The Exponentiated-Log-logistic Geometric Distribution: Dual Activation

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Abstract

The log-logistic distribution is commonly used to model lifetimes. We propose a wider distribution called the exponentiated log-logistic geometric distribution, based on a double activation approach. We obtain the quantile function, ordinary moments and generating function. The method of maximum likelihood is used to estimate the model parameters. We propose a new extended regression model based on the logarithm of the exponentiated log-logistic geometric distribution. This regression model can be very useful in the analysis of real data and could provide better fits than other special regression models. The potentiality of the new models is illustrated by means of two applications to real lifetime data sets.

Keywords: Geometric distribution; Log-logistic distribution; Maximum likelihood estimation; Regression model; Survival analysis.
1 Introduction

The log-logistic distribution is widely used in practice and it is an alternative to the log-normal distribution since it presents a failure rate function that increases, reaches a peak after some finite period and then declines gradually. The properties of the log-logistic distribution make it an attractive alternative to the log-normal and Weibull distributions in the analysis of survival data (Collet, 2003). This distribution can exhibit a monotonically decreasing failure rate function for some parameter values. It shares some properties of the log-normal and normal distributions (Ahmad et al., 1988), i.e., if \( Y \) has a logistic distribution, then \( T = e^Y \) has a log-logistic distribution. Some applications of the log-logistic distribution are discussed in economy to model the wealth and income (Kleiber and Kotz, 2003) and in hydrology to model stream flow data (Ashkar and Mahdi, 2006). Collet (2003) suggested the log-logistic distribution for modeling the time following a heart transplantation.

Alternatively, several authors proposed new distributions that are more flexible in modeling monotone or unimodal failure rates but they are not useful for modeling bathtub shaped failure rates. Adamidis and Loukas (1998) introduced the exponential geometric (EG) distribution to model lifetime data with decreasing failure rate function. Gupta and Kundu (1999, 2001, 2007) provided a comprehensive mathematical treatment of the so-called generalized exponential (GE) distribution, also referred to as the exponentiated exponential distribution, Carrasco et al. (2008) defined a generalized modified Weibull with applications in survival analysis and Silva et al. (2010) studied in details the beta modified Weibull distribution, which admits only increasing and decreasing failure rate functions. Following the same idea of the GE distribution, Silva et al. (2010) defined the generalized exponential geometric (GEG) distribution and showed that its failure rate function can be increasing, decreasing or unimodal. A Weibull geometric (WG) extension of the GE distribution was proposed by Barreto-Souza et al. (2011) for modeling monotone or unimodal failure rates and recently Cordeiro et al. (2011) proposed a new distribution, the so-called beta-Weibull geometric distribution, whose failure rate function can be decreasing, increasing or an upside-down bathtub.

Here, we introduce the exponentiated log-logistic geometric (ELLG) distribution based on a double activation mechanism that extends the exponentiated log-logistic (ELL) and log-logistic (LL) distribution. The ELL distribution ELL was introduced by Rosaiah et al. (2006), but the authors do not discuss its properties. In this paper, we propose and derive some structural properties of the ELLG and ELL distributions. The first model due to its flexibility in accommodating unimodal failure rate functions can be used in a variety of problems in modeling survival data. In addition, we propose a generalized regression model based on the logarithm of a random variable following the ELLG distribution with double activation. This regression model extends the exponentiated logistic and logistic regression models.

The rest of the paper is organized as follows. In Section 2, we define two ELLG models. In Section 3, we determine the moments and moment generating function (mgf) for both models. Maximum likelihood estimation of the model parameters and the observed information matrix are investigated in Section 4. We propose two extended exponentiated logistic geometric (ELG)
regression models in Section 5. In addition, we provide two applications to real data in Section 6. Finally, Section 7 offers some concluding remarks.

2 The exponentiated log-logistic geometric distribution with dual activation

Suppose that \( \{ Y_i \}_{i=1}^{Z_i} = 1 \) are independent and identically distributed (i.i.d.) random variables having the ELL distribution. Its cumulative distribution function (cdf) and probability density function (pdf) are defined by

\[
F_{a,\alpha,\beta}(y) = \left[ \left( \frac{y}{\alpha} \right)^{\alpha} \right]^\beta \quad \text{and} \quad f_{a,\alpha,\beta}(y) = \frac{a \beta}{\alpha} \left( \frac{y}{\alpha} \right)^{-\beta} \left[ 1 + \left( \frac{y}{\alpha} \right)^{-\beta} \right]^{-1} \left( \frac{y}{\alpha} \right)^{-\beta},
\]

respectively, where \( \beta > 0 \) is a scale parameter, and \( a > 0 \) and \( \alpha > 0 \) are shape parameters. For \( a = 1 \), we obtain as a special case the LL distribution. Basic properties of this distribution are given, for example, by Kleiber and Kotz (2003), Lawless (2003) and Ashkar and Mahdi (2006). The moments for \( a = 1 \) are easily derived as (Tadikamalla, 1980)

\[
E(Y^r) = \frac{\alpha \beta}{\alpha} \frac{\Gamma(2 - r \beta - 1)}{\Gamma(2 - r \beta)}, \quad r < \beta,
\]

where \( \Gamma(a, b) = \Gamma(a) \Gamma(b) / \Gamma(a + b) \) is the beta function and \( \Gamma(\cdot) \) is the gamma function. Hence,

\[
E(Y) = \frac{\pi \alpha \beta^{-1}}{\sin(\pi \beta^{-1})}, \quad \beta > 1 \quad \text{and} \quad \Var(Y) = \frac{2 \pi \alpha^2 \beta^{-1}}{\sin(2 \pi \beta^{-1})} - \left[ \frac{\pi \alpha \beta^{-1}}{\sin(\pi \beta^{-1})} \right]^2, \quad \beta > 2.
\]

Let \( Z \) be a geometric random variable with probability mass function \( P(z; p) = (1 - p)p^{z-1} \) for \( Z \in N \) and \( p \in (0, 1) \). Hereafter, let \( X_1 = \min(\{ Y_i \}_{i=1}^{Z_i}) \) and \( X_2 = \max(\{ Y_i \}_{i=1}^{Z_i}) \). In the latent competitive risks scenario, we consider two activations approaches:

- For \( X_1 = \min(\{ Y_i \}_{i=1}^{Z_i}) \):

  The conditional density function of \( X_1 \) given \( Z = z \) is

  \[
  f(x|z; a, \alpha, \beta) = \frac{z \beta a}{\alpha} \left( \frac{x}{\alpha} \right)^{\alpha-1} \left[ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right]^{-(a+1)} \left( \frac{x}{\alpha} \right)^{-\beta} \left( 1 - \left[ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right]^{-a} \right)^{z-1}.
  \]

  Then, the density function of the three-parameter log-logistic geometric type I (ELLGI) distribution reduces to

  \[
  f(x; a, p, \alpha, \beta) = \frac{(1-p) \beta a}{\alpha} \left( \frac{x}{\alpha} \right)^{\alpha-1} \left[ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right]^{-(a+1)} \left( 1 - p \left[ 1 - \left( \frac{x}{\alpha} \right)^{-\beta} \right]^{-a} \right)^{-2}, \quad (1)
  \]
where $x > 0$, $\beta > 0$ is a scalar parameter and $\alpha > 0$ and $p \in (0, 1)$ are shape parameters. The random variable $X_1$ having density function (1) is denoted by $X_1 \sim \text{ELLGI}(a, p, \alpha, \beta)$. The cdf of $X_1$ becomes

$$F(x; a, p, \alpha, \beta) = \left[ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right]^{-a} \left\{ 1 - p \left[ 1 - \left\{ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right\}^{-a} \right] \right\}^{-1}, \quad x > 0. \quad (2)$$

The hazard rate functions corresponding to (1) is

$$h(x; a, p, \alpha, \beta) = \frac{(1 - p) \beta a (x)}{\alpha} \left( \frac{x}{\alpha} \right)^{\beta a - 1} \left[ 1 + \left( \frac{x}{\alpha} \right)^{\beta (a+1)} \right]^{-a} \left\{ 1 - p \left[ 1 - \left\{ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right\}^{-a} \right] \right\}^{-1} \times$$

$$\left\{ 1 - p \left[ 1 - \left\{ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right\}^{-a} \right] \right\} - \left\{ 1 + \left( \frac{x}{\alpha} \right)^{-\beta} \right\}^{-a} \right\}^{-1}, \quad x > 0.$$

For $|z| < 1$ and $\rho > 0$, we work throughout with the power series

$$(1 - z)\rho = \sum_{j=0}^{\infty} \frac{\Gamma(\rho + j)}{\Gamma(\rho) j!} z^j. \quad (3)$$

Using (3) in equation (1), the density function of $X_1$ can be written as

$$f(x; a, p, \alpha, \beta) = (1 - p) f_{a,\alpha,\beta}(x) \sum_{j=0}^{\infty} (j + 1) p^j [1 - F_{a,\alpha,\beta}(x)]^j. \quad (4)$$

where $F_{a,\alpha,\beta}(x)$ is the ELL cumulative distribution. In addition, using the binomial expansion in (4), and changing $\sum_{j=0}^{\infty} \sum_{i=0}^{j} \sum_{i=0}^{\infty}$ by $\sum_{i=0}^{\infty} \sum_{j=i}^{\infty}$, we can obtain

$$f(x; a, p, \alpha, \beta) = \sum_{i=0}^{\infty} w_i f_{a(i+1),\alpha,\beta}(x), \quad (5)$$

where $w_i = (-1)^i (1 - p) (i + 1)^{-1} \sum_{j=i}^{\infty} (j + 1) p^j \binom{j}{i}$ and $f_{a(i+1),\alpha,\beta}(x)$ denotes the ELL($a(i+1), \alpha, \beta$) density function.

By inverting (2) we can obtain the quantile function of $X_1$ (for $0 < u < 1$)

$$Q(u) = F^{-1}(u) = \alpha \left\{ \left[ \frac{1 - up}{u(1-p)} \right]^{1/a} - 1 \right\}^{-1/\beta}. \quad (4)$$
For $X_2 = \max\{Y_i\}_{i=1}^{Z}$: The conditional density function of $X_2$ given $Z = z$ is

$$f(x|z; \alpha, \beta) = \frac{z^\beta a}{\alpha} \frac{(x/z)^{\beta a - 1}}{1 + (x/a)^{\beta}} \left[ 1 + \left( \frac{x}{a} \right)^{-\beta} \right]^{-a} (1 - p)^{-a} \left[ 1 + \left( \frac{x}{a} \right)^{-\beta} \right]^{-a}, \quad x > 0.$$  

Then, the density function of the three-parameter exponentiated log-logistic geometric type II (ELLGII) distribution reduces to

$$f(x; a, p, \alpha, \beta) = (1 - p) \frac{\beta a}{\alpha} \left( \frac{x}{a} \right)^{\beta a - 1} \left[ 1 + \left( \frac{x}{a} \right)^{\beta} \right]^{-(a+1)} \left( 1 - p \left[ 1 + \left( \frac{x}{a} \right)^{-\beta} \right]^{-a} \right)^{-2}, \quad x > 0. \quad (6)$$

The cdf corresponding to (6) becomes

$$F(x; a, p, \alpha, \beta) = (1 - p) \left[ 1 + \left( \frac{x}{a} \right)^{-\beta} \right]^{-a} \left( 1 - p \left[ 1 + \left( \frac{x}{a} \right)^{-\beta} \right]^{-a} \right)^{-1}, \quad x > 0. \quad (7)$$

A random variable $X_2$ having density function (6) is denoted by $X_2 \sim \text{ELLGII}(a, p, \alpha, \beta)$. The hazard rate function corresponding to (6) is

$$h(x; a, p, \alpha, \beta) = \frac{(1 - p) \beta a}{\alpha} \left( \frac{x}{a} \right)^{\beta a - 1} \left[ 1 + \left( \frac{x}{a} \right)^{\beta} \right]^{-(a+1)} \left( 1 - p \left[ 1 + \left( \frac{x}{a} \right)^{-\beta} \right]^{-a} \right)^{-2}, \quad x > 0.$$  

respectively.

Using (3) in equation (6), the density function of $X_2$ can be expressed as

$$f(x; a, p, \alpha, \beta) = (1 - p) \sum_{j=0}^{\infty} p^j f_{a, \alpha, \beta}(x) F_{a, \alpha, \beta}^{j}(x), \quad (8)$$

and, after some algebraic manipulation, we can write

$$f(x; a, p, \alpha, \beta) = \sum_{j=0}^{\infty} v_j f_{a(j+1), \alpha, \beta}(x), \quad (9)$$

where $v_j = (1 - p)^j$ and $f_{a(j+1), \alpha, \beta}(x)$ is the ELL($a(j + 1), \alpha, \beta$) density function.

We simulate the ELLGII distribution from the quantile function (for $0 < u < 1$)

$$Q(u) = F^{-1}(u) = a \left\{ \left[ \frac{(1 - p) + up}{u} \right]^{1/a} - 1 \right\}^{-1/\beta}.$$  

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The ELLGI and ELLGII distributions can be used for modeling lifetimes with large variability. Some special models of these distributions are obtained as follows. If $p = 0$, both models reduce to the ELL distribution. For $a = 1$, we have the log-logistic geometric type I (LLGI) and log-logistic geometric type II (LLGII) distributions. If $a = 1$ and $p = 0$, we obtain the classic log-logistic distribution. When $p \to 1^-$, it comes the degenerated distribution of a random variable $Y$ such that $P(Y = 0) = 1$. Hence, the parameter $p$ can be interpreted as a degeneration parameter. Plots of the ELLGI and ELLGII density functions for selected parameter values are displayed in Figures 1 and 1, respectively.

![Plots of the ELLGI density function](image)

Figure 1: Plots of the ELLGI density function for some parameter values. (a) For different values of $a$, $p$ and $\alpha$ with $\beta = 4.5$. (b) For different values of $a$, $p$ and $\beta$ with $\alpha = 1.5$. (c) For different values of $a$ and $p$ with $\alpha = 0.5$ and $\beta = 3.5$.

Plots of the ELLGI and ELLGII hazard rate functions for selected parameter values are displayed in Figures 3 and 4, respectively. Note that the failure rate of the new distributions have all types of shapes: increasing, decreasing, unimodal, U-shaped and bimodal.

### 3 Properties

We hardly need to emphasize the necessity and importance of moments and moment generating function in any statistical analysis especially in applied work. Some of the most important features
and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis).

3.1 Moments

**Theorem 1:** If \( Z \sim \text{ELL}(a, \alpha, \beta) \), the \((r, n)\)th probability weighted moment (PWM) of \( Z \) becomes

\[
M(r, n) = a \alpha^r B\left(a(n + 1) + \frac{r}{\beta}, 1 - \frac{r}{\beta}\right).
\]  

(10)

**Proof:**

The \((r, n)\)th probability weighted moment (PWM) of \( Z \) is

\[
M(r, n) = E[X^r F^n(x)] = \int_0^\infty x^r f_{a,\alpha,\beta}(x) F^n_{a,\alpha,\beta}(x) dx
\]

\[
= a \beta^\alpha \int_0^\infty x^{r-\beta-1} \left[1 + \left(\frac{x}{\alpha}\right)^{-\beta}\right]^{-a(n+1)-1} dx.
\]

Figure 2: Plots of the ELLGII density function for some parameter values. (a) For different values of \( a \), \( p \) and \( \alpha \) with \( \beta = 4.5 \). (b) For different values of \( a \), \( p \) and \( \beta \) with \( \alpha = 1.5 \). (c) For different values of \( a \) and \( p \) with \( \alpha = 0.5 \) and \( \beta = 3.5 \).
Figure 3: Plots of the ELLGI hazard rate function for some parameter values. (a) For different values of $a$, $p$ and $\beta$ with $\alpha = 1.5$. (b) For different values of $a$, $p$ and $\alpha$ with $\beta = 4.5$. (c) For different values of $a$, $p$ and $\beta$ with $\alpha = 1.5$.

Setting $k = \left[ 1 + \left( \frac{x}{\alpha} \right)^{\beta} \right]^{-a(n+1)-1}$ in the last equation yields

$$M(r, n) = \frac{\alpha^r}{(n+1)} \int_0^1 \left\{ k^{a(n+1)} \right\}^{r/\beta} \left\{ 1 - k^{a(n+1)} \right\}^{-r/\beta} dk,$$

and then transforming $u = k^{a(n+1)}$, we obtain

$$M(r, n) = a \alpha^r \int_0^1 u^{r\beta-1+a(n+1)-1} (1 - u)^{-r\beta-1} du.$$

Finally, we arrive at (10). For $a = 1$, it follows the $(r, n)$th PWM of the LL distribution as

$$M(r, n) = \alpha^r B\left( a(n+1) + \frac{r}{\beta}, 1 - \frac{r}{\beta} \right).$$

Further, if $a = 1$, in addition to $n = 0$, it reduces to the $r$th moment of the LL distribution

$$M(r, 0) = \mu'_r = \alpha^r B\left( 1 + \frac{r}{\beta}, 1 - \frac{r}{\beta} \right).$$
Theorem 2: If $X_1 \sim \text{ELLGI}(a, p, \alpha, \beta)$, the $r$th moment of $X_1$ is given by

$$
\mu'_r = E(X^r_1) = a a^r \sum_{i=0}^{\infty} w_i (i+1) B \left( a(i+1) + \frac{r}{\beta}, 1 - \frac{r}{\beta} \right),
$$

(11)

where $w_i$ is defined in equation (5).

Proof:

Using the expansion (5), we have

$$
\mu'_r = \int_0^\infty x^r f(x) dx = \sum_{i=0}^{\infty} w_i \int_0^\infty x^r f_{a(i+1),\alpha,\beta}(x).
$$

In addition, based on Theorem 1, $\mu'_r$ reduces to (11).

Theorem 3: If $X_2 \sim \text{ELGGII}(a, p, \alpha, \beta)$, the $r$th moment is given by

$$
\mu'_r = E(X^r) = a a^r \sum_{j=0}^{\infty} v_j B \left( a(j+1) + \frac{r}{\beta}, 1 - \frac{r}{\beta} \right),
$$

(12)
where $v_j$ is defined in (9).

**Proof:**

The proof is similar to Theorem 2 and then it is omitted.

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis for some choices of $a$ as function of $p$, and for some choices of $p$ as function of $a$, for different values of $\alpha$ and $\beta$, are displayed in Figures 5, 6, 7 and 8, respectively. These Figures immediately reveal that there is great flexibility in skewness and kurtosis curves.

![Figure 5: Skewness of the ELLGI distribution. (a) As function of $p$ for some values of $a$ with $\alpha = 1.5$ and $\beta = 3.5$. (b) As function of $a$ for some values of $p$ with $\alpha = 1.5$ and $\beta = 3.5$.](image)

3.2 Moment generating function

**Theorem 4:** If $Y \sim ELL(a, \alpha, \beta)$, the its moment generating function (mgf) is given by

$$M(t) = a \beta \left( V_{26} + V_{27} \right),$$
Figure 6: Kurtosis of the ELLGI distribution. (a) As function of $p$ for some values of $a$ with $\alpha = 1.5$ and $\beta = 4.5$. (b) As function of $a$ for some values of $p$ with $\alpha = 1.0$ and $\beta = 3.5$.

where

\[ V_{26} = \sum_{h=0}^{p_0-1} \frac{(-1)^h}{h!} E_{26} (-\alpha t)^h q_{0+1} F_{q_0+p_0} \left(1, \Delta(q_0, a_0); \Delta(p_0, 1 + h), \Delta(q_0, b); w_0 \right) \]

\[ V_{27} = \sum_{k=0}^{q_0-1} \frac{(-1)^k}{k!} E_{27} (-\alpha t)^v_{q_0+1} F_{q_{0+p_0}} \left(1, \Delta(q_0, a_1); \Delta(p_0, b_1), \Delta(q_0, 1 + k); w_0 \right) \]

\[ E_{26} = \frac{1}{\beta} B \left( \frac{a_0 + h}{\beta}, 1 + a - \frac{a_0 + h}{\beta} \right), \quad E_{27} = (1 + a)_k \Gamma \left[ a - \beta \left( 1 + a + k \right) \right], \]

\[ v_1 = \beta \left( 1 + k \right), \quad a_0 = \frac{(a_0 + h)}{\beta}, \quad b = -a + \frac{(a_0 + h)}{\beta}, \quad a_1 = 1 + a + k, \quad b_1 = 1 + \beta \left( 1 + k \right), \]

\[ w_0 = \left( -1 \right)^{p_0+q_0} \left( -\alpha t/p_0 \right)^{p_0}, \quad (1 + a)_k = (a + 1)(a + 2) \cdots (a + k), \]

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Figure 7: Skewness of the ELLGII distribution. (a) As function of $p$ for some values of $a$ with $\alpha = 1.5$ and $\beta = 3.5$. (b) As function of $a$ for some values of $p$ with $\alpha = 1.5$ and $\beta = 3.0$.

$$\Delta(t, s) = \frac{s}{t}, \frac{s+1}{t}, \cdots, \frac{s+l-1}{t},$$

$$\text{pFq}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k}{(b_1)_k(b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}$$

is a generalized hypergeometric function and $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the ascending factorial, $p_0$ and $q_0$ are relatively prime natural numbers, with the following restrictions $a \beta > 0$, $a t > 0$ and $\beta > 0$.

**Proof:** The mgf of the ELL distribution is calculated from the relation

$$M(t) = \int_0^{\infty} e^{yt} a \beta \left( \frac{y}{\alpha} \right)^{\beta-1} \left[ 1 + \left( \frac{y}{\alpha} \right)^{-\beta} \right]^{-a-1} \left[ 1 + \left( \frac{y}{\alpha} \beta \right) \right]^{-2} dy,$$

Substituting $z = \left( \frac{y}{\alpha} \right)$, we have

$$M(t) = a \beta \int_0^{\infty} z^{a \beta-1} \left( 1 + z^{\beta} \right)^{-a-1} e^{-(-\alpha t)z} dz.$$
Figure 8: Kurtosis of the ELLGII distribution. (a) As function of \( p \) for some values of \( a \) with \( \alpha = 1.5 \) and \( \beta = 3.0 \). (b) As function of \( a \) for some values of \( p \) with \( \alpha = 1.5 \) and \( \beta = 3.5 \).

Finally, using Eq. (3) (on page 319) of the book by Prudnikov et al. (1992) we obtain

\[
M(t) = a \beta \left( V_{26} + V_{27} \right).
\]

Using the results of the theorem 4, now we can calculate the mgf of the ELLGI and ELLGII distributions.

**Theorem 5:** If \( X_1 \sim ELLGI(a, p, \alpha, \beta) \), then its mgf is given by

\[
M(t) = E(\exp(tX)) = \sum_{i=1}^{\infty} w_i a(i + 1) \beta \left( V_{26}^* + V_{27}^* \right),
\]

where \( w_i \) is defined in equation (5),

\[
V_{26}^* = \sum_{h=0}^{p_0-1} \frac{(-1)^h}{h!} E_{26}^* (-\alpha t)^h q_{0+1} F_{q_0+p_0} \left( 1, \Delta(q_0, a_0^*); \Delta(p_0, 1 + h), \Delta(q_0, b^*); w_0 \right)
\]

\[
V_{27}^* = \sum_{k=0}^{q_0-1} \frac{(-1)^k}{k!} E_{27}^* (-\alpha t)^{q_1} q_{0+1} F_{q_0+p_0} \left( 1, \Delta(q_0, a_1^*); \Delta(p_0, b_1), \Delta(q_0, 1 + k); w_0 \right),
\]

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\[ E_{26}^* = \frac{1}{\beta} B\left(\frac{a(i + 1) \beta + h}{\beta}, 1 + a(i + 1) - \frac{a(i + 1) \beta + h}{\beta}\right), \]

\[ E_{27}^* = (1 + a(i + 1)) \Gamma \left[ a(i + 1) \beta - \beta \left(1 + a(i + 1) + k\right) \right], \]

\[ a_0^* = \frac{(a(i + 1) \beta + h)}{\beta}, \quad b^* = -a(i + 1) + \frac{(a(i + 1) \beta + h)}{\beta}, \quad a_1^* = 1 + a(i + 1) + k, \]

and the other terms are defined in the theorem 4.

**Proof:**
Using the expansion (5), we have

\[ M(t) = \int_0^\infty \exp(tx)f(x)dx = \sum_{i=0}^{\infty} w_i \int_0^\infty \exp(tx)f(a(i+1),\alpha,\beta)(x). \]

In addition, based on Theorem 4, \( M(t) \) reduces to (13).

**Theorem 6:** If \( X_2 \sim \text{ELGGII}(a, p, \alpha, \beta) \), then its mgf is given by

\[ M(t) = E(\exp(tX)) = \sum_{j=0}^{\infty} v_j a(j + 1) \beta \left( V_{26}^\dagger + V_{27}^\dagger \right), \quad (14) \]

where \( v_j \) is defined in equation (9),

\[ V_{26}^\dagger = \sum_{h=0}^{q_0-1} \frac{(-1)^h}{h!} E_{26}^* (-\alpha t)^h q_0 + 1 F_{q_0+p_0} \left( 1, \Delta(q_0, a_0^\dagger) ; \Delta(p_0, 1 + h), \Delta(q_0, b_0^\dagger) ; w_0 \right), \]

\[ V_{27}^\dagger = \sum_{k=0}^{q_0-1} \frac{(-1)^k}{k!} E_{27}^*(\alpha t)^{q_0} q_0 + 1 F_{q_0+p_0} \left( 1, \Delta(q_0, a_1^\dagger) ; \Delta(p_0, b_1^\dagger), \Delta(q_0, 1 + k) ; w_0 \right), \]

\[ E_{26}^\dagger = \frac{1}{\beta} B\left(\frac{a(j + 1) \beta + h}{\beta}, 1 + a(j + 1) - \frac{a(j + 1) \beta + h}{\beta}\right), \]

\[ E_{27}^\dagger = 1 + a(j + 1) \Gamma \left[ a(j + 1) \beta - \beta \left(1 + a(j + 1) + k\right) \right], \]

\[ a_0^\dagger = \frac{(a(j + 1) \beta + h)}{\beta}, \quad b^\dagger = -a(j + 1) + \frac{(a(j + 1) \beta + h)}{\beta}, \quad a_1^\dagger = 1 + a(j + 1) + k, \]

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and the other terms are defined in the theorem 4.

The proof is similar to Theorem 5 and then it is omitted.

We now derive an explicit expression for the incomplete moment of the LEE distribution. If $Y \sim \text{ELL}(a, \alpha, \beta)$, the its incomplete moment is given by

$$
\mu'_r(t) = \int_0^t y^r \frac{a \beta}{\alpha} \left(\frac{y}{\alpha}\right)^{-1} \left[1 + \left(\frac{y}{\alpha}\right)^{-\beta}\right]^{-(a-1)} \left[1 + \left(\frac{y}{\alpha}\right)^{\beta}\right]^{-2} dy,
$$

Substituting $z = \left(\frac{y}{\alpha}\right)^{\beta}$, we obtain

$$
\mu'_r(t) = a \alpha^r \int_0^{(t/\alpha)^{\beta}} z^{\pi+a-1} (1+z)^{-(a+1)} dz,
$$

finally, using mathematical software Maple, yields

$$
\mu'_r(t) = a \alpha^r \frac{(t/\alpha)^{\beta(2-a)-r}}{\Gamma(\frac{r}{\beta} + a)} \pi \csc \left[\pi (\frac{r}{\beta} + a - 1) - \pi(a+1)\right] \Gamma^{-1}(\frac{r}{\beta}) \Gamma^{-1}(a+1) + a \alpha^r \frac{(t/\alpha)^{\beta(1-a)}}{1 F_2\left(a+1; -\frac{r}{\beta} + 1; -\frac{r}{\beta} + 2; -\left(\frac{t}{\alpha}\right)^{-\beta}\right) \Gamma(a+1) (\frac{r}{\beta} - 1)^{-1} \Gamma^{-1}(a+1)}.
$$

where

$$
1 F_2(a; b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k(c)_k k!} x^k
$$

is a hypergeometric function and $(a)_k = a(a+1) \ldots (a+k-1)$ denotes the ascending factorial.

4 Maximum Likelihood Estimation

The parameters of the ELLGI and ELLGII distributions are estimated by maximum likelihood from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size $n$ from the ELLGI($a, p, \alpha, \beta$) or ELLGII($a, p, \alpha, \beta$) distribution. The log-likelihood function for the vector of parameters $\theta = (a, p, \alpha, \beta)^T$ can follow from (1) or (6) accordingly.

- For the ELLGI distribution:

$$
l(\theta) = n \log \left[\frac{(1-p)\beta}{\alpha}\right] + (\beta a - 1) \sum_{i=1}^{n} \log \left(\frac{x_i}{\alpha}\right) - (a+1) \sum_{i=1}^{n} \log \left[1 + \left(\frac{x_i}{\alpha}\right)^{\beta}\right] - 2 \sum_{i=1}^{n} \log \left\{1 - p \left[1 - \left\{1 + \left(\frac{x_i}{\alpha}\right)^{-\beta}\right\}^{-a}\right]\right\}.
$$

(16)
• For the ELLGII distribution:

\[ l(\theta) = n \log \left[ \frac{(1-p)\beta a}{\alpha} \right] + (\beta a - 1) \sum_{i=1}^{n} \log \left\{ 1 + \left( \frac{x_i}{\alpha} \right)^{-\beta} \right\} - (a + 1) \sum_{i=1}^{n} \log \left[ 1 + \left( \frac{x_i}{\alpha} \right)^{-\beta} \right] - \\
2 \sum_{i=1}^{n} \log \left\{ 1 - \frac{1}{1 + \left( \frac{x_i}{\alpha} \right)^{-\beta}} \right\} \right]. \] 

(17)

The components of the score vector \( U(\theta) \) are:

• For the ELLGI distribution:

\[
U_\alpha(\theta) = \frac{n}{a} - \sum_{i=1}^{n} \log[1 + (x_i/\alpha)^{-\beta}] + 2p \sum_{i=1}^{n} \frac{\log[1 + (x_i/\alpha)^{-\beta}] [1 + (x_i/\alpha)^{-\beta}]^{-a}}{[1 - p(1 - q_i)]},
\]

\[
U_\beta(\theta) = \frac{n}{\beta} - \sum_{i=1}^{n} \frac{1}{\alpha^{-\beta+1}} \sum_{i=1}^{n} \frac{x_i^{-\beta}}{[1 + (x_i/\alpha)^{-\beta}]^{-a}} + 2p \sum_{i=1}^{n} \frac{x_i^{-\beta} [1 + (x_i/\alpha)^{-\beta}]^{-a-1}}{[1 - p(1 - q_i)]},
\]

\[
U_p(\theta) = -\frac{n}{(1-p)} - 2 \sum_{i=1}^{n} \frac{1 + (x_i/\alpha)^{-\beta} - a}{[1 - p(1 - q_i)]},
\]

• For the ELLGII distribution:

\[
U_\alpha(\theta) = \frac{n}{a} - \sum_{i=1}^{n} \log[1 + (x_i/\alpha)^{-\beta}] - 2p \sum_{i=1}^{n} \frac{\log[1 + (x_i/\alpha)^{-\beta}] [1 + (x_i/\alpha)^{-\beta}]^{-a}}{[1 - pq_i]},
\]

\[
U_\beta(\theta) = \frac{n}{\beta} - \sum_{i=1}^{n} \frac{1}{\alpha^{-\beta+1}} \sum_{i=1}^{n} \frac{x_i^{-\beta}}{[1 + (x_i/\alpha)^{-\beta}]^{-a}} - 2p \sum_{i=1}^{n} \frac{x_i^{-\beta} [1 + (x_i/\alpha)^{-\beta}]^{-a-1}}{[1 - pq_i]},
\]

\[
U_p(\theta) = -\frac{n}{(1-p)} + 2 \sum_{i=1}^{n} \frac{1 + (x_i/\alpha)^{-\beta} - a}{[1 - pq_i]},
\]

\[
U_\alpha(\theta) = \frac{n}{\beta} - \sum_{i=1}^{n} \log(x_i/\alpha) + \frac{(a + 1)}{\alpha^{-\beta}} \sum_{i=1}^{n} \frac{x_i^{-\beta} \log(x_i/\alpha)}{[1 + (x_i/\alpha)^{-\beta}]^{-a}} + 2p \sum_{i=1}^{n} \frac{x_i^{-\beta} \log(x_i/\alpha) [1 + (x_i/\alpha)^{-\beta}]^{-a-1}}{[1 - pq_i]},
\]

\[
U_\beta(\theta) = \frac{n}{(1-p)} + 2 \sum_{i=1}^{n} \frac{1 + (x_i/\alpha)^{-\beta} - a}{[1 - pq_i]}.
\]
Let \( \hat{\theta} \) be the maximum likelihood estimate (MLE) of \( \theta \). The log-likelihood can be maximized either directly by using the Proc NLMixed of SAS or the sub-routine MaxBFGS of the program Ox (see, Doornik, 2007) or by solving the nonlinear likelihood equations \( U_a(\theta) = U_p(\theta) = U_\alpha(\theta) = U_\beta(\theta) = 0 \) simultaneously. For interval estimation and hypothesis tests on the model parameters, we require the 3x3 unit observed information matrix

\[
J(\theta) = \begin{pmatrix}
J_{aa} & J_{ap} & J_{a\alpha} & J_{a\beta} \\
J_{pa} & J_{pp} & J_{p\alpha} & J_{p\beta} \\
J_{\alpha a} & J_{\alpha p} & J_{\alpha\alpha} & J_{\alpha\beta} \\
J_{\beta a} & J_{\beta p} & J_{\beta\alpha} & J_{\beta\beta}
\end{pmatrix},
\]

whose elements are given in Appendix A for the proposed models.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of \( \sqrt{n}(\hat{\theta} - \theta) \) is \( N_4(0, I(\theta)^{-1}) \), where \( I(\theta) \) is the unit expected information matrix. Based on the multivariate normal \( N_4(0, J(\hat{\theta})^{-1}) \) distribution, we can construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard and survival functions.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain likelihood ratio (LR) statistics for testing some sub-models of the ELLGI or ELLGII distributions. So, we can construct LR statistics to check if the fit using the LGGI distribution is statistically “superior” to a fit using the logistic geometric (LG) distribution for a given data set. In any case, hypothesis tests of the type \( H_0 : \psi = \psi_0 \) versus \( H_1 : \psi \neq \psi_0 \), where \( \psi \) is a vector formed with some components of \( \theta \) and \( \theta_0 \) is a specified vector, can be performed using LR statistics.

5 The exponentiated logistic geometric regression model with censored data

From the random variables \( X_1 \) and \( X_2 \) having the ELLGI and ELLGII density function (1) and (6), respectively, we define the random variables \( Y_j = \log(X_j) \) (for \( j = 1, 2 \)) having the exponentiated logistic geometric type I (ELGI) and exponentiated logistic geometric type II (ELGII) distributions parameterized in terms of \( \beta = \sigma^{-1} \) and \( a = e^{\mu} \), where \( a > 0 \), \( 0 < p < 1 \), \( \sigma > 0 \) and \( -\infty < \mu < \infty \).

- The ELGI density function becomes

\[
f(y; a, p, \sigma, \mu) = \frac{(1-p)a}{\sigma} \exp\left[ a \left( \frac{y - \mu}{\sigma} \right) \right] \left[ 1 + \exp\left( \frac{y - \mu}{\sigma} \right) \right]^{-(a+1)} \times \\
\left\{ 1 - p \left[ 1 - \left( 1 + \exp\left[ -\left( \frac{y - \mu}{\sigma} \right) \right] \right)^{-a} \right] \right\}^{-2}, \quad -\infty < y < \infty. \quad (18)
\]
The ELGI density function becomes

\[
f(y; a, p, \mu, \sigma) = \frac{(1-p)}{\sigma} \exp \left[ a \left( \frac{y - \mu}{\sigma} \right) \right] \left[ 1 + \exp \left( \frac{y - \mu}{\sigma} \right) \right]^{-(a+1)} \times 
\left\{ 1 - p \left[ 1 + \exp \left\{ - \left( \frac{y - \mu}{\sigma} \right) \right\} \right]^{-a} \right\}^{-2}
\]

\[\times \left\{ 1 - p \left[ 1 + \exp \left\{ - \left( \frac{y - \mu}{\sigma} \right) \right\} \right]^{-a} \right\}^{-2}, \quad -\infty < y < \infty. \tag{19}\]

The ELGI and ELGII distributions contain as special models well-known distributions. They simplify to the exponentiated logistic (EL) distribution when \( p = 0 \). When \( a = 1 \) and \( p = 0 \), they reduce to the classic logistic distribution. Some studies in regression models, we have for example, Hashimoto et al. (2010) defined the log-exponentiated Weibull regression model for interval-censored data, Ortega et al. (2011a) studied in details the log-generalized modified Weibull regression model based on the modified Weibull distribution. Following the same idea, Ortega et al. (2011b) defined a log-linear regression model for the beta-Weibull distribution with censored data.

Plots of the density function (18) and (19) for selected parameter values are displayed in Figures 9 and 9, respectively. These plots indicate great flexibility for different values of \( a \) and \( b \).

We can obtain the survival functions corresponding to (18) and (19).

• For the ELGI distribution:

\[
S(y) = 1 - \left\{ 1 + \exp \left[ - \left( \frac{y - \mu}{\sigma} \right) \right] \right\}^{-a} \left\{ 1 - p \left[ 1 - \left\{ 1 + \exp \left[ - \left( \frac{y - \mu}{\sigma} \right) \right] \right\}^{-a} \right] \right\}^{-1}.
\]

• For the ELGII distribution:

\[
S(y) = 1 - (1-p) \left\{ 1 + \exp \left[ - \left( \frac{y - \mu}{\sigma} \right) \right] \right\}^{-a} \left\{ 1 - p \left[ 1 + \exp \left[ - \left( \frac{y - \mu}{\sigma} \right) \right] \right]^{-a} \right\}^{-1}.
\]

We define the standardized random variables \( Z_1 = (Y_1 - \mu)/\sigma \) and \( Z_2 = (Y_2 - \mu)/\sigma \).

• The density function of \( Z_1 \) (for \( -\infty < z < \infty \)) is

\[
\pi(z; a, p) = a (1 - p) \exp(az) \left[ 1 + \exp(z) \right]^{-(a+1)} \left\{ 1 - p \left[ 1 - \left\{ 1 + \exp(-z) \right\}^{-a} \right] \right\}^{-2}.
\]

• The density function of \( Z_2 \) (for \( -\infty < z < \infty \)) is

\[
\pi(z; a, p) = a (1 - p) \exp(az) \left[ 1 + \exp(z) \right]^{-(a+1)} \left\{ 1 - p \left[ 1 + \exp(-z) \right]^{-a} \right\}^{-2}.
\]
The standard logistic geometric type I and standard logistic geometric type II are obtained from (20) and (21) when \( a = 1 \), respectively. For \( p = 0 \), we obtain the standard exponentiated logistic. The special case for \( a = 1 \) and \( p = 0 \) corresponds to the standard logistic distribution.

In many practical applications, the lifetimes \( x_i \) are affected by explanatory variables such as the cholesterol level, blood pressure and many others. Let \( v_i = (v_{i1}, \ldots, v_{ip})^T \) be the explanatory variable vector associated with the \( i \)th response variable \( y_i \) for \( i = 1, \ldots, n \). Consider a sample \((y_1, v_1), \ldots, (y_n, v_n)\) of \( n \) independent observations, where each response is defined by \( y_i = \min\{\log(x_i), \log(c_i)\} \), and \( \log(x_i) \) and \( \log(c_i) \) are the log-lifetime and log-censoring, respectively. We consider non-informative censoring and that the observed lifetimes and censoring times are independent.

Based on the ELGI and ELGII distributions, we construct a linear regression model for the response variable \( y_i \) given by

\[
y_i = v_i^T \beta + \sigma z_i, \quad i = 1, \ldots, n,
\]

where the random error \( z_i \) can follow the density functions (20) or (21), \( \beta = (\beta_1, \ldots, \beta_s)^T \), \( \sigma > 0 \) and \( p \) are unknown positive shape parameters and \( v_i \) is the vector of explanatory variables modeling the location parameter \( \mu_i = v_i^T \beta \). Hence, the location parameter vector \( \mu = (\mu_1, \ldots, \mu_n)^T \) of the
ELGI and ELGII models has a linear structure $\mu = v^T \beta$, where $v = (v_1, \ldots, v_n)^T$ is a known design matrix.

Let $F$ and $C$ be the sets of individuals for which $y_i$ is the log-lifetime or log-censoring, respectively. The total log-likelihood function for the model parameters $\theta = (a, p, \sigma, \beta^T)^T$ can be obtained from (20), (21) and (22).

- For the ELGI distribution, we have

$$l(\theta) = r \log \left[ \frac{a(1-p)}{\sigma} \right] + a \sum_{i \in F} \left( \frac{y_i - v_i^T \beta}{\sigma} \right) - (a+1) \sum_{i \in F} \log \left[ 1 + \exp \left( \frac{y_i - v_i^T \beta}{\sigma} \right) \right] -$$

$$2 \sum_{i \in F} \log \left\{ 1 - p \left[ 1 - \{ 1 + \exp \left[ - \left( \frac{y_i - v_i^T \beta}{\sigma} \right) \right] \}^{-a} \right] \right\} +$$

$$+ \sum_{i \in C} \log \left\{ 1 - \frac{1 + \exp \left[ - \left( \frac{y_i - v_i^T \beta}{\sigma} \right) \right]^{-a}}{1 - p \left[ 1 - \{ 1 + \exp \left[ - \left( \frac{y_i - v_i^T \beta}{\sigma} \right) \right] \}^{-a} \right]} \right\}. \quad (23)$$
• For the ELGII distribution, we have

\[
 l(\theta) = r \log \left[ \frac{a(1-p)}{\sigma} \right] + a \sum_{i \in F} \left( \frac{y_i - \frac{v_i^T \beta}{\sigma}}{\sigma} \right) - (a + 1) \sum_{i \in F} \log \left[ 1 + \exp \left( \frac{y_i - \frac{v_i^T \beta}{\sigma}}{\sigma} \right) \right]
\]

\[
+ 2 \sum_{i \in F} \log \left\{ 1 - p \left[ 1 + \exp \left\{ - \left( \frac{y_i - \frac{v_i^T \beta}{\sigma}}{\sigma} \right) \right\} \right]^{-a} \right\} + \sum_{i \in C} \log \left\{ 1 - \frac{(1-p) \left[ 1 + \exp \left\{ - \left( \frac{y_i - \frac{v_i^T \beta}{\sigma}}{\sigma} \right) \right\} \right]}{1 - p \left[ 1 + \exp \left\{ - \left( \frac{y_i - \frac{v_i^T \beta}{\sigma}}{\sigma} \right) \right\} \right]^{-a}} \right\},
\]

(24)

where \( r \) is the observed number of failures.

The MLE \( \hat{\beta} \) of \( \beta \) is obtained by solving the nonlinear equations \( U_p(\beta) = 0, U_\sigma(\beta) = 0 \) and \( U_{\beta_1}(\beta) = 0 \). These equations cannot be solved analytically and statistical software can be used to solve them numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to calculate the estimate \( \hat{\beta} \). The elements of the observed information matrix corresponding to (23) and (24) can be obtained from the authors upon request. We use the subroutine NLMixed in SAS to compute \( \hat{\beta} \). Initial values for \( \beta \) and \( \sigma \) are taken from the fit of the ELGI or ELGII regression model with \( p = 0 \).

From the fitted model (22), the survival function for \( y_i \) can be easily estimated.

• For the ELGI distribution, we have

\[
 S(y_i; \hat{\theta}) = 1 - \left\{ 1 + \exp \left[ - \left( \frac{y_i - \frac{v_i^T \hat{\beta}}{\sigma}}{\sigma} \right) \right] \right\}^{-\hat{\alpha}} \left\{ 1 - \hat{p} \left[ 1 + \exp \left[ - \left( \frac{y_i - \frac{v_i^T \hat{\beta}}{\sigma}}{\sigma} \right) \right] \right]^{-\hat{\alpha}} \right\}^{-1}.
\]

• For the ELGII distribution, we have

\[
 S(y_i; \hat{\theta}) = 1 - \left\{ 1 + \exp \left[ - \left( \frac{y_i - \frac{v_i^T \hat{\beta}}{\sigma}}{\sigma} \right) \right] \right\}^{-\hat{\alpha}} \left\{ 1 - \hat{p} \left[ 1 + \exp \left[ - \left( \frac{y_i - \frac{v_i^T \hat{\beta}}{\sigma}}{\sigma} \right) \right] \right]^{-\hat{\alpha}} \right\}^{-1}.
\]

Under general regularity conditions, the asymptotic distribution of \( \sqrt{n}(\hat{\theta} - \theta) \) is multivariate normal \( N_{p+3}(0, K(\theta)^{-1}) \), where \( K(\theta) \) is the expected information matrix. The asymptotic covariance matrix \( K(\theta)^{-1} \) of \( \hat{\theta} \) can be approximated by the inverse of the \((p + 3) \times (p + 3)\) observed information matrix \( J(\theta) \) and then the inference on the parameter vector \( \theta \) can be based on the normal approximation \( N_{p+3}(0, J(\theta)^{-1}) \) for \( \hat{\theta} \). In fact, an \( 100(1-\alpha)\% \) asymptotic confidence interval for each parameter \( \theta_r \) is given by

\[
 ACI_r = \left( \hat{\theta}_r - z_{\alpha/2} \sqrt{J^{rr}}, \hat{\theta}_r + z_{\alpha/2} \sqrt{J^{rr}} \right),
\]

where \( J^{rr} \) represents the \( r \)th diagonal element of the inverse of the estimated observed information matrix \( J(\hat{\theta})^{-1} \) and \( z_{\alpha/2} \) is the quantile \( 1 - \alpha/2 \) of the standard normal distribution.
6 Applications

6.1 Contaminated data

In this section, we use a real data set \( n = 744 \) related to environmental contamination previously studied by Balakrishnan et al. (2009). Specifically, Santiago, Chile, is recognized as one of the most environmentally contaminated cities in the world. In order to obtain the level of air pollution and its associated adverse effects on humans in Santiago, the National Commission of Environment of the Government of Chile collects data on sulfur dioxide \((SO_2)\) concentrations in the air. We compare the fits of the ELLGI and ELLGII distributions to these data and of its two sub-models: the ELL \((p = 0)\) and LL \((p = 0, a = 1)\) distributions.

First, we describe the data set. Then, we report the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion) and BIC (Bayesian Information Criterion) statistics. The lower the value of these criteria, the better the fit. We note that over-parametrization is penalized in these criteria, so that the additional parameters in the ELLGI and ELLGII distributions do not necessarily lead to lower values of the AIC, CAIC or BIC statistics. In each case, the parameters are estimated by maximum likelihood (Section 4) using the subroutine NLMixed in SAS. Next, we perform the LR tests (Section 4) to verify if the third skewness parameter is really necessary. Finally, we give the histogram of these data and provide a visual comparison of the fitted density functions.

Table 1: Descriptive statistics.

<table>
<thead>
<tr>
<th>Mean</th>
<th>Median</th>
<th>Mode</th>
<th>SD</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.926</td>
<td>2.00</td>
<td>2.00</td>
<td>2.015</td>
<td>4.060</td>
<td>4.339</td>
<td>34.914</td>
<td>1.00</td>
<td>25.00</td>
</tr>
</tbody>
</table>

Table 1 gives a descriptive summary of the data which have positive skewness and kurtosis. We compute the MLEs of the model parameters and the AIC, CAIC and BIC statistics for each fitted model to these data. The ELLGI and ELLGII models were fitted and compared with the fits obtained from two sub-models cited before. Iterative maximization of the logarithm of the likelihood function is performed with initial values for \( \alpha \) and \( \beta \) taken from the fit of the LL distribution with \( a = 1 \) and \( p = 0 \). The results from the fitted models are reported in Table 2. The three information criteria agree on the model’s ranking in every case. The lowest values of the information criteria are obtained for the ELLGII distribution.

Clearly, the ELLGII model having three skewness parameters should be preferred in this case. Formal tests for the skewness parameters in the new distribution can be based on LR statistics (Section 4). These LR statistics for comparing the fitted models are shown in Table 3. We reject the null hypotheses in the three LR tests in favor of the wider model. The rejection is extremely highly significant and it gives clear evidence of the potential need for three skewness parameters.
Table 2: MLEs of the model parameters for the contaminated data and information criteria.

<table>
<thead>
<tr>
<th>Model</th>
<th>$a$</th>
<th>$p$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>AIC</th>
<th>CAIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELLGI</td>
<td>1.5314</td>
<td>0.6279</td>
<td>2.7022</td>
<td>2.8671</td>
<td>2512.1</td>
<td>2512.2</td>
<td>2530.5</td>
</tr>
<tr>
<td></td>
<td>(0.2191)</td>
<td>(0.3455)</td>
<td>(0.7396)</td>
<td>(0.2582)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ELLGII</td>
<td>19.8288</td>
<td>0.9190</td>
<td>0.4339</td>
<td>3.1975</td>
<td>2501.8</td>
<td>2501.9</td>
<td>2520.3</td>
</tr>
<tr>
<td></td>
<td>(18.99)</td>
<td>(0.0268)</td>
<td>(0.1383)</td>
<td>(0.1256)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ELL</td>
<td>1.5360</td>
<td>0</td>
<td>2.0317</td>
<td>3.0330</td>
<td>2511.0</td>
<td>2511.1</td>
<td>2524.9</td>
</tr>
<tr>
<td></td>
<td>(0.2384)</td>
<td>-</td>
<td>(0.1568)</td>
<td>(0.1562)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LL</td>
<td>1</td>
<td>0</td>
<td>2.4710</td>
<td>3.4285</td>
<td>2518.3</td>
<td>2518.4</td>
<td>2527.5</td>
</tr>
<tr>
<td></td>
<td>-</td>
<td>-</td>
<td>(0.0458)</td>
<td>(0.1053)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: LR tests.

<table>
<thead>
<tr>
<th>Contamination</th>
<th>Hypotheses</th>
<th>Statistic $w$</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELLGI vs ELL</td>
<td>$H_0 : p = 0$ vs $H_1 : p \neq 0$ is false</td>
<td>0.90</td>
<td>0.3428</td>
</tr>
<tr>
<td>ELLGI vs LL</td>
<td>$H_0 : p = 0$ and $a = 1$ vs $H_1 : H_0$ is false</td>
<td>11.20</td>
<td>0.0008</td>
</tr>
<tr>
<td>ELLGII vs ELL</td>
<td>$H_0 : p = 0$ vs $H_1 : p \neq 0$ is false</td>
<td>10.20</td>
<td>0.0061</td>
</tr>
<tr>
<td>ELLGII vs LL</td>
<td>$H_0 : p = 0$ and $a = 1$ vs $H_1 : H_0$ is false</td>
<td>20.5</td>
<td>&lt;0.00001</td>
</tr>
</tbody>
</table>

when modeling real data.

More information is provided by a visual comparison of the histogram of the data with the fitted density functions. The plot of the fitted ELLGII density function is displayed in Figure 11a.

In order to assess if the model is appropriate, plots of the cdf of the ELLGII distribution and the empirical cdf are displayed in Figure 11b. We conclude that the ELLGII distribution provides a good fit for these data.

6.2 Voltage data

Here, we illustrate the usefulness of the ELGI and ELGII distributions with one application. Lawless (2003) reported an experiment in which specimens of solid epoxy electrical-insulation were studied in an accelerated voltage life test. The sample size is $n = 60$, the percentage of censored observations was 10% and were considered three levels of voltage 52.5, 55.0 and 57.5. The variables involved in the study are: $t_i$ - failure times for epoxy insulation specimens (in min); $cens_i$ - censoring indicator (0=censoring, 1=lifetime observed); $v_{i1}$ - voltage (kV).

Now, we present results from the fit of the model

$$y_i = \beta_0 + \beta_1 v_{i1} + \sigma z_i,$$
where the random variable $z_i$ can follow the ELGI and ELGII distributions given by (18) and (19), respectively, for $i = 1, \ldots, 60$.

Table 4 lists the MLEs of the parameters for the ELGI and ELGII regression models fitted to the voltage data (using the procedure NLMixed in SAS) and the values of the AIC, BIC and CAIC statistics to compare the regression models. Iterative maximization of the logarithm of the likelihood function could come with initial values for $\beta$ and $\sigma$ taken from the fit of the ELGI or ELGII regression model with $a = 1$ and $p = 0$.

The results from the fitted models are reported in Table 4. The three information criteria agree on the model’s ranking in every case. The lowest values of the information criteria are associated with the ELGII distribution. The new ELGII model involves two extra parameters which yields more flexibility to fit these data. Note also that the estimation of the model parameter $p$ for the ELGI model is almost zero ($\hat{p} = 0$), thus indicating that the ELGU and EL models are almost equivalents. We note from the fitted ELGII regression model that $x_1$ is significant at 1% but there is a significant difference between the voltages 52.5, 55.0 and 57.5 for the survival times.
Table 4: MLEs of the parameters from the regression model fitted to the voltage data set, the corresponding SEs (given in parentheses), p-value in [ ], and the statistics AIC, CAIC and BIC.

<table>
<thead>
<tr>
<th>Model</th>
<th>$a$</th>
<th>$p$</th>
<th>$\sigma$</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>AIC</th>
<th>CAIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELGI</td>
<td>3.0931</td>
<td>1E-8</td>
<td>0.7057</td>
<td>15.0204</td>
<td>-0.1739</td>
<td>166.7</td>
<td>167.9</td>
<td>177.2</td>
</tr>
<tr>
<td></td>
<td>(4.3257)</td>
<td>(1E-9)</td>
<td>(0.1824)</td>
<td>(4.4597)</td>
<td>(0.0663)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[0.0013]</td>
<td>[0.0110]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ELGII</td>
<td>3.9296</td>
<td>0.9999</td>
<td>0.4396</td>
<td>13.0640</td>
<td>-0.2118</td>
<td>135.8</td>
<td>136.9</td>
<td>146.3</td>
</tr>
<tr>
<td></td>
<td>(0.3181)</td>
<td>(0.000092)</td>
<td>(0.0496)</td>
<td>(2.8459)</td>
<td>(0.0493)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[&lt;0.001]</td>
<td>[&lt;0.001]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL</td>
<td>3.0931</td>
<td>0</td>
<td>0.7057</td>
<td>15.0204</td>
<td>-0.1739</td>
<td>164.7</td>
<td>165.5</td>
<td>173.1</td>
</tr>
<tr>
<td></td>
<td>(4.3276)</td>
<td>-</td>
<td>(0.1824)</td>
<td>(4.4598)</td>
<td>(0.0663)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>[0.0013]</td>
<td>[0.0110]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Logistic</td>
<td>1</td>
<td>0</td>
<td>0.5296</td>
<td>17.8799</td>
<td>-0.2072</td>
<td>164.3</td>
<td>164.7</td>
<td>170.5</td>
</tr>
<tr>
<td></td>
<td>[&lt;0.001]</td>
<td>[0.0005]</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Table 5: LR tests.

<table>
<thead>
<tr>
<th>Voltage</th>
<th>Hypotheses</th>
<th>Statistic $w$</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELGII vs EL</td>
<td>$H_0 : p = 0$ vs $H_1 : H_0$ is false</td>
<td>30.90</td>
<td>&lt;0.00001</td>
</tr>
<tr>
<td>ELGII vs Logistic</td>
<td>$H_0 : p = 0$ and $a = 1$ vs $H_1 : H_0$ is false</td>
<td>32.50</td>
<td>&lt;0.00001</td>
</tr>
</tbody>
</table>

The LR statistics for comparing the fitted models are given in Table 5. We reject the null hypotheses for the three LR tests in favor of the wider distribution. We conclude that the ELGII distribution provides a good fit for these data. In order to assess if this model is appropriate, we plot in Figure 12a the empirical survival function and its estimated survival function given in Section (5). We can conclude that the ELGII regression model provides a good fit to these data.

7 Conclusions

In this paper, we propose and study two new lifetime distributions called the exponentiated log-logistic geometric distributions. We derive some general mathematical properties of these distributions based on simple expansions of their density functions. These properties include moments, mean deviations and generating functions. We define two new exponentiated logistic geometric regression models for censored data. We discuss maximum likelihood estimation of the model parameters and derive the observed information matrix. Two empirical applications to real data are presented in order to illustrate the flexibility of the new regression models.
Figure 12: Estimated survival functions for the ELGII distribution and some of its sub-models and the empirical survival for stress level data.

**Acknowledgments**

The financial support from CAPES and CNPq is gratefully acknowledged.
8 Appendix A

The elements of the observed information matrix $J(\theta)$ for the parameters $(a, p, \alpha, \beta)$ are obtained by differentiating (16) and (17) twice.

For the ELLGII distribution, we have:

\[
J_{aa} = -\frac{n}{a^2} - 2p \sum_{i=1}^{n} \left\{ \frac{\log(q_i) q_i^{-a}}{1 - p(1 - q_i^{-a})} - \frac{p \log(q_i) q_i^{-2a}}{[1 - p(1 - q_i^{-a})]^2} \right\},
\]

\[
J_{ap} = -2 \sum_{i=1}^{n} \left\{ -\frac{\log(q_i) q_i^{-a}}{1 - p(1 - q_i^{-a})} + \frac{p \log(q_i) q_i^{-a}(-1 + q_i^{-a})}{[1 - p(1 - q_i^{-a})]^2} \right\},
\]

\[
J_{aa} = -\frac{\beta}{\alpha} \sum_{i=1}^{n} \left( \frac{z_i}{\alpha} \right)^{-\beta} \frac{\log(z_i)}{q_i} - 2p \sum_{i=1}^{n} \left\{ \frac{a (\frac{z_i}{\alpha})^{-\beta} \log(z_i) q_i^{-(a+1)}}{1 - p(1 - q_i^{-a})} - \frac{a (\frac{z_i}{\alpha})^{-\beta} q_i^{-(a+1)}}{1 - p(1 - q_i^{-a})} \right\},
\]

\[
J_{a\beta} = -\sum_{i=1}^{n} \left( \frac{z_i}{\alpha} \right)^{-\beta} \frac{\log(z_i)}{q_i} - 2p \sum_{i=1}^{n} \left\{ \frac{a (\frac{z_i}{\alpha})^{-\beta} \log(z_i) q_i^{-(a+1)}}{1 - p(1 - q_i^{-a})} + \frac{a (\frac{z_i}{\alpha})^{-\beta} q_i^{-(a+1)}}{1 - p(1 - q_i^{-a})} \right\},
\]

\[
J_{pp} = -\frac{n}{(1 - p)^2} + 2 \sum_{i=1}^{n} \frac{(-1 + q_i^{-a})^2}{[1 - p(1 - q_i^{-a})]^2},
\]

\[
J_{pa} = -2a \sum_{i=1}^{n} \left\{ \frac{a (\frac{z_i}{\alpha})^{-\beta} q_i^{-(a+1)}}{1 - p(1 - q_i^{-a})} - \frac{p (\frac{z_i}{\alpha})^{-\beta} q_i^{-(a+1)}(-1 + q_i^{-a})}{[1 - p(1 - q_i^{-a})]^2} \right\},
\]

\[
J_{p\beta} = -2a \sum_{i=1}^{n} \left\{ \frac{(\frac{z_i}{\alpha})^{-\beta} \log(z_i) q_i^{-(a+1)}}{1 - p(1 - q_i^{-a})} - \frac{p (\frac{z_i}{\alpha})^{-\beta} \log(z_i) q_i^{-(a+1)}(-1 + q_i^{-a})}{[1 - p(1 - q_i^{-a})]^2} \right\},
\]
\[
J_{\alpha\alpha} = \frac{n\beta}{\alpha^2} - \frac{(a+1)\beta}{\alpha^2} \sum_{i=1}^{n} \left[ \frac{\beta \left( \frac{x_i}{\alpha} \right)^{-\beta}}{q_i} - \frac{\left( \frac{x_i}{\alpha} \right)^{-\beta}}{q_i^2} \right] \\
- 2pa\beta \sum_{i=1}^{n} \left[ \frac{a\beta \left( \frac{x_i}{\alpha} \right)^{-2\beta} q_i^{-(a+2)}}{1-p(1-q_i^{-a})} - \frac{\beta \left( \frac{x_i}{\alpha} \right)^{-\beta}}{q_i} - \frac{\left( \frac{x_i}{\alpha} \right)^{-\beta}}{q_i^2} \right] \\
+ \beta \left( \frac{x_i}{\alpha} \right)^{-2\beta} q_i^{-(a+2)} - \frac{pa\beta \left( \frac{x_i}{\alpha} \right)^{-2\beta} q_i^{-(a+2)}}{1-p(1-q_i^{-a})} \\
\left[ 1-p(1-q_i^{-a}) \right],
\]

\[
J_{\alpha\beta} = \frac{n}{\beta^2} - \frac{(a+1)}{\alpha} \sum_{i=1}^{n} \left[ \frac{(x_i/\alpha)^{-\beta} \log(x_i/\alpha)^2}{q_i} - \frac{(x_i/\alpha)^{-2\beta} \log(x_i/\alpha)^2}{q_i^2} \right] \\
- 2pa \sum_{i=1}^{n} \left[ \frac{a(x_i/\alpha)^{-2\beta} \log(x_i/\alpha)^2 q_i^{-(a+2)}}{1-p(1-q_i^{-a})} - \frac{(x_i/\alpha)^{-\beta} \log(x_i/\alpha)^2 q_i^{-(a+1)}}{1-p(1-q_i^{-a})} \\
+ \frac{(x_i/\alpha)^{-2\beta} \log(x_i/\alpha)^2 q_i^{-(a+2)}}{1-p(1-q_i^{-a})} - \frac{pa(x_i/\alpha)^{-2\beta} \log(x_i/\alpha)^2 q_i^{-(a+2)}}{1-p(1-q_i^{-a})^{2(a+1)}} \right],
\]

For the ELLGII distribution, we have:

\[
J_{\alpha \beta} = \sum_{i=1}^{n} \left[ \frac{a(x_i/\alpha)^{-\beta} \log(q_i) q_i^{-(a+1)}}{1-pq_i^{-a}} - \frac{(x_i/\alpha)^{-\beta} q_i^{-(a+1)}}{1-pq_i^{-a}} \right],
\]

\[
J_{\alpha \beta} = \sum_{i=1}^{n} \left[ \frac{a(x_i/\alpha)^{-\beta} \log(q_i) q_i^{-(2a+1)}}{1-pq_i^{-a}} - \frac{(x_i/\alpha)^{-\beta} q_i^{-(2a+1)}}{1-pq_i^{-a}} \right],
\]

For the ELLGII distribution, we have:

\[
J_{\alpha \beta} = \sum_{i=1}^{n} \left[ \frac{a(x_i/\alpha)^{-\beta} \log(q_i) q_i^{-(2a+1)}}{1-pq_i^{-a}} - \frac{(x_i/\alpha)^{-\beta} q_i^{-(2a+1)}}{1-pq_i^{-a}} \right].
\]

For the ELLGII distribution, we have:

\[
J_{\alpha \beta} = \sum_{i=1}^{n} \left[ \frac{a(x_i/\alpha)^{-\beta} \log(q_i) q_i^{-(2a+1)}}{1-pq_i^{-a}} - \frac{(x_i/\alpha)^{-\beta} q_i^{-(2a+1)}}{1-pq_i^{-a}} \right].
\]
\[ J_{\alpha \beta} = \sum_{i=1}^{n} \frac{(\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha})}{q_i} - 2p \sum_{i=1}^{n} \left[ a \frac{(\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha}) q_i}{1 - pq_i^{-a}} \right] \]

\[ = \frac{\sum_{i=1}^{n} (\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha}) q_i^{-(a+1)}}{1 - pq_i^{-a}} + q_i^{-2a} \left[ 1 - \frac{(\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha}) q_i^{-(a+1)}}{1 - pq_i^{-a}} \right] \]

\[ J_{pp} = -\frac{n}{(1-p)^2} + 2 \sum_{i=1}^{n} \frac{q_i^{-2a}}{1 - pq_i^{-a}} \]

\[ J_{pa} = -\frac{2a\beta}{\alpha} \sum_{i=1}^{n} \left[ \frac{(\frac{x_i}{\alpha})^{-\beta} q_i^{-(a+1)}}{1 - pq_i^{-a}} + \frac{p(\frac{x_i}{\alpha})^{-\beta} q_i^{-(2a+1)}}{1 - pq_i^{-a}} \right] \]

\[ J_{pb} = 2a \sum_{i=1}^{n} \left[ \frac{(\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha}) q_i^{-(a+1)}}{1 - pq_i^{-a}} + \frac{p(\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha}) q_i^{-(2a+1)}}{1 - pq_i^{-a}} \right] \]

\[ J_{aa} = -\frac{n\beta}{\alpha^2} - \frac{(a+1)\beta}{\alpha^2} \sum_{i=1}^{n} \left[ \frac{\beta (\frac{x_i}{\alpha})^{-\beta}}{q_i} - \frac{(\frac{x_i}{\alpha})^{-\beta}}{q_i} - \frac{\beta (\frac{x_i}{\alpha})^{-2\beta}}{q_i^2} \right] \]

\[ = -\frac{2pa\beta}{\alpha^2} \sum_{i=1}^{n} \left[ a \frac{\beta (\frac{x_i}{\alpha})^{-2\beta} q_i^{-(a+2)}}{1 - pq_i^{-a}} + \frac{\beta (\frac{x_i}{\alpha})^{-\beta} q_i^{-(a+1)}}{1 - pq_i^{-a}} \right] \]

\[ J_{a\beta} = \frac{n}{\alpha} \sum_{i=1}^{n} \left[ -\beta (\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha}) q_i^{-(a+1)} + \frac{(\frac{x_i}{\alpha})^{-\beta}}{q_i} + \frac{\beta (\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha})}{q_i^2} \right] \]

\[ = -\frac{2p\beta}{\alpha} \sum_{i=1}^{n} \left[ a \frac{\beta (\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha}) q_i^{-(a+2)}}{1 - pq_i^{-a}} - \frac{\beta (\frac{x_i}{\alpha})^{-\beta} q_i^{-(a+1)}}{1 - pq_i^{-a}} \right] \]

\[ + \frac{(\frac{x_i}{\alpha})^{-\beta} q_i^{-(a+1)}}{1 - pq_i^{-a}} + \frac{\beta (\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha}) q_i^{-(a+2)}}{1 - pq_i^{-a}} + \frac{p\beta (\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha}) q_i^{-(2a+1)}}{1 - pq_i^{-a}} \]
$$J_{\beta\beta} = \frac{-n}{\beta^2} - (a + 1) \sum_{i=1}^{n} \left[ \frac{(\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha})^2}{q_i} - \frac{(\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha})^2}{q_i^2} \right]$$

\[-2pa \sum_{i=1}^{n} \left[ \frac{a (\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha})^2 q_i^{-(a+2)}}{1 - pq_i^{-a}} + \frac{(\frac{x_i}{\alpha})^{-\beta} \log(\frac{x_i}{\alpha})^2 q_i^{-(a+1)}}{1 - pq_i^{-a}} \right]$$

\[-\frac{(\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha})^2 q_i^{-(a+2)}}{1 - pq_i^{-a}} - \frac{pa (\frac{x_i}{\alpha})^{-2\beta} \log(\frac{x_i}{\alpha})^2 q_i^{-2(a+1)}}{1 - pq_i^{-a}} \right]^2 \],

where $q_i = \left[ 1 + (\frac{x_i}{\alpha})^{-\beta} \right]$. 

References


