On the comparison of Schwartz and Smith’s two and three-factor models on commodity prices

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Abstract

Since Schwartz and Smith (2000) [5] published their study on two-factor model on commodities prices, many studies have used this model and others have extended it. The authors also proposed the three-factor model due to the poor fitting of the two-factor on long-term future prices. At that time the authors had only long-term prices from a private source to calibrate, test and compare these models. No public data on long-term future contracts were available. On the other hand, during the last decade the commodities prices soared as did the liquidity of long-term contracts. This means that the interest of the agents in the management of their risk on long-term positions increased the same way and this is the motivation for this study. In this paper we revisit the comparison between two and three-factor models using public data for short and long-term contracts. We also provide a detailed derivation of the three-factor model differently from that of the original article. We used oil future prices traded on NYMEX to calibrate the model. The results show a better fit of the three-factor model for the term structure of prices and volatilities mainly for long maturities contracts. The two-factor model in most portions of the curve underestimates the risk premiums compared to the three-factor model.

Keywords: Commodity models, future markets, Kalman filter, oil prices

JEL Classification: G13, G32, C33

Submission area: Finance

Abstract

Desde a publicação do modelo de preços de dois fatores para commodities em Schwartz e Smith (2000) [5], muitos outros artigos foram publicados fazendo aplicação do mesmo e outros buscaram estendê-lo. Em um adendo ao artigo, os autores estudaram o uso de um terceiro fator, modificando o modelo inicial, como um exemplo de extensão possível. Fizeram tal estudo em decorrência do pobre ajuste do modelo de dois fatores aos vértices longos da estrutura a termo. Na calibragem dos modelos, os autores dispunham de dados públicos para contratos
com maturidade até dezessete meses. A comparação com dados de longa maturidade somente foi possível com dados de uma empresa privada. Entretanto, na última década os contratos de mais longa maturação ganharam liquidez devido ao grande interesse dos agentes pelas commodities que tiveram seus preços impulsionadas pela alta demanda de países emergentes. O grande interesse dos agentes no gerenciamento de risco dos contratos de longa maturação é a maior motivação para este estudo. Neste artigo revisitamos a comparação dos modelos de dois e três fatores de Schwartz e Smith usando dados públicos (contratos futuros do NYMEX). Apresentamos também a derivação detalhada do modelo de três fatores diferente do artigo original. Como resultado encontramos um melhor ajuste para o modelo de três fatores nas estruturas a termo dos preços e volatilidades, com ênfase para os contratos de maior maturidade. Além disso, constatamos que ao longo da maior parte da curva futura, o modelo de dois fatores subestima o prêmio de risco comparado à resposta do modelo de três fatores.

Keywords: modelos em commodities, mercados futuros, filtro de Kalman, preços do petróleo

1 Introduction

The paper by Schwartz and Smith (2000) [5] provides a two-factor model for commodities prices. The importance of this type of analysis is twofold: (i) for pricing commodity linked securities and therefore managing the risk embedded in production and trading, (ii) pricing real options on investment decisions in new projects related to commodities production.

The literature on commodities before 1990 considered the behavior of prices based on a single factor evolving as a geometric Brownian motion. Works of Brennan and Schwartz (1985) [2] and Paddock, Siegel and Smith (1988) [10] followed this direction. The weakness of this approach in describing the specific characteristics was soon evident. For example, the convenience yield adopted in these models was deterministic. This is a simplified approach since empirical evidence shows its stochastic behavior.

In Gibson and Schwartz (1990) [5] the authors described the uncertainty using two stochastic factors: the spot price and the convenience yield. They were trying to fill the gap in previous studies. However the estimation proposed in their paper was complex and involved a solution of a PDE.

Later Schwartz (1997) [11] analyzed the behavior of commodities prices based on three different models. In the first model was used one single factor, the spot price. In the second he used the convenience yield and the spot price as driving factors. The third model included the interest rate as an additional factor to the previous one. These models were estimated through Kalman filter. The author concluded that the two-factor model was better than the one-factor describing more appropriately the term structure of future prices and volatilities. The three-factor model added no important effect on the two-factor. This is why the interest rate was discarded as playing a crucial role in these
types of models.

Schwartz and Smith (2000) modeled \[5\] the logarithm of the spot price as the sum of two components: the short-term variations \( \chi_t \) and the long-run equilibrium level \( \xi_t \). Both factors are not observables. Nonetheless they can be estimated through the filtering of future prices. The Kalman filter technique was used as in Schwartz (1997) \[11\]. The economic intuition behind this model is that the \( \chi_t \) (short-term variations) is the difference between the spot price and the long-run equilibrium level. It reflects the short-term decisions among producers related to the level of inventories or changes in demand due to weather conditions, for example. Its dynamics is modeled as an Ornstein-Uhlenbeck process reverting to zero (the equilibrium level of \( \chi_t \)). In this way, \( \chi_t \) fluctuates around zero according to economic forces in the near term. The long-run equilibrium price, \( \xi_t \), reflects the macro-economic issues such as advances in the technology of production, new discoveries, geopolitical changes, regulatory environment and new substitutes for the product, for example. It is modeled as geometric Brownian motion in which its drift represents these facts. One clear example of this issue is what happened recently to the natural gas prices in the USA. A new technology of hydraulic fracturing of the shale reservoir rock enabled the enhancement of production and reserves. A huge amount of gas volume could be considered recoverable and also the increase of production put a downward pressure on prices. These volumes of gas \textit{in place} that have been proven by the new technology have been named shale gas.

Schwartz and Smith’s model is exactly the same as in Gibson and Schwartz. The \( \chi_t \) factor is equivalent to the convenience yield. However the authors claim that it is more intuitive since it does not include the convenience yield directly which is a variable less clear to deal with. They also claim that their model is more orthogonal than the previous one, making the role of the two factors on the behavior of commodity prices more understandable.

Many studies were published on the commodities area as a consequence of the Schwartz and Smith paper. Manoliu and Tompaidis (2002) \[9\] used this model to analyze the natural gas prices. Sorensen (2002) \[13\] studied agricultural commodities with the same model. Lucia and Schwartz (2002) \[8\] and Villaplana (2004) \[14\] analyzed the electricity markets using the two-factor model. Schwartz (1998) proposed a one-factor model but retaining the good characteristics of the two-factor. Aiube, Baidya and Tito (2008) \[1\] extended the model introducing the jump component on the dynamics of the short-term factor.

In their original paper Schwartz and Smith were aware of the importance of capturing a good description of the long-term factor (\( \xi_t \)) through the inclusion of long maturities contracts. At that time these contracts were not publicly available at NYMEX. They estimated the model with two set of contracts. The first set was obtained from NYMEX with maturities ranging from one to seventeen months. The second set was private with maturities ranging from two months to nine years. Using the second, they could observe the behavior of the modeled prices compared to empirical data. As a result they concluded by a poor fit of the model on the long-term contracts of the
term structure. One natural way to deal with this fact is not considering the drift as fixed but stochastic. They did not do this in the original article, but in addendum to it available on the Smith’s page. Doing this they improved the model to fit the data better.

In this paper we highlight the comparison between the two and three-factor models. The importance of this study is related to the growth of commodities production that has been supplying the increased demand of emerging economies. This high demand pushed prices to levels never reached before (oil is an example). As producers and traders of commodities increased their demand for longer maturities contracts, they also became more interested in the management of the risk of their positions on investments or on long supply contracts of the physical commodity. Hence the liquidity of long contracts negotiated at exchanges increased rapidly the last decade. Therefore the long contracts of the term structure gained much more importance recently. As a consequence models that better fit the long contracts of the term structure are more appropriate currently. In this study we provide the comparison of both models presenting what is relevant in each one. We use public data of oil prices traded at NYMEX for up to six years. We also provide a detailed derivation of the model differently from that of the authors.

The remainder of the paper is organized as follows. Section two presents the model hypothesis adopted in the economy and a formal description of the two models detailing the term structure of prices, volatilities and risk premiums. Section three presents the results of the estimation (using public oil price data) based on the Kalman filter technique and the comparison of the two models. Section four concludes. The math involved on derivation of the three-factor is left to the Appendix.

2 The models

2.1 The model set up

Consider the probability space \((\Omega, \mathcal{F}, P)\) that models the uncertainty in the economy. This uncertainty is driven by the multivariate Brownian process \(B_t = (B_{1t}, B_{2t}, \ldots, B_{nt})\) of which the components are independent standard Brownian processes. The time when the events occur belongs to the interval \([0, \bar{T}]\) where \(\bar{T}\) is a fixed time. The information filtration \(\mathcal{F}_t\) is the natural filtration generated by \(B_t\).

Consider that in the economy there are a finite number of futures commodity contracts traded continuously. The maturity of such contracts are denoted by \(T_j\), \(j = 1, m\) and \(T_j \in [0, \bar{T}]\). Also, the interest rate is considered constant in \([0, \bar{T}]\). This assumption assures that future and forward prices are equals. The price of the contract \(j\) traded at \(t\) and maturing at \(T_j\) is denoted by \(F_{t,T_j}\). The spot price of a such commodity is \(S_t\) and it is not observable. This is a natural hypothesis since that even in physical markets the commodity is delivered upon the payment at a time in the future far from the moment of the negotiation. The spot price is decomposed in \(n\) factors which are described by vector \(X_t\). The dynamics of these factors follow a Markov process in a state space \(D \subset \mathbb{R}^n\) and evolves according to the stochastic differential equation

\[
dX_t = \mu (X_t) \, dt + \sigma (X_t) \, dB_t, \tag{1}
\]
where \( B_t \) is the multivariate Brownian in \( \mathbb{R}^n \), \( \mu : D \rightarrow \mathbb{R}^n \), \( \sigma : D \rightarrow \mathbb{R}^{n \times n} \). Also, we consider an affine structure for \( \mu, \sigma \sigma^\top \) and that \( X_0 \) is known.

Our economy is arbitrage-free where \( Q \) is the equivalent martingale measure (EMM) to \( P \). In this framework \( F_{t,T} \) is martingale. And also, there is a multivariate stochastic process \( \lambda_t = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) adapted to \( \mathcal{F}_t \) that describes the evolution of market prices of risks for each factor in which \( X_t \) is decomposed. We impose the usual regularity conditions for \( \mu, \sigma \) and \( \lambda \) for integrability. Under these specifications the Girsanov theorem assures that the \( \tilde{X}_t \) is the equivalent martingale measure (EMM) the two-factor model for \( \mathcal{F}_t \). In this framework \( \mathcal{F}_t \) and \( \lambda_t \) is considered constant in \( [0,T] \).

2.2 Two-factor model

The two-factor model is the case where \( n = 2 \) and \( X_t = (\chi_t, \xi_t)^\top \). The uncertainty is driven by \( B_t = (B_{\chi_t}, B_{\xi_t})^\top \) and \( \chi_t \), the first factor, follows an Orstein-Uhlenbeck reverting to zero with speed of reversion \( \kappa \) and volatility \( \sigma_\chi \), both constants in \([0,T]\). \( \chi_t \) is named the short-term variations and is a concept related to the convenience yield in Gibson and Schwartz (1990) [5] model. The second factor evolves according to a geometric Brownian motion with drift \( \mu_\xi \) and volatility \( \sigma_\xi \) admitted as constants in the same interval. It is called long-run equilibrium price.

Therefore for \( t \in [0,T] \) the two-factor model is written as

\[
\begin{align*}
\ln (S_t) &= f (t) + \chi_t + \xi_t \\
\frac{d\chi_t}{\chi_t} &= -\kappa \chi_t dt + \sigma_\chi dB_{\chi_t} \\
\frac{d\xi_t}{\xi_t} &= \mu_\xi dt + \sigma_\xi dB_{\xi_t},
\end{align*}
\]

where \( \kappa > 0, \sigma_\chi > 0, \sigma_\xi > 0 \), \( dB_{\chi_t} dB_{\xi_t} = \rho_{\chi\xi} dt \) and \( (\chi_0, \xi_0) \) is known. The function \( f (t) \) is deterministic and describes the seasonal effects on commodity prices.

Under the \( Q\)-EMM the two-factor model for \( t \in [0,T] \) is written as

\[
\begin{align*}
\frac{d\chi_t}{\chi_t} &= (-\kappa \chi_t - \lambda_\chi) dt + \sigma_\chi d\tilde{B}_{\chi_t} \\
\frac{d\xi_t}{\xi_t} &= (\mu_\xi - \lambda_\xi) dt + \sigma_\xi d\tilde{B}_{\xi_t},
\end{align*}
\]

where \( (\lambda_\chi, \lambda_\xi) \) is considered constant in \([0,T] \), \( d\tilde{B}_{\chi_t} d\tilde{B}_{\xi_t} = \rho_{\chi\xi} dt \) and we will use the definition \( \mu_\xi - \lambda_\xi = \mu^* \).

Proposition 1. Under the hypothesis above and under the general model set up we have:

(i) the term structure of future prices, which is the price at \( t \in [0,T] \) for a contract maturing at \( T_j \), is given by

\[
\ln (F_{t,T_j}) = f (T_j) + e^{-\kappa(T_j-t)} \chi_t + \xi_t + A (T_j - t), \quad j = 1, \ldots, m
\]

where

\[
A (T_j - t) = \mu^*_\xi (T_j - t) - (1 - e^{-\kappa(T_j-t)}) \frac{\lambda_\chi}{\kappa} \\
+ \frac{1}{2} \left[ (1 - e^{-2\kappa(T_j-t)}) \frac{\sigma^2_\chi}{2\kappa} + \sigma^2_\xi (T_j - t) + 2 (1 - e^{-\kappa(T_j-t)}) \frac{\rho_{\chi\xi} \sigma_\chi \sigma_\xi}{\kappa} \right].
\]
(ii) The term structure of volatilities is given by
\[
\sigma_{F_t,T_j} = \left[ \frac{1}{dt} \text{Var} \left( \frac{dF_t,T_j}{F_t,T_j} \right) \right]^{\frac{1}{2}} = \left[ \sigma_x^2 e^{-2\kappa(T_j-t)} + \sigma_\xi^2 + 2\rho_x\sigma_x\sigma_\xi e^{-\kappa(T_j-t)} \right]^{\frac{1}{2}}, \quad j = 1, \ldots, m. \tag{5}
\]

(iii) The term-structure of risk premium is
\[
RP_{t,T_j} = F_{t,T_j} - \mathbb{E}^P (S_{T_j} | \mathcal{F}_t) = \exp \left[ e^{-\kappa(T_j-t)} \chi_t + \xi_t \right] \{\exp [A(T_j-t)] - \exp [B(T_j-t)]\} \quad j = 1, \ldots, m, \tag{6}
\]
where \(B(T_j-t)\) is a similar expression to \(A(T_j-t)\) in which \(\lambda_x = \lambda_\xi = 0\).

\textbf{Proof.} The proof of items (i) and (ii) are in the Schwartz and Smith (2000) \cite{Schwartz:2000}. Item (iii) is a simpler case of the three-factor model which proof is in the Appendix. \(\square\)

We define the hyperparameters of the model by \(\Theta = (\mu_\xi, \lambda_\xi, \kappa, \lambda_x, \sigma_x, \sigma_\xi, \rho_x\sigma_\xi, s_j)\), where \(s_j\) is the standard deviation of the difference between observed and modeled future price of the \(j\)-th contract.

### 2.3 Three-factor model

In the case of the three-factor model we have \(n = 3\) and \(X_t = (\chi_t, \xi_t, \mu_t) \top\). The uncertainty is driven by \(B_t = (B_\chi_t, B_\xi_t, B_{\mu_t}) \top\). In this case the drift \(\mu_t\) of the geometric Brownian motion in equation (2c) evolves through a mean-reverting process. This is the third factor. The dynamics of the first and second factors remain the same. For \(t \in [0, \bar{T}]\) the three-factor model is described as

\[
\ln (S_t) = g(t) + \chi_t + \xi_t \tag{7a}
\]

\[
d\chi_t = -\kappa \chi_t dt + \sigma_\chi d\tilde{B}_\chi_t \tag{7b}
\]

\[
d\xi_t = \mu_t dt + \sigma_\xi d\tilde{B}_\xi_t \tag{7c}
\]

\[
d\mu_t = \eta (\bar{\mu} - \mu_t) dt + \sigma_\mu d\tilde{B}_{\mu_t} \tag{7d}
\]

where \(\eta > 0, \sigma_\mu > 0, dB_\chi_t dB_\xi_t = \rho_{x\xi} dt, dB_\chi_t dB_{\mu_t} = \rho_{x\mu} dt\) and \(dB_\xi_t dB_{\mu_t} = \rho_{\xi\mu} dt\). The parameter \(\bar{\mu}\) is the long-term equilibrium drift to which the \(\mu_t\) reverts. The deterministic function \(g(t)\) describes the seasonal effects. The remaining terms were previously defined.

Under the Q-EMM the three-factor model for \(t \in [0, \bar{T}]\) is written as

\[
d\chi_t = \kappa \left( -\frac{\lambda_x}{\kappa} - \chi_t \right) dt + \sigma_\chi d\tilde{B}_\chi_t \tag{8a}
\]

\[
d\xi_t = (\mu_t - \lambda_x) dt + \sigma_\xi d\tilde{B}_\xi_t \tag{8b}
\]

\[
d\mu_t = \eta (\bar{\mu} - \mu_t) dt + \sigma_\mu d\tilde{B}_{\mu_t} \tag{8c}
\]

\textsuperscript{1}The results from Proposition 1 can be deduced through the Duffie and Kan (1996) \cite{Duffie:1996} transform. This transform is a more general result useful for dynamics where the factors evolve through an affine jump diffusion.
where the vector of the market price of risk \((\lambda_\chi, \lambda_\xi, \lambda_\mu)\) is considered constant in \([0, \bar{T}]\), and \(\hat{\mu}\) is defined as \(\hat{\mu} = \bar{\mu} - \frac{\lambda_\mu}{\eta}\). The correlations among all three standard Brownian processes, under the EMM, are the same as above.

**Proposition 2.** Under the considerations of this section and under the general model set up we have

(i) the term structure of future prices is given by

\[
\ln(F_{t,T_j}) = g(T_j) + e^{-\kappa(T_j-t)} \chi_t + \xi_t + (\mu_t - \hat{\mu}) \left(1 - e^{-\eta(T_j-t)}\right) + C(T_j - t), \quad j = 1, \ldots, m, \tag{9}
\]

where the term \(C(T_j - t)\) is defined in the Appendix.

(ii) the term structure of variance \(\sigma^2_{F_t,T_j} = \frac{1}{\Delta t} \text{Var} \left(\frac{dF_{t,T_j}}{F_{t,T_j}}\right)\) is

\[
\sigma^2_{F_t,T_j} = \sigma_\chi^2 e^{-2\kappa(T_j - t)} + \sigma_\xi^2 + \frac{\sigma_\mu^2}{\eta^2} \left(1 - e^{-\eta(T_j-t)}\right)^2 + 2\rho_{\chi\xi}\sigma_\chi\sigma_\xi e^{-\kappa(T_j-t)} + \frac{2\rho_{\chi\mu}\sigma_\chi\sigma_\mu (1 - e^{-\eta(T_j-t)}) e^{-\kappa(T_j-t)}}{\eta} + \frac{2\rho_{\xi\mu}\sigma_\mu\sigma_\xi}{\eta} \left(1 - e^{-\eta(T_j-t)}\right), \quad j = 1, \ldots, m, \tag{10}
\]

and hence the square root of the above equation is \(\sigma_{F_t,T_j}\), the term structure of volatilities for \(j = 1, \ldots, m\).

(iii) the term structure of risk premium is

\[
RP_{t,T_j} = \exp\left[g(T_j) + e^{-\kappa(T_j-t)} \chi_t + \xi_t\right] \times \exp\left[(\mu_t - \hat{\mu}) \beta + C(T_j - t) - \exp[(\mu_t - \hat{\mu}) \beta + D(T_j - t)]\right] \quad j = 1, \ldots, m, \tag{11}
\]

where \(D(T_j - t)\) is a similar expression to \(C(T_j - t)\) in which \(\lambda_\chi = \lambda_\xi = \lambda_\mu = 0\) and \(\beta = \frac{1 - e^{-\eta(T_j-t)}}{\eta}\).

**Proof.** See the Appendix for a detailed proof of the Proposition 2.

The vector \(\Theta = (\kappa, \sigma_\chi, \lambda_\chi, \sigma_\xi, \eta, \bar{\mu}, \hat{\mu}, \lambda_\xi, \sigma_\mu, \rho_{\chi\xi}, \rho_{\chi\mu}, \rho_{\xi\mu}, \rho_{\xi\mu}, s_j)\) represents the hyperparameters of the three-factor model. The parameter \(s_j\) was already defined.

### 3 Empirical analysis

The factors \(\chi_t\) and \(\xi_t\) on two-factor model and \(\chi_t, \xi_t\) and \(\mu_t\) on three-factor model are non-observable. Hence the spot price \(S_t\) is also non-observable. Future prices \(F_{t,T_j}\) \(j = 1, \ldots, m\) are the variable that is observed in these models. This means that the spot price should be estimated through the filtering of future prices. On the other hand all these factors in both models have Gaussian dynamics. Furthermore, from equations (4) and (9) one can note that the logarithm of the future prices are linear with
their correspondent factors and we can also admit a Gaussian noise for these observed variables. In other words, both models are linear and Gaussian. This is a typical case where the Kalman filter is the appropriate methodology for estimation of non-observable variables. The details of the Kalman filter can be found on Harvey (1989) [6] and Durbin and Koopman (2001) [4] among others. An overview of filtering in finance can be found in Javaheri, Lautier and Galli (2003) [7].

\[
\begin{array}{l}
\text{Table 1: Main statistics of the data} \\
\hline
\text{F}_{t,1} & \text{F}_{t,4} & \text{F}_{t,24} & \text{F}_{t,30} & \text{F}_{t,60} & \text{F}_{t,67} \\
\hline
\text{Mean} & 82.82 & 84.53 & 86.70 & 86.77 & 87.66 & 88.71 \\
\text{Median} & 81.60 & 83.08 & 85.97 & 86.21 & 88.47 & 89.99 \\
\text{Maximum} & 145.29 & 146.43 & 144.26 & 143.53 & 141.62 & 141.94 \\
\text{Minimum} & 33.87 & 41.15 & 54.56 & 56.82 & 59.63 & 59.67 \\
\text{Std. dev.} & 20.98 & 19.69 & 16.10 & 15.58 & 14.71 & 14.79 \\
\text{Skewness} & 0.2079 & 0.3762 & 0.7049 & 0.7652 & 0.7675 & 0.6122 \\
\text{Kurtosis} & 3.0268 & 3.1125 & 3.8556 & 4.0885 & 4.5280 & 4.3634 \\
\text{Obs.} & 1388 & 1388 & 1388 & 1388 & 1388 & 1388 \\
\end{array}
\]

Therefore we proceed with the estimation writing the models on the space state form. In the Kalman filter language the factors are the state variables. Equations (2b) - (2c) and (7b) - (7d) are the state equations for two and three-factor models, respectively. The evolution of the state equations in the filtering process is taken under the \( P \) measure. At each time the algorithm of the Kalman filter estimates each state variable and hence \( S_t \) is also estimated. The Kalman filter allows the estimation of the hyperparameters of the model through the maximisation of the likelihood of the prediction error as following. On equations (4) and (9) we have the measurement equations through the inclusion of the error term. We call \( y_t = (y_{t,T_1}, \ldots, y_{t,T_m})^\top \) the observed multivariate time series with \( m \) elements, where each component is the logarithm of future price at time \( t \) and maturity \( T_j, j = 1, \ldots, m \). Consider the partition \( \tau : 0 = t_0 < t_1 \ldots < t_M = T_j \). Observations \( y_t \) are collected in a time interval \( \Delta t \) equally spaced, \( \Delta t = t_i - t_{i-1} \). At each time we have \( m \) observed future prices, one for each contract. Observed prices (signals) at \( t \) are denoted by \( F_{t,T_j} \) and we can write \( \ln (F_{t,T_j}) = \ln (F_{t,T_j}) + \epsilon_{t,j} \) where \( \epsilon_{t,j} \sim N (0, s_j^2) \), \( j = 1, \ldots, m \), here \( \epsilon_t \) accounts for errors of any kind in prices and it is assumed that it is normally and independently distributed. We write \( \mathcal{G}_t = \{ y_t, y_{t-\Delta t}, \ldots, y_{t_1}, y_{t_0} \} \) to describe the information available until \( t \). Each vector in \( \mathcal{G}_t \) is independent from the other. The joint density function that defines the likelihood function is given by

\[
L (\Theta; y) = \prod_t \ell (y_t | \mathcal{G}_{t-\Delta t}),
\]

where \( \ell (y_t | \mathcal{G}_{t-\Delta t}) \) means the density of \( y_t \) conditional on \( \mathcal{G}_{t-\Delta t} \).

Also, define \( \hat{y}_{t|t-\Delta t} = E^P (y_t | \mathcal{G}_t) \) with covariance matrix given by \( \Lambda_t \). Writing the prediction error as \( \nu_t = y_t - \hat{y}_{t|t-\Delta t} \), we have the logarithm of likelihood function as

\[
\ln [L (\Theta; y)] = - \frac{Mm}{2} \ln (2\pi) - \frac{1}{2} \sum_t \ln (|\Lambda_t|) - \frac{1}{2} \sum_t \nu_t^\top \Lambda_t \nu_t.
\]
To sum up, in each run of the filtering process all states variables at each instant are estimated. Since all the predicted signals are obtained, the hyperparameters are estimated based on the maximization of the likelihood function.

![Figure 1: F_{t,1} and F_{t,67} contracts traded on NYMEX](image)

We sampled oil futures prices traded at NYMEX from 11/06/2006 to 06/01/2012. The sample consists of a panel of closing daily prices with six different contracts. We used two short-term contracts, two middle-term and two long-term contracts. The panel is formed with $F_{t,1}$, $F_{t,4}$, $F_{t,24}$, $F_{t,30}$, $F_{t,60}$, $F_{t,67}$, where the subscript is the maturity in months of each contract. Table 1 shows the main statistics of the data used.

The panel is complete which means that there are no missing values. One can observe that the sample encompasses the period of the financial crisis of 2008 when oil prices peaked to approximately US$ 147/bbl right before Lehman Brothers collapsed. The oil prices fell down abruptly to US$ 40/bbl. Then they followed the recovery of American economy until the subsequent crises in Europe worsened. Figure 1 shows the first ($F_{t,1}$) and the last contract ($F_{t,67}$) of the data above. One can observe the behavior of the term structure on contango and on backwardation, but predominantly on contango.

The estimation results of the two-factor model is shown on Table 2. It is a well known fact that the oil prices are not affected by seasonality. Therefore in the estimation procedure we did not include the $f(t)$ function on equation (2a). We note that the parameters $\mu_\xi$ and $\lambda_\xi$ are not statistically significant making $\mu^*$ equal to zero. We believe that the ups and downs of oil prices in this turbulent period is one reason for this fact. A similar fact occurred in the original analysis of Schwartz and Smith. They found a negative expected growth rate $\mu_\xi$ due to price decline during the period sampled, making the risk premium negative. This was not likely to be the expectation of investors over that period. Furthermore this parameter was found not to be significant in the original article. Also, as mentioned by Schwartz and Smith paper, there is always one non-significant standard deviation $s_j$ in the set of measurement equations. In this case
Table 2: Estimation of two-factor model

<table>
<thead>
<tr>
<th>parameter</th>
<th>coefficient</th>
<th>std error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_\xi$</td>
<td>-0.018297</td>
<td>0.093725</td>
</tr>
<tr>
<td>$\lambda_\xi$</td>
<td>0.012220</td>
<td>0.093733</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.628309</td>
<td>0.004762</td>
</tr>
<tr>
<td>$\lambda_\chi$</td>
<td>-0.106721</td>
<td>0.002283</td>
</tr>
<tr>
<td>$\sigma_\chi$</td>
<td>0.209899</td>
<td>0.003326</td>
</tr>
<tr>
<td>$\sigma_\xi$</td>
<td>0.250754</td>
<td>0.003719</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.44670</td>
<td>0.017189</td>
</tr>
<tr>
<td>$s_1$</td>
<td>$1.21 \times 10^{-3}$</td>
<td>$1.6 \times 10^{-5}$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$1.73 \times 10^{-5}$</td>
<td>$2.4 \times 10^{-6}$</td>
</tr>
<tr>
<td>$s_{24}$</td>
<td>$2.61 \times 10^{-5}$</td>
<td>$1.2 \times 10^{-6}$</td>
</tr>
<tr>
<td>$s_{30}$</td>
<td>$2.89 \times 10^{-8}$</td>
<td>$1.3 \times 10^{-6}$</td>
</tr>
<tr>
<td>$s_{60}$</td>
<td>$5.38 \times 10^{-4}$</td>
<td>$3.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>$s_{67}$</td>
<td>$1.52 \times 10^{-3}$</td>
<td>$9.2 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>final state</th>
<th>value</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_t$</td>
<td>0.213169</td>
<td>0.00</td>
</tr>
<tr>
<td>$\xi_t$</td>
<td>4.219340</td>
<td>0.00</td>
</tr>
<tr>
<td>log likelihood</td>
<td>21623</td>
<td></td>
</tr>
</tbody>
</table>

is the $F_{t,30}$ contract in our analysis.

We ran the estimation procedure for the three-factor model with the same data we used in the two-factor case. The results are shown on Table 3. We note that $\Bar{\mu}$ is non-significant. The same reason that made the neutral drift not significant in the two-factor model can be the explanation of this case. We also note non-significance on standard deviation of $F_{t,30}$ contract. Comparing the results of both models one observes that the likelihood of the three-factor is greater.

On the top part of Figure 2 is shown for the first contract the filtered price and the observed price related to the two-factor model. On the bottom the same case for the three-factor model. It seems a better fit for the three-factor model.

Tables 4 and 5 present the error between the predicted and the observed prices for all maturities for both models respectively. These results confirm a smaller average error for the three-factor case. Also, the errors on the long portion of the curve are smaller in the three-factor model. Comparing the average errors of the two contracts in the middle of the curve one concludes that they are approximately the same. On short maturities contracts one observes minor errors on the three-factor model. If we isolated one single contract, for example $F_{t,1}$, and observe what happens with the errors for $t \in [0, T_1]$ we can note that the greatest errors are in the second semester of 2008, exactly on the period of the highest volatilities on prices. In months out of the crisis the errors are much smaller. The sample we used encompasses this turbulent period of high volatilities and this explains the large errors we found in both models for the term structure. This can be accounted by the standard deviation of the difference between the observed price (signal) and the filtered price (filtered signal), i.e. $F_{t,T_j} - F_{t,T_j}$ on the
Table 3: Estimation of two-factor model

<table>
<thead>
<tr>
<th>parameter</th>
<th>coefficient</th>
<th>std error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa$</td>
<td>1.174823</td>
<td>0.031353</td>
</tr>
<tr>
<td>$\sigma_\chi$</td>
<td>0.30003</td>
<td>0.006733</td>
</tr>
<tr>
<td>$\lambda_\chi$</td>
<td>-0.253517</td>
<td>0.007286</td>
</tr>
<tr>
<td>$\sigma_\xi$</td>
<td>0.271483</td>
<td>0.003820</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.125887</td>
<td>0.013557</td>
</tr>
<tr>
<td>$\bar{\mu}$</td>
<td>0.000789</td>
<td>0.028729</td>
</tr>
<tr>
<td>$\hat{\mu}$</td>
<td>0.040909</td>
<td>0.009284</td>
</tr>
<tr>
<td>$\hat{\mu} - \lambda_\xi$</td>
<td>0.019189</td>
<td>0.006440</td>
</tr>
<tr>
<td>$\sigma_\mu$</td>
<td>0.011570</td>
<td>0.000762</td>
</tr>
<tr>
<td>$\rho_{\chi\xi}$</td>
<td>0.216143</td>
<td>0.030344</td>
</tr>
<tr>
<td>$\rho_{\chi\mu}$</td>
<td>-0.275255</td>
<td>0.108010</td>
</tr>
<tr>
<td>$\rho_{\xi\mu}$</td>
<td>-0.695824</td>
<td>0.046622</td>
</tr>
<tr>
<td>$s_1$</td>
<td>$3.45 \times 10^{-5}$</td>
<td>$3.8 \times 10^{-6}$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>$4.14 \times 10^{-4}$</td>
<td>$1.3 \times 10^{-5}$</td>
</tr>
<tr>
<td>$s_{24}$</td>
<td>$1.38 \times 10^{-5}$</td>
<td>$9.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>$s_{30}$</td>
<td>$2.80 \times 10^{-8}$</td>
<td>$4.3 \times 10^{-7}$</td>
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<td>$s_{60}$</td>
<td>$8.68 \times 10^{-5}$</td>
<td>$6.0 \times 10^{-6}$</td>
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<tr>
<td>$s_{67}$</td>
<td>$2.42 \times 10^{-4}$</td>
<td>$1.6 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 4: Errors between predicted and observed prices: two-factor model

<table>
<thead>
<tr>
<th>contract maturity (months)</th>
<th>1</th>
<th>4</th>
<th>24</th>
<th>30</th>
<th>60</th>
<th>67</th>
<th>mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE (US$/bbl)</td>
<td>2.9072</td>
<td>1.8580</td>
<td>1.5954</td>
<td>1.4861</td>
<td>2.4596</td>
<td>3.6581</td>
<td>2.3274</td>
</tr>
<tr>
<td>MAE (US$/bbl)</td>
<td>2.1919</td>
<td>1.3629</td>
<td>1.1926</td>
<td>1.0644</td>
<td>1.9652</td>
<td>2.9135</td>
<td>1.7812</td>
</tr>
<tr>
<td>MAPE (%)</td>
<td>2.98</td>
<td>1.67</td>
<td>1.38</td>
<td>1.22</td>
<td>2.28</td>
<td>3.32</td>
<td>2.14</td>
</tr>
<tr>
<td>std dev (US$/bbl)</td>
<td>2.6950</td>
<td>1.8577</td>
<td>1.5784</td>
<td>1.4864</td>
<td>2.4502</td>
<td>3.4596</td>
<td>–</td>
</tr>
</tbody>
</table>

last line of these tables.

Equations (5) and (10) allow the comparison between the volatilities of both models. On Figure 3 we compare the term structure of volatilities, obtained from the estimated hyperparameters of both models, with the historical volatility during the same period. It is clear a better adjustment for the short-term and long-term contracts using the three-factor model. In the middle portion of the curve both models are approximately similar compared to historical volatilities. For sure the months of high volatility due to the financial crisis affected the term structure and who deals daily with oil prices can observe that this increased the volatility by a small amount in each contract.

On Figure 4 we plot the term structure of the risk premium for both cases on 09/01/2009. One can say that the models agree on positive risk premiums along all maturities. And
Table 5: Errors between predicted and observed prices: three-factor model

<table>
<thead>
<tr>
<th>contract maturity (months)</th>
<th>1</th>
<th>4</th>
<th>24</th>
<th>30</th>
<th>60</th>
<th>67</th>
<th>mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE (US$/bbl)</td>
<td>2.0380</td>
<td>2.1846</td>
<td>1.6496</td>
<td>1.6052</td>
<td>2.1057</td>
<td>2.4869</td>
<td>2.0118</td>
</tr>
<tr>
<td>MAE (US$/bbl)</td>
<td>1.4769</td>
<td>1.6149</td>
<td>1.1514</td>
<td>1.0779</td>
<td>1.2468</td>
<td>1.5153</td>
<td>1.3472</td>
</tr>
<tr>
<td>MAPE (%)</td>
<td>1.90</td>
<td>2.03</td>
<td>1.34</td>
<td>1.25</td>
<td>1.45</td>
<td>1.74</td>
<td>1.62</td>
</tr>
<tr>
<td>std dev (US$/bbl)</td>
<td>2.0373</td>
<td>2.1592</td>
<td>1.6501</td>
<td>1.6068</td>
<td>2.0925</td>
<td>2.4801</td>
<td>–</td>
</tr>
</tbody>
</table>

Figure 2: Filtered and observed prices of the first contract for both models

also both curves exhibit approximately the same shape. Except for the long-term portion of the curve, the two-factor model underestimates the risk premium compared to the three-factor model. We carried on the comparison for other dates and observed that the results change very little. In some cases, the three-factor model presents the long portion of the curve falling, but the long-term risk premium remains ranging from US$12/bbl to US$14/bbl. As far as we know, one issue that is still opened is how much the financial crisis affected the risk premium in oil prices (or in oil products) based on this type of modelling. Since it was an abnormal period, it seems a good issue for future research.

4 Conclusions

In this paper we analyzed the two and three-factor models on commodities proposed on Schwartz and Smith (2000) [5]. The importance of this type of study is related on pricing derivatives securities on commodity futures and real options on capital budget decisions. In the original article the authors claimed the weakness of the two-factor model to fit the long-term contracts. They proposed the three-factor model and compared them with private data in an addendum to the article. There were no long-term public contracts available. However in the last decade commodities prices reached high prices and also the liquidity increased making the long portion of the curve rich in information.
We provided a detail derivation of the three-factor model and proceeded with the empirical analysis using future oil prices. The non-observable state variables were estimated using the Kalman filter. The hyperparameters of the model resulted from the maximisation of the likelihood function of the prediction errors.

We focused on the term structure of prices, volatilities and risk premiums. We found a smaller average error in the three-factor model when comparing the observed and filtered prices. This fact is even more pronounced at the long portion of the term structure. We also observe that on the second semester of 2008 the error between the observed and filtered prices were greater than those out of crisis period. The term structure of volatilities on the three-factor model presented a better fitting on short and long-term portions of the curve. The risk premium on each day of the sample is greater for the three-factor model along the curve, except for longer maturities.

These types of model can be extended in different ways and this is a flexibility inherent of its formulation. Nevertheless, currently one should account for extensions that exhibit a good behaviour on the fitting long-term contracts. This is in accordance with the tendency of agents that are more heavily positioned on long maturities than ever before.

Figure 3: Term structure of volatility for both models compared with historical
5 Appendix

In this Appendix we will prove the proposition 2. To simplify the notation we will drop the time indices from the Brownians.

5.1 Preliminaries to the proof of proposition 2

In this section we compute the first and second moments of the variables in the three-factor model. These results will be useful for the next section.

The solutions for equations (7b) and (7d) are

\[
\chi_t = \chi_0 e^{-\kappa t} + \sigma \chi \int_0^t e^{\kappa(u-t)} dB_x. \tag{12a}
\]

\[
\mu_t = \bar{\mu} + (\mu_0 - \bar{\mu}) e^{-\eta t} + \sigma \mu \int_0^t e^{\eta(u-t)} dB_\mu. \tag{12b}
\]

The solution for equation (7c) is obtained introducing (12b) into (7c) and integrating. Then we get

\[
\xi_t = \xi_0 + \bar{\mu} t + (\mu_0 - \bar{\mu}) \frac{1 - e^{-\eta t}}{\eta} + \sigma \xi B_\xi + \int_0^t \left( \sigma \mu \int_0^t e^{\eta(u-t)} dB_\mu \right) du. \tag{13}
\]

Figure 4: Term structure of risk premium on 09/01/2009 for both models
The mean and variance under P-measure of $\chi_t$ and $\mu_t$ are straightforward using Itô’s isometry and are written as

\[
E^P(\chi_t) = \chi_0 e^{-\kappa t} \tag{14a}
\]
\[
\text{Var}^P(\chi_t) = \sigma^2 \kappa (1 - e^{-2\kappa t}) \tag{14b}
\]
\[
E^P(\mu_t) = \bar{\mu} + (\mu_0 - \bar{\mu}) e^{-\eta t} \tag{14c}
\]
\[
\text{Var}^P(\mu_t) = \sigma^2 \eta (1 - e^{-2\eta t}). \tag{14d}
\]

For $\xi_t$, the mean and variance are obtained through Fubini’s theorem and Itô’s isometry and then

\[
E^P(\xi_t) = \xi_0 + \bar{\mu} t + (\mu_0 - \bar{\mu}) \left(1 - e^{-\eta t}\right) / \eta \tag{15a}
\]
\[
\text{Var}^P(\xi_t) = \sigma^2 \eta + \frac{\sigma^2 t}{\eta^2} \left[t + \frac{(1 - e^{-2\eta t})}{2\eta} - 2 \frac{(1 - e^{-\eta t})}{\eta}\right] + \frac{2\rho \sigma_\xi \sigma_\mu}{\eta} \left[t - \frac{(1 - e^{-\eta t})}{\eta}\right]. \tag{15b}
\]

Now we compute the covariance among all the three variables. First let’s consider $\chi_t$ and $\mu_t$ on equations (12a) and (12b). Since we know the expected values from equations (14a) and (14c) we have

\[
\chi_t \mu_t = \left[E(\chi_t) + \sigma_\chi \int_0^t e^{\kappa(u-t)} dB_\chi\right] \left[E(\mu_t) + \sigma_\mu \int_0^t e^{\kappa(u-t)} dB_\mu\right].
\]

Multiplying the terms of the expression above, taking the expected value and noting that we have a quadratic covariation, we obtain

\[
\text{Cov}^P(\chi_t, \mu_t) = \rho \sigma_\chi \sigma_\mu \left[\frac{\kappa + \eta}{\eta} \left(1 - e^{-(\kappa+\eta) t}\right)\right]. \tag{16}
\]

In the same way, using equations (14a) and (15a) the product of $\chi_t$ and $\xi_t$ is

\[
\chi_t \xi_t = \left[E(\chi_t) + \sigma_\chi \int_0^t e^{\kappa(u-t)} dB_\chi\right] \left[E(\xi_t) + \sigma_\xi \int_0^t dB_\xi + \int_0^t \frac{\sigma_\mu}{\eta} (1 - e^{-\eta u}) dB_\mu\right].
\]

Multiplying all terms into parenthesis and taking the expected value we get

\[
E(\chi_t \xi_t) = E(\chi_t) E(\xi_t) + \sigma_\xi \sigma_\chi E(\int_0^t dB_\xi \int_0^t e^{\kappa(u-t)} dB_\chi)
\]
\[
+ \frac{\sigma_\chi}{\eta} \sigma_\mu E(\int_0^t e^{\kappa(u-t)} dB_\chi \int_0^t (1 - e^{-\eta u}) dB_\mu).
\]

\[\text{We are referring to expected value under the P-measure. To simplify we will not use the notation } E^P(\cdot). \text{ On the final covariance term we emphasize this notation.}\]
Noting again that we have a quadratic covariation and solving the integrals, we end up with
\[
\text{Cov}^P (\chi_t, \xi_t) = \frac{\rho_{\chi\xi} \sigma_{\chi} \sigma_{\chi}}{\kappa} (1 - e^{-\kappa t}) + \frac{\rho_{\chi\mu} \sigma_{\chi} \sigma_{\mu}}{\eta} \left[ \frac{(1 - e^{-\kappa t})}{\kappa} - \frac{(e^{-\eta t} - e^{-\kappa t})}{\kappa - \eta} \right].
\] (17)

Finally we compute the covariation between \(\xi_t\) and \(\mu_t\). Knowing the expected values in equations (15a) and (14c) we write
\[
\xi_t \mu_t = \left[ E (\xi_t) + \sigma_{\xi} \int_0^t dB_{\xi} + \int_0^t \frac{\sigma_{\mu}}{\eta} (1 - e^{-\eta u}) dB_{\mu} \right] \left[ E (\mu_t) + \sigma_{\mu} \int_0^t e^{\eta(u-t)} dB_{\mu} \right].
\]

Multiplying each term in both parenthesis, taking the expected value and noting the presence of quadratic covariation we get
\[
\text{Cov}^P (\xi_t, \mu_t) = \sigma_{\xi} \sigma_{\mu} \int_0^t e^{\eta(u-t)} \rho_{\xi\mu} du + \frac{\sigma_{\xi}^2}{\eta} \int_0^t (1 - e^{-\eta u}) e^{\eta(u-t)} du.
\]

Solving the integrals we end up with
\[
\text{Cov}^P (\xi_t, \mu_t) = \frac{\rho_{\xi\mu} \sigma_{\xi} \sigma_{\mu}}{\eta} (1 - e^{-\eta t}) + \frac{\sigma_{\mu}^2}{\eta} \left[ \frac{(1 - e^{-\eta t})}{\eta} - e^{-\eta t} \right].
\] (18)

Now we have completed the variance-covariance matrix (\(\Sigma\)) computed in the P-measure. That is
\[
\Sigma^P_t = \begin{bmatrix}
\text{Var} (\chi_t) & \text{Cov} (\chi_t, \xi_t) & \text{Cov} (\chi_t, \mu_t) \\
\text{Cov} (\xi_t, \chi_t) & \text{Var} (\xi_t) & \text{Cov} (\xi_t, \mu_t) \\
\text{Cov} (\mu_t, \chi_t) & \text{Cov} (\mu_t, \xi_t) & \text{Var} (\mu_t)
\end{bmatrix}
\]

5.2 Proof of the proposition 2

Remind the three-factor model under the EMM on equation (8). Following the same steps as before, we have the solutions of these stochastic differential equations
\[
\chi_t = -\frac{\lambda_{\chi}}{\kappa} + \left( \chi_0 + \frac{\lambda_{\chi}}{\kappa} \right) e^{-\kappa t} + \sigma_{\chi} \int_0^t e^{\kappa(u-t)} dB_{\chi}.
\] (19a)
\[
\xi_t = \xi_0 - \lambda_{\chi} t + \hat{\mu} t + (\mu_0 - \hat{\mu}) \frac{1 - e^{-\eta t}}{\eta} + \sigma_{\xi} \tilde{B}_{\xi} + \sigma_{\mu} \int_0^t \left( \int_0^t e^{\eta(u-t)} dB_{\mu} \right) du
\] (19b)
\[
\mu_t = \hat{\mu} + (\mu_0 - \hat{\mu}) e^{-\eta t} + \sigma_{\mu} \int_0^t e^{\eta(u-t)} dB_{\mu}.
\] (19c)

Now we compute the expected values and get
\[
E^Q (\chi_t) = \chi_0 e^{-\kappa t} - \frac{\lambda_{\chi}}{\kappa} (1 - e^{-\kappa t})
\] (20a)
\[
E^Q (\xi_t) = \xi_0 + (\hat{\mu} - \lambda_{\chi}) t + (\mu_0 - \hat{\mu}) \frac{1 - e^{-\eta t}}{\eta}
\] (20b)
\[
E^Q (\mu_t) = \hat{\mu} + (\mu_0 - \hat{\mu}) e^{-\eta t}.
\] (20c)
From equation (7a) we can write for the $j$-th contract that
\[
\ln(S_{T_j}) = g(T_j) + \chi_{T_j} + \xi_{T_j}.
\]  

(21)

Next we compute the conditional expected values of $\ln(S_T)$ and $S_T$:
\[
E^Q \left[ \ln(S_{T_j}) \mid \mathcal{F}_t \right] = g(T_j) + E^Q (\chi_{T_j} \mid \mathcal{F}_t) + E^Q (\xi_{T_j} \mid \mathcal{F}_t)
\]
(22a)

\[
E^Q (S_{T_j} \mid \mathcal{F}_t) = E^Q \left[ \exp \left( \ln(S_{T_j}) \right) \mid \mathcal{F}_t \right]
= \exp \left[ E^Q \left( \ln(S_{T_j}) \mid \mathcal{F}_t \right) + \frac{1}{2} \text{Var}^Q \left( \ln(S_{T_j}) \mid \mathcal{F}_t \right) \right]
\]

(22b)

\[
\ln \left[ E^Q \left( S_{T_j} \mid \mathcal{F}_t \right) \right] = \left[ E^Q \left( \ln(S_{T_j}) \mid \mathcal{F}_t \right) + \frac{1}{2} \text{Var}^Q \left( \ln(S_{T_j}) \mid \mathcal{F}_t \right) \right].
\]

We have previously computed the unconditional expected values and variances. This is equivalent to compute them conditional on $\mathcal{F}_0$. Therefore we have similar results for expected values and variances of the terminal value conditional on $\mathcal{F}_t$.

Introducing equations (20a) and (20b) into equation (22a), we get
\[
E^Q \left[ \ln(S_{T_j}) \mid \mathcal{F}_t \right] = g(T_j) + \chi_t e^{-\kappa(T_j - t)} - \frac{\lambda}{\kappa} (1 - e^{-\kappa(T_j - t)})
+ \xi_t + (\hat{\mu} - \lambda T_j) (T_j - t) + (\mu_t - \hat{\mu}) \left( \frac{1 - e^{-\kappa(T_j - t)}}{\eta} \right).
\]

(23)

Now we need the $\text{Var}^Q \left[ \ln(S_{T_j}) \mid \mathcal{F}_t \right]$ to use into equation (22b). Since we have $\Sigma^p_t = \Sigma^Q_t$ we can use the results from previous section. Working with equation (21) we write
\[
\text{Var}^Q \left[ \ln(S_{T_j}) \mid \mathcal{F}_t \right] = \text{Var}^Q (\chi_{T_j} \mid \mathcal{F}_t) + \text{Var}^Q (\xi_{T_j} \mid \mathcal{F}_t) + 2 \text{Cov}^Q (\chi_{T_j}, \xi_{T_j} \mid \mathcal{F}_t).
\]

(24)

Using the results on equations (14b), (15b) and (17) into equation (24) we have
\[
\text{Var}^Q \left[ \ln(S_{T_j}) \right] = \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa(T_j - t)}) + \sigma^2 T_j - t
+ \frac{\sigma^2}{\eta^2} \left[ (T_j - t) + \frac{1}{2\eta} (1 - e^{-2\eta(T_j - t)}) - \frac{2}{\eta} (1 - e^{-\eta(T_j - t)}) \right]
+ \frac{2\rho_{\xi \mu} \sigma_{\xi} \sigma_{\mu}}{\eta} \left[ (T_j - t) - \frac{1}{\eta} (1 - e^{-\eta(T_j - t)}) \right] + \frac{2\rho_{\xi \mu} \sigma_{\xi} \sigma_{\chi}}{\kappa} (1 - e^{-\kappa(T_j - t)})
+ \frac{2\rho_{\chi \mu} \sigma_{\chi} \sigma_{\mu}}{\eta} \left[ \frac{1}{\kappa} (1 - e^{-\kappa(T_j - t)}) - \frac{1}{\kappa} - \frac{1}{\eta} (e^{-\eta(T_j - t)} - e^{-\kappa(T_j - t)}) \right].
\]

(25)
Using equations (25) and (23) into equation (22b) we get

\[
\ln \left[ E^Q (S_{T_j} | \mathcal{F}_t) \right] = g(T_j) + \chi_t e^{-\kappa(T_j-t)} + \xi_t + (\mu_t - \hat{\mu}) \left( \frac{1 - e^{-\kappa(T_j-t)}}{\eta} \right) + \left( \hat{\mu} - \lambda \xi \right) (T_j - t) \\
- \frac{\lambda \kappa}{\kappa} (1 - e^{-\kappa(T_j-t)}) + \frac{\sigma^2}{4\kappa} (1 - e^{-2\kappa(T_j-t)}) + \frac{1}{2} \sigma^2 (T_j - t) \\
+ \frac{\sigma^2}{2\eta^2} \left[ (T_j - t) + \frac{1}{2\eta} (1 - e^{-2\eta(T_j-t)}) - \frac{2}{\eta} (1 - e^{-\eta(T_j-t)}) \right] \\
+ \frac{\rho \xi \sigma \xi \sigma}{\eta} \left[ (T_j - t) - \frac{1}{\eta} (1 - e^{-\eta(T_j-t)}) \right] + \frac{\rho \xi \sigma \xi \chi}{\kappa} (1 - e^{-\kappa(T_j-t)}) \\
+ \frac{\rho \xi \sigma \rho \sigma}{\eta} \left[ \frac{1}{\kappa} (1 - e^{-\kappa(T_j-t)}) - \frac{1}{\kappa - \eta} (e^{-\eta(T_j-t)} - e^{-\kappa(T_j-t)}) \right].
\]

(26)

We know that the future prices is given by \( F_{t,T_j} = E^Q (S_{T_j} | \mathcal{F}_t) \) and therefore \( \ln \left( F_{t,T_j} \right) = \ln \left( E^Q (S_{T_j} | \mathcal{F}_t) \right) \). Hence we prove the first item of the proposition. And the term structure of future prices is written as

\[
\ln \left( F_{t,T_j} \right) = g(T_j) + \chi_t e^{-\kappa(T_j-t)} + \xi_t + (\mu_t - \hat{\mu}) \left( \frac{1 - e^{-\kappa(T_j-t)}}{\eta} \right) + C (T_j - t),
\]

(27)

where

\[
C (T_j - t) = (\hat{\mu} - \lambda \xi) (T_j - t) - \frac{\lambda \kappa}{\kappa} (1 - e^{-\kappa(T_j-t)}) + \frac{\sigma^2}{4\kappa} (1 - e^{-2\kappa(T_j-t)}) + \frac{1}{2} \sigma^2 (T_j - t) \\
+ \frac{\sigma^2}{2\eta^2} \left[ (T_j - t) + \frac{1}{2\eta} (1 - e^{-2\eta(T_j-t)}) - \frac{2}{\eta} (1 - e^{-\eta(T_j-t)}) \right] \\
+ \frac{\rho \xi \sigma \xi \sigma}{\eta} \left[ (T_j - t) - \frac{1}{\eta} (1 - e^{-\eta(T_j-t)}) \right] + \frac{\rho \xi \sigma \xi \chi}{\kappa} (1 - e^{-\kappa(T_j-t)}) \\
+ \frac{\rho \xi \sigma \rho \sigma}{\eta} \left[ \frac{1}{\kappa} (1 - e^{-\kappa(T_j-t)}) - \frac{1}{\kappa - \eta} (e^{-\eta(T_j-t)} - e^{-\kappa(T_j-t)}) \right].
\]

(28)

The second item is a straightforward application of Ito’s lemma. From equation (27) we can write that

\[
\ln \left( F_{t,T_j} \right) = h (\chi_t, \xi_t, \mu_t, T - t).
\]

Ito’s lemma gives

\[
d \left[ \ln \left( F_{t,T_j} \right) \right] = -h_t dt + h_\chi d\chi + h_\xi d\xi + h_\mu d\mu + \frac{1}{2} h_{\chi \chi} (d\chi)^2 \\
+ \frac{1}{2} h_{\xi \xi} (d\xi)^2 + \frac{1}{2} h_{\mu \mu} (d\mu)^2 + h_\chi d\chi d\xi + h_\mu d\chi d\mu + h_\xi d\mu d\xi,
\]

where the underlying indicate the partial derivatives. We can compute these derivatives as

\[
h_t = \kappa e^{-\kappa(T_j-t)} \chi_t - e^{\eta(T_j-t)} (\mu_t - \hat{\mu}) - B' (T_j - t),
\]

\[
h_\chi = e^{-\kappa(T_j-t)}, \quad h_\xi = 1, \quad \text{and} \quad h_\mu = \frac{1 - e^{-\eta(T_j-t)}}{\eta}.
\]

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All the second and cross derivatives are zero. Inserting all these results into equation (29) and collecting the similar terms we can write

$$\frac{dF_{t,T_j}}{F_t,T_j} = \ldots$$

The variance $\sigma^2_{F_t,T_j} = \text{Var}^Q \left( \frac{dF_{t,T_j}}{F_t,T_j} \right)$ is given by equation (10) and this proves the second item.

The last item results from the computation of the risk premium $RP_{t,F_{T_j}} = F_{t,T_j} - E^P \left( S_{T_j} | F_t \right)$. Since $F_{t,T_j} = E^P \left( S_{T_j} | F_t \right)$, we can write

$$RP_{t,T_j} = \exp \left[ g (T_j) + \chi_t e^{-\kappa(T_j-t)} + \xi_t + (\mu_t - \tilde{\mu}) \frac{1 - e^{-\eta(T_j-t)}}{\eta} + C (T_j - t) \right] - \exp \left[ g (T_j) + \chi_t e^{-\kappa(T_j-t)} + \xi_t + (\mu_t - \tilde{\mu}) \frac{1 - e^{-\eta(T_j-t)}}{\eta} + D (T_j - t) \right],$$

where $C (T_j - t)$ was given in equation (28), $D (T_j - t)$ is a similar expression considering $\lambda_\chi = \lambda_\xi = \lambda_\mu = 0$ and hence $\tilde{\mu} = \bar{\mu}$.

Rearranging the last equation we finally get the term structure of risk premium

$$RP_{t,T_j} = \exp \left[ g (T_j) + e^{-\kappa(T_j-t)} \chi_t + \xi_t \right] \times \{ \exp [ (\mu_t - \tilde{\mu}) \beta + C (T_j - t) ] - \exp [ (\mu_t - \tilde{\mu}) \beta + D (T_j - t) ] \} \quad j = 1, \ldots, m,$$

where $\beta = \frac{1 - e^{-\eta(T_j-t)}}{\eta}$. 

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References


