Equilibrium in stochastic economies with incomplete financial markets

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Abstract

We construct a representative agent supporting regular equilibria in a stochastic economy with more than two agents. Then we give conditions for the existence of equilibria. In this way we extend the results of Cuoco and He (1994).

Resumo

Construímos um agente representativo que suporta equilíbrios regulares numa economia estocástica com mais de dois agentes. Depois damos condições para a existência de equilíbrio. Desta forma estendemos os resultados obtidos por Cuoco e He (1994).

Key Words: Representative Agent; Incomplete Markets, Stochastic Economy.

JEL Code: D52.
1. Introduction.

The dynamic equilibrium problem in stochastic economies has been studied by many authors. Huang (1987), Dumas (1989) and Karatzas, Lehoczky and Shreve (1990) have studied the complete markets case, whereas Cox, Ingersoll and Ross (1985) have worked with incomplete markets. In the case of complete markets, the utility of the representative agent is constructed as a linear combination of individual utility functions using constant weights; in the case of incomplete markets that utility is assumed to be given exogenously. When the markets are complete or the equilibrium allocation is Pareto-efficient, Negishi (1960), Constantinides (1982) and Huang (1987) have shown that this type of aggregation is possible. Cuoco and He (1994) have shown that this aggregation is also possible for an economy with two agents, incomplete markets and whose representative agent's utility is given endogenously. They construct this utility as a linear combination of individual utility functions, but with weights represented by stochastic processes, they transform the equilibrium problem into the characterization of the weighting processes. So the optimal policies can be obtained from the problem with the representative agent. In this paper we show how to carry out this aggregation when there are more than two agents in the economy.

The paper is organized as follows: in section 2 we introduce the stochastic economy; in section 3 we present the consumption problem and define a rational expectation equilibrium for our economy. In section 4 we characterize the optimal policies and in section 5 we present our main results, constructing the representative agent; in section 6 we analyze the existence of equilibria and in section 7 we present our conclusions. The last section consists of an appendix with some useful results.
2. Economy.

We consider a continuous-time economy at time interval $[0, T]$, with $T < \infty$, there is a single perishable consumption good (the numeraire) and a financial market $\mathcal{M}$ consisting of $n + 1$ assets. The first will be called bond (riskless asset) and the remaining $n$ will be called stocks (risky assets). The bond price and stock prices will be denoted by $B(t)$ and $P_i(t)$ ($1 \leq i \leq n$), respectively. The evolution of these prices are modeled, respectively, by the following equations:

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1,$$

$$dP_i(t) = P_i(t)[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}dW_j(t)], \quad P_i(0) \in (0, \infty), \forall i = 1, \ldots, n.$$  

These two processes are denominated in units of the consumption good. In this economy the sources of risk are modeled by the independent components of a $d$–dimensional Brownian motion $W(t) = (W_1, \ldots, W_d)'$, $0 \leq t \leq T$. With this interpretation $\sigma_{ij}$ models the instantaneous intensity in which the $j$ source of risk influence the price of the $i^{th}$ stock at time $t$. The Brownian motion $W$ is defined on a given complete probability space $(\Omega, \mathcal{F}, P)$; we will denote by $\mathcal{F} = \{\mathcal{F}(t), 0 \leq t \leq T\}$ the $\sigma$-augmentation\(^1\) of the natural filtration generated by $W$:

\(^1\)The augmented Brownian filtration $\mathcal{F}$ is defined by $\mathcal{F}(t) = \sigma(\mathcal{F}_w(t) \cup \mathcal{N})$, where $\mathcal{N} = \{E \subset \Omega: \exists G \in \mathcal{F} \text{ with } E \subseteq G, P(G) = 0\}$ denotes the set of $P$-null events. It is well known that the augmented filtration is continuous and $W$ is still a Brownian motion with respect to it (Karatzas and Shreve (1988), Corollary 2.7.8 and Prop. 2.7.9).
Equilibrium in stochastic economies with incomplete financial markets

\[ \mathcal{F}_W(t) = \sigma(W(s), 0 \leq s \leq t), \ 0 \leq t \leq T. \]

The interest rate process \( \{r(t) : 0 \leq t \leq T\} \), the vector process of appreciation rates \( \{b(t) = (b_1(t), \ldots, b_n(t))', 0 \leq t \leq T\} \), and the matrix process of volatilities \( \sigma(t) = \{(\sigma_{ij}(t))_{1 \leq i \leq n, 1 \leq j \leq d}, 0 \leq t \leq T\} \)
will be referred to as the coefficients of the financial market \( \mathcal{M} \). We will assume that these coefficients are progressively measurable with respect to \( \mathcal{F} \) and that they satisfy the condition :

\[
\int_0^T \left( |r(t)| + \|b(t)\| + \|\sigma(t)\|^2 \right) dt < \infty,
\]

with \( \|\cdot\| \) the Euclidean norm.

The requirement that the coefficients \( r(t), b(t) \) and \( \sigma(t) \) be progressively measurable with respect to \( \mathcal{F} \), essentially makes them functionals of the Brownian path \( \{W(s), 0 \leq s \leq t\} \) up to time \( t, \forall t \in [0, T] \). This assumption avoids anticipation of the future, but allows for dependence on the past of the driving Brownian motion, or of the stock prices; both features are desirable.

All the agents in this economy are endowed with the same information represented by \( \mathcal{F} \) and they have the same beliefs represented by \( \mathcal{P} \). These agents are considered small investors (their decisions do not affect the market prices) and each of them will decide at each moment \( t \in [0, T] \):

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2 A \( \mathbb{R}^n \)-valued process \( X = \{X(t) : t \in [0, T]\} \) is said to be progressively measurable with respect to the filtration \( \mathcal{F} = \{\mathcal{F}_t\} \) if for every \( t \in [0, T] \) the map \( (s, w) \mapsto X(s, w) \) from \( ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}(t)) \) into \( (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \) is measurable, where \( \mathcal{B}([0,t] \otimes \mathcal{F}(t)) \) denotes the product \( \sigma \)-field of the Borel \( \sigma \)-field on \([0,T]\) and \( \mathcal{F}(t) \).
1. How much money \((\alpha, \theta)\) he wants to invest. Where \(\alpha(t)\) and \\
\(\theta(t) = (\theta_1(t), ..., \theta_n(t))^t\) denote the number of shares of bond \\
and stocks, respectively.

2. what his cumulative consumption \(C(t)\) should be.

Of course these decisions must be made in a non-anticipative 
way, then \(C\) and \((\alpha, \theta)\) must be adapted processes. In order to know 
the feasible decision of our investor, let us characterize the consump­
tion space \(C\) by the set of adapted consumption rate process \(c\) with 
\[
\int_0^t |c(s)| ds < \infty \text{ for all } t \in [0, T].
\]
The individual consumption sets 
are subsets of the non-negative orthant \(C_+ = \{c \in C : c(t) \geq 0 \forall t\}\). 
In the security price processes, \(\sigma\) is an exogenously given map, the 
interest rate process and the \(n\)-dimensional vector of appreciation 
rates will be determined endogenously in equilibrium. Now in order 
to rule out redundant securities we have our first assumption

**Assumption 1** The diffusion matrix \(\sigma(t, w)\) is continuous in its first 
arguments and full row rank almost surely for all \(t \in [0, T]\).

Now an admissible trading strategy for this investor must satisfy 
for all \(t \in [0, T]\) the following equation

\[
\int_0^t \left( |\alpha(s)r(s)| + |\theta'(s)b(s)|| + |\theta'(s)\sigma(s)|^2 \right) ds < \infty,
\]

The set of admissible strategies will be denoted by \(\Theta\). We con­
sider that there are a finite number \(K \geq 2\) of investors in the econ­
omy. The preference of each investor is characterized by the time­
additive, state-independent utility function
Equilibrium in stochastic economies with incomplete financial markets

\[ U_k(c) = E \left[ \int_0^T u_k(c(t), t) dt \right], k \in \{1, \ldots, K\}. \]

In order to solve the problem of maximizing the expected utility from consumption we will consider that for each agent \( k \) the set \( C_k \) of consumption processes satisfying

\[ E \left[ \int_0^T u_k(c(t), t)^{-} dt \right] < \infty, \]

Where \( x^- = \max(0, -x) \) and to solve the individual optimization problem we need to impose some smooth properties on preferences, as the following assumption

**Assumption 2** The functions \( u_i(\cdot, t) \) are strictly increasing, strictly concave and three times continuously differentiable on \((0, \infty)\) for all \( t \in [0, T] \). Moreover, they satisfy the Inada’s conditions

\[ \lim_{x \to 0} u_k(c, t) = \infty \quad \text{and} \quad \lim_{x \to \infty} u_k(c, t) = 0, \quad (3) \]

and there exist constants \( \delta_k \in (0, 1) \) and \( \gamma_k \in (0, \infty) \) such that

\[ \delta_k u_k(c, t) \geq u_k(\gamma_k c, t) \quad \forall (c, t) \in (0, \infty) \times [0, T]. \quad (4) \]

Finally, \( u_k(c, \cdot) \) is continuously differentiable on \([0, T]\) with

\[ \int_0^T |u_i(c, t)| dt < \infty, \quad (5) \]
for all \( c \in (0, \infty) \).

The condition (3) implies that derivative function \( u_{kc}(\cdot, t) \) has a continuous and strictly decreasing inverse \( f_k(\cdot, t) \) mapping \((0, \infty)\) onto itself. Condition (4) is technical and will be used to guaranteeing that certain integral functional can be differentiated under the integral sign. We could verify that the utility functions \( u(c, t) = \varrho(t) \log(c) \) and \( u(c, t) = \varrho(t) \frac{c^{1-b}}{1-b} \) with \( b > 0, b \neq 1 \) satisfy this condition.

Each investor \( k \) is endowed with an income process \( e_k \in C_+ \), with \( e_k \neq 0 \). Now denoting \( \text{The aggregate income process} \) by "\( e \)" \( i.e. \)

\[
e = \sum_{k=1}^{K} e_k,
\]

we have the following

**Assumption 3** \( \text{The process \"e\" is an Ito process,} \)

\[
de(t) = \mu(t) dt + \rho(t) dW_t,
\]

for some continuous, adapted, bounded processes \( \mu, \rho \). Moreover, there exist constants \( 0 < e' \leq e'' \) such that

\[
e' \leq e(t) \leq e'' \quad \forall t \in [0, T],
\]

In what follows we denote by \( \mathcal{E} = (\Omega, \mathcal{F}, \mathcal{F}, \mathcal{P}, \sigma, \{u_k, e_k\}_{k=1}^{K}) \) the primitives for the above economy, and by \( \mathcal{P} = (\tau, b) \) the parameters defining the security price process. We will refer to \( \mathcal{E} \) as \( \text{the economy} \) and to \( \mathcal{P} \) as \( \text{the price system} \).
3. Individual consumption.

Given a price system $\mathcal{P}$ each investor $k$ chooses a consumption process $c_k \in C_k$ and an admissible strategy $(\alpha_k, \theta_k) \in \Theta$, subject to $\alpha_k(0) = 0$, $\theta_k(0) = 0$,

$$X_k(t) \equiv \alpha_k(t)B(t) + \theta_k^t P(t)$$

$$= \int_0^t \alpha_k(s)dB(s) + \int_0^t \theta_k'(s)dP(s) - \int_0^t (c_k(s) - e_k(s))ds,$$ \hfill (7)

$$X_k(t) \geq -KB(t),$$ \hfill (8)

$$X_k(T) \geq 0,$$ \hfill (9)

for all $t \in [0, T]$ and some $K \in \mathbb{R}_+$, where $\{X_k(t), 0 \leq t \leq T\}$ denotes the wealth process. The equation (7) is the well known dynamic budget constraint: current wealth equals the trading gains, plus the cumulative income, minus the cumulative consumption. Now, since we allow investors to borrow against future income, we will need a liquidity constraint (equation (9)) in order to rule out default possibilities and to avoid arbitrage opportunities, such as the doubling strategies, we need a condition like equation (8), the sufficiency of this condition to rule out free lunches was proved by Dybvig and Huang (1989).

**Definition 1** Given the price system $\mathcal{P}$, a consumption process $c_k \in \mathcal{C}$ is said to be feasible from income $e_k$ if there exists an admissible trading strategy $(\alpha_k, \theta_k) \in \Theta$ such that (7)-(9) are satisfied. Then we said that $(\alpha_k, \theta_k)$ finances $c_k$. 
e will denote the set of feasible consumption processes for income $e_k$ given $\mathcal{P}$ by $\mathcal{B}(e_k, \mathcal{P})$. Now, for any given price system $\mathcal{P}$ define the standardized risk premium process:

$$\eta_0 = -\sigma'(t) (\sigma(t)\sigma'(t))^{-1} (b(t) - r(t)),$$  \hspace{1cm} (10)

and the exponential process

$$Z_0(t) = \exp \left( \int_0^t \eta_0'(s)dW(s) - \frac{1}{2} \int_0^t |\eta_0(s)|^2 ds \right),$$  \hspace{1cm} (11)

with these two processes we have the admissible price system

**Definition 2** A price system $\mathcal{P} = (r, b)$ is admissible if:

a) the interest rates process satisfies

$$\int_0^t |r(s)| ds < \infty,$$  \hspace{1cm} (12)

for all $t \in [0, T]$ and there exists a constant $K_1 > 0$ such that

$$\int_0^T r(t)^{-1} dt < K_1,$$  \hspace{1cm} (13)

b) the standardized risk premium process $\eta_0$ of (10) satisfies the Novikov condition:

$$E \left[ \exp \left( \frac{1}{2} \int_0^T |\eta_0(t)|^2 dt \right) \right] < \infty,$$  \hspace{1cm} (14)

\footnote{ \(1 = (1, \ldots, 1) \in \mathbb{R}^n\)}
Equilibrium in stochastic economies with incomplete financial markets

c) there is a unique strong solution to the stochastic integral equation (2).

Condition a) ensures that bond price is well defined and bounded away from zero and condition b) is needed to ensure the existence of equivalent martingale measure\(^5\) and to rule out arbitrage opportunities.

With this definition of admissible price system we can proceed to define an equilibrium

**Definition 3** A rational expectations equilibrium for the economy \( \mathcal{E} \) is an admissible price system \( \mathcal{P} \) and a set \( \{ c_k, (\alpha_k, \theta_k) \} \) of admissible consumption and trading strategies such that:

(i) \( c_k \) maximize \( U_k \) on \( B(e_k, \mathcal{P}) \cap \mathcal{C}_k \)

(ii) \( (\alpha_k, \theta_k) \) finances \( c_k \)

(iii) the security and consumption markets clear i.e.

\[
\sum_{k=1}^{K} \alpha_k = 0, \quad \sum_{k=1}^{K} \theta_k = 0, \quad \text{and} \quad \sum_{k=1}^{K} c_k = e.
\]

4. Optimal policies.

In the literature, there exist results that characterize the optimal policies without applying dynamic programming (they didn’t use any Markovian hypothesis). Instead they used martingale techniques.

\(^5\) An equivalent martingale measure is a probability \( Q \) which is equivalent to the original \( \mathcal{P} \), i.e. they have the same null events, and such that the discounted price process \( \{ e^{-\int_{0}^{t} \tau(s)ds} P_t \}_{t \geq 0} \) is a martingale and a process \( \{ X_t \} \) is said to be a \( \{ \mathcal{F}_t \} \)-martingale(submartingale, supermartingale) if the following conditions are satisfied

i) \( E|X_t| < \infty \).

ii) \( E[X_t/\mathcal{F}_s] = X_s \) a.s. \( \forall s \leq t \) \(( \geq, \leq \) respectively).
These results are due to Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989,1991) in the case of complete markets and Karatzas, Lehoczky, Shreve and Xu (1991) and Cuoco (1997) in the case of incomplete markets. Now we review some of these results, which we will need later to state our main results.

First, suppose that markets are complete \((n = d)\). In this case the problem of maximizing expected utility subject to the dynamic budget constraints (7)-(9) is equivalent to the problem of maximizing the expected utility subject to the single Arrow-Debreu budget constraint

\[
E \left[ \int_0^T \gamma(t)Z_0(t)(c(t) - e_k(t)) \, dt \right] \leq 0, \quad (15)
\]

where \(Z_0\) is the density process for the unique equivalent martingale measure or risk-neutral probability; it is unique since markets are complete; its expression is given by equation (11) and the process \(H_0(\cdot) = \gamma(\cdot)Z_0(\cdot)\) is the unique state-price density for the economy, in the sense that the value at time zero of any consumption process \(c\) is given by \(E \left[ \int_0^T H_0(t)c(t) \, dt \right] \).

Now, by Lagrangian theory, we know that if the optimal solution \(c_{k0}\) to the \(k^{th}\) individual optimization problem exists, then it satisfies the first order condition

\[
u_{kc}(c_{k0}(t), t) = \psi_k H_0(t), \quad (16)
\]

For some Lagrangian multiplier \(\psi_k\) such that (15) is satisfied as an equality.

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\(6\)See Karatzas and Shreve (1998).
To obtain similar characterization in incomplete markets we need to generalize the definition of an equivalent martingale measure.

**Definition 4** A probability measure $Q$ on $(\Omega, \mathcal{F})$ is "locally equivalent" to $P$ if the restriction of $Q$ to $\mathcal{F}_t$ is equivalent to the restriction of $P$ to $\mathcal{F}_t$ for all $t \in [0, T]$. A locally equivalent probability measure $Q$ is said to be a locally equivalent martingale measure if $Q$ is absolutely continuous with respect to $P$ and the discounted price process is a local martingale under $Q$.

The set of locally equivalent martingale measures has an explicit structure. Let $\mathcal{L}^2$ denote the set of adapted $n$-dimensional processes $\xi$ such that

$$\int_0^t |\xi(s)|^2 ds < \infty,$$

for all $t \in [0, T]$ and decompose $\mathcal{L}^2$ in the two subspaces

$$K(\sigma) = \{ \xi \in \mathcal{L}^2 : \sigma \xi = 0 (P \times l) - a.e \}$$

$$S(\sigma) = \{ \xi \in \mathcal{L}^2 : \xi \in R(\sigma')(P \times l) - a.e \}$$

where $l$ denotes the Lebesgue measure on $[0, T]$ and $R$ denotes the range. Now for each $\nu \in K(\sigma)$ define where $\eta_\nu(t) = \eta_0 + \nu(t)$, then the exponential process

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A process $\{X_t\}$ is said to be a local martingale if there exists a sequence of $\{\mathcal{F}_t\}$-stopping times $\tau_n$ such that

i) $\tau_n \to \infty$ a.s

ii) $\forall n \quad \{X_{t \wedge \tau_n}\}$ is a $\{\mathcal{F}_t\}$-martingale.
\[ Z_\nu(t) = \exp \left( \int_0^t \eta_\nu(s) dW(s) - \frac{1}{2} \int_0^t |\eta_\nu(s)|^2 ds \right), \quad (17) \]

is well defined and is a strictly positive continuous local martingale. Now denote by \( \mathcal{N} \) the set of \( \nu \in K(\sigma) \) for which the process \( Z_\nu \) is a uniformly integrable martingale. Notice that the set \( \mathcal{N} \) is nonempty for any admissible price system \( \mathcal{P} \) since (14) implies that \( 0 \in \mathcal{N} \). This set will be very useful to understand how the equivalent martingale measures change in the presence of incomplete markets, as the following proposition shows.

**Proposition 1** A probability measure \( Q \) is a locally equivalent martingale measure if and only if \( \frac{dQ_t}{dP_t} = Z_\nu(t) \) for some \( \nu \in \mathcal{N} \) and all \( t \in [0,T] \), where \( P_t(Q_t) \) denotes the restriction of \( P(Q) \) to \( \mathcal{F}_t \).

*Proof.* See Cuoco and He (1994). \[ \square \]

Then it is easy to see that when markets are complete \( (n = d) \), \( \nu = 0 \) must hold \( (P \times I) \) a.e. for all \( \nu \in K(\sigma) \) and then there is a unique locally equivalent martingale measure with density \( Z_0 \).

The following lemma is a useful result that characterizes the state price densities using the locally equivalent martingale measures above.

**Lemma 1** If \( c_k \) is feasible for income \( e_k \), then

\[ \sup_t \int_{|X_t| > c} |X_t| dP \rightarrow 0 \text{ as } c \rightarrow \infty \]

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9A martingale \( \{X_t\}_{t \geq 0} \) is said to be uniformly integrable if
Equilibrium in stochastic economies with incomplete financial markets

\[ E \left[ \int_0^T H_\nu(t)(c_k(t) - e_k(t)) dt \right] \leq 0, \]  \hspace{1cm} (18)

holds for all \( \nu \in \mathcal{N} \), where \( H_\nu(\cdot) = \gamma(\cdot)Z_\nu(\cdot) \). And conversely, let \( c_k \in C_k \) and suppose that there exists a process \( \nu_k \in \mathcal{N} \) such that

\[ E \left[ \int_0^T H_\nu(t)(c_k(t) - e_k(t)) dt \right] \leq E \left[ \int_0^T H_{\nu_k}(t)(c_k(t) - e_k(t)) dt \right] = 0, \]  \hspace{1cm} (19)

for all \( \nu \in \mathcal{N} \). Then \( c_k \) is feasible for income \( e_k \).

**Proof**

The first part is a consequence of Ito's lemma\(^{10}\) and equation (7). Then you have to use the fact that the expectation of a stochastic integral with respect to a Brownian motion is zero. Finally using (9) the result follows. For the converse see Theorem 1 in Cuoco (1997).\(\square\)

We will refer to (18) as the *static budget constraint*. The above lemma implies that our optimization problem can be reformulated as the maximization of expected utility subject to a sequence of budget constraints associated with the densities \( H_\nu \) with \( \nu \in \mathcal{N} \). Then we could expect that the utility gradients at the optimal policies be a positive linear combination of such \( H_\nu \). Now by Ito’s lemma we have that the set \( \{H_\nu : \nu \in \mathcal{N}\} \) is a convex set, then the utility gradient of agent \( k \) at the optimal policy is proportional to a state-price

\(^{10}\)See appendix.
density $H_{\nu_k}$ with $\nu_k \in \mathcal{N}$, This implies that the solution of the individual consumption problem for agent $k$ coincides with the solution of the problem of maximizing expected utility over the consumption processes $c \in C_k$ satisfying the single budget constraint

$$E \left[ \int_0^T H_{\nu_k}(t) (c_k(t) - e_k(t))dt \right] \leq 0. \quad (20)$$

The process $\nu_k$ satisfying the above conditions is necessarily unique, since $H_{\nu}(0) = 1$, $\forall \nu \in \mathcal{N}$, the corresponding process $H_{\nu_k}$ has been defined as the minimax state-price density by He and Pearson (1991), since it can be characterized as the solution of a dual minimization problem (see theorem 4 in the appendix). Karatzas, Lehoczky, Shreve and Xu (1991) interpret the minimax state-price density as the unique state price density that would prevail in an artificial market obtained by adding additional securities in such a way that the agent would not want to invest in them.

**Remark.**

Notice that in general the minimax state-price densities for the $K$ agents will be different, unless the markets are complete or the allocation is Pareto-efficient.

In analogy with (16) our solution is given by

$$c_{\nu_k}(t) = f_k (\psi_k H_{\nu_k}(t), t), \quad (21)$$

Where $\psi_k$ is the Lagrangian multiplier such that the constraint (20) is satisfied as an equality. We can formalize our intuition with the following result

**Theorem 1.** Suppose that the consumption process $c_{\nu_k}$ of (21) satisfies the budget constraint in (20) as an equality for some $\psi_k > 0$,
Equilibrium in stochastic economies with incomplete financial markets

$\nu_k \in \mathcal{N}$. If there exists strategy financing $c_{\nu_k}$, then $H_{\nu_k}$ is the minimax state-price density, $c_{\nu_k}$ is the optimal consumption policy for the agent $k$, and the corresponding wealth process is given by

$$X_{\nu_k}(t) = H_{\nu_k}^{-1}(t)E \left[ \int_{t}^{T} H_{\nu_k}(s) \left( c_{\nu_k}(s) - e_k(s) \right) ds / \mathcal{F}_t \right]$$  \hspace{1cm} (22)

Proof

From the definition of $c_{\nu_k}$ and the continuity of $f_k$ and $H_{\nu_k}$ we have

$$\int_{0}^{t} c_{\nu_k}(s) ds < \infty, \forall t \in [0, T].$$

And from the inequality

$$u_k(1, t) - y \leq \max_{c > 0} [u_k(c, t) - yc] = u_k(f_k(y, t), t) - yf_k(y, t),$$  \hspace{1cm} (23)

we obtain

$$E \left[ \int_{0}^{T} u_k(c_{\nu_k}(t), t)^{-} dt \right] \leq \int_{0}^{T} u_k(1, t)^{-} dt$$

$$+ \psi_k E \left[ \int_{0}^{T} \gamma(t) Z_{\nu_k}(t) dt \right] < \infty,$$

the last inequality follows from (5), (13) and the martingale property of $Z_{\nu_k}$. Therefore $c_{\nu_k} \in C_k$.

Now take an arbitrary $c \in \mathcal{B}(e_k, \mathcal{P})$. By concavity
Now taking expectation and \( x = \psi_k H_{\nu_k} \), we have

\[
U_k(c_{\nu_k}) - U(c) \geq E \left[ \int_0^T \psi_k H_{\nu_k}(t)(c_{\nu_k}(t) - \alpha(t)) dt \right] \geq 0,
\]

by the definition of \( \psi_k \) we obtain

\[
E \left[ \int_0^T H_{\nu_k}(s)c_{\nu_k}(s) ds \right] = E \left[ \int_0^T H_{\nu_k}(s)e_k(s) ds \right].
\]

Now by lemma 1 we know that \( c \) satisfies (18). Then we obtain the last inequality in (25), i.e. \( c_{\nu_k} \) is optimal. To prove the last part, let \((\alpha_k, \theta_k)\) be the trading strategy financing \( c_{\nu_k} \) and we will find the expression of the optimal wealth associated with this optimal policy. Let \( X_k(t) = \alpha_k(t)B(t) + \theta_k(t)P(t) \). Now from (7) and applying Ito's lemma we have

\[
H_{\nu_k}(t)X_k(t) = \int_0^t H_{\nu_k}(s)(e_{\nu_k}(s) - \alpha_k(s)) ds
\]

\[
+ \int_0^t H_{\nu_k}(s)(\theta_k'(s)\sigma(s) + X_k(s)\eta_{\nu_k}(s)) dW(s)
\]

from here we obtain
Equilibrium in stochastic economies with incomplete financial markets

\[ M(t) = H_{\nu_k}(t)X_k(t) + \int_0^t H_{\nu_k}(s)(c_{\nu_k}(s) - e_k(s))ds \]

\[ = \int_0^t H_{\nu_k}(s)(\theta_k^i(s)\sigma(s) + X_k(s)\eta_k^i(s))dW(s) \quad (28) \]

from (13), (8) and the fact that \( \nu_k \in \mathcal{N} \), we have that \( M(t) \) is uniformly integrable from below and from its representation as a stochastic integral, we have that it is a local martingale. Then from Fatou’s lemma \( M(t) \) is a supermartingale, it implies

\[ EM(T) \leq EM(t) \leq M(0) = 0 \quad \forall t \in [0, T]. \]

Also from (26) and (9), we have

\[ EM(T) = E[H_{\nu_k}(T)X_k(T)] \geq 0, \quad (29) \]

then \( EM(T) = EM(t) = M(0) = 0 \quad \forall t \in [0, T] \). Then it is a martingale. Now from (9) and \( E[H_{\nu_k}(T)X_k(T)] = 0 \), we obtain \( X_k(T) = 0 \). Then

\[ H_{\nu_k}(t)X_k(t) + \int_0^t H_{\nu_k}(s)(c_{\nu_k}(s) - e_k(s))ds = M(t) \]

\[ = EM(T) | \mathcal{F}_t = E \left[ \int_0^T H_{\nu_k}(s)(c_{\nu_k}(s) - e_k(s))ds / \mathcal{F}_t \right] \]

\[ ^{11} \text{See Karatzas and Shreve (1988), 1.3.25.} \]
finally from the fact that \( \int_0^t H_{\nu_k}(s)(c_{\nu_k}(s) - e_k(s))ds \) is \( \mathcal{F}_t \)- measurable, we obtain (22).\( \Box \)

We have seen in the previous theorem implications of the existence of the minimax state-price density for the agent \( k \), i.e. the existence of a process \( \nu_k \in \mathcal{N} \) such that the associated policy \( c_{\nu_k} \) is marketed. Then we will be interested in guaranteeing the existence of this minimax state price density, in the appendix we present some results on this existence problem. The results above motivate the following definition

**Definition 5** An equilibrium \( (\mathcal{P}, c_1, \ldots, c_K) \) is a regular equilibrium if there exists a minimax state-price density \( H_{\nu_k} \) for each agent \( k \).

With this definition in mind, in the next section, we will construct a representative agent.

5. Representative agent.

In this section we will construct a representative agent supporting a regular equilibrium \( (\mathcal{P}, c_1, \ldots, c_k), \) for our economy \( \mathcal{E} \), i.e. we will construct a utility function \( U \) such that \( (\mathcal{P}, e) \) is a no-trade equilibrium for the single-agent economy \( ((\Omega, \mathcal{F}, \mathcal{F}, \mathcal{P}), \sigma, U, e) \).

Define the function \( u(c, \lambda, t) : (0, \infty) \times (0, \infty)^K \times [0, T] \rightarrow \mathbb{R} \) by

\[
    u(c, \lambda, t) = \max_{\sum_{k=1}^K e_k = c} \left[ \sum_{k=1}^K \lambda_k \cdot u_k(c_k, t) \right]
\]

We know that for each \( \lambda \in (0, \infty)^K \) fixed, the function \( u \) is strictly increasing, concave and continuously differentiable in its first argument; and satisfies the Inada’s condition (3) for all \( \lambda \in (0, \infty)^K \) and \( t \in [0, T] \), these properties are easy to be verified, since they
Equilibrium in stochastic economies with incomplete financial markets

are inherited from the individual utilities \( u_k \). The solution of the allocation problem (30) is

\[
c_k = f_k \left( \frac{u_c(c, \lambda, t)}{\lambda_k}, t \right), \quad \forall k \in \{1, ..., K\}.
\]

Summing (31) over all \( k \), we obtain

\[
\sum_{k=1}^{K} f_k \left( \frac{u_c(c, \lambda, t)}{\lambda_k}, t \right) = c,
\]

then the function

\[
f(x, \lambda, t) = \sum_{k=1}^{K} f_k \left( \frac{x}{\lambda_k}, t \right),
\]

and \( u_c \) are inverses, i.e.

\[
f(u_c(c, \lambda, t), \lambda, t) = c.
\]

From this last equation and from the implicit function theorem we have that \( U_c \) is two times continuously differentiable with respect to \( \lambda \).

Now we will show that any regular equilibrium of our economy can always be supported by a representative agent with the following state-dependent utility function

\[
U(c, \lambda) = E \left[ \int_0^T u(c(t), \lambda(t), t) \, dt \right],
\]

with \( u \) given by (30) and \( \lambda \) being an adapted process.
Proposition 2  Suppose that \((P, \bar{c}_1, \ldots, \bar{c}_k)\) is a regular equilibrium for the economy \(E\). Then there exists a continuous and adapted process \(\lambda\) such that the aggregate income process \(e\) maximizes \(U(c, \lambda)\) over \(B\left(\sum_{k=1}^{K} e_k, P\right)\) and the equilibrium policies \((\bar{c}_1, \ldots, \bar{c}_k)\) solve the representative agent's allocation problem in (30) with \(c = e(t)\) and \(\lambda = \lambda(t)\) for all \(t \in [0, T]\). With

\[
\lambda_k(t) = \frac{\psi_1 H_{\nu_1}(t)}{\psi_k H_{\nu_k}(t)}, \quad \forall \ k \in \{1, \ldots, K\},
\]

(34)

where \(H_{\nu_k}\) denotes the minimax state-price density for agent \(k\).

Proof

First we verify the feasibility of \(e\), take \(\alpha = \theta = 0\) in (7)-(9). Then we have \(e \in B\left(\sum_{k=1}^{K} e_k, P\right)\).

Now the optimality of \(e\), let \(c_1, \ldots, c_K\) be arbitrary non-negative processes with \(c = \sum_{k=1}^{K} c_k \in B\left(\sum_{k=1}^{K} e_k, P\right)\). By Lemma 1 we have

\[
E \left[ \int_0^T H_{\nu_1}(t) \left( \sum_{k=1}^{K} c_k(t) - \sum_{k=1}^{K} e_k(t) \right) dt \right] \leq 0.
\]

(35)

Then taking \(\lambda\) as in (34)

\[
U(e, \lambda) - E \left[ \int_0^T \left( \sum_{k=1}^{K} \lambda_k u_k(c_k(t), t) \right) dt \right]
\]
Equilibrium in stochastic economies with incomplete financial markets

\[ \geq E \left[ \int_0^T \left( \sum_{k=1}^K \lambda_k(t) u_k(\bar{c}_k(t), t) \right) dt \right] - E \left[ \int_0^T \left( \sum_{k=1}^K \lambda_k(t) u_k(c_k(t), t) \right) dt \right] \]

we obtained the first inequality from the definition of \( U \) and the market clearing condition, i.e. \( \sum_{k=1}^K \bar{c}_k = e \), the second one follows from the optimality of \( \bar{c}_k \) and (24) and the last one from (35) and the market clearing condition above. Since \((c_1, c_2, \ldots, c_K)\) were arbitraries we have

\[ U(e, \lambda) \geq U(c, \lambda), \quad \forall c \in B \left( \sum_{k=1}^K e_k, \mathcal{P} \right), \]

hence the optimality of \( e \). Now, since in our model the first order conditions are necessary and sufficient; and by strictly concavity, we have a unique solution; if \((c_1, \ldots, c_K)\) is the solution of (30) then the first order condition

\[ \frac{u_{1c}(c_1, t)}{u_{kc}(c_k, t)} = \frac{\lambda_k(t)}{u_{kc}(\bar{c}_k, t)} = \frac{u_{1c}(\bar{c}_1, t)}{u_{kc}(\bar{c}_k, t)} \]
where the second equality follows from the regularity of the equilibrium, the last equation implies \((c_1, \ldots, c_K) = (\tilde{c}_1, \ldots, \tilde{c}_K). \square\)

The following result shows how the risk tolerance coefficients are related in equilibrium

**Corollary 1** The representative agent’s Arrow-Pratt coefficient of absolute risk tolerance at the aggregate consumption is the sum of the risk tolerance coefficients of each agent at their optimal consumption policies:

\[
\frac{u_c(e(t), \lambda(t), t)}{u_{cc}(e(t), \lambda(t), t)} = - \sum_{k=1}^{K} \frac{u_c(\tilde{c}_k(t), t)}{u_{cc}(\tilde{c}_k(t), t)}
\]

**(36)**

**Proof**

First, we have to differentiate (33) with respect to \(c\) and use (32). Finally, by the second claim of the last proposition, we have that \((\tilde{c}_1, \ldots, \tilde{c}_K)\) solves the representative agent’s allocation problem, then using the first order conditions we obtain the result. \(\square\)


In the last section we have constructed a representative agent for our economy \(\mathcal{E}\) by using an adapted process \(\lambda\), now we will use this process to characterize the equilibrium of \(\mathcal{E}\). And finally we will give sufficient conditions for every process with such characterization to be an equilibrium.

By (17) and the definition of \(H_{\nu_k}\) we have

\[
dH_{\nu_k}(t) = H_{\nu_k}(t) \left(-r(t)dt + \eta'_{\nu_k}(t)dW(t)\right),
\]

**(37)** by using Ito’s lemma, we have that the process \(\lambda_k\) defined in (34) solves the following stochastic differential equation for all \(k \in \{1, \ldots, K\}\).
\begin{align*}
\lambda_k(t) &= -\nu'_k(t) (\nu_1(t) - \nu_k(t)) \lambda_k(t) dt + (\nu_1(t) - \nu_k(t))' \lambda_k(t) dW(t) \\
& \quad \quad \text{And to facilitate the notation, let } \mathcal{G} \text{ denote the following operator} \\
(\mathcal{G}u)(c, \lambda, t) &= \mu(t) u(c, \lambda, t) - \sum_{k=1}^{K} \nu'_k(t) (\nu_1(t) - \nu_k(t)) \lambda_k(t) u_{\lambda_k}(c, \lambda, t) \\
& \quad \quad + \frac{1}{2} u_{cc}(c, \lambda, t) |b(t)|^2 + \sum_{k=1}^{K} b(t) (\nu_1(t) - \nu_k(t))' \lambda_k u_{c\lambda_k}(c, \lambda, t) \\
& \quad \quad + \frac{1}{2} \sum_{k,j=1}^{K} (\nu_1 - \nu_k)' (\nu_1 - \nu_j) \lambda_k \lambda_j u_{\lambda_k \lambda_j}(c, \lambda, t)
\end{align*}

Then we have the following characterization

**Theorem 2** Suppose that \((P, \bar{c}_1, \ldots, \bar{c}_K)\) is a regular equilibrium for the economy \(E\). Define the representative agent’s utility and the process \(\lambda\) as in the Proposition 2. Then the equilibrium prices system \(P = (r, b)\) and consumption policies are given in terms of \(\lambda\) by

\begin{align*}
r(t) &= -\mathcal{G} u_c(e(t), \lambda(t), t) + u_{ct}(e(t), \lambda(t), t) u_c(e(t), \lambda(t), t), \\
b(t) &= r(t) 1 - \frac{u_{cc}(e(t), \lambda(t), t)}{u_c(e(t), \lambda(t), t)} \sigma(t) \rho(t),
\end{align*}
\[ \bar{c}_k(t) = f_k \left( \frac{u_c(e(t), \lambda(t), t)}{\lambda_k(t)}, t \right), \quad (41) \]

And the minimax state-price densities of the \( K \) agents are related in equilibrium by

\[ \sum_{k=2}^{K} (\nu_1(t) - \nu_k(t)) \lambda_k(t) u_{c\lambda_k}(t) = u_c(t) \nu_1(t) - u_{cc}(t) L(t) \rho(t), \quad (42) \]

where \( L(t) = I - \sigma'(t)(\sigma(t)\sigma'(t))^{-1} \sigma(t) \) and \( I \) denotes the \( n \times n \) identity matrix.

Proof

First, it is easy to see that (41) follows from Proposition 2 and (31). Now, since \( \lambda_1(t) = 1 \), we have

\[ u_c(e(t), \lambda(t), t) = u_{1c}(\bar{c}_1(t), t) = \psi_1 H_{\nu_1}(t). \]

Then by (37), we obtain

\[ du_c(t) = -r(t) u_c(t) dt + u_c(t) \eta_{\nu_1}'(t) dW(t) \]

where we put \( u_c(t) \) instead of \( u_c(e(t), \lambda(t), t) \) and the same will be done in the sequel. Now by Assumption 3 and (38) we know that \( e(t) \) and \( \lambda(t) \) are Itô processes. Then applying Itô's lemma, we have

\[ du_c(t) = (G u_c(t) + u_{ct}(t)) dt \]

\[ + \left( u_{cc}(t) \rho(t) + \sum_{k=1}^{K} (\nu_1(t) - \nu_k(t)) \lambda_k(t) u_{c\lambda_k}(t) \right) dW(t) \]
Equilibrium in stochastic economies with incomplete financial markets

By matching the terms, we obtain

\[-r(t)u_c(t) = G u_c(t) + u_{ct}(t)\]

\[u_c(t)\eta_1(t) = u_{cc}(t)\rho(t) + \sum_{k=1}^{K} (\nu_1(t) - \nu_k(t))\lambda_k(t)u_{c\lambda_k}(t)\]

Then (39) follows from the first equality and by decomposing the second one, we obtain

\[u_c(t)\eta_0(t) + u_c(t)\nu_1(t) = u_{cc}(t)\Gamma(t)\rho(t)\]

\[+ \left[u_{cc}(t)L(t)\rho(t) + \sum_{k=1}^{K} (\nu_1(t) - \nu_k(t))\lambda_k(t)u_{c\lambda_k}(t)\right]\]

where \(\Gamma(t) = o'(t)(o(t)o'(t))^{-1}o(t)\). By the definition of \(\mathcal{N}\) and \(L\), we have that the second terms in each member of the equation belong to \(K(o)\) and by (10), we have that the first terms belong to \(S(o)\). By matching terms and using the definition of \(\eta_0\), we obtain

\[-u_c(t)(b(t) - r(t)1) = u_{cc}(t)\sigma(t)\rho(t)\]

\[u_c(t)\nu_1(t) = u_{cc}(t)L(t)\rho(t) + \sum_{k=1}^{K} (\nu_1(t) - \nu_k(t))\lambda_k(t)u_{c\lambda_k}(t)\]

these two equations give us (40) and (42). \(\square\)

We have seen that the real interest rate is equal to minus the expected rate of growth of the representative agent’s marginal utility of consumption and the equilibrium excess rate of return on risky securities is proportional to the instantaneous conditional covariance.
between changes in consumption and security returns. These relations establish the Consumption based Capital Asset Pricing Model (CCAPM) of Breeden (1979) for our economy. However Breeden’s result assumes Markovian equilibria and the existence of smooth solutions for Bellman equations. Duffie and Zame (1989) and Karatzas, Lehoczky and Shreve (1990) obtained this CCAPM for complete markets, results with incomplete markets have been obtained by Grossman and Shiller (1982) and Back (1991). Our results give sufficient condition for the CCAPM to hold in equilibrium for incomplete markets.

Now we will solve the following question: Provided that the process $\lambda$ satisfies (38) and (42) for some $\nu = (\nu_1, \ldots, \nu_k) \in (\mathcal{L}^2)^K$ and appropriate initial condition $\lambda(0)$, when will a process such (39)-(41) be an equilibrium for our economy $\mathcal{E}$? To solve this question, let us change our variables: $\lambda_k(t) = e^{\beta_k(t)}$ with this, we guarantee $\lambda_k(t) \in (0, \infty)$. By applying Ito’s lemma, we obtain

$$d\beta_k(t) = -\frac{1}{2}(|\nu_1(t)|^2 - |\nu_k(t)|^2)dt + (\nu_1(t) - \nu_k(t))'dW(t)$$

For a given process $\nu_1 \in \mathcal{N}$, define the price system $\mathcal{P}_{\nu_1}$ by (39)-(40). Now, doing $\beta = (\beta_1, \ldots, \beta_K)$, we could write the stochastic differential equations of $S = (P, \beta)$ as follows

$$dS(t) = b_{\nu_1}(P(t), \beta(t), t, w)dt + \sigma_{\nu_1}(P(t), \beta(t), t, w)dW(t)$$

(43)

Since coefficients $b_{\nu_1}$ and $\sigma_{\nu_1}$ are continuous in their first two arguments. By a theorem of stochastic differential equations\(^\text{12}\) we

\(^{12}\)See Protter (1990), Theorem 7.38.
Equilibrium in stochastic economies with incomplete financial markets

know that the solution of (43) is unique and exists up to an explosion time depending on the initial condition $\beta(0)$.

Now we respond our question

**Theorem 3** Suppose that there exist $\beta(0) \in \mathbb{R}^K$, $\nu_1 \in \mathcal{N}$, and $\lambda$ satisfy (38) and (42) for some $\nu = (\nu_1, \ldots, \nu_K) \in (L^2)^K$ such that

(a) The equation (43) has a strong solution on $[0, T]$ and

$$
E \left[ \int_0^T u_c(e(t), \lambda(t), t) \left( f_1(u_c(e(t), \lambda(t), t), t) - e_1(t) \right) dt \right] = 0,
$$

(b) for all $k$ the consumption policy $\bar{c}_k$ of (41) is marketed for the price system $\mathcal{P}_{\nu_1}$

(c) the interest rate satisfy (13)

If $\forall k \in \{2, \ldots, K\}$, $E[Z_{\nu_k}(T)] = 1$ and $\sigma(t)\nu_k(t) \geq 0$, $\forall t \in [0, T]$, then $\nu_k \in \mathcal{N}$, $\forall k \in \{2, \ldots, K\}$ and the price system $\mathcal{P}_{\nu_1}$ with the consumption policies in (41) define a regular equilibrium for the economy $E$.

**Proof**

To prove our theorem we will show that the consumption policies $\bar{c}_k$ of (41) satisfy the market clearing condition for goods. By (41) and (33), we obtain

$$
\sum_{k=1}^K \bar{c}_k(t) = \sum_{k=1}^K f_k \left( \frac{u_c(e(t), \lambda(t), t)}{\lambda_k(t)}, t \right) = f \left( u_c(e(t), \lambda(t), t), t \right) = e(t)
$$

Now to see the optimality of this consumption policies, observe that by Ito’s lemma and the choice of $\mathcal{P}_{\nu_1}$:
\[ du_c(t) = (G u_c(t) + u_{ct}(t)) \, dt \]
\[ + \left( u_{cc}(t) \rho(t) + \sum_{k=1}^{K} (\nu_1(t) - \nu_k(t)) \lambda_k(t) u_c \lambda_k(t) \right) \, dW(t) \]

Now by (42), we have

\[ u_{cc}(t) \rho(t) + \sum_{k=1}^{K} (\nu_1(t) - \nu_k(t)) \lambda_k(t) u_c \lambda_k(t) \]
\[ = u_{cc}(t) \rho(t) + u_c(t) \nu_1(t) - u_{cc}(t) L(t) \rho(t) \]

from the definition of \( L \)

\[ = u_{cc}(t) \Gamma(t) \rho(t) + u_c(t) \nu_1(t) \]

By (40)

\[ = -u_c(t) \sigma'(t) (\sigma(t) \sigma'(t))^{-1} (b(t) - r(t) 1) + u_c(t) \nu_1(t) = u_c(t) \eta_{\nu_1} \]

Now from (39) we obtain

\[ du_c(t) = -r(t) u_c(t) \, dt + u_c(t) \eta'_{\nu_1}(t) \, dW(t) \]

then \( u_c(t) = u_c(0) Z_{\nu_1} \) and since \( \bar{c}_1 \) is feasible, by theorem 1 we have that \( \bar{c}_1 \) is optimal.

Now by applying \( \sigma \) to (42), since \( \sigma L = 0 \) and \( \sigma \nu_1 = 0 \), we obtain

\[ \sum_{k=2}^{K} \sigma(t)(\nu_k(t)) \lambda_k(t) u_c \lambda_k(t) = 0 \]
Equilibrium in stochastic economies with incomplete financial markets

And by the change of variable we know that $\lambda_k(t) > 0$ and by (33), we have $u_c\lambda_k = -\frac{f_{\lambda_k}}{f_c} > 0$. With the assumptions, it implies $\nu_k \in K(\sigma)$, $\forall k \in \{2, ..., K\}$. Now from the fact that $Z_{\nu_k}$ is a non-negative local martingale, and hence a supermartingale, the condition $E[Z_{\nu_k}(T)] = 1$ is sufficient to conclude that $Z_{\nu_k}$ is a uniformly integrable martingale, then $\nu_k \in \mathcal{N}$, $\forall k \in \{2, ..., K\}$. And by using (38) and Ito's lemma, we have

$$d\left(\frac{u_c(t)}{\lambda_k(t)}\right) = -r(t)u_c(t)dt + u_c(t)\eta_{\nu_k}'(t)dW(t)$$

It implies that

$$u_{kc}(\tilde{c}_k(t), t) = \frac{u_c(t)}{\lambda_k(t)} = \frac{u_c(0)}{\lambda_k(0)} Z_{\nu_k}.$$ 

Since $\frac{u_c(0)}{\lambda_k(0)} > 0$ and $\tilde{c}_k(t)$ is feasible, we have by theorem 1 that $\tilde{c}_k(t)$ is optimal $\forall k \in \{2, ..., K\}$.

Since $\nu_k \in \mathcal{L}^2$, $\forall k \in \{1, ..., K\}$, by continuity the integrability condition (12) is satisfied and to verify that the price system $\mathcal{P}_{\nu_1}$ is admissible, we have to verify only (14). To achieve this goal, observe that by using (40)

$$|\eta_0(t)| = |\sigma'(t)(\sigma(t)\sigma'(t))^{-1}\sigma(t)\rho(t)\frac{u_{cc}(t)}{u_c(t)}| \leq K_\sigma |\frac{u_{cc}(t)}{u_c(t)}| |\rho(t)|.$$ 

Now from (36) and based on the fact that (6) implies that the aggregate consumption is bounded from above and below, we have that $u_{cc}(t)/u_c(t)$ is uniformly bounded, finally from the boundeness of $\rho(t)$, we obtain (14). $\square$
7. Conclusions.

We have shown that it is possible to construct a representative agent for a stochastic economy with incomplete markets and a finite number $K$ of agents; with this representative agent we have analyzed the characterization and existence of equilibria. Many extensions of this model can be easily presented, as introducing jumps or dividends into asset prices. An interesting model considering restrictions on stock-market participation and using the same techniques of this paper is based on Başak and Cuoco (1998), who consider just two agents.

In order to make our model more realistic, the equilibria analyzed in this paper should be extended to models with market frictions, (e.g. short-sale constraints, transactions cost), since Shreve and Xu (1992a,b) and Cvitanić and Karatzas (1996) considered that they can solve the optimal consumption and investment problem. For a methodology that solves the competitive equilibria of economies with dynamically incomplete markets and heterogeneous agents we refer the reader to Dumas and Maenhout (2002).

A more difficult problem is the analysis of existence of equilibria that allow the agents to dishonor their commitments. Araujo, Monteiro and Páscoa (1998) solve this problem in a discrete time setting and we would like to extend this result to continuous time setting in a future work, using the results presented in these papers.

8. Appendix.

Ito's Lemma

Suppose that $\{W_t\}_{t \geq 0}$ is a standard Brownian Motion and $X$ is an Ito process, i.e. $dX_t = \mu_t dt + \sigma_t dW_t$, and let $f : (0, \infty) \times [0, T] \rightarrow \mathbb{R}$ be such that $f \in C^{2,1}( (0, \infty) \times [0, T] )$. Then the process $Y_t = f(X_t, t)$, is an Ito process with
Equilibrium in stochastic economies with incomplete financial markets

\[ dY_t = \left[ f_x(X_t, t)\mu_t + f_t(X_t, t) + \frac{1}{2} f_{xx}(X_t, t)\sigma_t^2 \right] dt + f_x(X_t, t)\sigma_t dW_t \]

This expression is known as Ito’s formula.

**Minimax State Price Density**

The next results, which is due to He and Pearson (1991) and Karatzas, Lehoczky, Shreve and Xu (1991) for the case of no income stream, provides a characterization of such process.

Denote the convex conjugate of \( u_k \) by \( \hat{u}_k(y, t) \), i.e.

\[ \hat{u}_k(y, t) \equiv \max_{c > 0} [u_k(c, t) - yc] = u_k(f_k(y, t), t) - yf_k(y, t) \]

and for any \((\psi, \nu) \in (0, \infty) \times \mathcal{N}\) define

\[ J_k(\psi, \nu) = E \left[ \int_0^T \hat{u}_k(\psi H_\nu(t), t) dt + \psi \int_0^T H_\nu(t) e_k(t) dt \right] , \]

The above expectation is well defined, since we have from (23)

\[ E \left[ \int_0^T \hat{u}_k(\psi H_\nu(t), t)^- dt \right] \leq \int_0^T u_k(1, t)^- dt + \psi E \left[ \int_0^T H_\nu(t) dt \right] < \infty \]

**Theorem 4** If \((\psi_k, \nu_k) \in (0, \infty) \times \mathcal{N}\) solves the problem
\[ \inf_{\psi \in (0, \infty)} \inf_{\nu \in \mathcal{N}} J_k(\psi, \nu), \]  

(45)

and

\[ E \left[ \int_0^T H_{\nu_k}(t) \left( f_k(\psi_k H_{\nu_k}(t), t) - e_k(t) \right) dt \right] < \infty \]  

(46)

then \( c_{\nu_k} \) is optimal for agent \( k \) and the optimal wealth process is given by \( X_{\nu_k} \). In particular \( H_{\nu_k} \) identifies the minimax state-price density for agent \( k \).

The next result is due to Cuoco and He (1994) and gives sufficient conditions for assumptions of the previous theorem to hold.

**Theorem 5** Assume that

a) \( u_k(\infty, t) = \infty \) for all \( t \in [0, T] \) and \( u_k(c, t) \leq k(1 + c^{1-b}) \) on \( (0, \infty) \times [0, T] \) for some \( k \geq 0, b \geq 1 \);

b) \( y/B > \epsilon (1 \times \mathbb{P}) - \text{a.e. for some } \epsilon > 0 \);

c) \( \forall \psi \in (0, \infty), \exists \nu \in \mathcal{N} \text{ such that } J_k(\psi) \),

then the minimum in (45) is attained and hence a minimax state-price density for agent \( k \) exists. If in addition

d) \( cu_{k,c}(c, t) \leq a + (1 - b)u_k(c, t) \) on \( (0, \infty) \times [0, T] \) for some \( a \geq 0, b > 0 \),

then the condition (46) is also satisfied, and hence there exists an optimal consumption/investment policy for agent \( k \).


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Brazilian Review of Econometrics 22 (1) May 2002 99
Equilibrium in stochastic economies with incomplete financial markets


