A Note on Inverse Moments of Binomial Variates

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Abstract

We show that the inverse moments of a binomial random variable are such that

$$E[(1 + X)^{-\alpha}] = O((np)^{-\alpha}) \quad \text{as } n \rightarrow \infty$$

for all $\alpha \in \mathbb{R}$.

Resumo

Mostramos que os momentos inversos de uma variável aleatória binomial são tais que

$$E[(1 + X)^{-\alpha}] = O((np)^{-\alpha}) \quad \text{quando } n \rightarrow \infty$$

para todo $\alpha \in \mathbb{R}$.

Key Words: inverse moments, binomial random variable, moment generating function.

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1. Introduction.

Oftentimes in econometrics, statistics and probability one needs to evaluate inverse moments of random variables. For instance, Bock, Judge and Yancey (1984) and Xie (1988) obtain expressions for inverse moments of non-central chi-squared variates. In this note we address the issue of binomial inverse moments evaluation.

The binomial distribution was derived by James Bernoulli in 1713 for the case where the probability of success p equals $r/(r+s)$, where r and s are positive integers, and had been previously considered by Pascal for the case where $p = 1/2$. McKendrick (1914) obtained the binomial distribution as the solution to the birth-and-emigration process, and Haight (1957) has shown that the distribution arises in the context of the $M/M/1$ queue with balking, provided that the arrival rate of customers in the queue is $(N-n)/[N(n+1)]$, where there are n customers and N denotes the maximum queue size. The binomial distribution can also be viewed as the stationary distribution for the Ehrenfest model. It has a number of applications on many different fields. For instance, in genetics it arises in the study of the number of nucleotides that are in the same state in two DNA sequences (Kaplan and Risko, 1982). The binomial distribution has also applications in ecology (e.g., Seber, 1982), mortality data (e.g., Seal, 1949), etc. It should also be remarked that the number of defective items in random samples of size n from a stable production process follows a binomial distribution. For that, it has a number of applications in the area of quality control. The binomial distribution is also used for distribution-free inference on distribution values, distribution quantiles, and tolerance limits for distributions. For further details on this distribution and its applications, see Bury (1999, Chapter 6) and Johnson, Kotz and Kemp (1993, Chapter 3).

Inverse moments for binomial random variables appear in several

contexts. The evaluation of moments of the form

$$\mathbb{E} \left[\frac{1}{(1+X)^\alpha} \right] \quad (1)$$

is necessary for bounds used in asymptotic results. For example, in Garcia and Palacios (2000) bounds for the mixing time of a random walk are estimated in terms of the above moments for $p = 1/2$. As early as Stephan (1945) and as recent as Marciniak and Wesolowski (1999), there has been interest in finding closed-form expressions, approximations or bounds for inverse moments, especially for the binomial distribution with view toward applications in life testing, Bayesian post-stratification and Stein estimators among others. In particular, when α is a non-negative integer it is possible to use a strategy described in Chao and Strawderman (1972) and Cressie et al. (1981). It consists of successively integrating the moment generating function $M_X(t)$ of X , namely:

$$\mathbb{E} \left[\frac{1}{(1+X)^k} \right] = \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty M_X(-t_k) e^{-t_k} dt_k \cdots dt_2 dt_1. \quad (2)$$

We either make the changes of variables $e^{-t_i} = s_i, 1 \leq i \leq k$, or exchange the order of integration in (2) to obtain the desired expressions. However, even in the binomial case, where $M_X(t) = (1 - p + pe^t)^n$, there are no closed-form expressions except when $\alpha = 1$, as was observed in Chao and Strawderman (1972), in which case we get

$$\mathbb{E} \left[\frac{1}{(1+X)} \right] = \frac{1}{p(n+1)} [1 - (1-p)^{n+1}].$$

Nevertheless, there are situations in which the exact expression is not important and where asymptotic results can be useful. Lower

bounds can be obtained easily by Jensen inequality. Bounds have been studied extensively in several cases. Wooff (1985) and Pittenger (1990) develop different distribution-free methods and get the same upper bound when α is a positive integer. However, for large n they do not achieve the right order of magnitude $O((np)^{-\alpha})$. One important information is to get the order of the expression in (1) and the leading constant. Garcia and Palacios (2001) present a proof for $\alpha > 0$ based on a normal approximation. We offer an alternative proof for $\alpha > 0$ based on the moment generating function, and argue that the result also holds for $\alpha < 0$.

2. Approximating $E\left[\frac{1}{(1+X)^\alpha}\right]$

Consider the function $f(x) = (1+x)^{-\alpha}$. Our interest lies in $E[f(X)]$ where $X \sim \text{Bin}(n, p)$, $n \in \mathbb{N}$ and $0 < p < 1$, with n assumed large. At the outset, let $\alpha > 0$. We write

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(1+x)t} t^{\alpha-1} dt,$$

where $\Gamma(\cdot)$ is the gamma function. Thus,

$$\begin{aligned} E[f(X)] &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} E[e^{-tX}] dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} (1 - p + pe^{-t})^n dt, \end{aligned}$$

where the last equality follows from the moment generating function for the binomial distribution. Now use the change of variable $(1 - p + pe^{-t}) = e^{-r}$. Thus $t = -\log[p^{-1}(e^{-r} + p - 1)]$ and $e^{-t} dt = p^{-1} e^{-r} dr$. Therefore,

$$\mathbb{E}[f(X)] = \int_0^{-\log(1-p)} g(r) e^{-rn} dr,$$

where

$$g(r) = p^{-1} e^{-r} \frac{1}{\Gamma(\alpha)} \left\{ -\log[p^{-1}(e^{-r} + p - 1)] \right\}^{\alpha-1}.$$

We have the following series expansion for $g(r)$:

$$g(r) = p^{-\alpha} r^{\alpha-1} \frac{1}{\Gamma(\alpha)} \left[1 + \frac{1}{2} \left(\frac{\alpha-1}{p} - (\alpha+1) \right) r + O(r^2) \right]. \quad (3)$$

Expansion (3) was obtained from the following expressions: $\log(1+z) = z - z^2/2 + O(z^3)$ for $|z| < 1$, $e^{-z} = 1 - z + O(z^2)$, and $(1+z)^\nu = 1 + \nu z + O(z^2)$ for $z > -1$.

Fix any $0 < \delta < \min\{1, -\log(1-p)\}$. For $0 < r < \delta$,

$$\int_0^\delta r^j e^{-rn} dr = \frac{\Gamma(j+1)}{n^{j+1}} + O(e^{-\delta n}). \quad (4)$$

Equation (4) can be established as follows:

$$\begin{aligned} \int_0^\delta r^j e^{-rn} dr &= \int_0^\infty r^j e^{-rn} dr - \int_\delta^\infty r^j e^{-rn} dr \\ &= \frac{\Gamma(j+1)}{n^{j+1}} - \frac{\Gamma(j+1, \delta n)}{n^{j+1}}, \end{aligned}$$

where $\Gamma(\cdot, \cdot)$ denotes the incomplete gamma function. Now, expand $\Gamma(j+1, \delta n)$ as (see Abramowitz and Stegun 1965, §6.5.32, p. 263)

$$\Gamma(j+1, \delta n) = (\delta n)^j e^{-\delta n} \left[1 + \frac{j}{\delta n} + \frac{j(j-1)}{\delta^2 n^2} + \dots \right],$$

and the result follows. Moreover,

$$\int_{\delta}^{-\log(1-p)} g(r) e^{-rn} dr = O(e^{-\delta n}). \quad (5)$$

Noticing that the remaining terms in (4) and (5) go to 0 exponentially fast, we have

$$\begin{aligned} \mathbb{E}[f(X)] &= \int_0^{\delta} g(r) e^{-rn} dr + \int_{\delta}^{-\log(1-p)} g(r) e^{-rn} dr \\ &= \int_0^{\delta} p^{-\alpha} r^{\alpha-1} \frac{1}{\Gamma(\alpha)} \left[1 + \frac{r}{2} \left(\frac{\alpha-1}{p} - (\alpha+1) \right) + \dots \right] e^{-rn} dr + O(e^{-\delta n}) \\ &= \frac{1}{\Gamma(\alpha)} p^{-\alpha} \left[\frac{\Gamma(\alpha)}{n^{\alpha}} + \left(\frac{\alpha-1}{2p} - \frac{\alpha+1}{2} \right) \frac{\Gamma(\alpha+1)}{n^{\alpha+1}} \right] + O(n^{-(\alpha+2)}) \\ &= (np)^{-\alpha} + O(n^{-(\alpha+1)}). \end{aligned}$$

Notice now that the upper bound given by Wooff (1985) and Pittenger (1990) is

$$\frac{(1-p)}{(1-p) + np} + \frac{np}{(1-p) + np} (1 + np + (1-p))^{-\alpha}$$

which is greater than the previous equation for large n .

Also, not only do we have the leading term $n^{-\alpha}$, but also higher order terms. For example, the term of order $n^{-(\alpha+1)}$ is

$$\begin{aligned} E[(1+X)^{-\alpha}] &= (np)^{-\alpha} + \left(\frac{\alpha-1}{2p} - \frac{\alpha+1}{2} \right) \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{1}{n^{\alpha+1} p^{\alpha}} \\ &\quad + O(n^{-(\alpha+2)}). \end{aligned}$$

Now let $\alpha < 0$ and $\gamma = -\alpha$. We have that $(1 + X)/n$ has the same distribution as $1/n + \bar{Y}$, where $\bar{Y} = n^{-1} \sum_{i=1}^n Y_i$ with $Y_1, \dots, Y_n$ being a sequence of independent Bernoulli trials with probability of success equal to p . But

$$\left(\frac{1}{n} + \bar{Y}\right)^\gamma \leq 2^\gamma$$

since $0 < n^{-1} + \bar{Y} \leq 2$. We also know by the Strong Law of Large Numbers that $\bar{Y} \xrightarrow{\text{a.s.}} p$. Therefore, by the Bounded Convergence Theorem,

$$\mathbb{E} \left(\frac{1}{n} + \bar{Y} \right)^\gamma \rightarrow p^\gamma.$$

More specifically,

$$\left| \mathbb{E} \left(\frac{1}{n} + \bar{Y} \right)^\gamma - p^\gamma \right| = o(1)$$

as $n \rightarrow \infty$.

Equivalently,

$$\mathbb{E} \left[\frac{1}{(1 + X)^\alpha} \right] = O((np)^{-\alpha}) + o(n^{-\alpha})$$

as $n \rightarrow \infty$.

The result thus holds for all $\alpha \in \mathbb{R}$.

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