Assessing Debt Sustainability in Brazil*

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Abstract

We study Ramsey model for an open economy applied to the case of Brazil. We find optimal trajectories of consumption, investment, capital stock and net foreign liabilities and, with this, analyze debt sustainability in Brazil. A new approach to capturing optimal trajectories is presented. The main tools are the Stable Manifold Theorem and the Lambda Lemma, normally used in the field of dynamical systems.

Keywords: Debt Sustainability, Optimal Control, Numerical Approach.

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1. Introduction

During the 1980's external debt crisis, Brazilian government adopted fiscal and monetary policies to reduce the level of economic activity, the increasing current account deficits and, consequently, the stock of external debt, which was considered excessively high by most economists. The tight fiscal and monetary policies caused a painful effect on output: GDP decreases of 1.6% in 1981.

Blanchard (1983) questioned the excess of external debt stock, and simulated a Ramsey model for an open economy to obtain sustainable debt/GDP ratios for Brazil. Results showed that the current account deficits could be kept, since Brazilian external debt stock was compatible with a solvency situation in the long run.

Now Brazil is in a new external debt cycle, with different features from the previous one. The economic conditions are also quite different from the ones observed in the past. Nevertheless, there is a visible rise in the Net Foreign Liabilities/GDP ratio, a measure of external indebtedness. This ratio increased from 0.25 in 1995 to 0.76 in 2002. In 2004, with the drop in external debt, the ratio decreased to 0.52, but it was still considerable. An old question is raised again: is the external debt accumulation compatible with Brazil's ability to pay it?

To answer this question, we simulate the same macroeconomic model used by Blanchard, changing the capital installation cost function and the parameter values. The solution to this version of Ramsey model includes the analysis of a nonlinear system of ordinary differential equations in the variables (k, q) (k is capital and q is the market value of capital / capital replacement cost ratio). The optimal solution is a convergent trajectory to the equilibrium (k^*, q^*), i.e., it is a branch of the stable manifold of (k^*, q^*). We present some tools from the dynamical systems theory in order to capture this solution. The Stable Manifold Theorem and the Lambda Lemma are the main tools used. Our results indicate that Brazil's Net Foreign Liabilities/GDP ratio is sustainable, i.e., the levels of consumption and investment do not need to be reduced in the future to assure the payment of Net Foreign Liabilities. The results also show some sensitivity to changes in interest rate, capital depreciation rate and capital share in the production function.

2. Literature

The definition of sustainability used here refers to a situation in which a borrower is expected to be able to continue servicing his debt without future reductions in consumption and investment spending. Assessments of debt sustainability is a leading component of the International Monetary Fund's work in member countries. The assessments, mostly based on medium-term current account and balance of payments projections, support decisions about the appropriate level of debt financing and debt restructuring.[1]

As an example of an application of the IMF framework, the Argentine external debt to GDP ratio was predicted to increase (in a baseline projection) from 47% in 1998 to 50% in 2001. Sensitivity analysis with different assumptions about policy variables, growth rates and interest rates (out of the baseline) predicted even higher debt to GDP ratios, close to 60%. Although no single indicator is expected to precisely capture the probability of a country's debt becoming unsustainable, an indicative threshold approach based on observed episodes of debt correction considered this level worrisome. In the 1979-2001 period, the IMF identified 53 worldwide episodes of debt corrections (a debt correction is defined as a sharp decline (above some threshold) or "correction" of the debt to GDP ratio, due to debt default, debt restructuring or corrective adjustment policies). Considering this set, two thirds of the episodes occurred at debt to GDP ratios below 60%, three quarters occurred at debt to GDP ratios below 70% and 20% occurred at debt to GDP ratios above 100% (some debt corrections occurred at ratios close to 150%).

The empirical analysis of debt experience in developing countries by the IMF and World Bank has provided some other important threshold indicators of debt sustainability. The data have revealed that export earnings / current debt service and net present value of future debt-service payments / export ratios above levels of 0.20 −0.25 and 2.0 −2.5, respectively, indicate high probability of hard times in debt payments. See Cohen (1985) and Underwood (1990).

Detragiache and Spilimbergo (2001), even though they did not directly address questions about debt sustainability, identified important indicators to predict debt crisis. The authors analyzed 54 episodes of debt crisis in a panel sample of 69 countries between 1971 and 1998. A debt crisis is identified whenever a country falls into arrears with the principal or interest on external obligations towards commercial creditors or signs an agreement of debt restructuring or debt reduction

[1] The IMF framework for assessing debt sustainability includes the following usual equation to project debt dynamics, $d_{t+1} = \frac{(1+r)}{(1+g)(1+\rho)} d_t - tb_{t+1}$, where g denotes the real GDP growth rate, ρ is the growth rate of US dollar value of the GDP deflator, r stands for the interest rate, d denotes the external debt to GDP ratio and tb is the debt-creating component of the balance of goods and services (in percent of GDP). The complete description of the IMF framework to assess debt sustainability is presented in "Assessing Sustainability – IMF", prepared by the Policy and Review Department, 2002.

with them. The results of a probit estimation indicated that the stock of external debt, the share of short-term debt and debt service due are factors that increase the probability of debt crisis. Foreign reserves and the level of international openness, variables that affect a country's ability to service debt payments, decrease the probability of a country ending up with a debt crisis.

The rise in the Net Foreign Liabilities / GDP ratio during the 1990s motivated some studies of debt sustainability in Brazil. The main concern was about the possibility of an explosive trajectory. To simulate the trajectory of future current account deficits and future Net Foreign Liabilities/GDP ratios, Castro et al. (1998) used short and long run elasticities of imports and exports related to the following variables: GDP growth rate, worldwide trade growth rate and real exchange rate. These elasticities were estimated with different assumptions about real exchange devaluation rate and GDP growth rate (Castro and Cavalcanti, 1997). Simulations did not indicate an explosive trajectory of the Net Foreign Liabilities / GDP ratio, i.e., the external obligations could be paid in the long run without reductions in consumption and investment.

Maka (1997), considering the same scenario of increasing current account deficits, applied Cohen's original work (1985) in order to obtain an index of sustainability to Brazil's current account deficits. Starting from an intertemporal budget constraint of an infinite horizon economy, which establishes that the external debt must be at most equal to the present value of the real future transfers to the rest of the world, the author obtained an index that indicates the minimum fraction of trade balance surplus which must be reserved to the burden of Net Foreign Liabilities, in such a way to assure external obligations compatible with a solvency condition in the long run. The application of this index as a reference for external payment would avoid Ponzi schemes and keep the levels of consumption and investment within the limits of the intertemporal budget constraint. By assuming different hypotheses about GDP growth rate and the rate of remuneration of Net Foreign Liabilities, the author calculated indices of sustainability of current account deficits between 1.43% and 1.83% of GDP, which would imply, respectively, Net Foreign Liabilities / GDP ratios between 0.36 and 0.48, in steady state. These numbers indicated that Brazil should reduce the trade balance deficits to assure debt sustainability in the long run.

In the following sections we analyze the sustainability of the Net Foreign Liabilities/GDP ratio through a simulation of Ramsey model for an open economy. We verify the optimal trajectories of consumption, investment and the maximum level of the Net Foreign Liabilities/GDP ratio compatible with the country's ability to repay the debt in the future, and compare the results with the values of the variables recently observed in Brazil.

3. The Model

Ramsey (1928) considers a problem of optimal intertemporal allocation of resources. In the subsequent applications of this model to an open economy, the level of consumption and investment is not limited by the output, since the country is allowed to borrow resources from the rest of the world. The solution of the model allows us to know the optimal path of debt as a fraction of GDP.

The objective of the central planner is to maximize the following infinite utility:

$$J(C,I) = \int_0^\infty L(t)U(\frac{C(t)}{L(t)})e^{-\theta t}dt \tag{1}$$

subject to the following ordinary differential equations

$$\dot{B}(t) = rB(t) + C(t) + I(t)(1 + \psi(\frac{I(t)}{K(t)})) - F(K(t), L(t)) \tag{2}$$

$$\dot{K}(t) = I(t) - \delta K(t) \tag{3}$$

for the constraints given by

$$C(t), I(t), K(t) > 0 \tag{4}$$

to the initial conditions

$$\begin{aligned} B(0) &= B_0 \\ K(0) &= K_0 \end{aligned} \tag{5}$$

and satisfying the transversality condition

$$\lim_{t \to +\infty} e^{-rt}B(t) = 0 \tag{6}$$

which is assumed to avoid Ponzi schemes. All functions are assumed to be smooth. The notion of optimality will be given later.

The utility function U is a strictly increasing and concave ($U'' < 0$) function and depends on *per capita* consumption.

The control variables $C(t)$ and $I(t)$ represent consumption and investment, respectively, while the state variables $B(t)$ and $K(t)$ represent net foreign liabilities and capital, respectively.

The population (or labor) is given by $L(t) = L_0 e^{nt}$ where n is the growth rate, which is assumed to be constant. The discount rate or time-preference factor is also assumed to be constant in time and is equal to θ.

The function ψ is called the capital installation cost function and depends on the fraction of investments in capital. In this paper we use the function ψ proposed by Summers (1981), that is

$$\psi(x) = \frac{\chi}{2}\frac{(x-\delta-n)^2}{x} \tag{7}$$

for all $x > 0$, where χ is a positive constant.[2]

The function F is the production function and its arguments are capital and labor. We assume that F is a Cobb-Douglas production function, i.e, $F(x,y) = x^{\alpha}y^{1-\alpha}$, where $0 < \alpha < 1$.

The constants r and δ are the world interest rate and capital depreciation, respectively. We assume that $r > n$.

Now we divide the variables $C(t)$, $I(t)$, $B(t)$ and $K(t)$ by L_0e^{nt} and represent these new variables by lower case letters $c(t)$, $i(t)$, $b(t)$ and $k(t)$, respectively.

The modified objective function is now given by

$$J_0(c,i) = \int_0^{\infty} e^{(n-\theta)t}U(c(t))dt \tag{8}$$

subject to the ordinary differential equations

$$\dot{b}(t) = (r-n)b(t) + i(t)[1+\psi(\frac{i(t)}{k(t)})] + c(t) - f(k(t)) \tag{9}$$

$$\dot{k}(t) = i(t) - (\delta+n)k(t) \tag{10}$$

for the constraints

$$c(t), i(t), k(t) > 0 \tag{11}$$

to the initial conditions

$$\begin{aligned} b(0) &= b_0 \overset{def}{=} \frac{B_0}{L(0)} \\ k(0) &= k_0 \overset{def}{=} \frac{K_0}{L(0)} \end{aligned} \tag{12}$$

and the transversality condition

$$\lim_{t\to+\infty} e^{-(r-n)t}b(t) = 0 \tag{13}$$

[2]In Blanchard (1983), ψ is defined by $\psi(x) = 2x$.

The function f, defined by $f(x) = F(x,1) = x^\alpha$ is the reduced form of the production function. It is a strictly increasing and concave function ($f'' < 0$) and satisfies Inada's conditions, i.e., $f(0) = 0$, $\lim\limits_{x\to 0} f'(x) = \infty$ and $\lim\limits_{x\to\infty} f'(x) = 0$.

The constraint (9) is the balance of payments identity: the variation in Net Foreign Liabilities is equal to the current account deficit, or the total expenses (consumption, investments, capital installation costs and interest payments) minus the output of the economy.

We say that $A(t) \overset{def}{=} (b(t),\, k(t),\, c(t),\, i(t))$, $t \geq 0$, is an admissible trajectory if equations (9)−(13) are satisfied. We recall that, for an admissible trajectory, the state variables $b(t)$ and $k(t)$ are determined by the choices of the control variables $c(t)$ and $i(t)$. We also assume that for each $A(t)$ the state variables $b(t)$ and $k(t)$ are bounded. This automatically avoids Ponzi schemes.

We want to find trajectories $\bar{c}(t)$ and $\bar{i}(t)$, $t \geq 0$, such that $\bar{A}(t) \overset{def}{=} (\bar{b}(t),\, \bar{k}(t),\, \bar{c}(t),\, \bar{i}(t))$ is an admissible trajectory and (8) is maximum in the following sense (see Seierstad and Sydsaeter (1997)): Defining $J_0^T(c,i) = \int\limits_0^T e^{(n-\theta)t} U(c(t))dt$ as the accumulated utility in the interval $[0,T]$, we say that $\bar{A}(t)$ is optimal if $\liminf\limits_{t\to\infty} (J_0^T(\bar{c},\bar{i}) - J_0^T(c,i)) \geq 0$ for any other admissible trajectory $A(t)$. This criterion is called the catching-up criterion and may appear in the literature as the overtaking criterion.

Remark 1 *We remark that necessary and sufficient conditions for optimality in an infinite horizon problem are more subtle than in the finite horizon case, meaning that in general they are not analogous and many counter-examples related to this question are found in the literature. See Cartigny and Michael (2003) for a more detailed discussion.*

The Hamiltonian of the problem is defined by

$$\begin{aligned} H(b,k,c,i,p_1,p_2,t) &\overset{def}{=} U(c)e^{(n-\theta)mt} + p_2\,(i - (\delta+n)k) \\ &+ p_1 \left((r-n)b + i(1+\psi(\frac{i}{k})) + c - f(k) \right) \end{aligned} \tag{14}$$

Let us also define the function

$$H^*(b,k,p_1,p_2,t) \overset{def}{=} \max_{c,i} H(b,k,c,i,p_1,p_2,t) \tag{15}$$

We have the following theorem which is a sufficient condition for an admissible trajectory to be optimal.

Theorem 1 *Let $R(t) \stackrel{def}{=} (b(t),\, k(t),\, c(t),\, i(t))$ be an admissible trajectory. Let H and H^* be as defined in (14) and (15). Assume that there exist C^1 functions $p_1(t)$ and $p_2(t)$ such that for all $t \geq 0$ the following conditions are satisfied:*

$$\begin{aligned} \dot{p}_1(t) &= -\frac{\partial H}{\partial b}(b(t), k(t), c(t), i(t), p_1(t), p_2(t), t) \\ \dot{p}_2(t) &= -\frac{\partial H}{\partial k}(b(t), k(t), c(t), i(t), p_1(t), p_2(t), t) \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial H}{\partial c}(b(t), k(t), c(t), i(t), p_1(t), p_2(t), t) &= 0 \\ \frac{\partial H}{\partial i}(b(t), k(t), c(t), i(t), p_1(t), p_2(t), t) &= 0 \end{aligned} \tag{17}$$

and

$$\lim_{t \to +\infty} p_1(t) = \lim_{t \to +\infty} p_2(t) = 0. \tag{18}$$

Then $R(t)$ is optimal.

Proof The strict concavity of H in (c, i) is given by

$$\begin{aligned} \det \begin{pmatrix} H_{cc} & H_{ci} \\ H_{ci} & H_{ii} \end{pmatrix} &= \det \begin{pmatrix} U''(c)e^{(n-\theta)t} & 0 \\ 0 & \chi\frac{p_1}{k} \end{pmatrix} > 0 \\ tr \begin{pmatrix} H_{cc} & H_{ci} \\ H_{ci} & H_{ii} \end{pmatrix} &< 0 \end{aligned}$$

and, therefore, by (17), we have

$$H^*(b(t), k(t), p_1(t), p_2(t), t) = H(b(t), k(t), c(t), i(t), p_1(t), p_2(t), t)$$

The concavity of H in (b, k) is given by

$$\begin{aligned} \det \begin{pmatrix} H_{bb} & H_{bk} \\ H_{bk} & H_{kk} \end{pmatrix} &= \det \begin{pmatrix} 0 & 0 \\ 0 & \chi\frac{p_1 i^2}{k^3} - p_1 f''(k) \end{pmatrix} = 0 \\ tr \begin{pmatrix} H_{bb} & H_{bk} \\ H_{bk} & H_{kk} \end{pmatrix} &= \chi\frac{p_1 i^2}{k^3} - p_1 f''(k) < 0 \end{aligned}$$

and it implies the concavity of H^* in (b, k).

The hypotheses on $p_1(t)$ and $p_2(t)$ imply that for any other admissible trajectory $(\bar{b}(t), \bar{k}(t), \bar{c}(t), \bar{i}(t))$, we have $\lim\limits_{t \to \infty} [p_1(t)(\bar{b}(t) - b(t)) + p_2(t)(\bar{k}(t) - k(t))] = 0$.

A direct application of Theorems 9 and 10 of Seierstad and Sydsaeter (1997) finishes the proof. ■

Let $q(t) = -\frac{p_2(t)}{p_1(t)}$ be the shadow price of capital, also called Tobin's q.

From equations (17) we find

$$p_1(t) = -e^{(n-\theta)t}U'(c(t))$$

and

$$q(t) = 1 - \chi\left(\frac{i(t)}{k(t)} - (\delta + n)\right) \tag{19}$$

Thus, from (16), we obtain

$$\dot{c}(t) = (\theta - r)\frac{U'(c(t))}{U''(c(t))} \tag{20}$$

and

$$\dot{q} = (\delta + r)q - (q-1)(\delta + n + \frac{q-1}{2\chi}) - \alpha k^{\alpha-1} \tag{21}$$

To prevent the trajectory of consumption c from going to 0 or $+\infty$ we assume from now on that $\theta = r$. From (20) we find that $c(t) = c^*$ is constant in time.

From (19) we have

$$i(t) = k(t)\left(\frac{q(t)-1}{\chi} + (\delta + n)\right) \tag{22}$$

Finally, using (10), (21) and (22) we get the following two-dimensional system of ordinary differential equations

$$\begin{aligned} \dot{k} &= \frac{k(q-1)}{\chi} \stackrel{def}{=} f(q,k) \\ \dot{q} &= (\delta + r)q - (q-1)(\delta + n + \frac{q-1}{2\chi}) - \alpha k^{\alpha-1} \stackrel{def}{=} g(q,k) \end{aligned} \tag{23}$$

As we will see in the next section, the system (23) has an equilibrium point $(k^*, 1)$, $k^* > 0$, which is a hyperbolic saddle. Therefore, there exist convergent trajectories for it. We assume that there exists $q_0 > 1$ such that (k_0, q_0) is contained in one of them.

Now we construct our admissible trajectory. By solving system (23) through the initial condition (k_0, q_0), we find trajectories of $k(t)$ and $q(t)$, for $t \geq 0$, which are convergent to $(k^*, 1)$.

The trajectory of the control variable $i(t)$ is defined by (22) and is therefore convergent. Let $i^* \stackrel{def}{=} \lim_{t\to\infty} i(t) = (\delta + n)k^*$.

By integrating (9) and using (13), we obtain the constant c^*, which is the value of the control variable $c(t)$ for all $t \geq 0$.

$$c^* = (r-n)(\int_0^\infty \left(f(k(t)) - i(t)(1+\psi(\frac{i(t)}{k(t)})) \right) e^{-(r-n)t} dt - b_0) \tag{24}$$

Remark 2 *We point out that since $k(t)$, $q(t)$ and $i(t)$ converge as $t \to \infty$, the integral in the definition of $\widetilde{c}$ is convergent. The equation (24) reveals that the constant level of consumption is equal to a fraction $(r-n)$ of the present value of the real output discounted to the rate $(r-n)$ minus the initial stock of Net Foreign Liabilities.*

Finally, equation (9) gives the state variable $b(t)$, i.e.,

$$b(t) = e^{(r-n)t} \int_t^\infty \left(f(k(s)) - i(s)(1+\psi(\frac{i(s)}{k(s)})) \right) e^{-(r-n)s} ds - \frac{c^*}{r-n}$$

for all $t \geq 0$. After some calculations, we check that it converges to

$$b^* \stackrel{def}{=} \lim_{t\to\infty} b(t) = \frac{f(k^*) - i^*(1+\psi(\frac{i^*}{k^*})) - c^*}{r-n} \tag{25}$$

A direct calculation shows that the convergent trajectory $(b(t), k(t), c(t), i(t))$ defined above satisfies equations $(9)-(13)$ and is therefore an admissible trajectory. Also, the functions $p_1(t)$ and $p_2(t)$ satisfy the hypotheses of Theorem 1 for the admissible trajectory $(b(t), k(t), c(t), i(t))$. Thus, $(b(t), k(t), c(t), i(t))$ is optimal. Afterwards, we will do a qualitative study of the dynamics near the equilibrium of point $(k^*, 1)$ of the system (23) and find numerically one of its convergent trajectories.

4. Equilibrium Point and Qualitative Analysis of the System of Ordinary Differential Equations

To start the qualitative study of system (23), we first find its equilibrium points, i.e., points (q,k) satisfying $f(q,k) = g(q,k) = 0$. Since $k > 0$, there is only one equilibrium point $x^* = (q^*, k^*)$ given by

$$q^* = 1 \tag{26}$$

$$k^* = \left(\frac{\delta + r}{\alpha} \right)^{\frac{1}{\alpha - 1}} \tag{27}$$

To describe the topological behavior of the system in the neighborhood of x^*, we must find the eigenvalues of the Jacobian of the system at x^*, which is given by the following matrix

$$\begin{pmatrix} \frac{\partial f(q^*,k^*)}{\partial k} & \frac{\partial f(q^*,k^*)}{\partial q} \\ \frac{\partial g(q^*,k^*)}{\partial k} & \frac{\partial g(q^*,k^*)}{\partial q} \end{pmatrix} = \begin{pmatrix} 0 & \frac{k}{\chi} \\ (1-\alpha)\alpha {k^*}^{\alpha-2} & r-n \end{pmatrix} \tag{28}$$

The real eigenvalues $\lambda_1 > 0$ and $\lambda_2 < 0$ are given by

$$\lambda_{1,2} = \frac{(r-n) \pm \sqrt{(r-n)^2 + 4u}}{2}$$

where $u = \frac{(\delta+r)(1-\alpha)}{\chi} > 0$, since $0 < \alpha < 1$. It follows that the equilibrium point is a hyperbolic saddle. The eigenspaces associated with λ_1 and λ_2 are generated by $v_1 = \left(1, \frac{(r-n)}{k^*}\right)$ and $v_2 = \left(1, \frac{\chi\lambda_2}{k^*}\right)$, respectively.

The Hartman-Grobman Theorem states that once we have a hyperbolic Jacobian matrix, that is, with associated eigenvalues with nonzero real parts, the orbit of the linear system is topologically equivalent to the orbit of the original nonlinear system in a sufficiently small neighborhood of the equilibrium point x^*. The straight lines that cross through the origin, determined by the directions of v_1 and v_2, are invariants in the linear system and their points tend to the origin as t tends to $-\infty$ or $+\infty$, respectively. They correspond to the local unstable and stable manifolds of x^* in the nonlinear system. See Palis and Melo (1977) for further details.

5. Capture of the Stable Manifold

The phase portrait of the system and some non-convergent trajectories can be seen in Figure 1. The straight lines have the directions of the eigenvectors associated with the linearized system and the point where they cross is the equilibrium point x^*. The curves are examples of non-convergent trajectories with arbitrary initial conditions.

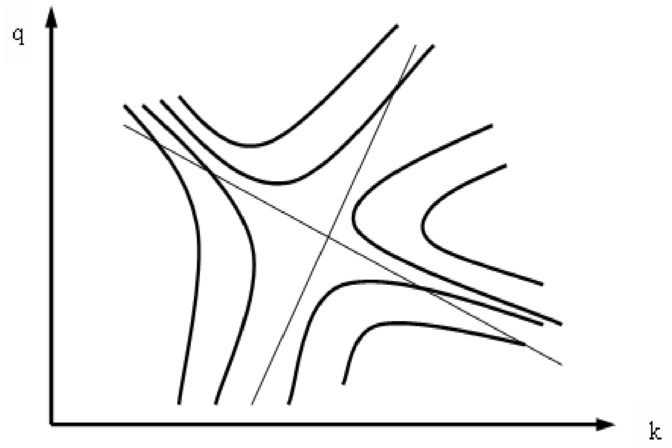

Figure 1
Phase portrait of variables q and k

A hyperbolic saddle equilibrium point has a stable and an unstable manifold. The stable manifold of a hyperbolic saddle equilibrium point x^* is the set of points which tend to x^* as $t \to \infty$. Analogously, the unstable manifold of an equilibrium point x^* is the set of points which tend to x^* as $t \to -\infty$. For a hyperbolic saddle, it is known that locally the stable and unstable manifolds are each one given by two non-constant trajectories and the equilibrium itself.

Our aim is to find the local branch of the stable manifold where there is an accumulation of capital, i.e., k is increasing with time. We base on two results of dynamical systems to approximate it numerically. They are the Stable Manifold Theorem and the Lambda Lemma.

Theorem 2 *(Stable Manifold Theorem) – Let x^* be a hyperbolic saddle equilibrium point of the system of ordinary differential equations $\dot{x} = F(x)$ and $E^s(E^u)$ the stable (unstable) subspace of $DF(x^*)$. Then the subspace tangent to the stable (unstable) manifold in x^* is $E^s(E^u)$.*

The Stable Manifold Theorem guarantees that the invariant manifolds are tangent (at the equilibrium point) to the straight lines that pass through the equilibrium point and that have the directions of the eigenvectors of the linearized system. Particularly, the stable manifold is tangent (at the equilibrium point) to the straight lines that cross through x^* and that have the direction of the eigenvector associated with the negative eigenvalue. For further details, see Palis and Melo (1977).

Lemma 1 *(Lambda Lemma) – Let x^* be a hyperbolic saddle equilibrium point of the system of ordinary differential equations $\dot{x} = F(x)$. Let W^s_{loc} and W^u_{loc} be its local stable and unstable manifolds, respectively. Let γ be a closed curve transversal to W^u_{loc} and $q \in W^u_{loc} \cap \gamma$. Let B^s and B^u be closed curves contained in W^s_{loc} and W^u_{loc}, respectively. Let $V = B^s \times B^u$. Then, to a given $\varepsilon > 0$, there is t_0 such that, if $t > t_0$ and γ_t is a connected component of $\varphi(-t, \gamma) \cap V$ that contains $\varphi(-t, q)$, then D^s_t is $\varepsilon - C^1 -$ closed to B^s.*[3]

Hence, the Lambda Lemma states that, given a transversal curve to the unstable (stable) manifold, then the set of final points of the backward (forward) flow of this curve, to a sufficiently large t, locally approaches the stable (unstable) manifold. Further details also in Palis and Melo (1977).

According to this lemma, we take a transversal curve to the local unstable manifold and iterate backward through every point of this curve. The resulting set of points will be a very good approximation to the local stable manifold. However, as we do not know exactly where the stable manifold is, we cannot know where the unstable manifold is. Nevertheless, the Stable Manifold Theorem guarantees that this one is tangent, at the equilibrium point, to the straight line z, through x^* in the direction of the eigenvector of the linearized system associated with the positive eigenvalue. Thus, curves transversal to z and sufficiently close to x^* are also transversal to the local unstable manifold. These curves are schematically represented in Figure 2. In the numerical simulation, the transversal curve is drawn in a perpendicular way to both branches of the unstable manifold. The backward iteration of this transversal curve to one branch of the unstable manifold gives the results presented in Figure 3.

[3] Two curves are closed in C^1- topology when they are closed both in the Euclidean distance and in the derivative.

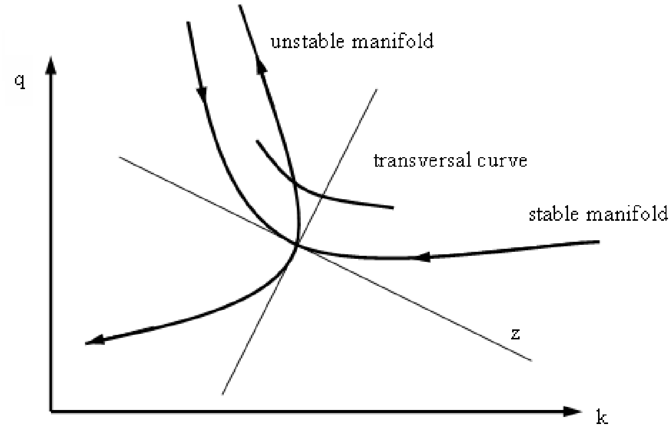

Figure 2
The stable and unstable manifolds of the saddle equilibrium point and a transversal curve

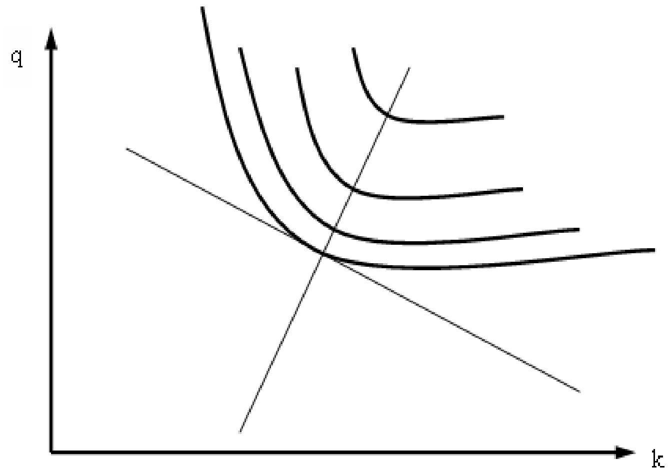

Figure 3
Backward iteration of a curve transversal to one branch of the unstable manifold

In Judd (1998), the Projection Method is presented as an alternative method for capturing the stable manifold. We can see the local stable manifold as the graph of a function ψ defined near k^* with values near $q^* = 1$. This function ψ in general does not have a simple expression, although information about the slope at each point is obtained from (23), which can be represented as an operator O between two different function spaces with the property that $O(\psi) = 0$. This method consists in choosing a finite dimensional space of functions $\mathcal{F}$, such as polynomials with a fixed maximum degree, as an approximation of the whole space of functions. We consider O as acting on $\mathcal{F}$ and find a function f^* in $\mathcal{F}$ which gives the least norm of $O(f)$, $f \in \mathcal{F}$. The function f^* is the best approximation for ψ in this setup. This method may provide fast and very accurate solutions and depends on two main choices: the space $\mathcal{F}$ (for instance, which polynomials should be used as its spanning basis) and the metric for evaluating the norm of the residual functions.

Our method seems to provide similar results when compared to the Projection Method. Its main advantage lies in its implementation. While in the Projection Method some previous experience is necessary for making the starting choices of the finite dimensional family of functions and the norm for evaluating the discrepancies (which may differ according to the problem), the Lambda Lemma approach does not make any distinction regarding the complexity of the problem, being always applied in the same way, and with the computation time being the main constraint for obtaining better and better approximations. We remark that the Projection Method is a powerful tool that can be used in a more general setup while the Lambda Lemma approach, as used here, has its use mainly restricted to localizing the stable/unstable manifolds of hyperbolic equilibrium points of ordinary differential equations.

6. Steady States of Investment, Consumption and Net Foreign Liabilities

The resolution of Ramsey model allows us to find out the steady state values of capital, output, investment, consumption and Net Foreign Liabilities, given by k^*, i^*, c^* and b^* (see previous section for their expressions). According to (25), the steady state of Net Foreign Liabilities is equal to the present value of future trade surplus, discounted to the rate $(r - n)$. This expression reveals that we just need to know the steady state of consumption to find the steady state of Net Foreign Liabilities, once we already know k^* and i^* and there is no capital installation cost in steady state. Nevertheless, as the expression of c^* shows, the constant level of consumption also depends on the convergent path of capital and investment. So, to calculate c^*, we need to know the stable path of capital that takes to the hyperbolic saddle point of system (23) and, afterwards, do the numerical integration of (24).

Based on the results from the previous section, we know that there is only one path that takes the variables k and q to the steady state equilibrium. System (23) has only one value of q_0 that takes the economy under the saddle path to the initial capital stock k_0. If, for example, the economy initially has a smaller capital stock than the steady state capital stock, the marginal productivity of capital will be higher than the interest rate, which is equal to the time-preference factor θ. In this case, the capital stock will not adjust instantaneously, due to capital installation costs. Instead, the capital stock will grow gradually towards the steady state. We point out that we can assume that q_0, the initial value of Tobin's q, is such that (k_0, q_0) is in the branch of the stable manifold of (k^*, q^*).

7. Comparative Static Analysis

Now we present a comparative static analysis of the variables, in steady state, with respect to the parameters r (interest rate), δ (capital depreciation rate), α (share of the capital in the Cobb-Douglas production function) and n (population growth rate). We will start with the capital stock:

$$k^* = \left(\frac{\delta + r}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{29}$$

$$\frac{\partial k^*}{\partial r} = \frac{\partial k^*}{\partial \delta} = \frac{1}{\alpha^2 - \alpha}\left(\frac{\delta + r}{\alpha}\right)^{\frac{-\alpha+2}{\alpha+1}} < 0$$

with $0 < \alpha < 1$ and

$$\frac{\partial k^*}{\partial \alpha} = -\left(\frac{\delta + r}{\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{1}{(\alpha-1)^2}\ln\left(\frac{\delta + r}{\alpha}\right) + \frac{1}{(\alpha^2 - \alpha)}\right)$$

The negative sign of the partial derivatives of capital with respect to interest rate and capital depreciation rate indicates that increases in these rates, *ceteris paribus*, decrease the steady state of capital. The partial derivative of capital with respect to the share of capital in the production function depends on the interest rate and capital depreciation rate values. Considering 1/3 or 0.4 and interest rate and capital depreciation rate satisfying $5.26\% < r < 11.11\%$ and $5\% < \delta < 6.56\%$ (these intervals were obtained from the literature, see next section), capital and output in steady state are increased.

The investment in the steady state is given by $i^* = \left(\frac{\delta+r}{\alpha}\right)^{\frac{1}{\alpha-1}}(\delta + n)$. We have

$$\frac{\partial i^*}{\partial r} = \frac{1}{\alpha^2 - \alpha}\left[\frac{\delta + r}{\alpha}\right]^{\frac{-\alpha+2}{\alpha-1}}(\delta + n) < 0$$

$$\frac{\partial i^*}{\partial \delta} = \frac{1}{\alpha^2 - \alpha}\left[\frac{\delta + r}{\alpha}\right]^{\frac{-\alpha+2}{\alpha-1}}(\delta + n) + \left[\frac{\delta + r}{\alpha}\right]^{\frac{1}{\alpha-1}}$$

$$\frac{\partial i^*}{\partial n} = \left[\frac{\delta + r}{\alpha}\right]^{\frac{1}{1-\alpha}} > 0$$

Increases in the interest rate, *ceteris paribus*, diminish the investment, due to the reduction in the capital stock. Increases in the capital depreciation rate, in turn, can increase or diminish the investment, since in this case there are two effects of opposite signs: the reduction in the capital stock, which diminishes the investment in the steady state, and the increase in the investment necessary to keep the level of capital per capita constant. The latter effect is also responsible for the increase in investment, *ceteris paribus*, after an increase in the population growth rate.

To know the sign of $\frac{\partial c^*}{\partial r}$, we assume that the variables inside the integral in (24) are in steady state. By solving the integral and deriving the resulting expression with respect to r, we get:

$$\frac{\partial c^*}{\partial r} = \frac{\partial f(k^*)}{\partial r} - \frac{\partial i^*}{\partial r} = \frac{\partial k^*}{\partial r}(r - n) < 0$$

since $r > n$.

This approach is valid for trajectories with initial conditions close to the steady state. It is also possible to numerically verify the effects of changes in interest rate on consumption considering the complete optimal path. In this case, the sign of the derivative depends on the parameters and initial conditions, but it is sometimes positive. These ambiguous results are coherent with the economic theory.

8. Parameters

In the numerical simulations, we used estimates of capital depreciation rate, population growth rate, capital share in the production function and interest rate available in the literature.

Carvalho (1996) analyzed the capital/output ratio and potential output for Brazil in the 1977-1995 period and estimated a capital depreciation rate between 3.56% and 5.88%. The results were extremely sensitive to the capital/output ratio. With a capital/output ratio of 2.57, the estimated depreciation rate was 4.32%. Carvalho considered that this rate would be a reasonable capital depreciation rate for the 1977-1995 period as a whole. However, the studies carried out by the author suggest a smaller capital/output ratio in recent years, which would imply a higher capital depreciation rate for the 1990s.

Araújo and Ferreira (1999), based on capital/output ratios and investment/output ratios, adopted a capital depreciation rate of 6.56% per year in a simulation carried out to investigate the dynamic effects of fiscal policies in Brazil. Nevertheless, they recognized the absence of a consensus in the literature about this rate. In our simulations, we assume capital depreciation rates of 5% and 6.56% to also verify the sensitivity of the results.

Brazil's annual population growth rate is approximately 1.93%, according to the IBGE census of 2000. The capital share in production function usually used in the literature is 1/3, but we also simulated the model with a capital share of 0.4.

The interest rate of our model is equal to the intertemporal discount rate of the representative consumer. Issler and Piqueira (2000) estimated this rate for Brazilian consumers in an infinite horizon model using three types of utility function: CRRA (constant relative risk aversion), utility with external habit formation, and Kreps-Porteus utility. With Euler equations and time series of aggregate consumption and real interest rate, the authors applied GMM (Generalized Method of Moments) and found intertemporal discount rates between 5.25% and 11.11% (which imply intertemporal discount factors between 0.90 and 0.95). We consider this interval for the long-term real interest rate in the simulations.

The parameter χ of the capital installation cost function – equal to 32 – is the inverse of the derivative of investment per capital unit with respect to Tobin's q, and was estimated by Summers (1981) using investment and capital depreciation data from the 1931-1978 period in the United States. Hayashi (1982) estimated this parameter based on the marginal q instead of on the average q, and found it to be close to 24. In the system of ordinary differential equations of the model, increases in χ imply decreases in capital variation per unit of time.

The capital initial condition k_0 is such that the marginal product of capital is 22%, a little less than 25% as assumed by Blanchard (1983), but above the marginal product of capital in the United States, estimated by Hall (1999) to be equal to 17% in the 1990s.

9. Results

In this section we present the results of the numerical simulation of the model. The trajectories of capital and investment with capital installation costs verified in numerical simulations generally indicate the path of net output depicted in Figure 4. We will use this figure, as Blanchard and Fischer (1996), to explain the dynamics of the current account deficits. For the sake of illustration, we also present the paths of capital, Tobin's q and investment with installation costs.

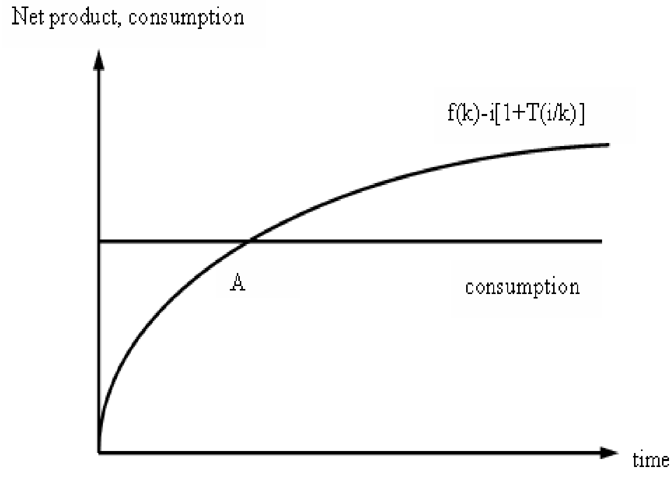

Figure 4
Net output and consumption trajectories

In Figure 4 we can see the path of net output from the initial condition to the steady state. Initially, the economy presents a trade balance deficit, consumption is higher than the net output, and there is an invisible trade balance deficit due to interest payments, once excess consumption is financed with external debt. As the net output grows, the trade balance deficit diminishes. In point A, the trade balance is zero, but the economy still presents current account deficit due to external debt services. After A, the economy starts to present trade balance surplus, which grows until it equals the deficits in the invisible trade balance. In steady state, trade surpluses are equal to debt services, and the current account balance is zero.

In the simulations we assumed interest rates of 5.26%, 7.0%, 9.0% and 11.11%, the upper limit of Issler and Piqueira's interval for the intertemporal discount rate. For the first set of results we considered a capital share in the production function equal to 1/3 and a capital depreciation rate equal to 5.0%. In the second set of results we assumed a capital share equal to 0.4, and kept the other rates unchanged. In the third set, the capital depreciation rate is equal to 6.56% and the capital share in the Cobb-Douglas function is 1/3 again. In each column we have, respectively, consumption (c), investment (i), invisible trade balance deficit (dib), trade balance deficit (dtb) and Net Foreign Liabilities (b), in terms of GDP (y).

Table 1
Results

α	δ	r	c/y	i/y	dib/y	dbt/y	b/y
1/3	5.00%	5 .26%	0.66	0.23	0.11	-0.11	3.27
1/3	5.00%	7.00%	0.69	0.19	0.12	-0.12	2.43
1/3	5.00%	9.00%	0.73	0.16	0.11	-0.11	1.57
1/3	5.00%	11.11%	0.77	0.15	0.08	-0.08	0.91
40%	5.00%	5.26%	0.60	0.27	0.13	-0.13	4.04
40%	5.00%	7.00%	0.61	0.23	0.16	-0.16	3.08
40%	5.00%	9.00%	0.66	0.20	0.14	-0.14	1.98
40%	5.00%	11.11%	0.72	0.17	0.11	-0.11	1.19
1/3	6.56%	5.26%	0.67	0.24	0.09	-0.09	2.75
1/3	6.56%	7.00%	0.70	0.21	0.09	-0.09	1.77
1/3	6.56%	9.00%	0.74	0.18	0.08	-0.08	1.12
1/3	6.56%	11.11%	0.78	0.16	0.06	-0.06	0.70

Figures 5, 6 and 7 show, respectively, the trajectories of capital Tobin's q and investment with capital installation costs when $r = 9\%$ and $\delta = 5\%$.

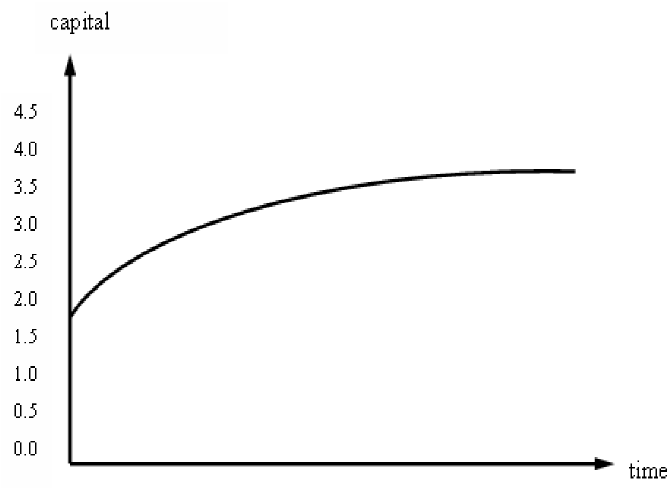

Figure 5

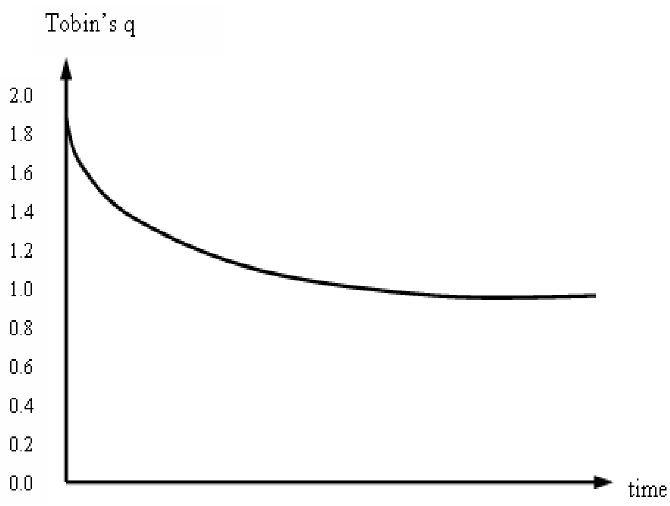

Figure 6

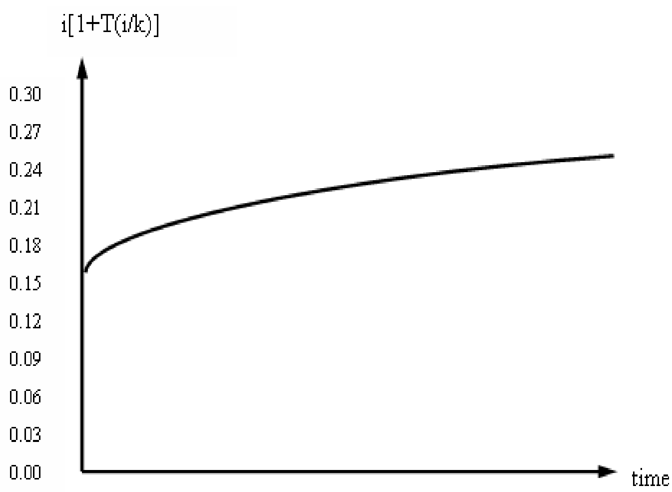

Figure 7

The results show the highest feasible level of constant consumption and Net Foreign Liabilities. The current account deficit in steady state is always zero because all trade balance surpluses are used to the burden of Net Foreign Liabilities. The trade surplus is equal to output minus consumption and investment, while the deficit in the invisible trade balance is equal to Net Foreign Liabilities in steady state multiplied by $(r - n)$.

We observe that low interest rates imply very high levels of Net Foreign Liabilities, of about three times those of GDP, in the simulations where the capital share is 1/3 and capital depreciation rate is 5.0%. Even with high interest rates the level of indebtedness remains significantly high. Interest rates of 9.0% and 11.11% allow the economy to hold Net Foreign Liabilities/ GDP ratios close to 1 in steady state if the capital share is 1/3. Increases in interest rates, *ceteris paribus*, decrease investment in steady state through the reduction in the equilibrium capital stock, as we verified in the comparative static analysis. The effect of high interest rates on consumption is ambiguous, but we generally observed positive results. As we verified in the comparative static analysis, increases in capital share also bring about an increase in capital and Net Foreign Liabilities in steady state.

Net Foreign Liabilities in steady state are very sensitive to changes in the interest rate and capital depreciation rate. Assuming a capital share equal to 1/3 and a capital depreciation rate of 6.56%, the Net Foreign Liabilities ratio in steady state is equal to 0.70 with an interest rate of 11.11% and 2.75 with an interest rate of 5.26%. The ratio grows 40% if we change the capital depreciation rate from 5.0% to 6.56%, with interest rates of 9.0% and a capital share of 1/3.

Numerical simulations with higher interest rates, out of the long-term interval, resulted in lower levels of indebtedness. Interest rates of 14.5%, for example, and a capital depreciation rate of 5% imply Net Foreign Liabilities of 68% of GDP.

These results indicate that Brazil's current Net Foreign Liabilities/GDP ratio, 0.52, is sustainable in the long run. The observed paths of consumption and investment do not need to be reduced to assure debt payment.

By comparing our results with Blanchard's, we perceived two similarities: high and very sensitivity levels of steady state debt. In Blanchard's simulations, the most reasonable results indicate steady state Net Foreign Liabilities GDP/ratios between 0.60 and 1.40. The results are also very sensitive to the intertemporal discount rate adopted. Optimistic simulations indicate even higher debt ratios, nearly three times those of the GDP.

We noticed that this is a characteristic of this type of one-sector model, where all the output is tradable, generating high levels of debt/GDP ratios in steady state. A natural extension is the analysis of a two-sector model, with tradable and non-tradable goods, as presented in Pessôa (1999). Probably, a two-sector model would reveal smaller sustainable debt/GDP ratios. Moreover, in this paper we consider the terms of trade, a relevant variable to characterize debt dynamics, to be constant. A simulation of a two-sector model would also present the variability of the results with respect to changes in this variable.

10. Conclusion

In this paper we studied debt sustainability in Brazil, simulating the Ramsey model for an open economy and applying the Stable Manifold Theorem and the Lambda Lemma – two useful tools of the dynamical systems theory – to obtain optimal paths of consumption, investment, capital stock and Net Foreign Liabilities. The results of simulations suggest high Net Foreign Liabilities/GDP ratios and indicate that Brazil's current ratio is sustainable. Considering a capital share of 1/3, a capital depreciation rate of 5% and interest rates of 11.11%, for example, the Net Foreign Liabilities in steady state are equivalent to 91% of GDP. The results, however, are very sensitive to the choice of parameters.

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