Cycles, Steady States and Sunspot Equilibrium in Models with Memory

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Abstract

In this paper I provide necessary and sufficient conditions for the existence of Markovian sunspot equilibrium around a cycle or multiple steady states in models with lagged variables. These conditions are numerically tested in three endogenous growth models and it is shown the non-existence of these stochastic equilibria in a neighborhood of the cycle or multiple steady states.

Resumo

Neste artigo mostro condições necessárias e suficientes para a existência de equilíbrios sunspot Markovianos em vizinhanças de ciclos ou múltiplos estados estacionários em modelos com variáveis defasadas. Estas condições são numericamente testadas em três modelos de crescimento endógeno e prova-se a não existência destes equilíbrios estocásticos em vizinhanças dos ciclos ou múltiplos estados estacionários.

Key Words: Sunspot equilibrium, models with lagged variables, cycles and multiple steady states .

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1. Introduction.

In most of the intertemporal economic models where it is proved the existence of sunspot equilibrium (SE), the structural equations depend on the current value of the state variable and the expectations for the next period (Chiappori, Geoffard and Guesnerie (1992), Chiappori and Guesnerie (1991)). In these models the existence of regular cycles of multiple steady states is a sufficient condition for the existence of SE in a neighborhood of these equilibria (Azariadis and Guesnerie (1986), Peck (1988)). When the models has lagged variables in the structural equations (models with memory) the indeterminacy of the steady state is a sufficient condition for existence of SE (Woodford (1986), Spear (1991), Dávila (1997)).

In this work, I show necessary and sufficient conditions for the existence of Markovian SE in models with memory whose deterministic dynamics exhibits cycles or multiple steady states. For this result to hold, the regularity of the economy must be strengthened by a condition related with the cycle or multiple steady state. This condition is numerically tested in three endogenous growth models and it is concluded for each model the non-existence of Markovian SE in a neighborhood of the cycle.

Section 2 introduces the framework and the main theorem. In section 3 the existence of Markovian SE in neighborhoods of the cycles and/or multiple steady state is analyzed in three models. In section 4 some conclusions are drawn, while proofs of statements are given in the appendix.

2. The framework and main results.

The structural equations are defined from: $\tilde{Z} : X \times X \times \mathcal{P}(X) \rightarrow \mathbb{R}^n$, where $X \subset \mathbb{R}^n$ is the state variable set and $\mathcal{P}(X)$ is the set of probability measures defined on the Borel sets of X . If agents

observe $x_{t-1} \in X$ (the state variable value in period $t - 1$) and they have expectations for the next period $t + 1$ defined by $\mu_{t+1} \in \mathcal{P}(X)$ then $x_t \in X$ is the current optimal decision for this variable that produces market equilibrium, if and only if $\tilde{Z}(x_{t-1}, x_t, \mu_{t+1}) = 0$. The perfect foresight dynamics is defined by $Z(x_{t-1}, x_t, x_{t+1}) = \tilde{Z}(x_{t-1}, x_t, \delta_{x_{t+1}})$ (where δ_z is the Dirac measure concentrated at z).

Definitions

- 1) A *steady state* is a $\bar{x} \in X$ such that: $Z(\bar{x}, \bar{x}, \bar{x}) = 0$.
- 2) A *perfect foresight equilibrium* is a sequence $(x_t^*)_{t \geq 0}$ such that $Z(x_{t-1}^*, x_t^*, x_{t+1}^*) = 0$.
- 3) A *k-cycle* is a perfect foresight equilibrium such that: a) $x_{t+k}^* = x_t^*, \forall \geq 1$. b) k is the smallest integer with the property a).

The existence of cycles in models with this structure was proven by several authors (Reichlin (1986), Farmer (1986), de Vilder (1996), Michel and Venditti (1997)). I will provide conditions for the existence of stationary sunspot equilibrium (SSE) in a neighborhood of these cycles.

Definition 4) A *Markovian k stationary sunspot equilibrium (k-SSE)* for $\tilde{Z}$ is a k -tuple $x = (x^1, \dots, x^k) \in X^k$ with all of them distinct and k^2 vectors in the interior of S^{k-1} (the $k - 1$ dimensional simplex) denoted by $m_{ij}; i = 1, \dots, k; j = 1, \dots, k$ such that:

$$\tilde{Z}(x^i, x^j, (x, m_{ij})) = 0, \quad \forall i, j = 1, \dots, k,$$

where (x, m_{ij}) represents the probability distribution measure with support $x^1, \dots, x^k$ and with probabilities given by m_{ij} (i.e. $\text{Prob}[x^l] = m_{ij}^l$).

It is easy to prove that any Markovian process with memory is stationary.

Lemma. *Let m be a k -Markovian process with memory, i.e. $m = (m_{ij})_{i,j=1}^k \in (S^{k-1})^{k^2}$. Then there exists $\bar{s} \in S^{k-1}$ such that:*

$$\bar{s}^l = \sum_{i,j=1}^k m_{ij}^l \bar{s}^i \bar{s}^j, \forall l = 1, \dots, k.$$

The structural equations of macro models frequently presents linearity with respect to the probability measure and differentiability with respect to the state variable. I will use the following assumptions:

(L) Linearity: $\tilde{Z}(x, y, \mu) = \int_X Z(x, y, z) \mu(dz)$.

(D) Differentiability: Z is a C^1 function.

I will suppose the regularity of cycles in the following sense: Let $\mathbf{x} = (x^1, \dots, x^k)$ be an interior k -cycle and let us define $F : X^k \times (S^{k-1})^k \rightarrow \mathfrak{R}$

$$F(\mathbf{y}, \mathbf{s}) =$$

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where $\mathbf{y} = (y^1, \dots, y^k)$ and $\mathbf{s} = (s^1, \dots, s^k)$. It is easy to see that $F(\mathbf{x}, \mathbf{e}) = 0$, where $\mathbf{e} = (e^l)_{l=1}^k$ and e^l is the l -th canonical vector of $\mathfrak{R}^k$. The regularity condition is:

(R) The interior k -cycle $\mathbf{x}$ is regular if $D_y F(\mathbf{x}, \mathbf{e})$ is a non-singular matrix.

Theorem. *Suppose that $\tilde{Z}$ satisfies assumptions (L) and (D). If there exists an interior k -cycle $(x^1, \dots, x^k)$ which satisfies condition (R) and the following condition:*

(S) *The convex hull $\mathcal{C}(\{Z(x^i, x^j, x^l) : l = 1, \dots, k\})$ contains "0" in its interior for all $i, j = 1, \dots, k; j \neq i + 1; (i, j) \neq (k, 1)$.*

Then there exists a k -SSE in a neighborhood of the cycle. Furthermore, if the complement of any of the sets in condition (S) contains "0" in its interior then there exists a neighborhood of the cycle where there is no k -SSE.

Remarks: 1) The (S) condition (sign condition) is easy to verify for the case $n = 1$. For example if we have a 2-cycle the (S) condition is:

$$Z(x^1, x^1, x^1) Z(x^1, x^1, x^2) < 0 \text{ and } Z(x^2, x^2, x^1) Z(x^2, x^2, x^2) < 0.$$

2) The (S) condition is an open condition in the sense that if some parameter values of the model satisfy this condition then there exists an open set of these parameters that satisfies the same condition.

An analogous result can be obtained if there exist two regular steady states. Let $\bar{x}^1, \bar{x}^2 \in X$ be two interior steady states. Let us define $G : X^2 \times (S^1)^2 \rightarrow \mathbb{R}^n$ as:

$$G(y^1, y^2, s^1, s^2) = \left(\tilde{Z}(y^1, y^1, (y, s^1)), \tilde{Z}(y^2, y^2, (y, s^2)) \right).$$

Then $G(\bar{x}, e) = 0$. In this case the regularity condition will be:

(R') $(\bar{x}^1, \bar{x}^2)$ are two regular steady states if $D_y G(\bar{x}^1, \bar{x}^2, e^1, e^2)$ is a non-singular matrix.

Corollary. Suppose that $\tilde{Z}$ satisfies conditions (L) and (D). If there exist two steady states $\bar{x}^1$ and $\bar{x}^2$ which satisfy (R') and:

$$Z(\bar{x}^1, \bar{x}^2, \bar{x}^1) Z(\bar{x}^1, \bar{x}^2, \bar{x}^2) < 0 \text{ and } Z(\bar{x}^2, \bar{x}^1, \bar{x}^1) Z(\bar{x}^2, \bar{x}^1, \bar{x}^2) < 0$$

then there exists a 2-SSE in a neighborhood of $(\bar{x}^1, \bar{x}^2)$. Furthermore, if some of the inequalities above are ">" then there exists a neighborhood of $(\bar{x}^1, \bar{x}^2)$ where there is no 2-SSE.

3. Some examples.

In this section I will analyze some models with memory in order to verify if there exists k -SSE in a neighborhood of a regular cycle or multiple steady state. Since most of the cycles are attractive or repellent for the perfect foresight dynamics, I use these properties to compute them. Namely, if the cycle is an attractor we can write the deterministic dynamics as $x_{t+1} = \emptyset(x_t, x_{t-1})$ and the iterations of this function will converge to the cycle; analogously if the cycle is a repeller.

Example 1) Farmer (1986) analysed the dynamics of the rate of return of the asset in an OLG economy. In this model the government follows a policy of maintaining a constant budget deficit which is equal to zero. He considered an utility function $U(c_{t+1}, l_t) = c_{t+1}^s (1 - l_t)^{1-s}$ and technology given by $f(k) = (1/\alpha)(1 + \alpha k^\gamma)^{1/\gamma}$. If x_{t+1} is the gross real rate of return on assets held between periods t and $t + 1$ then the perfect foresight dynamics is given by the function:

$$Z(x_{t-1}, x_t, x_{t+1}) = \frac{s}{\alpha} (1 - \alpha x_t^{-d})^{-\frac{1}{d}} - (x_{t+1}^d - \alpha)^{-\frac{1+d}{d}} \\ - x_t \left[\frac{s}{\alpha} (1 - \alpha x_{t-1}^{-d})^{-\frac{1}{d}} - (x_t^d - \alpha)^{-\frac{1+d}{d}} \right],$$

where $d = \frac{\gamma}{1-\gamma}$. So, for the following parameter values $\alpha = 0.5$, $s = 0.5$, $d = -0.84$, I obtain the attractive 6-cycle:

$$x^1 = 1.2806; x^2 = 2.2075; x^3 = 2.2745; x^4 = 1.8974; x^5 = 0.18271; \\ x^6 = 0.44863.$$

In this case, the (S) condition fails because $\mathcal{C}\{Z(x^1, x^3, x^l); l = 1, \dots, 6\}$ does not contain "0". Then, there exists a neighborhood of the cycle where there is no Markovian SE. For these parameter values there is only one steady state. However, if we put $d = -0.84$,

$\alpha = 0.3$ and $s = 0.5$ two interior steady states appear $\bar{x}^1 = 0.047559$ and $\bar{x}^2 = 1.590101$. However, $Z(\bar{x}^1, \bar{x}^2, \bar{x}^1) Z(\bar{x}^1, \bar{x}^2, \bar{x}^2)$ and $Z(\bar{x}^2, \bar{x}^1, \bar{x}^1) Z(\bar{x}^2, \bar{x}^1, \bar{x}^2)$ are positive numbers, then there is no 2-SSE in a neighborhood of the steady states.

If we consider the indirect utility function $V(w_t, R_{t+1}) = w_t \exp(sR_{t+1})$ and a Leontieff technology then the following excess demand functions results.

$$Z(x_{t-1}, x_t, x_{t+1}) = sx_{t+1}(\alpha - x_t) - 1 - x_t[sx_t(\alpha - x_{t-1}) - 1].$$

For $\alpha = 2.01$ and $s = 0.5$ I obtain the repellent 6-cycle:

$$x^1 = 1.0708; x^2 = 0.90679; x^3 = 0.80012; x^4 = 0.92072; x^5 = 1.1689; x^6 = 1.196.$$

The (S) condition is not fulfilled for $\mathcal{C}\{Z(x^2, x^5, x^1); l = 1, \dots, 6\}$.

For this specification of the model we can obtain two interior steady states. If $s\alpha^2 > 4$ then $\bar{x}^1 = \alpha/2 - \sqrt{(\alpha/2)^2 - 1/s}$ and $\bar{x}^2 = \alpha/2 + \sqrt{(\alpha/2)^2 - 1/s}$ are two interior steady states for the model. Since the function $g(s, \alpha) = Z(\bar{x}^1, \bar{x}^2, \bar{x}^1) Z(\bar{x}^1, \bar{x}^2, \bar{x}^2)$ is strictly positive in the domain $\{(s, \alpha) \in \mathbb{R}_+^2 / s\alpha^2 > 4\}$ then there is no 2-SSE in a neighborhood of $\{\bar{x}^1, \bar{x}^2\}$ for these parameter values.

Example 2) de Vilder (1996) describes an OLG model with two sectors and he showed the existence of complicated and periodic attractors for the deterministic dynamics. The individuals have utility function $U(c_{t+1}, l_t) = \frac{(l^* - l_t)^{1-\alpha_1}}{1-\alpha_1} + \frac{c_{t+1}^{1-\alpha_2}}{1-\alpha_2}$ and the technology is given by $f(k_t - 1, l_t) = \text{Min}\{l_t, k_{t-1}/a\}$ with depreciation rate of capital equal to δ . If x_t is the production in period t then the dynamics are given by:

$$Z(x_{t-1}, x_t, x_{t+1}) = x_{t-1}(l^* - x_{t-1})^{-\alpha_1} - [(1 + a(1 - \delta))x_t - ax_{t+1}]^{1-\alpha_2}.$$

For $a = 3$, $l^* = 2$, $\alpha_1 = 0.5$, $\alpha_2 = 0.77$ and $\delta = 0.1$ I found an attractive 11-cycle:

$$x^1 = 0.92821; x^2 = 1.076; x^3 = 1.1197; x^4 = 0.83669; x^5 = 0.31298; \\ x^6 = 0.27551; x^7 = 0.3391; x^8 = 0.41785; x^9 = 0.51435; x^{10} = 0.6316; x^{11} = 0.77114.$$

We can conclude the non existence of Markovian 11-SE since $\mathcal{C}\{Z(x^1, x^3, x^l); l = 1, \dots, 6\}$ does not contain zero.

Example 3) Michel and Venditti (1997) analyzed the solutions of the central planner problem in an OLG economy when the utility function of the representative agent is non-separable. For their model, let us consider the utility function $U(c, d) = c^\alpha d^\beta$, where c and d are the consumptions when the individual is young and old respectively. The production *per capita* is $f(k) = F(k, 1) = k^\lambda$. The indirect utility function is defined as:

$$V(x_t, x_{t+1}) = \max\{U(c, d); x_{t+1} = f(x_t - c) - d\}$$

The Euler equations will characterize the Z function:

$$Z(x_{t-1}, x_t, x_{t+1}) = V_2(x_{t-1}, x_t) + \delta V_1(x_t, x_{t+1}).$$

For the specified preferences and technology it results:

$$Z(x_{t-1}, x_t, x_{t+1}) = -\beta c^\alpha(x_{t-1}, x_t) [(x_{t-1} - c(x_{t-1}, x_t))^\lambda - x_t]^{\beta-1} + \\ \beta \delta \lambda c^\alpha(x_t, x_{t+1}) [(x_t - c(x_t, x_{t+1}))^\lambda - x_{t+1}]^{\beta-1}$$

Where $c(x, y)$ is the function implicitly defined by:

$$\alpha (x - c(x, y))^\lambda - \lambda \beta c(x, y) (x - c(x, y))^{\lambda-1} - \alpha y = 0$$

For $\lambda = 1/3$, $\alpha = 0.5356$, $\beta = 0.4643$ and $\delta = 0.858$, I found the two cycles:

$$x^1 = 0.342902; x^2 = 0.342871,$$

and the (S) condition fails because $Z(x^1, x^1, x^1)$ and $Z(x^1, x^1, x^2)$ are negative numbers.

4. Conclusions.

This work shows a necessary and sufficient condition (the (S) condition) for the existence of a Markovian sunspot equilibrium in a neighborhood of a regular cycle or multiple steady states for models with lagged variables. Such condition is also applied to three well known models in order to check for the existence of that type of stochastic equilibrium. I conclude that the regularity of the cycle/multiple steady state is not sufficient to guarantee the existence of Markovian sunspot equilibrium while it is in models without memory as shown by Azariadis and Guesnerie (1986) and Peck (1988). This, however, does not preclude the existence of other types of SSE in some neighborhood of a k -cycle. For example, Woodford (1986) proved the existence of SE, when the steady state is indeterminate, considering that the equilibrium depends on the history of observations of the sunspot variable.

Appendix.

Proof of Lemma:. Let us define $T : S^{k-1} \rightarrow S^{k-1}$ as $T(s) = \left[\sum_{i,j=1}^k m_{ij}^l s^i s^j \right]_{l=1,\dots,k}$. It is a continuous map from a compact and convex set into itself. The Brouwer's Theorem allow us to conclude that there exists $\bar{s} \in S^{k-1}$ such that $T\bar{s} = \bar{s}$. Let us note that if m is in the interior of $(S^{k-1})^{k^2}$ the fixed point of T is unique.

Proof of Theorem:. From $F(x, e) = 0$ and the non-singularity of the matrix $D_y F(x, e)$, there exist neighborhoods $\mathcal{O}_1, \dots, \mathcal{O}_k$ of $e^1, \dots, e^k$ respectively and there exist functions $g_l : \mathcal{O}_1 \times \dots \times \mathcal{O}_k \rightarrow (\mathbb{R}^n)^k, l = 1, \dots, k$ such that $g_l(e) = x^l, \forall l = 1, \dots, k$ and for all $s \in \mathcal{O}_1, \dots, \mathcal{O}_k; t = 1, \dots, k-2$ we have:

$$\bar{Z}(g_t(s), g_{t+1}(s), (g(s), s^{t+2})) = 0,$$

$$\bar{Z}(g_{k-1}(s), g_k(s), (g(s), s^1)) = 0 \text{ and } \bar{Z}(g_k(s), g_1(s), (g(s), s^2)) = 0.$$

Let us define $m_{i(i+1)} = s^i, i = 1, \dots, k-1$ and $m_{k1} = s^k$. To define the other transition probabilities let us use the (S) condition. By continuity, there exist neighborhoods of $x^1, \dots, x^k$ where (S) holds and we can redefine the neighborhoods $\mathcal{O}_1, \dots, \mathcal{O}_k$ (if it is necessary) in order to obtain that $\mathcal{C}(\{Z(g_i(s), g_j(s), g_l(s)); l = 1, \dots, k\})$ contains "0" in its interior $\forall i, j = 1, \dots, k; j \neq i+1; (i, j) \neq (k, 1)$ and $\forall s \in \mathcal{O}_1 \times \dots \times \mathcal{O}_k$. Then for each (i, j) with the above characteristics and $s \in \mathcal{O}_1 \times \dots \times \mathcal{O}_k$ there exists $m_{ij} \in S^{k-1}$ such that $\bar{Z}(g_i(s), g_j(s), (g(s), m_{ij})) = 0$; therefore $\forall s \in \mathcal{O}_1 \times \dots \times \mathcal{O}_k, (g_1(s), \dots, g_k(s), (m_{ij})_{i,j=1}^k)$ is a k -SSE. The last part of the theorem follows from continuity of F .

Proof of collorary:. It is analogous to the proof above.

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