on nontraded goods, or alternatively a higher equilibrium value of the real exchange rate.

4. DYNAMICS AND STABILITY

Having characterized the steady-state equilibrium of the system, we turn to the question of whether such equilibrium is attainable, that is, we are interested in studying the stability and
dynamic properties of the system. In this context, the questions to be answered are the following: Given initial values for the state variable: \([K(0), D(0)]\) in equilibrium at some point along the \(\lambda = 0\) locus that characterizes the steady state? And if so, what can be said about the dynamic interaction between the capital stock and the level of indebtedness as the system evolves towards that steady-state equilibrium? And further, how do changes in the initial conditions \([K(0), D(0)]\) affect the steady-state equilibrium values of those variables?

To start out we note that, given the characteristics of the control problem we are attempting to solve, with two state variables and two costate variables, our task in conducting stability analysis will be that of studying the signs of the characteristic roots of a system of four differential equations. This would in principle constitute a very difficult task; also, in spite of the fact that the Pontryagin formulation is fully general in scope, the phase diagram technique is hopelessly incapable of generalization when there are more than three relevant variables. Fortunately, from the F.O.C. for this problem we have already determined that one of the characteristic roots must necessarily be equal to zero. This fact will enable us to focus on a reduced system characterized by three differential equations, in order to infer the stability properties of the whole system. We note, however, that we are not expecting to claim stability in its classical sense of convergence towards a unique steady-state equilibrium independent of initial conditions on the primal-state variables, for we have shown that the steady state is characterized by a locus of equilibrium points. Thus, stability in the present context involves determining whether there is a well defined optimal path converging to a steady-state equilibrium, recognizing that such a path depends on the initial values of the state variables \([K, D(0)]\).

From the necessary conditions for optimality (13) - (19), we write:\(^8\)

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\(^8\) An asterisk (*) denotes optimal value.
as the choice of instruments (controls) that maximize the Hamiltonian \( H(X, M, I; K, D, \lambda, \theta) \).

Thus the optimal Hamiltonian is given by:

\[
H^* = H[
\]

The paths of \( \lambda(t) \), \( K(t) \), \( \theta(t) \) and \( \phi(t) \) satisfying the necessary conditions for optimality as determined by equations (16) to (19) are described by the differential equations\(^9\):

\[
\dot{\lambda} = r\lambda - H^*_K(K, D, \lambda, \theta) \tag{25}
\]

\[
\dot{K} = H^*_\lambda(K, D, \lambda, \theta) \tag{26}
\]

\[
\dot{D} = H^*_\theta(K, D, \lambda, \theta) \tag{27}
\]

\[
\dot{\theta} = r\theta - H^*_D(K, D, \lambda, \theta) = 0 \text{ since } H^*_D = r\theta. \tag{28}
\]

---

\(^9\) Let \( H_i \) and \( H_{ij} \) denote the first and second partials of \( H \): thus

\[
H_i = \frac{\partial H}{\partial i} \quad \text{and} \quad H_{ij} = \frac{\partial^2 H}{\partial i \partial j}.
\]
The associated linear system to the nonlinear system (25) -- (28), in the neighborhood of a resting point, in matrix form is:

\[
\begin{bmatrix}
    r - H_{\lambda}^* & -H_{KK}^* & -H_{KD}^* \\
    H_{\lambda\lambda}^* & H_{\lambda K}^* & H_{\lambda D}^* \\
    H_{\theta\lambda}^* & H_{\theta K}^* & H_{\theta D}^*
\end{bmatrix}
\begin{bmatrix}
    \lambda - \lambda \\
    K - \bar{K} \\
    D - \bar{D}
\end{bmatrix} =
\begin{bmatrix}
    \lambda \\
    \bar{K} \\
    \bar{D}
\end{bmatrix}
\]  

(29)

We will focus on the above reduced system to study the local stability properties of the whole system. But our task may still be simplified since we have more specific information about some of the elements of the above matrix, namely:

- \( H_{KK}^* < 0 \), by concavity of the Hamiltonian function with respect to state variables (see Arrow 1968),

- \( H_{\lambda\lambda}^* > 0 \), by convexity of the Hamiltonian function with respect to costate variables (ibid.),

- \( H_{KD}^* = 0 \), since \( \frac{\partial}{\partial K} (H_{D}^*) = \frac{\partial}{\partial K} (\theta r) = 0 \)

- \( H_{\lambda D}^* = 0 \), since \( \frac{\partial}{\partial \lambda} (H_{D}^*) = \frac{\partial}{\partial \lambda} (\theta r) = 0 \)

- \( H_{\theta D}^* = r \), since \( \frac{\partial}{\partial \theta} (H_{D}^*) = \frac{\partial}{\partial \theta} (\theta r) = r \).

There the matrix of the reduced system simplifies to:

\[
\begin{bmatrix}
    r - H_{\lambda\lambda}^* & -H_{KK}^* & 0 \\
    H_{\lambda\lambda}^* & H_{\lambda K}^* & 0 \\
    H_{\theta\lambda}^* & H_{\theta K}^* & r
\end{bmatrix}
\]  

(29')

Call this Matrix A.
The trace of $A = 2r > 0$, therefore the sum of the characteristic roots is a positive real number, which means that at least one root must be positive. This is readily verifiable from the above matrix. Let $\mu$ denote a characteristic root of this system, therefore the characteristic equation for the system is:

$$(r - \mu)(r - H_{\lambda K}^\ast - \mu)(H_{\lambda K}^\ast - \mu) + H_{\lambda K}^\ast H_{\lambda K}^\ast = 0. \quad (30)$$

Thus one immediately obtains that $\mu = r$ is one characteristic root of the system (call it $\mu'$), and one that is a positive real number. The fact that $r$ is a characteristic root of the system was expected since $r$ is the rate at which debt service grows through time.

Rearranging the term in brackets of equation (30) we write the quadratic equation from which the two remaining characteristic roots will be determined:

$$\mu^2 - \mu r + H_{\lambda K}^\ast (r - H_{\lambda K}^\ast) + H_{\lambda K}^\ast H_{\lambda K}^\ast = 0 \quad (31)$$

therefore we obtain:

$$\mu', \mu'' = \frac{r \pm \sqrt{r^2 - r(H_{\lambda K}^\ast (r - H_{\lambda K}^\ast) + H_{\lambda K}^\ast H_{\lambda K}^\ast)}}{2} \quad (32)$$

Hence, if we can show that $H_{\lambda K}^\ast < 0$, we will have shown that one of the remaining roots is a positive and the other is a negative real number. Thus the system will have two positive real roots, one zero root and one negative real root, which is what we need to claim that the system is stable.

So now we examine the sign of $H_{\lambda K}^\ast$. We know that:

$$H_{\lambda K}^\ast = \frac{\partial}{\partial K} (H_{\lambda}^\ast) = \frac{\partial}{\partial K} [I(K,D,\lambda,\theta) - \delta K]$$
or

\[ \lambda_{AK} = \frac{\partial I^*}{\partial K} - \delta. \]

Therefore, if \( \frac{\partial I^*}{\partial K} < \delta \) then \( \lambda_{AK} < 0. \)

But recalling equation (20):

\[ I = \psi K^{-1} \left( - \frac{\lambda}{\theta} \right) \]

we obtain \( \frac{\partial I^*}{\partial K} = 0 \) and therefore \( \lambda_{AK} = 0 - \delta < 0. \) Thus having determined that \( \lambda_{AK} < 0, \) we have assured stability. But it still remains to establish how the capital stock and the level of external debt vary along an optimal path converging to a steady-state equilibrium. Technically this amounts to examining the eigenvector \( (V) \) of the matrix \( A \) belonging to the eigenvalue \( \mu, \) when the latter equals the negative characteristic root of \( A, \) which satisfies the relation,

\[ AV = \mu V, \quad \mu < 0. \]

(33)

To do so, we must first determine the sign of the two remaining elements of Matrix \( A, \) namely, \( \lambda_{\theta} \) and \( \lambda_{K}, \) where:

\[ \lambda_{\theta} = \frac{\partial}{\partial \theta} \left( \lambda_{\theta}^{*} \right) = \frac{\partial}{\partial \theta} \left[ I(K,D,\lambda,\theta) - \delta K \right] = \frac{\partial I^{*}}{\partial \theta} > 0 \]

which is intuitive since a rise in the shadow price of external debt (a fall in its absolute value) reduces the opportunity cost of foreign borrowing, thus reducing the marginal cost of investment, and:
that is, the higher the capital stock at a particular point in time (ceteris paribus), the lower is the economy’s optimal trade-account deficit, which is intuitive since for larger capital stocks the desired rate of investments should fall, and thus inducing a decline in imports demand.

We are now ready to examine the relative $AV = \mu V$. Let us denote the nonzero elements of $A$ by $\gamma_{ij}$. Thus we write:

$$
\begin{vmatrix}
\gamma_{11} & \gamma_{12} & 0 \\
\gamma_{21} & \gamma_{22} & 0 \\
\gamma_{31} & \gamma_{32} & \gamma_{33}
\end{vmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 
\begin{bmatrix}
\mu v_1 \\
\mu v_2 \\
\mu v_3
\end{bmatrix}
$$

(34)

where $v_i, i = 1, 2, 3$ are the elements of $V$ and as we have established in the text:

$$
\gamma_{11} > 0; \gamma_{12} > 0; \gamma_{21} > 0; \gamma_{22} < 0; \gamma_{31} > 0; \gamma_{32} < 0; \gamma_{33} > 0.
$$

What we are looking for are the signs of $v_2$ and $v_3$. If they are equal we will have established that the capital stock and external debt are positively related along an optimal path, and negatively related if $v_2$ and $v_3$ are of opposite signs.

We first establish the relation between $v_1$ and $v_2$, that is, how the movement in the shadow price of capital ($\lambda$) and the capital stock ($K$) are related along an optimal path. Interpreting equation (17), we suggested that they should be negatively related and this is confirmed by the fact that $v_1$ and $v_2$ must be of opposite signs, since from (34) we get:

Derivation of this result is available from the Author.
The term in parentheses above is negative since $\mu < 0$ and $\gamma_{11} > 0$, therefore if $v_1 > 0$, the right hand side must be negative, thus $v_2 < 0$ since $\gamma_{12} > 0$ and conversely for $v_1 < 0$.

To look at the relation between $v_2$ and $v_3$ we obtain from (34):

\[ v_3(\mu - \gamma_{33}) = \gamma_{31}v_1 + \gamma_{32}v_2. \]  

(36)

Since $\gamma_{31} > 0$, $\gamma_{32} < 0$, $v_1$ and $v_2$ are of opposite signs, and $(\mu - \gamma_{33}) < 0$, we can establish that if:

\[ v_3 < 0 \text{ then } (\gamma_{31}v_1 + \gamma_{32}v_2) > 0 \]

therefore:

\[ v_1 > 0 \text{ and } v_2 < 0. \]

Conversely, if:

\[ v_3 > 0 \text{ then } (\gamma_{31}v_1 + \gamma_{32}v_2) < 0 \]

therefore:

\[ v_1 < 0 \text{ and } v_2 > 0. \]

Thus we have established that along an optimal path, the capital stock and the level of external indebtedness are positively related. Figure 2, which was drawn under the assumption that $\lambda$ was being projected on the $(K,D)$ plane, is reproduced in Figure 3 with the added motion laws just derived.

This is an intuitive result, and one which is in accordance with the view that current-account imbalances may reflect optimal behavior on the part of economic agents, as predicted by recent
intertemporal models that investigate optimal consumption paths and external indebtedness. That result is expanded here to include optimal capital accumulation. Thus, a country that finds itself with a capital stock below its steady-state equilibrium, will find it optimal to incur external debt as it builds the economy's capital stock. By doing so, it does not have to rely entirely on reduction of domestic consumption in order to release resources for its capital accumulation effort. Therefore, by incurring external debt it is also allowing for domestic consumption to be smoothed over time. Note, however, that by incurring external debt to partially finance its growth process, the country places itself on a "constrained" growth path that affects its optimal long run level of the capital stock due to allocative effects exerted by the need to service that external debt in the long run.

Figure 3
The relation between the capital stock and external debt along an optimal path.
Another implication of this result concerns the evolution of a country's current account and balance of indebtedness over time. Kindleberger (1968, p. 485) characterizes the balance of payments on the current account through the various stages of the development of the balance of indebtedness. In this argument he states that: "In the early stages of growth when investment opportunities exceed savings, a country may make up the gap with international borrowing. At a later stage as its income and savings rise beyond its investment requirements, it pays back debt and accumulates foreign investments of its own" (ibid., p. 583). Therefore, in the early stages a country should be a net debtor, and the current account on the balance of payments negative. As the country matures the current account eventually changes to positive; the debtor balance of indebtedness diminishes as the country eventually turns into a net creditor.

By inspection of Figure 3 above, we observe that this characterization does not hold in general in the context of the present model. If a country starts at time t=0 with a K(0) and D(0) combination such as implied by point A in Figure 3, Kindleberger's characterization of the evolution of the balance of indebtedness would follow through as the country starts out as a net debtor, and achieves its steady-state equilibrium position at point B, as a net creditor. But, for a country that inherits a K(0) and D(0) combination like that of point C in Figure 3, its optimal behavior will be to further increase its net debtor position, and reach a steady-state equilibrium at point E.

The existence of stages in the balance of payments accounts, through which every economy goes as part of its development process, has already been examined in the context of an intertemporal optimization model by Bazdarich (1978). In his framework of analysis he concludes that a developing country will always be a net debtor and a net borrower, with its net debt position monotonically approaching some long-term steady-state level. However, in the present context, by allowing each country to be characterized by both its predetermined capital stock K(0) and external indebtedness D(0), we are able to expand on Bazdarich's results, and take explicit account of situations where optimal behavior may imply
capital accumulation effort together with a net creditor position on the country's balance of indebtedness.

Thus, the present model highlights the importance of initial conditions characterizing the state of the economy \([K(O), D(O)]\) in determining a country's optimal evolution of the stock of capital and balance of indebtedness, and the configuration of the steady-state equilibrium between these variables. We therefore close this section with two comparative dynamics exercises to further emphasize this point.

It can be determined that:

\[
\frac{\partial K(\infty)}{\partial D(0)} < 0, \tag{37}
\]

that is, for a given initial capital stock \(K(0)\), the higher the initial external debt \(D(0)\), the lower will be the country's optimal capital stock in the steady state \(K(\infty)\).

In understanding the reasoning behind this result, it is instructive to compare two countries, identical in all respects except for their initial external debt \(D(0)\). Let A and B in Figure 4 denote \([K(O), D(0)]\) for country 1 and 2 respectively. Thus, \(K_1(0) = K_2(0)\), and \(D_1(0) < D_2(0)\). As Figure 4 shows, these countries will be moving along different paths towards their steady-state equilibrium. Specifically, for every \(K(t)\), where \(K_1(t) = K_2(t)\), country 2 will face a higher stock of external debt \(D(t)\) than country 1. This has two implications: country 2 will face a relatively higher shadow price of external debt throughout its optimal path\(^{11}\), and in the steady-state equilibrium it will produce relatively more

\(^{11}\) Remember that the optimal Hamiltonian is concave in \(D\) and that \(\partial H*/\partial D = \theta\). Thus, holding \(K\) constant, as \(D\) increases \(\theta\) necessarily falls. But by (14) \(\theta\) is negatively defined, therefore as \(\theta\) falls its absolute value rises.
traded goods than country 1. These two facts combined determine that incentives for additional investment in country 2 should be relatively lower than in country 1, thus explaining the negative relation expressed in (37) above and also highlighting the fact already mentioned that, even though the incurrence of external debt for a growing economy is optimal, its presence constrains the growth path of the economy through its long-run allocative effects on the productive structure of the country.

Figure 4
A comparative dynamics exercise.

Finally, we also observe:

\[ 0 < \frac{\partial D(\infty)}{\partial D(0)} < 1. \]  

That is, for a given initial capital stock \( K(0) \), an increase in the initial level of external debt \( D(0) \) induces a less-than-proportional...
tional rise in the steady-state level of external indebtedness $D(\infty)$.

By inspection of Figure 4 this result should be obvious, and its rationale is the following: Let A and B in Figure 4 again denote initial conditions for the two similar countries 1 and 2. Thus, as before, in comparing points along these countries' optimal paths for which $K_1^1(t) = K_2^2(t)$, we observe $D_2^2(t) > D_1^1(t)$ for all $t$, and hence, the shadow price of external debt ($\theta$) faced by country 2 is greater than that faced by country 1 throughout, meaning that each unit of additional debt incurred by country 2 costs relatively more than that same unit incurred by country 1, and therefore country 2 will borrow relatively less from abroad. Thus, despite identical initial capital stocks, the higher external debt endowment $D(0)$ of country 2 puts it on a relatively more "severely constrained" growth path, which leads to a lower steady-state capital stock than that of country 1.

5. CONCLUSION

Recent intertemporal optimizing models in an open economy have emphasized the relation between optimal consumption choices and external indebtedness. Though recognizing the importance of capital accumulation motives in explaining persistent current-account imbalances, these models do not explicitly incorporate this element within their theoretical framework of analysis. Therefore, the principal objective of this paper has been to develop an intertemporal model from which an analytical solution characterizing the optimal paths of capital accumulation and external indebtedness can be explicitly derived. This objective has placed our analysis in the realm of optimal growth models. Section 2 presented a simple model that enabled us to characterize both the steady-state equilibrium of the system and the dynamic interaction between capital accumulation and external indebtedness. Regarding the steady-
state equilibrium, contrary to previous optimal growth models, we showed it to be characterized by a locus of points depicting a negative relation between the steady-state capital stock and the level of external indebtedness. We also proved that for given initial conditions \( K \) capital accumulation and external indebtedness results as the economy moves along its optimal path converging to a steady-state equilibrium. This result is in accordance with the view that external imbalances may be the outcome of optimal household behavior, and it also stresses the importance of the initial conditions characterizing the state of the economy in determining the optimal evolution of the balance of external indebtedness and the configuration (net creditor vs. net debtor) of the economy's steady-state equilibrium position.

Of course, in any theoretical analysis, conclusions are contingent on the assumptions which have been used in deriving them. The reader is referred to Nunes (1984) for a study of the robustness of the theoretical model, through several generalizations and modifications to both its production and preferences structure. In concluding, I would like to point out two related areas of research that would benefit from the general structure of the model developed here. First, the recent modeling of default risk in international lending and borrowing relations has emphasized, among other things, a particular country's propensity to invest as being one of its major determinants. However, under the present structure where the economy is disaggregated into its tradeables and non-tradeables sectors, this factor must be broadened to encompass the sectoral composition of investment as a major determinant of default risk, given the "nontransferable" nature of one of the goods produced. Second, once we are able to impose adjustment costs in the reallocation of resources across the two sectors, we should be better equipped to analyze the behavior of the system when subject to shocks that affect the economy's external indebtedness profile and hence trigger costly resource movements as the economy adjusts its productive structure to meet a heavier external debt service.
Here we derive the transformation function \( T \) and its properties. From equation (1.1) we obtain:

\[
K_X = F^{-1}(x^g). \tag{A.1}
\]

Using (1.4) to substitute for \( x^g \) we get:

\[
K_X = F^{-1}(X + \psi(I)) \tag{A.2}
\]

rewriting (1.2) as:

\[
Z = T = G(K - K_X) \tag{A.3}
\]

and substituting (A.2) for \( K_X \) we obtain:

\[
Z = T(X,K,I) = G(\cdot) \tag{A.4}
\]

from where we obtain properties (a) - (g) as:
\[ T_X = -\frac{G'}{F'} < 0 \]

\[ T_K = G' > 0 \]

\[ T_I = -\frac{G'}{F'} \psi' = T_X \psi' < 0 \]

\[ T_{XX} = \frac{G''}{F'^2} + \frac{G'F'}{F'^3} < 0 \]

\[ T_{XK} = -\frac{G''}{F'} > 0 \]

\[ T_{XI} = \frac{G''}{F'^2} \psi' + \frac{G'F''}{F'^3} \psi' < 0 \]

\[ T_{KK} = G'' < 0 \]

\[ T_{II} = \frac{G''}{F'^2} \psi' + \frac{G'F''}{F'^3} \psi' - \frac{G'}{F'} \psi'' = T_{XX}(\psi')^2 + T_X \psi'' < 0 \]

\[ T_{IK} = -\frac{G''}{F'} \psi' = T_{XK} \psi' > 0. \]

II

Equations (13') and (17'), after substituting for \( T, T_K \) and \( T_X \), can be written as:

\[
\frac{G'[K-F^{-1}(X+\psi(\delta K))]}{X-xD} \left(\frac{\alpha}{1-\alpha}\right) - \frac{G'[K-F^{-1}(X+\psi(\delta K))]}{F'[F^{-1}(X+\psi(\delta K))]^2} = 0 \quad (A.5)
\]

\[
(x + \delta) - \frac{F'[F^{-1}(X+\psi(I))]}{\psi'(\delta K)} = 0. \quad (A.6)
\]
Totally differentiating (A.5), we get:

\[
\frac{dx}{dk} \bigg|_{\lambda=0} = \frac{\alpha}{(X - rD)(1-\alpha)} \frac{[TK + \delta T_I] + TXK + \delta TXI}{(X - rD)^2} > 0
\]

In obtaining this result we note that in the numerator of the expression above, \(TK + \delta T_I > 0\) and \(TXK + \delta TXI > 0\) since in the steady state we can write:

\[TK = -T_I(r + \delta).\]  \hspace{1cm} (A.7)

Adding \(T_I\delta\) on both sides, we obtain:

\[TK + \delta T_I = -T_I r > 0\]

and also differentiating (A.7) w.r.t. \(X\) we get:

\[TXX + T_{IX} \delta = -T_{IX} r > 0.\]

Totally differentiating (A.6), we get:

\[
\frac{dx}{dk} \bigg|_{\lambda=0} = -\frac{\psi}{F''(F'\psi)^{-1}} < 0. \hspace{1cm} (A.8)
\]
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