A General Approach for Pricing Rollover Options^{*}

Christian Johannes Zimmer^{**}

Abstract

When an insurance company sells a mutual fund with death and maturity guarantees to its client, it may consider allowing the client to extend the guarantee for some more years. If the renewal only happens once, a so-called rollover option is implied in the contract. In this paper, we show how the generalized Bermudan option can be applied to the special case of the rollover option.

By avoiding the heavy mathematical tools which are necessary to prove the existence of a hedging strategy, we will focus on the calculations that are common in the Black-Scholes-type analysis. Contrary to Bilodeau (1997) who analyzed the one-time renewal, we can refer to the results on the (generalized) Bermudan option for which the existence of a hedging strategy was already proved. We will see that the strike price has to be adjusted if the contract is renewed in order to explicitly calculate the price of the contract.

 $Keywords\colon$ Rollover Option, Bermudan Option, Renewal of Options, No-Arbitrage Pricing.

JEL Codes: G10, G12.

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^{**}Banco Itaú S.A., Área de Mercado de Capitais, Diretoria de Administração de Carteiras, Superintendência de Pesquisa Quantitativa, São Paulo, Brasil.

E-mail: christian.zimmer@itau.com.br

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1. Introduction

The basic idea of contract renewal is of interest for several industries. Besides specific insurance contracts, even investment vehicles for the general public may have this feature. Take a look, for example, at the Brazilian PIBB fund, where one invests in the IBOVESPA stock benchmark and receives a put option to hedge against downside moves. This means that at expiration date t_1 of the put, the fund's value is given by the underlying's final value S_{t_1} plus the price of a put option with strike K:

$$S_{t_1}^{\text{Guar}} = S_{t_1} + (K - S_{t_1})^+$$

Let us focus on the put option $(K - S_{t_1})^+$ first and assume that the buyer has the additional right to decide at expiration date whether to stop the contract and receive $S_{t_1}^{\text{Guar}} = S_{t_1} + (K - S_{t_1})^+$ or to continue the contract until final time T, where he receives $S_T^{\text{Guar}} = S_T + (K - S_T)^+$.¹

The option with such a renewal feature is called rollover option. In the case of continuing the contract, the initial strike price will be adjusted, if the underlying is in the money, for the option to be at the money. In Bilodeau (1997), a way to price this option was proposed, assuming that a hedging strategy exists. We verify this existence by generalizing the rollover option to an option that allows the buyer to decide whether he executes his right at several preset times. This is the specification of a Bermudan option. The existence of a hedging strategy for a generalized Bermudan option was shown in Zimmer (2000) and we use these results to verify the pricing formula of Bilodeau (1997).²

After an introduction of the model settings, we analyze the rollover option from two points of view. First, we follow the original paper, clarifying some steps which explicitly use the Markovian framework. It will then become clear that the possibility of an explicit pricing formula is caused not only by the Markovian assumption, but also by the crucial feature of an adjustment of the previous strike price. Then, in a second part, we show how to get the evaluation formula even if more than two execution times are allowed. Here we use the approach of Bermudan options.

As the rollover option was first analyzed in the framework of a complete Black-Scholes model, we start with $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \ge 0}, P)$ as a filtered probability space. On this space lives the price process of the risky asset, $S = (S_t)_{t \in [0,T]}$, which follows a geometric Brownian motion:

¹Actually, the PIBB does not have this additional right, but one might understand that such a feature could be of interest for the general investor.

 $^{^{2}}$ Schweizer (2002) presented a nice summary of the main results and of the pricing of the rollover option. In the present paper we provide the missing details for the rollover option.

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1}$$
$$S_0 > 0$$

where $W = (W_t)_{t \in [0,T]}$ is a standard Brownian motion under P, $\mu, \sigma \in \mathbb{R}, \sigma > 0$. We assume the interest rate $r \in \mathbb{R}^+$ to be constant and positive. A riskless asset, $B = (B_t)_{t \in [0,T]}$, shall be deterministic and remunerated with interest rate r:

$$\frac{dB_t}{B_t} = rdt \tag{2}$$

Setting $\widetilde{W}_t = W_t + \left[\left(\mu - r \right) / \sigma \right] t$, we have

$$\frac{dS_t}{S_t} = rdt + \sigma d\widetilde{W}_t \tag{3}$$

For the discounted risky asset $S^{\text{disk}} = (S_t^{\text{disk}})_{t \in [0,T]}$ we then have

$$S_t^{\text{disk}} = \frac{S_t}{B_t} = S_0 \cdot e^{\left(\mu - r - \frac{\sigma^2}{2}\right)t + \sigma W_t} = S_0 \cdot e^{\left(-\frac{\sigma^2}{2}\right)t + \sigma \widetilde{W}_t} \tag{4}$$

Now, let \tilde{P} be a *P*-equivalent probability measure, under which $\left(\widetilde{W}_t\right)_{t\in[0,T]}$ is a standard Brownian motion.³ Under this measure (which differs from the original measure if $\mu \neq r$), the discounted price process S^{disk} is a martingale. To ease the notation, we just set $\mu = r$ and hence have $\tilde{P} = P$, which implies that

$$S_t^{\text{disk}} = S_0 \cdot e^{\left(-\frac{\sigma^2}{2}\right)t + \sigma \widetilde{W}_t} = S_0 \cdot e^{\left(-\frac{\sigma^2}{2}\right)t + \sigma W_t}$$
(5)

We adopt the following notation: $(P_x)_{x\geq 0}$ is a family of probability measures on (Ω, \mathbb{F}) . Then the triple $(S_t, \mathcal{F}_t, P_x)$ is a homogenous Markovian process, *i.e.*

- 1. S_t is \mathcal{F}_t -measurable for every $t \ge 0$;
- 2. $P_x (S_t \in \Gamma)$ is measurable with respect to x for $t \ge 0, \Gamma \in \mathcal{B}$;
- 3. $P_x (S_0 \in \mathbb{R}^+ \setminus \{x\}) = 0$ for every $x \in \mathbb{R}^+ \setminus \{0\}$;
- 4. $P_x (S_t \in \Gamma | \mathcal{F}_s) = P_{S_s} (S_{t-s} \in \Gamma) P_x$ -almost sure.

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³The formal conditions, the formulation, and the proof of the necessary Girsanov's theorem may be found, *e.g.* in Karatzas and Shreve (1991), Section 3.5.

The rest of the paper is structured as follows. We start with a review of the original Bilodeau (1997) approach and show how the author calculates the price of a rollover option with two execution times: $t_1 < T$. Then, we show how to use the general approach of a Bermudan option and derive the same price. In both parts, it is important to understand why the original strike price K has to be adjusted at t_1 , in order to allow for explicit results. The third part (Section 4) derives the hedging strategy for the special case of two execution times.

2. The Original Approach – A Direct Way to Price the Rollover Option

Bilodeau (1997) assumes that there exists a hedging strategy for the rollover option with only one additional execution time t_1 before final time T. The author then evaluates the payoff, under this hypothesis, and calculates the value of a European put option with execution time t_1 and the value of a European put option with execution time $t_2 - t_1 =: \Delta t \ge 0$. After having done this, one just has to multiply the value of the second option by the probability of renewing the contract (hence the validity of the second option), and finally the two values have to be added. One might want to evaluate the put option as:⁴

$$E_{t_1}\left[\left(K - S_{t_1}^{\text{disk}}\right)^+ + \mathbf{1}_{\{\text{renewal}\}}\left(K - S_{\Delta t}^{\text{disk}}\right)^+\right], \Delta t \ge 0$$
(6)

This payoff can then be evaluated within the Black-Scholes framework, where the author analyzes two cases on how to determine the rule for renewing the contract or not. The first case uses the rule $K < S_{t_1}$ (naive behavior). In the second case, named optimal behavior, the execution of the option (and hence no renewal) is determined by the fact that the price process S is below a certain (deterministic) critical value $S_{t_1} < S_{t_1}^*$ with $S_{t_1}^* < K$. We will directly take a look at the optimal behavior as the naive one can be shown to open arbitrage possibilities.

In the case of optimal behavior (similarly to the well-known decision rule for an American put option), the holder of the option does not directly execute his right when the price of the underlying drops below the strike price. Rather, the holder compares the payoff he would receive with the remaining value of the option. Bilodeau (1997) argues in a heuristic way that the holder of the option is indifferent about execution at t_1 when the price of the underlying at t_1 satisfies:

$$\left(K - S_{t_1}^*\right)^+ = S_{t_1}^* \cdot \operatorname{Put}\left(1, \frac{K}{x}, \Delta t\right) \tag{7}$$

⁴Here it is important to note that we wrote the payoffs with a fixed strike price $K \in \mathbb{R}^+$. See below for a further discussion about this important issue.

where

$$\operatorname{Put}\left(1, \frac{K}{x}, \Delta t\right) \tag{8}$$

$$= E_{P_x} \left[\left(\frac{K}{x} - 1 \cdot e^{\left(r - \sigma^2/2\right)\Delta t + \sigma W_{\Delta t}} \right)^+ \middle| \mathcal{F}_{t_1} \right]$$
(9)

is the value at time t_1 of a put option with strike price⁵ $\frac{K}{x}$ on an underlying price process with initial value of 1 (at t_1). The time to maturity of the option is $\Delta t > 0$. The objective is to find the deterministic value $S_{t_1}^*$ that yields equality. Having found this value,⁶ the holder of the option should execute his right if the price process falls below it. Bearing in mind that Put $\left(1, \frac{K}{x}, \Delta t\right) > 0$, we get, under $K > S_{t_1}^*$, that

$$S_{t_1}^* = \frac{K}{1 + \operatorname{Put}\left(1, \frac{K}{x}, \Delta t\right)} \tag{10}$$

Altogether, we can calculate the value of the rollover option at t_0 with optimal behavior following the formula

$$V_{t_0}^{\text{RO,opt}} = E_{P_x} \left[e^{-rt_1} \cdot (K - S_{t_1}) \cdot \mathbf{1}_{\{S_{t_1} < K\}} \cdot \mathbf{1}_{\{S_{t_1} < S_{t_1}^*\}} \right]$$
(11)
+
$$E_{P_x} \left[e^{-r\Delta t} \cdot (K - S_{\Delta t}) \cdot \mathbf{1}_{\{S_{\Delta t} < K\}} \cdot \mathbf{1}_{\{S_{t_1} \ge S_{t_1}^*\}} \right]$$

where $V_{t_0}^{\text{RO,opt}}$ stands for the value of the rollover option at t_0 when the holder acts optimally.

Assuming that $K > S_{t_1}^*$ we then have

$$E_{P_{x}}\left[e^{-rt_{1}} \cdot (K - S_{t_{1}}) \cdot \mathbf{1}_{\left\{S_{t_{1}} < K\right\}} \cdot \mathbf{1}_{\left\{S_{t_{1}} < S_{t_{1}}^{*}\right\}}\right]$$

= $E_{P_{x}}\left[e^{-rt_{1}} \cdot (K - S_{t_{1}}) \cdot \mathbf{1}_{\left\{S_{t_{1}} < S_{t_{1}}^{*}\right\}}\right]$ (12)

with

$$d_{1}^{*}(x,t) := d_{1}\left(\frac{x}{S_{t_{1}}^{*}}\right) = \frac{\ln\left(\frac{x}{S_{t_{1}}^{*}}\right) + \left(r + \frac{1}{2}\sigma^{2}\right)t}{\sigma\sqrt{t}}$$
(13)

$$d_{2}^{*}(x,t) := d_{1}^{*}(x,t) - \sigma\sqrt{t} = d_{2}\left(\frac{x}{S_{t_{1}}^{*}}\right) - \sigma\sqrt{t}$$
(14)

⁵Observe that $P_x(S_0 \in \mathbb{R}^+ \setminus \{x\}) = 0$ from the Markovian assumption (3).

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 $^{^{6}}$ With this idea, Bilodeau (1997) follows the argumentation for the American put option, where the stopping boundary resolves a free boundary problem and this equation is one of the additional conditions. For this idea, see also Carr et al. (1992), p. 90.

and by using the notations of the Black-Scholes formula, we deduce:

$$V_{t_{0}}^{\text{RO,opt}} = Ke^{-rt_{1}} \cdot N\left(-d_{2}^{*}(x,t_{1})\right) - xN\left(-d_{1}^{*}(x,t_{1})\right) + N\left(d_{1}^{*}(x,t_{1})\right) + \left\{Ke^{-r(\Delta t)} \cdot N\left(-d_{2}\left(\frac{x}{K},\Delta t\right)\right) - xN\left(-d_{1}\left(\frac{x}{K},\Delta t\right)\right)\right\}$$
(15)

Now, we add the term

$$0 = S_{t_1}^* e^{-rt_1} \cdot N\left(-d_2^*\left(x, t_1\right)\right) - S_{t_1}^* e^{-rt_1} \cdot N\left(-d_2^*\left(x, t_1\right)\right)$$
(16)

and set

$$V^{\mathrm{Eu}}\left(S_{t_{1}}^{*}, t_{1}\right) := S_{t_{1}}^{*} e^{-rt_{1}} \cdot N\left(-d_{2}^{*}\left(x, t_{1}\right)\right) - xN\left(-d_{1}^{*}\left(x, t_{1}\right)\right)$$
(17)

$$V^{\mathrm{Eu}}(K,\Delta t) := K e^{-r\Delta t} \cdot N\left(-d_2\left(\frac{x}{K},\Delta t\right)\right) - xN\left(-d_1\left(\frac{x}{K},\Delta t\right)\right)$$
(18)

These are the values of a European put option at t_0 with strike price $S_{t_1}^*$ and time t_1 , respectively with strike price K and time Δt . Altogether, we then have the value proposed by Bilodeau (1997)

$$V_{t_0}^{\text{RO,opt}} = V^{\text{Eu}}(S_{t_1}^*, t_1) + e^{-rt_1} \cdot N(-d_2^*(x, t_1)) \cdot (K - S_{t_1}^*)$$
(19)
+ $N(-d_2^*(x, t_1)) \cdot V^{\text{Eu}}(K, \Delta t)$

3. Evaluating the Rollover Option Using the Bermudan Approach

In the previous section, we analyzed the rollover option with two execution times. Nevertheless, more execution times $t_1, ..., t_N, N > 1$ may be possible. It is possible to tackle the pricing problem of such a Bermudan option, once again, with backward induction. In the case of a general pricing process, the existence of a hedging strategy and the associated price was developed in Zimmer (2000) and summarized, *e.g.*, in Schweizer (2002). In the complete Markovian case used by Bilodeau (1997), one expects to get the same results. In what follows, we will show how to carry out the details when using the Bermudan approach. We give the price at and between execution times.

As already mentioned, the approach called *naive* behavior does not lead to an arbitrage-free price. The comparison with the approach called optimal behavior turns out to be more interesting, though. The evaluation at any time $t \in [t_0, t_2] = [0, T]$ is of special interest for the hedging strategy developed in Section 4. We have to pay attention that we need to differentiate explicitly between strike prices K_{t_1} and K_{t_2} , a fact that does not become very clear in the original approach.

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3.1 Determination of the stopping boundary and of the payoff process

According to Theorem 16 in Zimmer (2000) we can ensure the existence of a stopping boundary. This boundary shall now be determined for our special case. The idea is that at the boundary price the buyer of the option is indifferent about whether to execute the option or to continue the contract. This happens at time t_1 if and only if we have

$$(K_{t_1} - S_{t_1})^+ = e^{-r(t_2 - t_1)} \cdot E_{P_x} \left[(K_{t_2} - S_{t_2})^+ \middle| \mathcal{F}_{t_1} \right]$$
(20)

As a consequence of the Markovian property of S, we can write this as

$$(K_{t_1} - S_{t_1})^+ = e^{-r(t_2 - t_1)} \cdot E_{P_y} \left[(K_{t_2} - S_{t_2 - t_1})^+ \right] \Big|_{y = S_{t_1}}$$
(21)

Now, let $s_{t_1}^{\text{Ber}}$ be that deterministic value which implies equality. We then have:

$$\left(K_{t_1} - s_{t_1}^{\text{Ber}}\right)^+ = e^{-r(t_2 - t_1)} \cdot E_{P_{s_{t_1}}^{\text{Ber}}} \left[\left(K_{t_2} - S_{t_2 - t_1}\right)^+ \right]$$
(22)

The right side is the value of a European put option at t_0 with time $t_2 - t_1$, strike price K_{t_2} and start of the price process in $s_{t_1}^{\text{Ber}}$. We can write this as

$$e^{-r(t_2-t_1)} \cdot E_{P_{s_{t_1}^{\text{Ber}}}} \left[(K_{t_2} - S_{t_2-t_1})^+ \right]$$

= $s_{t_1}^{\text{Ber}} e^{-r(t_2-t_1)} \cdot E_{P_1} \left[\left(\frac{K_{t_2}}{s_{t_1}^{\text{Ber}}} - 1 \cdot e^{\left(r - \frac{\sigma^2}{2}\right)(t_2-t_1) + \sigma\left(W_{t_2-t_1}\right)} \right)^+ \right]$ (23)

Altogether, by considering $K_{t_1} > s_{t_1}^{\text{Ber}}$ and $\text{Put}\left(1, \frac{K_{t_2}}{s_{t_1}^{\text{Ber}}}, t_2 - t_1\right) > 0$ from formula (22) we get:

$$s_{t_1}^{\text{Ber}} = \frac{K_{t_1}}{1 + \text{Put}\left(1, \frac{K_{t_2}}{s_{t_1}^{\text{Ber}}}, t_2 - t_1\right)}$$
(24)

We now see that a constant strike price does not allow us to explicitly determine the stopping boundary. But, if we use

$$K_{t_2} := \frac{K_{t_1}}{x} S_{t_1} \tag{25}$$

as the strike price valid for t_2 (and if there was no execution at t_1), we will get the value indicated in (10):⁷

⁷Here, we also have to set $s_{t_1}^{\text{Ber}} = S_{t_1}^*$, $K_{t_1} = K$ and $t_2 - t_1 = \Delta t$ in formula (24).

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$$s_{t_1}^{\text{Ber}} = \frac{K}{1 + \text{Put}\left(1, \frac{K}{x}, \Delta t\right)} \tag{26}$$

Hence, we note that it is necessary to adjust the strike price in order to explicitly determine the stopping boundary. In the next section, it will become clear that we also have to adjust the strike price K_{t_2} in order to get an explicit evaluation formula.

3.2 Evaluation at t_0 and equivalence to the Bilodeau (1997) approach

The fair price of the rollover option at starting time t_0 can be determined from the Bermudan approach, as follows:

$$V_{t_0}^{\text{RO,Ber}} = \operatorname{ess} \sup_{\tau \in \mathcal{T}_{\text{Ber}}^{>t_0}} E_{P_x} \left[e^{-r\tau} \cdot f_\tau \middle| \mathcal{F}_{t_0} \right]$$
(27)

where we already considered that the option should not be executed at t_0 , but only at times t_1 and $t_2 = T$. $V_{t_0}^{\text{RO,Ber}}$ is the value of a rollover option (with two execution times in this case) using the Bermudan approach. Bearing in mind the representation of a time point between two execution times and the fact that \mathcal{F}_{t_0} is trivial, this term may be written as

$$V_{t_0}^{\text{RO,Ber}} = E_{P_x} \left[V_{t_1}^{\text{RO,Ber}} \right]$$
(28)

$$= E_{P_x} \left[e^{-rt_1} \cdot \max\left\{ f_{t_1}, E_{P_x} \left[f_{t_2} | \mathcal{F}_{t_1} \right] \cdot e^{-r(t_2 - t_1)} \right\} \right]$$
(29)

Explicitly, we get

$$V_{t_{0}}^{\text{RO,Ber}} = e^{-r \cdot t_{1}} \cdot E_{P_{x}} \\ \left[\left((K - S_{t_{1}}) \cdot \mathbf{1}_{\left\{ S_{t_{1}} < K \right\}} \right)^{*} \right] \\ \cdot \mathbf{1}_{\left\{ (K - S_{t_{1}})^{+} > e^{-r \cdot t_{2}} \cdot E_{P_{x}} \left[\left(K \cdot \frac{S_{t_{1}}}{x} - S_{t_{2}} \right)^{+} \middle| \mathcal{F}_{t_{1}} \right] \right\} \\ + \left(E_{P_{x}} \left[e^{-r(t_{2} - t_{1})} \cdot \left(K \cdot \frac{S_{t_{1}}}{x} - S_{t_{2}} \right) \cdot \mathbf{1}_{\left\{ K \cdot \frac{S_{t_{1}}}{x} > S_{t_{2}} \right\}} \middle| \mathcal{F}_{t_{1}} \right]$$
(30)
$$\cdot \mathbf{1}_{\left\{ (K - S_{t_{1}})^{+} \le e^{-r \cdot t_{2}} E_{P_{x}} \left[\left(K \cdot \frac{S_{t_{1}}}{x} - S_{t_{2}} \right)^{+} \middle| \mathcal{F}_{t_{1}} \right] \right\} \right) \right]$$

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Here, we substitute the stopping boundary $s^{\rm Ber}_{t_1}$ and get the formula

$$V_{t_{0}}^{\text{RO,Ber}} = e^{-r \cdot t_{1}} \cdot E_{P_{x}} \left[(K - S_{t_{1}}) \cdot \mathbf{1}_{\left\{S_{t_{1}} < K\right\}} \cdot \mathbf{1}_{\left\{S_{t_{1}} < s_{t_{1}}^{\text{Ber}}\right\}} \right]$$
(31)
+ $e^{-r \cdot t_{1}} \cdot E_{P_{x}} \left[E_{P_{x}} \left[e^{-r(t_{2} - t_{1})} \left(K \cdot \frac{S_{t_{1}}}{x} - S_{t_{2}} \right)^{+} \middle| \mathcal{F}_{t_{1}} \right]$
 $\cdot \mathbf{1}_{\left\{S_{t_{1}} \ge s_{t_{1}}^{\text{Ber}}\right\}} \right]$

Considering $s_{t_1}^{\text{Ber}} = S_{t_1}^*$, the first term in (31) can be written as in the case of the optimal behavior:⁸

$$e^{-rt_{1}} \cdot E_{P_{x}} \left[(K - S_{t_{1}}) \cdot \mathbf{1}_{\left\{S_{t_{1}} < K\right\}} \cdot \mathbf{1}_{\left\{S_{t_{1}} < s_{t_{1}}^{\text{Ber}}\right\}} \right]$$

= $E_{P_{x}} \left[e^{-rt_{1}} \left(K - S_{t_{1}}\right) \cdot \mathbf{1}_{\left\{S_{t_{1}} < s_{t_{1}}^{\text{Ber}}\right\}} \right]$ (32)

$$= Ke^{-rt_1} \cdot N\left(-d_2\left(\frac{x}{s_{t_1}^{\text{Ber}}}, t_1\right)\right) - x \cdot N\left(-d_1\left(\frac{x}{s_{t_1}^{\text{Ber}}}, t_1\right)\right)$$
(33)

$$= V^{\mathrm{Eu}}\left(s_{t_{1}}^{\mathrm{Ber}}, t_{1}\right) + N\left(-d_{2}\left(\frac{x}{s_{t_{1}}^{\mathrm{Ber}}}, t_{1}\right)\right) \cdot \left(K - s_{t_{1}}^{\mathrm{Ber}}\right)e^{-rt_{1}}$$
(34)

The second term in (31), we first rewrite as:

$$e^{-rt_{1}} \cdot E_{P_{x}} \left[E_{P_{x}} \left[e^{-r(t_{2}-t_{1})} \cdot \left(K \cdot \frac{S_{t_{1}}}{x} - S_{t_{2}} \right)^{+} \middle| \mathcal{F}_{t_{1}} \right] \cdot \mathbf{1}_{\left\{ S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}} \right\}} \right]$$
(35)
$$= e^{-rt_{1}} \cdot E_{P_{x}} \left[E_{P_{x}} \left[e^{-r(t_{2}-t_{1})} \cdot K \cdot \frac{S_{t_{1}}}{x} \cdot \mathbf{1}_{\left\{ K \cdot \frac{S_{t_{1}}}{x} > S_{t_{2}} \right\}} \middle| \mathcal{F}_{t_{1}} \right] \cdot \mathbf{1}_{\left\{ S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}} \right\}} \right]$$
$$-e^{-rt_{1}} \cdot E_{P_{x}} \left[E_{P_{x}} \left[e^{-r(t_{2}-t_{1})} \cdot S_{t_{2}} \cdot \mathbf{1}_{\left\{ K \cdot \frac{S_{t_{1}}}{x} > S_{t_{2}} \right\}} \middle| \mathcal{F}_{t_{1}} \right] \cdot \mathbf{1}_{\left\{ S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}} \right\}} \right]$$

There are now two equivalent possibilities to derive a more detailed form of this term. The first alternative uses the tower property of the conditional expectation. The second approach uses the Markovian property of the price process. It seems that Bilodeau (1997) has followed one of these alternatives, too. We will restrict ourselves to the first alternative using the tower property of the conditional expectation.

As an example, we will calculate the first term of (35). It is possible to rewrite this term (recalling that $\mathbf{1}_{\left\{S_{t_1} \geq s_{t_1}^{\text{Ber}}\right\}}$ is \mathcal{F}_{t_1} -measurable) in the form

⁸Compare with equalities (12) and (19).

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$$e^{-rt_{1}} \cdot E_{P_{x}} \left[E_{P_{x}} \left[e^{-r(t_{2}-t_{1})} K \cdot \frac{S_{t_{1}}}{x} \cdot \mathbf{1}_{\left\{K \cdot \frac{S_{t_{1}}}{x} > S_{t_{2}}\right\}} \middle| \mathcal{F}_{t_{1}} \right] \cdot \mathbf{1}_{\left\{S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}}\right\}} \right]$$

$$= e^{-rt_{1}} \cdot E_{P_{x}} \left[E_{P_{x}} \left[e^{-r(t_{2}-t_{1})} K \cdot \frac{S_{t_{1}}}{x} \cdot \mathbf{1}_{\left\{K \cdot \frac{S_{t_{1}}}{x} > S_{t_{2}}\right\}} \cdot \mathbf{1}_{\left\{S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}}\right\}} \middle| \mathcal{F}_{t_{1}} \right] \right] (36)$$

$$= \frac{K}{x} \cdot e^{-r(t_{2}-t_{1})} \cdot E_{P_{x}} \left[e^{-rt_{1}} S_{t_{1}} \cdot \mathbf{1}_{\left\{S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}}\right\}} \cdot \mathbf{1}_{\left\{K \cdot \frac{S_{t_{1}}}{x} > S_{t_{2}}\right\}} \right]$$

$$(37)$$

Now we take the terms for S_{t_1} and S_{t_2} and substitute them into $K \cdot \frac{S_{t_1}}{x} > S_{t_2}$, which leads us to:

$$K \cdot \frac{x}{x} \exp\left\{\sigma W_{t_1} + \left(r - \frac{\sigma^2}{2}\right)t_1\right\} > x \exp\left\{\sigma W_{t_2} + \left(r - \frac{\sigma^2}{2}\right)t_2\right\}$$
(38)

This can be formulated as:

$$\sigma \left(W_{t_1} - W_{t_2} \right) + \left(r - \frac{\sigma^2}{2} \right) \left(t_1 - t_2 \right) > \ln \frac{x}{K}$$
(39)

or even as

$$\sigma \left(W_{t_2} - W_{t_1} \right) + \left(r - \frac{\sigma^2}{2} \right) \left(t_2 - t_1 \right) < \ln \frac{K}{x}$$
(40)

Now we introduce S_{t_1} into the equality (37) and have the result

$$\frac{K}{x} \cdot e^{-r(t_2 - t_1)} \cdot E_{P_x} \left[e^{-rt_1} S_{t_1} \cdot \mathbf{1}_{\left\{S_{t_1} \ge s_{t_1}^{\mathrm{Ber}}\right\}} \cdot \mathbf{1}_{\left\{K \cdot \frac{S_{t_1}}{x} > S_{t_2}\right\}} \right] \quad (41)$$

$$= K \cdot e^{-r(t_2 - t_1)} \cdot E_{P_x} \left[e^{-rt_1} \exp \left\{ \sigma W_{t_1} + \left(r - \frac{\sigma^2}{2}\right) t_1 \right\} \right] \quad (41)$$

$$\cdot \mathbf{1}_{\left\{W_{t_1} \ge \ln \frac{\frac{s_{\mathrm{Ber}}}{x} - \left(r - \frac{\sigma^2}{2}\right) t_1}{\sigma}\right\}} \cdot \mathbf{1}_{\left\{(W_{t_2} - W_{t_1}) < \frac{\ln \frac{K}{x} - \left(r - \frac{\sigma^2}{2}\right) (t_2 - t_1)}{\sigma}\right\}} \right]$$

As we know that W is a Brownian motion, we also know that the random variables $(W_{t_2} - W_{t_1})$ and W_{t_1} are independent. This leads us to

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$$\frac{K}{x} \cdot e^{-r(t_{2}-t_{1})} \cdot E_{P_{x}} \left[e^{-rt_{1}} S_{t_{1}} \cdot \mathbf{1}_{\left\{S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}}\right\}} \cdot \mathbf{1}_{\left\{K \cdot \frac{S_{t_{1}}}{x} > S_{t_{2}}\right\}} \right]$$

$$= K \cdot e^{-r(t_{2}-t_{1})} \cdot E_{P_{x}} \left[e^{-rt_{1}} \exp\left\{\sigma W_{t_{1}} + \left(r - \frac{\sigma^{2}}{2}\right)t_{1}\right\} \cdot \mathbf{1}_{\left\{W_{t_{1}} \ge \ln\frac{\frac{S_{\mathrm{Ber}}}{x} - \left(r - \frac{\sigma^{2}}{2}\right)t_{1}\right\}} \right]$$

$$\cdot P_{x} \left((W_{t_{2}} - W_{t_{1}}) < \frac{\ln\frac{K}{x} - \left(r - \frac{\sigma^{2}}{2}\right)(t_{2} - t_{1})}{\sigma} \right)$$

$$(42)$$

The term (42) can now be calculated in the following manner: First, we set again

$$Y_{t_1} := \sigma W_{t_1} + \left(r - \frac{\sigma^2}{2}\right) t_1 \tag{43}$$

As $Y_{t_1} \sim \mathcal{N}\left(\left(r - \frac{\sigma^2}{2}\right)t_1, \sigma^2 t_1\right)$ and with the transformation

$$z := \frac{Y_{t_1} - \left(r - \frac{\sigma^2}{2}\right)t_1}{\sigma\sqrt{t_1}} \tag{44}$$

we arrive at the following equation

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$$E_{P_{x}}\left[e^{-rt_{1}}\exp\left\{\sigma W_{t_{1}}+\left(r-\frac{\sigma^{2}}{2}\right)t_{1}\right\}\cdot\mathbf{1}_{\left\{W_{t_{1}}\geq\ln\frac{s_{t_{1}}^{\mathrm{Ber}}-\left(r-\frac{\sigma^{2}}{2}\right)t_{1}}{\sigma}\right\}}\right]$$
$$=\int_{\ln\frac{s_{t_{1}}^{\mathrm{Ber}}-\left(r-\frac{\sigma^{2}}{2}\right)t_{1}}{\ln\frac{s_{t_{1}}^{\mathrm{Ber}}-\left(r-\frac{\sigma^{2}}{2}\right)t_{1}}{\sigma}}}\left(45\right)$$

$$= \int_{\frac{\ln \frac{s_{\text{Er}}^{\text{Ber}}}{x} - \left(r - \frac{\sigma^2}{2}\right)t_1}}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\left(z - \sigma\sqrt{t_1}\right)^2}{2}} dz$$
(46)

$$= \int_{-\infty}^{\frac{\sigma\sqrt{t_1}}{\sigma\sqrt{t_1}}} \frac{\frac{\ln \frac{x}{s_{t_1}^{\text{Ber}}} + \left(r + \frac{\sigma^2}{2}\right)t_1}{\sigma\sqrt{t_1}}}{\int_{-\infty}^{\sigma\sqrt{t_1}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}dz}$$
(47)

$$= N\left(d_1\left(\frac{x}{s_{t_1}^{\text{Ber}}}, t_1\right)\right) \tag{48}$$

In the case of setting $S^{\ast}_{t_1}=s^{\rm Ber}_{t_1},$ analogously to Bilodeau (1997) we would have:

$$N\left(d_1\left(\frac{x}{s_{t_1}^{\text{Ber}}}, t_1\right)\right) = N\left(d_1^*\left(x, t_1\right)\right)$$
(49)

Altogether, for (42) we get:

$$\frac{K}{x} \cdot e^{-r(t_2 - t_1)} \cdot E_{P_x} \left[e^{-rt_1} S_{t_1} \cdot \mathbf{1}_{\left\{S_{t_1} \ge s_{t_1}^{\mathrm{Ber}}\right\}} \cdot \mathbf{1}_{\left\{K \cdot \frac{S_{t_1}}{x} > S_{t_2}\right\}} \right] \quad (50)$$

$$= K \cdot e^{-r(t_2 - t_1)} \cdot N \left(d_1 \left(\frac{x}{s_{t_1}^{\mathrm{Ber}}}, t_1 \right) \right) \cdot N \left(-d_2 \left(\frac{x}{K}, t_2 - t_1 \right) \right)$$

This is the same term as the third term in (15) from the "optimal" approach with $\Delta t = t_2 - t_1$ and $S_{t_1}^* = s_{t_1}^{\text{Ber}}$:

$$K \cdot e^{-r(\Delta t)} \cdot N\left(d_1^*\left(x, t_1\right)\right) \cdot N\left(-d_2\left(\frac{x}{K}, \Delta t\right)\right)$$
(51)

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The second sum of (35) can be calculated in an analogous way and hence results in:

$$E_{P_x}\left[e^{-rt_1} \cdot E_{P_x}\left[e^{-r(t_2-t_1)} \cdot S_{t_2} \cdot \mathbf{1}_{\left\{K \cdot \frac{S_{t_1}}{x} > S_{t_2}\right\}} \middle| \mathcal{F}_{t_1}\right] \cdot \mathbf{1}_{\left\{S_{t_1} \ge s_{t_1}^{\mathrm{Ber}}\right\}}\right] (52)$$
$$= x \cdot N\left(d_1\left(\frac{x}{s_{t_1}^{\mathrm{Ber}}}, t_1\right)\right) \cdot N\left(-d_1\left(\frac{x}{K}, t_2 - t_1\right)\right)$$

By substituting (52) and (35) in (31), we can specify the value of the rollover option with the Bermudan approach as:

$$V^{\text{RO,Ber}} = V^{\text{Eu}}\left(s_{t_{1}}^{\text{Ber}}, t_{1}\right) + N\left(-d_{2}\left(\frac{x}{s_{t_{1}}^{\text{Ber}}}, t_{1}\right)\right) \cdot \left(K - s_{t_{1}}^{\text{Ber}}\right)e^{-rt_{1}}$$
(53)
+ $N\left(d_{1}\left(\frac{x}{s_{t_{1}}^{\text{Ber}}}, t_{1}\right)\right) \cdot \left\{K \cdot e^{-r(t_{2}-t_{1})} \cdot N\left(-d_{2}\left(\frac{x}{K}, t_{2}-t_{1}\right)\right)\right)$
- $x \cdot N\left(-d_{1}\left(\frac{x}{K}, t_{2}-t_{1}\right)\right)\right\}$

So, using the alternative of the tower property, bearing in mind the slightly different notations, we have that

$$V_{t_0}^{\text{RO,Ber}} = V_{t_0}^{\text{RO,opt}} \tag{54}$$

3.3 Evaluation for any time

As the final task of pricing the rollover option, we will specify its values for the relevant regions $[t_0, t_1), \{t_1\}, (t_1, t_2)$. The time t_2 is obvious and will be included in the analysis of the interval (t_1, t_2) . We use the notation V^{RO} for the value process of the rollover option, whose value at t_0 is given by (54).

1. Case: $t \in (t_1, t_2]$

It is true for $t \in (t_1, t_2]$, as S_{t_1} is \mathcal{F}_{t_1} -measurable that S_t is \mathcal{F}_t -measurable for every $t \in (t_1, t_2]$. Thus, also $\widetilde{K} := K \cdot \frac{S_{t_1}}{x}$ is measurable with respect to \mathcal{F}_t . Now, we can calculate the value

$$V_t^{\text{RO}} = E_{P_x} \left[\left(K \cdot \frac{S_{t_1}}{x} - S_{t_2} \right)^+ \middle| \mathcal{F}_t \right] \cdot e^{-r(t_2 - t)}$$
(55)

In Zimmer (2000), Section 6.1, Lemma 6.1.2, it was shown that the value process in the interval $(t_1, t_2]$ is a martingale. The term (55) is hence given by the well-known formula of a European put option:

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$$V_t^{\text{RO}} = \widetilde{K} \cdot N\left(-d_2\left(\frac{S_t}{\widetilde{K}}, t_2 - t\right)\right) \cdot e^{-r(t_2 - t)}$$

$$- S_t \cdot N\left(-d_1\left(\frac{S_t}{\widetilde{K}}, t_2 - t\right)\right) e^{-r(t_2 - t)}, t \in (t_1, t_2]$$
(56)

2. Case: $t = t_1$

For $t = t_1$ it holds that

$$V_{t_1}^{\text{RO}} = \max\left\{ \left(K - S_{t_1}\right)^+, E_{P_x} \left[\left(K \cdot \frac{S_{t_1}}{x} - S_{t_2}\right)^+ \cdot e^{-r(t_2 - t_1)} \middle| \mathcal{F}_{t_1} \right] \right\}$$
(57)

which we can rewrite, as in the above cases, as

$$V_{t_{1}}^{\text{RO}} = (K - S_{t_{1}}) \cdot \mathbf{1}_{\left\{S_{t_{1}} < s_{t_{1}}^{\text{Ber}}\right\}}$$

$$+ E_{P_{x}} \left[\left(K \cdot \frac{S_{t_{1}}}{x} - S_{t_{2}} \right)^{+} \cdot e^{-r(t_{2} - t_{1})} \middle| \mathcal{F}_{t_{1}} \right] \cdot \mathbf{1}_{\left\{S_{t_{1}} \ge s_{t_{1}}^{\text{Ber}}\right\}}$$
(58)

As in the second term, we only have to calculate the value of the payoff at t_1 , and as the process $(E_{P_x} [f_{t_2} \cdot e^{-r(t_2-t)} | \mathcal{F}_t])_{t \in [t_1, t_2]}$ is a martingale at $[t_1, t_2]$, analogously to the first case, we have:

$$E_{P_x}\left[\left(\widetilde{K} - S_{t_2}\right)^+ \cdot e^{-r(t_2 - t_1)} \middle| \mathcal{F}_{t_1}\right] = \widetilde{K} \cdot N\left(-d_2\left(\frac{S_{t_1}}{\widetilde{K}}, t_2 - t_1\right)\right) \cdot e^{-r(t_2 - t_1)} - S_{t_1} \cdot N\left(-d_1\left(\frac{S_{t_1}}{\widetilde{K}}, t_2 - t_1\right)\right) e^{-r(t_2 - t_1)} e^{-r(t_2 - t_1)}$$

$$(59)$$

Altogether, we can write:

$$V_{t_{1}}^{\text{RO}} = (K - S_{t_{1}}) \cdot \mathbf{1}_{\left\{S_{t_{1}} < s_{t_{1}}^{\text{Ber}}\right\}} + \mathbf{1}_{\left\{S_{t_{1}} \ge s_{t_{1}}^{\text{Ber}}\right\}}$$

$$\cdot \left\{e^{-r(t_{2} - t_{1})}\widetilde{K} \cdot N\left(-d_{2}\left(\frac{S_{t_{1}}}{\widetilde{K}}, t_{2} - t_{1}\right)\right)\right\}$$

$$-S_{t_{1}} \cdot N\left(-d_{1}\left(\frac{S_{t_{1}}}{\widetilde{K}}, t_{2} - t_{1}\right)\right)\right\}$$
(60)

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3. Case: $t \in [t_0, t_1)$

This case was already proven in the previous section, and hence, for every $t \in [t_0, t_1)$, we analogously have:

$$V_t^{\text{RO}} = V_t^{\text{Eu}} \left(s_{t_1}^{\text{Ber}}, t_1 - t \right) + N \left(d_1^* \left(S_t, t_1 - t \right) \right) \cdot V_t^{\text{Eu}} \left(K, t_2 - (t_1 - t) \right)$$
(61)
+ $N \left(-d_2^* \left(S_t, t_1 - t \right) \right) \cdot \left(K - s_{t_1}^{\text{Ber}} \right) \cdot e^{-r(t_1 - t)}$

4. Determining a Hedging Strategy

As the rollover option is a special case of the Bermudan option, we can apply the theorem of the existence of an optional decomposition, which gives us the existence of a hedging strategy. During the derivation of the original approach, this existence was only assumed because the Bermudan approach was yet unknown. Now, we will explicitly specify the replicating trading strategy $\phi = (\phi_t)_{t \in [0,T]} = (\phi_t^{\rm S}, \phi_t^{\rm K})_{t \in [0,T]}$ abusing the special structure of the Black-Scholes model. In order to do this, we have to analyze different cases: At time $t = t_2$ it holds that $V_{t_2}^{\text{RO}} = (K_{t_2} - S_{t_2})^+$,

hence we immediately calculate $\phi_{t_2}^{\mathrm{S}} = \frac{(K_{t_2} - S_{t_2})^+}{S_{t_2}}$ and $\phi_{t_2}^{\mathrm{K}} = 0$. Let $I_1 = (t_1, t_2)$, $I_2 = \{t_1\}$, $I_3 = [t_0, t_1)$ be subsets of \mathbb{R}^+ , $I = I_1 \cup I_2 \cup I_3$ and $\mathcal{D} = (0, +\infty) \times I$. Let further $v : \mathcal{D} \to \mathbb{R}$ be given by $v(S_t, t) = V_t^{\mathrm{RO}}$. We then know that $v|_{I_1 \cup I_2} \in C^{1,2}(I_1 \cup I_2, \mathbb{R})$ and $v|_{I_3} \in C^{1,2}(I_3, \mathbb{R})$, which

means that v is sufficiently smooth on the considered intervals. We will write

$$v_s(s,t) = \frac{\partial v(s,t)}{\partial s} \tag{62}$$

for the partial derivative of v in the first variable. Again, we use the fact that the value processes between two execution times are martingales. We do not only know that a hedging strategy exists, but we can also derive it directly from the martingale representation theorem. If we ignore, in a first step, the question about whether the holder of the option executed his right or not, then we can formulate the strategy for the stock respectively for the savings account for times $t \in (t_{i-1}, t_i)$ for i = 1, 2 as:

$$\phi_t^{\rm S} = v_s \left(S_t, t \right) \tag{63}$$

$$\phi_t^{\rm K} = e^{-rt} \left(V_t^{\rm RO} - \phi_t^{\rm S} \cdot S_t \right)$$

As another example, let us derive the strategy for the case $t = t_1$. It also indicates how to use the knowledge about whether the holder of the option used his right of execution. First of all, analogously to the above case, we have:

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$$\phi_{t_1}^{S} = v_s \left(S_{t_1}, t_1 \right)$$

$$= -\left(\mathbf{1}_{\left\{ S_{t_1} < s_{t_1}^{\text{Ber}} \right\}} + N\left(-d_1 \left(\frac{S_{t_1}}{\widetilde{K}}, t_2 - t_1 \right) \right) \cdot \mathbf{1}_{\left\{ S_{t_1} \ge s_{t_1}^{\text{Ber}} \right\}} \right)$$
(64)

and

$$\phi_{t_{1}}^{\mathrm{K}} = e^{-rt_{1}} \left(V_{t_{1}}^{\mathrm{RO}} - \phi_{t_{1}}^{\mathrm{S}} \cdot S_{t_{1}} \right)$$

$$= \widetilde{K} \cdot \left(e^{-rt_{1}} \cdot \mathbf{1}_{\left\{ S_{t_{1}} < s_{t_{1}}^{\mathrm{Ber}} \right\}} + e^{-rt_{2}} \cdot N \left(-d_{2} \left(\frac{S_{t_{1}}}{\widetilde{K}}, t_{2} - t_{1} \right) \right)$$

$$\cdot \mathbf{1}_{\left\{ S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}} \right\}} \right)$$
(65)

Here we have to pay special attention to the possibility of execution of the option. The buyer of the option will behave optimally, if he executes his right as soon as $S_{t_1} \geq s_{t_1}^{\text{Ber}}$ becomes true. If he does not execute his right following this rule, the seller of the option will realize a riskless gain at t_1 with the value of

$$C_{t_1} := f_{t_1} - E_{P_x} \left[f_{t_2} | \mathcal{F}_{t_1} \right] e^{-r(t_2 - t_1)}.$$
(66)

For simplicity, we assume that the decision of execution (which obviously can be suboptimal) is taken independently of the price of the underlying S. We will model this event on the same probability space and define it as \mathcal{F}_{t_1} -measurable: $A := \{$ "The buyer of the option does not execute his right at t_1 " $\}$.

The additional consumption arising for the seller of the option in the case of a suboptimal execution will be accounted for the savings account and will only have a value for the seller if the residual value of the option is smaller than the actual payoff at t_1 :

$$\phi_{t_{1}}^{\mathrm{K}} = e^{-rt_{1}} \left(V_{t_{1}}^{\mathrm{RO}} - \phi_{t_{1}}^{\mathrm{S}} \cdot S_{t_{1}} \right) + \mathbf{1}_{A} \cdot \mathbf{1}_{\left\{ S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}} \right\}} \left(C_{t_{1}} \right)$$

$$= \widetilde{K} \cdot \left(e^{-rt_{1}} \cdot \mathbf{1}_{\left\{ S_{t_{1}} < s_{t_{1}}^{\mathrm{Ber}} \right\}} + N \left(-d_{2} \left(\frac{S_{t_{1}}}{\widetilde{K}}, t_{2} - t_{1} \right) \right) \cdot \mathbf{1}_{\left\{ S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}} \right\}} \right)$$

$$+ \mathbf{1}_{A} \cdot \mathbf{1}_{\left\{ S_{t_{1}} \ge s_{t_{1}}^{\mathrm{Ber}} \right\}}$$

$$\left(\left(K - S_{t_{1}} \right)^{+} - E_{P_{x}} \left[\left(K \frac{S_{t_{1}}}{x} - S_{t_{2}} \right)^{+} \middle| \mathcal{F}_{t_{1}} \right] e^{-r(t_{2} - t_{1})} \right)$$
(67)

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5. Conclusion

In this paper, we showed how the Bermudan option can be used to generalize the rollover option. We showed how the Bermudan approach replicates the results of Bilodeau (1997) to price the rollover option, and provided additional insights into the problems of pricing and hedging of this option. We explicitly carried out the calculations following either the original approach of Bilodeau (1997) or the steps necessary to price a Bermudan option. Both approaches yielded the same result. We made it also clear that when the strike prices were readjusted at every possible execution time, an explicit result could be obtained.

This approach may be of practical interest for investment vehicles, such as the Brazilian PIBB, when further attractiveness via a renewal option is desired.

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