A Dynamic Price-Setting Mechanism for a Hybrid Matching Market

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Abstract

After observing the problem of the intimate wear market of Nova Friburgo with the distribution of its product both in Brazil and abroad, we designed a dynamic allocation mechanism that sets the prices according to the demand from the buyers, based on their preferences, and that yields an allocation for the core of the market game in a finite number of steps. With this scheme, all agents are simultaneously present and all sellers sell to both national and international markets. We prove that the core allocation produced by this mechanism provides the lowest price among all outcomes for the core that maintains the same allocations of objects for international buyers as the final allocation. In addition, it coincides with the competitive allocation of minimum price equilibrium, when restricted to the national market, and with the allocation produced by the Gale-Shapley algorithm (1962), in which buyers make proposals and where there is a convenient tie-breaking rule, when all buyers are from abroad.

Key Words: stable allocation, core, competitive price.

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Resumo

Motivados pela problemática detectada no pólo de confecções de moda íntima de Nova Friburgo quanto à distribuição do seu produto, tanto a nível nacional como internacional, desenhamos um mecanismo centralizado de alocação que usa as preferências dos compradores e os preços para exportação fixados exogenamente como input, e cujo output é uma alocação do núcleo do jogo de mercado. No novo esquema, todos os agentes estão presentes ao mesmo tempo e todos os vendedores vendem para ambos os mercados, nacional e internacional. Provamos que a alocação produzida pelo mecanismo não somente está no núcleo do jogo de mercado, mas dá o menor preço dentre todos os resultados estáveis que mantém as mesmas alocações dos objetos para os compradores internacionais que a alocação final. Além disso, ela coincide com a alocação competitiva de equilíbrio de preço mínimo quando restrito ao mercado nacional e com a alocação produzida pelo algoritmo de Gale e Shapley (1962), com os compradores propondo e uma conveniente regra de desempate, quando todos os compradores são internacionais.

1. Introduction.

The present study focuses on the problem detected by Isleide R. Maeda (2003) in her field research, regarding the current organization of the intimate wear market of Nova Friburgo.

Companies, mostly the small-sized ones, choose between selling to national or international buyers, through a decentralized market mechanism. The buyers get information on the product and choose from the samples the companies provide them with. A wholesale order corresponds to a lot. A lot consists of several different models, with a large amount of items. Each clothing item has its own set of models, thus the lots differ in terms of items and their amount. To meet the demand from the wholesale sector the clothing factories have to devote themselves exclusively to the production of the requested lot, for some time, due to their restricted size.

Since objects are heterogeneous, the traditional model for this market is structured as a general multi-market equilibrium model.
However, as markets are thin, the appropriate modeling has consisted in treating all markets together as a single market game, in which the terms of partnerships are determined endogenously, along with the allocation of objects, through negotiations between prospective partners. Thus, we may regard the market as a market of multiple indivisible objects, where each seller owns only one object and each buyer is interested in purchasing at most one object.

Outside Brazil there are other markets where a buyer can import products similar to the ones negotiated in the intimate wear market of Nova Friburgo. The prices of the products in these markets are obtained in the Stock Market, where the goods are negotiated and international sales prices are set. These prices are accepted by international buyers and are called spot prices. This way, for international buyers of the intimate wear market of Nova Friburgo, the prices attributed to a lot are the same for all buyers. In the national market, however, the prices are negotiated between the involved agents.

In practice, the sellers have attempted to organize themselves into export consortia, and the agents operate in two disjoint markets. This has some disadvantages, since the pooled companies are reluctant to give up their autonomy and many sellers wish to negotiate with both types of buyers.

This makes us think of a single buying and selling market in which all agents could negotiate freely, preserving some of their original features, such as the sales to national and international buyers, fixed prices for the international market and negotiable prices for the national one.

Allowing all sellers to negotiate in the international market offers great advantages. Companies are encouraged to improve their product so as to meet the demands of the international market. Obviously, those companies that are not truly qualified will not have their products accepted by international buyers and will consequently op-
erate only in the national market. International buyers will compete with national ones for the products that are acceptable to them. The difference between these two types of agents lies in the fact that international buyers pay the prices preset by the international market, where these prices are competitive. In the market of Nova Friburgo, these prices are not competitive (if two of these buyers have the same preference over the available lots, and if these preferences are strict, then their favorite object will be overdemanded, causing an imbalance between supply and demand). On the other hand, for national buyers, prices are competitively readjusted according to supply and demand.

The modified market has some of the characteristics of the hybrid model developed by Eriksson and Karlander (2000), inspired in the unified model due to Roth and Sotomayor (1996). It is analyzed in Sotomayor (2000), in the context of firms and workers. Presumably all agents are simultaneously present in the market and are well informed. Thus, the appropriate solution concept is that of stable allocation: once a transaction is made between a buyer and a seller, none of them may establish a new agreement with another market participant after reneging on the current one, in such a way that both will benefit from the new agreement. For this market, this concept is equivalent to that of core allocation, which satisfies the properties of fairness and efficiency.

The idea is then to design a centralized mechanism that generates stable allocations for the hybrid matching market proposed here. Instead of a sealed bid auction in which buyers submit their valuation on each object, we propose a dynamic mechanism which produces a stable allocation in a finite number of steps, and captures the interactions between the existing agents in real markets. In every step, the buyers submit their demands according to the prices announced and readjusted by the auctioneer. National buyers demand
all objects that will maximize their surplus, which is the difference between the highest price they are willing to pay for the object and the announced price; with regard to international buyers, the prices are fixed and coincide with their value for the object. Therefore, their demands are determined by their preferences over the various acceptable products and will be defined in the text. We should note that all sales for both types of buyers occur by means of the centralized mechanism. Thus, as in usual auctions, there is no direct negotiation between buyers and sellers.

Without international buyers, the market model is the well known Assignment Game of Shapley and Shubik (1972) in which the negotiated prices may vary continuously in the set of real numbers. In this market, the concept of stable allocation coincides with that of competitive equilibrium: each buyer receives an object from her demand set and the price of each unsold object is equal to the reservation price. By using linear programming techniques, these authors proved not only that the set of stable allocations is nonempty, but also that there is a stable allocation able to provide each buyer (respectively, seller) the highest payoff obtainable in any stable allocation - optimal stable allocation for buyers (respectively, seller) or minimum (respectively, maximum) competitive equilibrium price. We proved that our mechanism, when restricted to the national market, produces minimum competitive equilibrium price and coincides with the dynamic mechanism proposed by Demange, Gale and Sotomayor (1986).

The market model for the case in which all buyers are international is the Marriage Market, by Gale and Shapley (1962). In this market, the prices are fixed and regarded as one of the factors that determine the preferences of buyers for the products. Gale and Shapley proved, through an algorithm, and by using only combinatorial arguments, the existence of stable allocations and of an optimal
stable allocation for each side of the market\textsuperscript{2}. We showed that, limited to this market, the mechanism outlined herein coincides with the Gale-Shapley algorithm (1962), in which the buyers make the offers their valuations. We cannot claim that the final outcome for the hybrid market consists of the buyers’ favorite stable allocation, as in this market such allocation does not always exist, due to the indifferences allowed by the model (every seller is indifferent between two international buyers). Nevertheless, we showed that the final allocation is always stable and yields the lowest price among all the stable outcomes that maintain the same allocations of objects for international buyers as the final allocation does.

The mechanism developed herein applies to a wide range of other markets. In the car market, for instance, every seller usually has one car for sale and every buyer wishes to buy only one car. The prices of second-hand cars are negotiable while those for new cars are non-negotiable, closely resembling list prices.

The labor market is another application. An example of this is the market for academic positions and professors. Every university offers a position for full-time head professor, thus preventing each applicant from taking any other job. The salaries paid by some universities (e.g.: Brazilian public universities) are fixed, whereas those paid by other universities (e.g.: American private universities) are negotiable. In practice, a professor may want to negotiate with both types of universities at the same time, which characterizes the hybrid market. Managing positions are usually taken by a single person. This way, it is reasonable that each firm will offer one managing position, and that applicants will only accept one job. When the position is concerned with a government office, the salary is not negotiable.

\textsuperscript{2}A non-constructive demonstration for the existence theorem is due to Sotomayor (1996).
Theoretically, the results shown in the present article represent, in our point of view, a significant contribution to the yet underexplored development of hybrid matching games, and are an unusual generalization of the Gale-Shapley algorithm (1962) and of the Demange, Gale and Sotomayor mechanism (1986).

Aside from the introduction, the paper is organized as follows. Section 2 presents the formal model. Section 3 describes the dynamic price-setting mechanism, shows the properties mentioned above and gives an example of its operation. Section 4 discusses related works and open questions.


2.1 Description of the Hybrid Market Model.

The model presented herein is adapted from the model unified by Sotomayor (2000). The concept of quasi-stability is introduced here.

The organization of the market of Nova Friburgo as a hybrid market is described by two finite and disjoint sets of agents, $P$ and $Q : P = \{p_1, p_2, \ldots, p_i, \ldots, p_m\}$ is the set of buyers and $Q = \{q_1, q_2, \ldots, q_j, \ldots, q_n\}$ is the set of sellers. For simplicity, we will use the same notation for a seller and his object. Each buyer is interested in buying at most one object and each seller has only one object available for sale. The set of buyers consists of two disjoint subsets $P^I$ and $P^N$, such that $P^I \cup P^N = P$. $P^I$ is the set of buyers of the international market. $P^N$ is the set of national buyers. For each object $q_j$ there is a number $b_j \geq 0$. The number $b_j$ represents the sales price of the object $q_j$ for international buyers. For each buyer $p_i \in P^N$ there is a number $\alpha_{ij}$ that represents the valuation of object $q_j$ by buyer $p_i$. Therefore, if object $q_j$ is sold to $p_i \in P^N$ for price $v_j$, the payoff of $p_i$ will be $u_i = \alpha_{ij} - v_j$ and the payoff of seller $q_j$ will...
be given by \( v_j \). For simplification of the model, we are normalizing, considering the reservation price of each seller for national buyers as being zero and \( \alpha_{ij} \geq 0 \) for all \( p_i \in P^N \). We will assume that \( Q \) contains a null object, \( q_0 \), with \( \alpha_{i0} = b_0 = 0 \) for all \( p_i \in P^N \). Object \( q_0 \) may be allocated to any amount of buyers.

Every buyer \( p_i \in P^I \) lists the objects she would like to buy in a strict order of preference. Thus, the list of preferences of buyer \( p_i \) given by \( L(p_i) = q_h, q_j, ..., q_0, q_p \) means that \( p_i \) prefers \( q_h \) to \( q_j \). Object \( q_p \) is unacceptable to \( p_i \), such that \( p_i \) prefers not to buy any object to buying \( q_p \). Obviously, a seller is indifferent between two international buyers, since the price of his object is the same for both. According to the construction of the model, if the price of object \( q_j \) is higher than that of \( b_j \) it can only be sold to a national buyer.

In order to standardize the notation and simplify the formulations of concepts, it is sometimes more convenient to represent the utility payoff of an international buyer by the numerical values \( a_{ij} \). Therefore, \( p_i \) prefers object \( q_j \) to \( q_k \) if and only if \( a_{ij} > a_{ik} \) and \( q_j \) is acceptable to \( p_j \) if and only if \( a_{ij} \geq a_{i0} = 0 \). With this notation, if \( p_i \in P^I \) buys object \( q_j \) then her payoff will be \( u_i = a_{ij} \) and the seller’s payoff will be given by \( v_j = b_j \). For the same reasons, for each pair \( (p_i, q_j) \in P^N \times Q \), we will write \( a_{ij} = \alpha_{ij} - b_j \). Thus, the number \( a_{ij} + b_j \) will represent the gain from trade between \( p_i \in P \) and \( q_j \) if \( p_i \) buys object \( q_j \) at price \( v_j \). In this case, \( u_i + v_j = a_{ij} + b_j \).

We will set \( a_i \equiv (a_{i1}, ..., a_{in}) \), for all buyer \( p_i \). The hybrid market is then given by \( (P, Q, a, b) \), where \( a \) is the matrix of \( a_{ij} \)'s and \( b \) is the vector \( (b_1, ..., b_n) \). The market \( (P^N, Q, a^N, b) \) will be called purely national market, where \( a^N \) is the matrix involving only the payoffs of national buyers. The market \( (P^I, Q, a^I, b) \) will be called purely international market, where \( a^I \) is the matrix that corresponds to international buyers. By abusing notation and for
simplicity, we will use the same notation \( a \) for both markets, when there is no confusion. Given a price vector \( v \), we say that \( q_j \) is acceptable to \( p_i \) at prices \( v \) if \( a_{ij} + b_j - v_j \geq 0 \).

An allocation of objects for buyers will be denoted by \( x \). An allocation is feasible if no buyer is assigned to more than one object and no non-null object is assigned to more than one buyer. If object \( q_j \) is assigned to buyer \( p_i \) by \( x \) then we denote \( q_j = x(p_i) \) and \( p_i = x(q_j) \). If \( p_i \) is assigned to the null object then we may say sometimes that \( p_i \) is unallocated. Object \( q_0 \) may be allocated to any amount of buyers. Given a set \( S \) of agents that belong to the same side of the market, we denote \( x(S) \equiv \{ x(s) ; s \in S \} \).

**Definition 1.** An outcome is an allocation \( x \) of objects and a pair of vectors \((u, v)\) called payoff, with \( u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \). It will be denoted by \((u, v; x)\).

**Definition 2.** The outcome \((u, v; x)\) is feasible if \( x \) is feasible,

a) \( u_i \geq 0 \) and \( v_j \geq 0 \) for all \((p_i, q_j) \in P \times Q\);

b) \( u_i = 0 \) if \( p_i \) is not assigned by \( x \);

c) \( v_j = 0 \) if \( q_j \) is not allocated by \( x \);

d) \( u_i + v_j = a_{ij} + b_j \) if \( q_j \) is allocated to \( p_i \) by \( x \);

e) \( u_i = a_{ij} \) and \( v_j = b_j \) if \( q_j \) is allocated to \( p_i \) by \( x \) and \( p_i \in P^f \).

Condition a) (individual rationality) means that an agent has the option not to make any deal. Conditions b) and c) give the reservation payoffs to the agents. Conditions d) and e) require that the gain from trade be actually splitted between the partners.

A feasible allocation is a pair \((v, x)\), with \( v \in \mathbb{R}^n \) and \( x \) a feasible allocation of objects, such that for some \( u \in \mathbb{R}^m \), the corresponding outcome \((u, v; x)\) is feasible. In this case, we say that \( x \) is compatible with \( v \). An allocation \((v, x)\) is quasi-feasible if \( x \) is feasible and the corresponding outcome \((u, v; x)\) satisfies conditions...
a), b), d) and e). In other words, in a quasi-feasible allocation, an unallocated object may have a price greater than zero.

The solution concept for this type of game is that of stability.

**Definition 3.** The outcome \((u, v; x)\) is **stable** if it is feasible and for all \((p_i, q_j) \in P \times Q\) we have

a) \(u_i + v_j \geq a_{ij} + b_j\) if \(p_i \in P^N\) and

b) \(u_i \geq a_{ij}\) or \(v_j \geq b_j\) if \(p_i \in P^I\).

If a) or b) is not satisfied for some pair \((p_i, q_j)\), then these agents would do better if they could break their current transactions and make a new one together, since this would give them a higher payoff. In this case, we say that the pair blocks the outcome. Thus, an outcome is stable if it is feasible and has no blocking pair.

If \((u, v; x)\) is stable we say that allocation \((v, x)\) is **stable** and the payoff pair \((u, v)\) is a **stable payoff**. (Allocation \((v, x)\) is quasi-stable if it is quasi-feasible and if \((u, v; x)\) satisfies conditions a) and b) of Definition 3).

As pairs \((p_i, q_j)\) are the only essential coalitions, an outcome is stable if and only if it is in the core of the cooperative game induced by the market \((P, Q, a, b)\).

### 2.2.1 The International Market.

In this market, the set of buyers is \(P^I\) and the set of sellers is \(Q\). This market is a special case of the well-known Marriage Market, introduced by Gale and Shapley in 1962.

In this model the allocation \((v, x)\) is stable if each object assigned to a buyer is acceptable to her, each unsold object has zero price, and if no pair \((p_i, q_j) \in P^I \times Q\) exists such that \(q_j\) is not sold and \(p_i\) prefers \(q_j\) to the object assigned to her by \(x\) and \(b_j > 0\). When the condition that every unsold object
has zero price is relaxed, we obtain the concept of quasi-stable allocation.

The stable allocation preferred by all the agents on the same side of the market is not always available, since the sellers’ preferences are not strict.

2.2.2 The National Market.

In this market, the set of buyers is $P^N$ and the set of sellers is $Q$. According to our previous assumption, $\alpha_{ij} = a_{ij} + b_j \geq 0$ for all $(p_i, q_j) \in P^N \times Q$ and the reservation price of each seller is 0. This is the Assignment Game introduced by Shapley and Shubik (1972).

An outcome $(u, v; x)$ is feasible if it satisfies a), b), c) and d) of Definition 2. An outcome $(u, v; x)$ is stable if it is feasible and if it satisfies a) of Definition 3. Allocation $(v, x)$ is stable if $(u, v; x)$ is stable for some vector $u \in R^m$. Therefore, an allocation $(v, x)$ is stable if $v_j \geq 0$ for all $q_j \in Q, v_j = 0$ if $q_j$ is not allocated by $x$ and for all pair $(p_i, q_k), \alpha_{ij} - v_j \geq a_{ik} - v_k$, where $q_j$ is allocated to $p_i$ by $x$. That is, if $(v, x)$ is stable then each $p_j$ is maximizing the utility payoff at prices $v$ and every object not allocated by $x$ has price zero. This concept coincides with that of competitive equilibrium allocation in this market. In this case, $v$ is called a competitive equilibrium price vector. When the condition that every unsold object has zero price is relaxed, the resulting concept is that of a quasi-stable allocation or competitive allocation. In this case, we say that $v$ is a competitive price vector.

For this market there is always a stable outcome that is the most weakly preferred by all buyers and the least weakly preferred by all sellers (optimal stable outcome for buyers) and a stable outcome that is the most weakly preferred by sellers and the least weakly preferred by buyers (optimal stable outcome for sellers). The competitive equi-
librium allocations are respectively the minimum competitive price equilibrium and the maximum competitive price equilibrium.

3. Dynamic Price-Setting Mechanism.

This section describes the dynamic price-setting mechanism that produces a sales price for each product and the allocation of various lots to buyers, respecting the agents’ quotas: each object is allocated to at most one buyer and no buyer receives more than one object. We will call the operator of the mechanism “auctioneer”. Subsection 3.1 reveals that the final outcome is not reliant on the auctioneer’s choices and is always a stable allocation. Furthermore, this outcome restricted to the national market coincides with the minimum competitive price equilibrium, and when restricted to the international market, coincides with the allocation produced by the Gale-Shapley algorithm, in which all buyers submit their valuations, with a tie-breaking rule determined by the reindexation of international buyers established by a criterion selected by the auctioneer (e.g.: alphabetical order). In addition, the allocation produced yields the lowest price vector among all stable outcomes that maintain the same allocations of objects to international buyers as the final allocation does.

Some more notations and terminology are necessary. As $b_j$’s are determined exogenously, we may consider them known. Given a price vector $v$ and $p_i \in P^N$, $D_i(v)$ will denote the set of objects $q_j$ such that $a_{ij} + b_j - v_j \geq a_{ik} + b_k - v_k$ for every object $q_k$. In other words, the payoff of $p_i$ when she buys object $q_j \in D_i(v)$ at price $v_j$ is greater than or equal to the payoff of $p_i$ when she buys any other object $q_k$ at price $v_k$. There may be more than one object in $D_i(v)$. The null object is always an option for $p_i$, therefore $D_i(v) \neq \emptyset$ for all $p_i \in P^N$.

For international buyers, given a price vector $v$, we will use the
following procedure to compute $D_i(v)$ for all $p_i \in P^I$. Index the international buyers: $P^I = \{p_1, p_2, ..., p_r\}$. Consider the lists of preference of all buyers $p_i \in P^I$ over the objects. Remove all objects with prices $v_j > b_j$ from the list of each $p_i$. Also remove any object $q_j \neq q_0$ with price $v_j = b_j$ and which belongs to a set $S$ such that the set $T$ of national buyers with $D_h(v) \subseteq S$ for all $p_h \in T$, is such that $|T| \geq |S|$ and $S$ does not have any proper subset that contains $q_j$, with this property. We will call $S$ a minimal quasi-overdemanded set. The new lists of preferences will be referred to as reduced lists. We will define $D_1(v)$ as the favorite object of $p_1$ according to its reduced list. Remove the object of $D_1(v)$ from the lists of the other buyers. We will define $D_2(v)$ as the favorite object of $p_2$ in the resulting list. Remove $D_2(v)$ from the lists of $\{p_3, ..., p_r\}$. We will define $D_3(v)$ as the favorite object of $p_3$ in the resulting list, and so on and so forth. Thus, $D_i(v)$ is the favorite object of $p_i$ in the resulting list after the objects of $D_1(v) \cup D_2(v) \cup ... \cup D_{i-1}(v)$ are removed from her reduced list.

Observe that the null object is never removed from the list of international buyers, so $D_i(v) \neq \emptyset$ for all $p_i \in P^I$.

Abusing language, we will call $D_i(v)$ a demand set of buyer $p_i$ at prices $v$, although international buyers are not necessarily maximizing their payoffs in this set.

Given a set $S$ of objects, the set $T$ of buyers with $D_i(v) \subseteq S$ for all $p_i \in T$ is called set of loyal demanders of $S$. A set $S$ of objects is overdemanded if it has cardinality lower than its set of loyal demanders. An overdemanded set is minimal if it does not have any proper overdemanded subset.

To describe the mechanism, we will use the following result known in the Graph Theory, subdivision of the Combinatorial Theory. Let $B$ and $C$ be two finite and disjoint sets, for instance, buyers and objects, respectively. For each $p_i$ in $B$, let $D_i$ be a subset of $C$. 
A simple allocation is an allocation of objects to the buyers such that each buyer $p_i$ is exactly assigned to an object $q_j$ and $q_j$ is in $D_i$. In addition, each object is allocated to at most one buyer. (That is, a simple allocation allocates an object to every buyer, but may not assign every object to one buyer). So, if there exists a simple allocation, every buyer in all subset $B'$ of $B$ must be assigned to a different object, and there must therefore exist at least as many objects in $D(B') \equiv \bigcup_{B'} D_i$ as the number of buyers in $B'$. Hall’s theorem (1935) states that this necessary condition is also sufficient.

**Theorem 1** *(Hall’s theorem)* A simple allocation exists if and only if the number of objects in $D(B')$ is greater than or equal to the number of buyers in $B'$ for every subset $B'$ of $B$.

We will consider all numbers $a_{ij}$ and $b_j$ as integers. The unit of price could be, say, one hundred reais, etc. At the first stage of the mechanism each buyer specifies her demand set (including object $q_0$) at price $v(1) = (0, ..., 0)$ announced by the auctioneer.

**Stage $(t + 1)$**: After the demand sets are specified, the mechanism halts if it is possible to assign each buyer to an object in her demand set at price $v(t)$. The final price will be $v_j(t + 1) = v_j(t)$ if $q_j$ is allocated to a national buyer and will be $v_j(t + 1) = b_j$ if $q_j$ is allocated to an international buyer. If such an allocation does not exist, Hall’s theorem implies that there exists an overdemanded set of objects. The auctioneer selects a minimal overdemanded set, e.g. $S$, and the price $v_j(t + 1)$ of each object $q_j$ in the set is obtained as follows: if $v_j(t) \geq b_j$ then $v_j(t + 1) = v_j(t) + 1$ (recall that in this case $q_j$ is only demanded by elements of $P^N$ once $S$ is overdemanded); if $v_j(t) < b_j$ and $q_j$ has a loyal demander of $P^I$ then $v_j(t + 1) = b_j$. Otherwise, $v_j(t + 1) = v_j(t) + 1$. All the other prices remain unchanged.

Obviously, the mechanism comes to a halt at some stage, since as soon as the price of an object becomes higher than the valuation
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of any national buyer, then it will not be demanded by any national buyer, and as soon as the price of an object \( q_j \) is higher than \( b_j \) it will not be demanded by any international buyer. According to the construction of the mechanism, the allocation produced is quasi-stable. It is still unclear and not so simple to prove that it is possible for the auctioneer to allocate the objects in such a way that every unsold object has zero price and that the final price is lower than or equal to the price of any stable allocation that maintains the same allocations to international buyers as the final allocation does. For this reason, when \( P^I = \emptyset \) the final prices coincide with the minimum competitive prices and when \( P^N = \emptyset \) the allocation of objects is obtained using the Gale-Shapley algorithm with the buyers proposing and with a tie-breaking rule determined by the sequence obtained by reindexation of international buyers. These results will be shown in the next subsection.

### 3.1 Properties of the Mechanism.

The minimality of final prices obtained through the mechanism is shown in Theorem 1.

**Theorem 1.** Let \((v, x)\) be the allocation obtained through the dynamic price-setting mechanism. Therefore, for all \( q_j \in Q, v_j \leq v'_j \) for all quasi-stable allocation \((v', x')\) so \( x(p_i) = x'(p_i) \) for all \( p_i \in P^I \).

**Proof.** Suppose, by way of contradiction, that there is a quasi-stable allocation \((v', x')\) such that \( x(p_i) = x'(p_i) \) for all \( p_i \in P^I \) but \( v \) is not lower than or equal to \( v' \). At stage \( t = 1 \) we have \( v(1) = (0, \ldots, 0) \), so \( v(1) \leq v' \). Let \( t \) be the last stage of the mechanism in which \( v(t) \leq v' \) and consider \( S_1 = \{ q_j ; v_j(t + 1) > v'_j \} \). Let \( S \) be the minimal overdemanded set whose prices are raised at stage \( t = 1 \). Thus, \( S = \{ q_j ; v_j(t + 1) > v_j(t) \} \) and then \( S_1 \subseteq S \). Define \( T = \{ p_i ; D_i(v(t)) \subseteq S \} \). The fact that \( S \) is overdemanded means that
Now observe that if \( q_j \in S_1 \) and \( q_j \) is not demanded by any international buyer then \( v_j' = v_j(t) \), due to the fact that prices are integer-valued; if \( q_j \in S_1 \) and \( q_j \) is not demanded by any international buyer then \( v_j(t + 1) = b_j, v_j(t) < b_j \) and \( v_j' < b_j \). We will show that \( S - S_1 \) is nonempty and overdemanded and therefore \( S \) is not a minimal overdemanded set, contradicting the rules of the mechanism.

Define \( T_1 = \{ p_i \in T; \text{the set of objects in } S_1 \text{ demanded by } p_i \text{ at prices } v(t) \text{ is nonempty} \} \). We claim that \( D_i(v') \subseteq S_1 \) for all \( p_i \in T_1 \). In fact, take \( p_i \in T_1 \). Select \( q_j \in S_1 \cap D_i(v(t)) \). Let us first show that \( p_i \in P^N \). Actually, if \( p_i \in P^I \) then \( D_i(v(t)) = \{ q_j \} \), \( v_j(t) < v_j(t + 1) = b_j \) and \( v_j' < b_j \), so \( q_j \) is not allocated to \( p_i \) by \( x' \). As \( x(p_i) = x'(p_i) \) it follows that \( q_j \) is not allocated to \( p_i \) by \( x \). It also follows that \( p_i \) should be assigned to some object \( q_k \) by \( x \) and \( x' \), which she prefers to \( q_j \), otherwise the pair \( (p_i, q_j) \) would block the allocation \( (v', x') \). Therefore we should have \( v_k = b_k \) (recall that \( v \) is the final price), so \( v_k(t) \leq b_k \). On the other hand, as \( p_i \) prefers \( q_k \) to \( q_j \) and demands \( q_j \) at prices \( v(t) \), it follows that \( q_k \) had been removed from her list of preferences at stage \( t \).

Define \( A \equiv \{ p_h \in P^I; p_h \text{ prefers } x(p_h) \text{ to the object of } D_h(v(t)) \} \). We have \( A \neq \emptyset \), as \( p_i \in A \). Using the indexation of international buyers, let \( d \) be the lowest index of the elements in \( A \). Since \( p_{d'} \) prefers \( x(p_{d'}) \) to the object of \( D_{d'}(v(t)) \) it is implied that \( x(p_{d'}) = q_f \) for some object \( q_f \neq q_0 \), so \( v_f = b_f \). However, at stage \( t \), \( q_f \) is not demanded by any international buyer preceding \( p_{d'} \), because if this occurred for some \( p_{d''} \), with \( d' < d \), we would have \( p_{d''} \notin A \), by definition of \( p_{d'} \), so \( p_{d''} \) prefers \( q_f \) to \( x(p_{d''}) \) and \( v_f = b_f \), which goes against the rules of the mechanism. This means that at stage \( t \) the object \( q_f \) belongs to some minimal quasi-overdemanded set \( S' \), demanded by national buyers and has price \( v_f(t) = b_f \). Let \( T' \) be
the set of loyal demanders of $S'$. Then $|T'| \geq |S'|$. Now define $S'_1 = \{q_s \in S'; v_s(t) = v_s\}$ and $T'_1 = \{p_h \in T'; D_h(v(t)) \cap S'_1 \neq \emptyset\}$. Note that if $p_h \in T'_1$ then $D_h(v(t)) \subseteq S'_1$. In fact, if $q_s \in D_h(v(t)) \cap S'_1$ and $q_r \notin S'$ then $p_h$ prefers $q_s$ to $q_r$ at prices $v(t)$ as $p_h \in T'_1$, so $D_h(v(t)) \subseteq S'$. However, $v_r(t) \leq v_r$ and $v_s(t) = v_s$, then $p_h$ prefers $q_s$ to $q_r$ at price $v$. On the other hand, if $q_r \in S' - S'_1$ then $p_h$ likes $q_s$ at least as much as she likes $q_r$ at prices $v(t)$, but $v_r(t) < v_r$ (as $q_r \notin S_1$) and $v_s(t) = v_s$, then $p_h$ prefers $q_s$ to $q_r$ at prices $v$. Therefore $T'_1$ is a set of loyal demanders of $S'_1$ at prices $v$. Since $(v,x)$ is a quasi-stable allocation then $|T'_1| \leq |S'_1|$. We claim that $|T'_1| = |S'_1|$. In fact, if $|T'_1| < |S'_1|$ then $|T' - T'_1| > |S' - S'_1|$. However, $T' - T'_1 = \{p_h \in T'; D_h(t) \subseteq S' - S'_1\}$, which means that $S' - S'_1$ is overdemanded, contradicting the fact that $S'$ is minimal.

Now use the fact that $p_d \notin T'_1$, $p_d$ demands $q_f$ at prices $v$ (as $x(p_d) = q_f$) and $q_f \in S'_1$ to conclude that $T'_1 \cup \{p_i\}$ is a set of loyal demanders of $S'_1$ and $|T'_1 \cup \{p_i\}| > |S'_1|$, so $S'_1$ is overdemanded at prices $v$, which contradicts the quasi-stability of $(v,x)$. Therefore $p_i$ is a national buyer.

If $q_k \notin S$, then $p_i$ prefers $q_j$ to $q_k$ at prices $v(t)$ because $p_i \in T$, but $v_k(t) \leq v'_k$ and $v_j(t) = v'_j$. So $p_i$ prefers $q_j$ to $q_k$ at prices $v'$. On the other hand, if $q_k \in S - S_1$, then $p_i$ likes $q_j$ at least as much as she likes $q_k$ at prices $v(t)$, but $v_k(t) < v_k(t+1) \leq v'_k$, so $p_i$ prefers $q_j$ to $q_k$ at prices $v'$. Therefore $D_i(v') \subseteq S_1$. Now, providing $(v',x)$ is quasi-stable, there is no overdemanded set at prices $v'$ and therefore

$$|T1| \leq |S1|$$

From (1) and (2), $|T - T_1| > |S - S_1|$ so $T - T_1 \neq \emptyset$ and $T - T_1 = \{p_i \in T; D_i(v(t)) \subseteq S - S_1\}$. Then $S - S_1 \neq \emptyset$ and $S - S_1$ is overdemanded, which is a contradiction.

Hence, $v \leq v'$, and the proof is complete. \[\]
The stability of the allocation produced by the mechanism is guaranteed by the following result:

**Theorem 2.** If \( v \) is the price obtained through the dynamic price-setting mechanism, then there exists an allocation of objects \( x' \) such that \( (v, x') \) is a stable allocation.

**Proof.** Let \( x \) be the allocation of objects compatible with \( v \). Call an object \( q_j \) “expensive” if it is not allocated by \( x \) but \( v_j > 0 \). From the fact that \( (v, x) \) is quasi-stable it follows that if \( (v, x) \) is not stable then there exists at least one expensive object. The plan of the proof is to use a procedure to change \( x \) so as to eliminate the expensive objects. For that define \( D_i^*(v) = \{ q_j; q_j \) is not allocated to \( p_i \) by \( x \) but \( q_j \in D_i(v) \} \) if \( p_i \in P^N \), and \( D_i^*(v) = \{ q_j; q_j \) is not allocated to \( p_i \) by \( x \) but \( v_j = b_j \) and \( p_i \) prefers \( q_j \) to the object allocated to her by \( x \} \). Construct a direct graph whose vertices are \( P \cup Q \). There are two types of arches. If \( q_j \) is allocated to \( p_i \) by \( x \) there is an arc from \( p_i \) to \( q_j \). If \( q_j \in D_i^*(v) \) there is an arc from \( q_j \) to \( p_i \).

Let \( q_k \) be an expensive object. Then \( q_k \notin D_i^*(v) \) for \( p_i \in P^I \). In fact, if \( q_k \in D_i^*(v) \) and \( p_i \in P^I \) then \( p_i \) prefers \( q_k \) to the object allocated to her by the mechanism and \( v_k = b_k \). This means that there exists a set \( S \) that is quasi-overdemanded by national buyers, with \( q_k \in S \). Let \( T \subseteq P^N \) be the set of loyal demanders of \( S \). Since there is no overdemanded set at prices \( v \) it follows that \( |T| = |S| \). According to Hall’s theorem, all objects of \( S \) will be allocated to buyers of \( T \) by any simple allocation, particularly by \( x \), which contradicts the fact that \( q_k \) is not allocated to any buyer.

Therefore \( q_k \) is in \( D_i^*(v) \) for some \( p_i \in P^N \), otherwise we could decrease \( v_k \) and still obtain a quasi-stable allocation compatible with \( x \), which contradicts theorem 1. Let \( P^* \cup Q^* \) be the set of all vertices that can be reached by a direct path, starting with \( q_k: (q_k, p_{i1}, q_{j2}, p_{i2}, q_{j3}, p_{i3}, \ldots); p_{i1} \in P^N \). We claim that \( P^* \subseteq P^N \).
fact, suppose, by way of contradiction that there exists an international buyer in $P^*$. Take $(q_k, p_{i1}, q_{j2}, p_{i2}, q_{j3}, p_{i3}, \ldots)$ a direct path that contains this buyer. Now select $p_{ir}$ as the first international buyer of this path. We have that $p_{ir}$ prefers $q_{jr}$ to $q_{jr+1}$, the object allocated to her by the mechanism, and $v_{jr} = b_{jr}$. This means that there is a set $S$ that is quasi-overdemanded by national buyers, with $q_{jr} \in S$ (recall that $p_{ir}$ is the first international buyer of the path, so $p_{ir-1} \in P^N$). Let $T \subseteq P^N$ be the set of loyal demanders of $S$. Since, according to Hall's theorem, there is no overdemanded set at prices $v$ it follows that $|T| = |S|$. According to the construction of the mechanism, all buyers of $T$ will be assigned to the objects of $S$ by $x$. Therefore, using the fact that $\{p_{i1}, p_{i2}, \ldots, p_{ir-1}\} \subseteq P^N$ and all the objects demanded by the buyers of $T$ are in $S$, we may conclude that $p_{ir-1} \in T$, since she is assigned to $q_{jr}$ by $x$, so $q_{jr-1} \in S$ because it belongs to the demand set of $p_{ir-1}$, and then $p_{ir-2} \in T$, so $q_{jr-2} \in S, \ldots, p_{i1} \in T$, so $q_k \in S$, which contradicts the fact that $q_k$ is not allocated to any buyer. Therefore $P^* \subseteq P^N$. We have two cases.

**Case 1.** $P^*$ contains a buyer assigned to the null object by $x$, say $p_i$.

Let $(q_k, p_{i1}, q_{j2}, p_{i2}, q_{j3}, p_{i3}, \ldots, q_{js}, p_i)$ be a path from $q_k$ to $p_i$. Then we can change $x$ by allocating $q_k$ to $p_{i1}$, $q_{j2}$ to $p_{i2}$, ..., $q_{js}$ to $p_{i}$. The resulting allocation has the same price $v$ and is still quasi-stable. This way, $q_k$ is not an expensive object anymore. Therefore the number of expensive objects was reduced.

**Case 2.** All the buyers of $P^*$ are assigned to some object by $x$.

Then we can claim that there must be some $q_j$ in $Q^*$ such that $v_j = 0$. Suppose there is not, that is, $v_j > 0$ for all $q_j \in Q^*$. By definition of $P^* \cup Q^*$ we know that if $p_i \notin P^*$ then $p_i$ does not demand any object in $Q^*$. Thus, provided that every buyer in $P^*$ is in $P^N$, we can reduce the price of each object in $Q^*$ by
some sufficiently small amount and still have a quasi-stable allocation, compatible with the same allocation \( x \), thus contradicting the minimality of \( v \). Then select \( q_j \) in \( Q^* \) such that \( v_j = 0 \) and let \((q_k, p_{i1}, q_{j2}, p_{i2}, q_{j3}, p_{i3}, \ldots, q_{js}, p_{is}, q_j)\) be a path from \( q_k \) to \( q_j \). Again, change \( x \) by assigning \( p_{i1} \) to \( q_k, p_{i2} \) to \( q_{j2}, \ldots \), leaving \( q_j \) unallocated. Once again the number of expensive objects has been reduced. □

### 3.2 Example.

The following example illustrates the mechanism.

**Example 1.** Consider the market given by \( P^I = \{p_1, p_2\}, P^N = \{p_3, p_4, p_5, p_6\} \) and \( Q = \{q_1, q_2, q_3\} \). The matrix of \( a_{ij} + b_j \)'s corresponding to national buyers is as follows:

\[
\begin{pmatrix}
q_1 & q_2 & q_3 & q_4 & q_5 \\
p_3 & 4 & 3 & 5 & 5 & 0 \\
p_4 & 2 & 1 & 3 & 3 & 0 \\
p_5 & 4 & 3 & 4 & 3 & 0 \\
p_6 & 4 & 2 & 2 & 4 & 0
\end{pmatrix}
\]

The matrix of \( a_{ij} \)'s for international buyers is given by:

\[
\begin{pmatrix}
q_1 & q_2 & q_3 & q_4 \\
p_1 & 4 & 5 & 6 & 3 \\
p_2 & 2 & 1 & 4 & 3
\end{pmatrix}
\]

If we prefer, we may work with the lists of preferences:

\[
L(p_1) = q_3, q_2, q_1, q_4, q_0 \\
L(p_2) = q_3, q_4, q_1, q_2, q_0
\]

The \( b_j \)'s are given by vector: \( b = (2, 4, 6, 4) \).
The initial price vector announced by the auctioneer is $v(1) = (0,0,0,0)$. The values corresponding to the demanded objects are in boldface:

\[
\begin{align*}
q_1 & q_2 & q_3 & q_4 & q_0 \\
p_3 & (4 & 3 & 5 & 5 & 0) \\
p_4 & (2 & 1 & 3 & 3 & 0) \\
p_5 & (4 & 3 & 4 & 3 & 0) \\
p_6 & (4 & 2 & 2 & 4 & 0)
\end{align*}
\]

Since the prices are lower than $b_j$'s, $p_1$ demands $q_3$. According to the rules, $p_2$ cannot demand $q_3$. Then, she demands $q_4$. The set \{q_3,q_4\} is minimal overdemanded (it is demanded by $p_1,p_2,p_3$ and $p_4$). The prices of $q_3$ and $q_4$ are raised to 6 and 4 respectively. Thus, $v(2) = (0,0,6,4)$. We recalculate the values $a_{ij} + b_j - v_j(2)$ and the new demand sets of national buyers in boldface in the table below. The numbers $a_{ij} + b_j - v_j(2) < 0$ will be replaced with $z$:

\[
\begin{align*}
q_1 & q_2 & q_3 & q_4 & q_0 \\
p_3 & (4 & 3 & z & 1 & 0) \\
p_4 & (2 & 1 & z & z & 0) \\
p_5 & (4 & 3 & z & z & 0) \\
p_6 & (4 & 2 & z & 0 & 0)
\end{align*}
\]

Buyer $p_1$ demands $q_3$. Buyer $p_2$ demands $q_4$ and \{q_1\} is overdemanded and minimal. The price of $q_1$ is then increased to 1. Thus, $v(3) = (1,0,6,4)$. We recalculate the values $a_{ij} + b_j - v_j(3)$ and the new demand sets of national buyers in boldface in the table below.
Buyer $p_1$ demands $q_3$ and $p_2$ demands $q_4$. The set $\{q_1, q_2\}$ is minimal overdemanded. Since no international buyer is demanding these objects, their prices are increased by one unit. Therefore, $v(4) = (2, 1, 6, 4)$. The new matrix of the values of national buyers is given by:

$$
\begin{pmatrix}
q_1 & q_2 & q_3 & q_4 & q_0 \\
p_3 & 3 & 3 & z & 1 & 0 \\
p_4 & 1 & 1 & z & z & 0 \\
p_5 & 3 & 3 & z & z & 0 \\
p_6 & 3 & 2 & z & 0 & 0
\end{pmatrix}
$$

Buyer $p_1$ demands $q_3$ and $p_2$ demands $q_4$. The set $\{q_1, q_2\}$ remains minimal overdemanded. The new price vector is $v(5) = (3, 2, 6, 4)$ and the corresponding matrix of values of national buyers is given by:

$$
\begin{pmatrix}
q_1 & q_2 & q_3 & q_4 & q_0 \\
p_3 & 2 & 2 & z & 1 & 0 \\
p_4 & 0 & 0 & z & z & 0 \\
p_5 & 2 & 2 & z & z & 0 \\
p_6 & 2 & 1 & z & 0 & 0
\end{pmatrix}
$$

Buyer $p_1$ demands $q_3$ but $p_2$ cannot demand $q_4$ because $q_4$ belongs to the set $\{q_1, q_2, q_4\}$, which is minimal quasi-overdemanded.
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At this stage, the reduced list of $p_2$ contains only $q_2$. ($q_3$ is demanded by a predecessor and the price of $q_1$ is higher than $b_1$). Therefore $p_2$ demands $q_2$ whose price is lower than $b_2$. The set $\{q_1, q_2\}$ is minimal overdemanded. So the price of $q_1$ is increased to 4 and that of $q_2$ is increased to $b_2 = 4$. The new price vector is $v(6) = (4, 4, 6, 4)$. The matrix of values of the corresponding national buyers is given by:

\[
\begin{pmatrix}
q_1 & q_2 & q_3 & q_4 & q_0 \\
p_3 & (0 & z & z & 1 & 0) \\
p_4 & (z & z & z & z & 0) \\
p_5 & (0 & z & z & z & 0) \\
p_6 & (0 & z & z & 0 & 0)
\end{pmatrix}
\]

Buyer $p_1$ demands $q_3$, $p_2$ demands $q_2$, $p_3$ demands $q_4$, $p_4$ demands $q_0$, $p_5$ demands $q_1$ and $q_0$ and $p_6$ demands $q_1$, $q_4$ and $q_0$. There is no overdemanded set, so there exists a simple allocation of objects. In fact, there are two such allocations: $x$ and $x'$. In $x$, $q_1$ is allocated to $p_5$ and $q_0$ is allocated to $p_4$ and $p_6$. In $x'$, $q_1$ is allocated to $p_6$ and $q_0$ is allocated to $p_4$ and $p_5$. In both, $q_2$ is allocated to $p_2$, $q_3$ is allocated to $p_1$ and $q_4$ is allocated to $p_3$. The final price is $v = (4, 4, 6, 4)$. The reader is encouraged to verify that $(v, x)$ and $(v, x')$ are stable allocations and that using the mechanism only for the purely national market the allocation obtained is given by $v^* = (1, 0, 1, 1)$ and $x^*(q_1) = p_6$, $x^*(q_2) = p_5$, $x^*(q_3) = p_4$ and $x^*(q_4) = p_3$, which is the minimum competitive price equilibrium for the market $(P^N, Q, a, b)$.

4. Related Works and Open Questions.

The mechanism presented herein is a variant of the mechanism developed by Eriksson and Karlander (2000), who consider only the special case in which the number of buyers is equal to the number of objects.
of sellers, by the addition of dummy-agents when necessary, and every agent from one side is acceptable to any agent of the other side. Thus, their mechanism cannot be used in every marriage market as they claim. We allow any number of sellers and buyers and that some rigid buyers have unacceptable objects, so our mechanism may produce a quasi-stable allocation, while the procedure of Eriksson and Karlander always produce a stable allocation. The concept of quasi-stability is different from that of stability as it allows an unallocated object to have a price higher than zero. Consequently, our proof is not so straightforward as the one of these authors, since in order to prove that the outcome yielded by the auction is stable we also have to show that the auctioneer may choose the final allocation of the objects so that every unsold object has zero price. By using a different line of arguments from that employed by Eriksson and Karlander, we also characterize the allocation yielded by the mechanism.

The use of the core as allocation mechanism, thus satisfying the desirable properties of fairness and efficiency, has been extensively explored in the literature. In the second price (respectively, first price) auction, for example, first described by Vickrey (1961), the buyers submit their valuations (sealed bids) and the bidder who submitted the highest bid is awarded the object being sold and pays a price equal to the second highest bid (respectively, highest bid). The outcome of the second price auction (respectively, first price auction) is the optimal stable allocation for the buyers (respectively, sellers).

In case of a multi-item auction, Demange (1982) and Leonard (1983) consider an allocation mechanism where each buyer submits a sealed bid, listing her valuations of all objects. The items are then allocated according to the optimal stable allocation to the buyers (see also Roth and Sotomayor [1990 - Theorem 8.16]). Demange, Gale and Sotomayor (1986) consider a dynamic ascending mechanism that
produces the minimum competitive equilibrium price, that is, the optimal stable allocation for buyers (see also Roth and Sotomayor [1990- Theorems 8.13, 8.14]). In Sotomayor (2002), a descending bid auction mechanism produces the maximum competitive price equilibrium.

For the *Marriage market* with strict preferences, the Gale-Shapley algorithm (1961) produces the optimal stable allocation for men. By reversing the gender roles in the algorithm, we obtain the optimal stable allocation for women (see also Roth and Sotomayor [1990 - Theorem 2.12]).

The analysis of the strategic behavior of agents in the game induced by a centralized allocation mechanism has also been studied by several authors. For the *Assignment Game*, Demange and Gale (1985), Demange (1981) and Leonard (1982) prove that it is a dominant strategy for buyers to reveal their true valuations when the mechanism that produces optimal core allocation for buyers is used. An analogous result for the *Marriage Game* is obtained by Dubins and Freedman (1981), Roth (1982) and Gale and Sotomayor (1985-a). It is also known that the sincere strategy is not the best policy for the sellers in any of the two markets. An analysis of the strategic behavior of sellers for the continuous market is made by Demange and Gale (1985) and for the discrete market by Gale and Sotomayor (1985-b) and Roth (1984).

Very likely, when the mechanism discussed in the present paper is used, the results for the discrete and continuous models mentioned above may be extended and thus allow the agents of these two markets to negotiate between themselves. However, the line of arguments does not apply anymore, rendering the analysis correspondingly more difficult.

Taking into account that the companies belonging to the intimate wear market of Nova Friburgo require a relatively long time
to meet their output demand, a new approach, yet not used by the existing models, is to model the market game as a repeated game, in which the mechanism repeats itself periodically. In a short time period the agents are always the same. Thus, the sellers could use the information obtained from the previous sales in order to estimate the values of the objects by the buyers as well as their preferences, to strategically determine their reservation prices.

It should be noted that in Example 1, the price vector for the hybrid market obtained through the mechanism, $v = (3, 2, 4, 7)$, is greater in each component than the price obtained for the purely national market, $v^* = (1, 0, 1, 2)$. This means that, in this example, all sellers benefit from the entry of international buyers into the market, although some have continued to sell to the national market. Intuition tells us that the entry of new agents into one side of the market weakens the competitive positions of the agents on the same side and strengthens the competitive positions of the agents on the other side. The development of a theory that allows us to confirm or not such intuition, and also to derive other results of comparative statics that are important for the market, is still a matter to be investigated.

Finally, we believe that the mathematical treatment given to the hybrid market studied here may be extended in order to allow for more general models - each seller can offer more than one object or each buyer can buy any amount of objects she wants - in which the preferences of national buyers are linear and additive and those of international buyers correspond to their preferences over individual objects.


References


A Dynamic Price-Setting Mechanism for a Hybrid Matching Market


