On the Consistency of Arbitrary Money–Demand Functions with the Sidrauski and the Shopping–Time Models

Rubens Penha Cysne, David Turchick

Janeiro de 2008
Os artigos publicados são de inteira responsabilidade de seus autores. As opiniões neles emitidas não exprimem, necessariamente, o ponto de vista da Fundação Getulio Vargas.
On the Consistency of Arbitrary Money-Demand Functions with the Sidrauski and the Shopping-Time Models\textsuperscript{1}

Rubens Penha Cysne and David Turchick\textsuperscript{2}

EPGE - Fundação Getulio Vargas

December 31, 2007

\textsuperscript{1}Key Words: Money Demand, Rationalization, Integrability, Sidrauski, Shopping-time, Interest-Bearing Assets. JEL: C0, E40, D50, D60.

\textsuperscript{2}Rubens Penha Cysne is a Professor at the Graduate School of Economics of the Getulio Vargas Foundation (EPGE/FGV). David Turchick is a researcher at EPGE/FGV. E-mails: rubens@fgv.br; davidturchick@fgvmail.br
Abstract

This paper investigates which properties money-demand functions have to satisfy to be consistent with multidimensional extensions of Lucas’ (2000) versions of the Sidrauski (1967) and the shopping-time models. We also investigate how such classes of models relate to each other regarding the rationalization of money demands. We conclude that money demand functions rationalizable by the shopping-time model are always rationalizable by the Sidrauski model, but that the converse is not true. The log-log money demand with an interest-rate elasticity greater than or equal to one and the semi-log money demand are counterexamples.
1 Introduction


Investigating if a certain money-demand specification can be rationalized by such models is a problem in general-equilibrium applied monetary theory which, somehow, parallels the well-known "integrability problem" in standard partial-equilibrium microeconomics.¹

In microeconomics, given a system of demand functions, one investigates the existence of a utility function from which such functions can be derived. Here, our objective is to investigate the existence of a utility function and of a transacting technology from which some usual money-demand specifications can be obtained. This problem can be called a rationalization problem or, more specifically, an integrability problem.

The investigation of integrability, in some cases, arises as a theoretical complement to applied studies. For example, Lucas (2000) derives expressions for the welfare costs of inflation using unidimensional versions of each one of the models we study here (the dimension refers to the number of monetary assets in the economy). Next, assuming money demands to be given, respectively, by a semi-log and a log-log specification he provides estimates of such welfare costs. However, except for the case of the log-log specification with an interest-rate elasticity equal to −0.5, Lucas does not investigate if such money-demand specifications are indeed

¹See, e.g., Varian (1992, section 8.5).
consistent with the models at hand. This is one of the tasks we take to ourselves here.

Other papers in the literature using versions of the Sidrauski and of the shopping-time models which fall into the category we deal with here and which could, therefore, benefit from our results, are Fischer (1979), Siegel (1983), Asako (1983), Weil (1991) and Goodfriend (1997).

Our analysis starts with the unidimensional case, where only one type of money is available in the economy. In a second step, we extend our results to the multidimensional case, where the existence of several assets performing monetary functions is taken into consideration [as in Cysne (2003) and Cysne and Turchick (2007)]. Whatever the dimension, we shall show that when interest rates are allowed to vary in the \((0, +\infty)\) interval, both the semi-logarithmic money demand and the log-log money demand with (absolute-value) elasticity greater than or equal to 1 cannot be rationalized by the shopping-time model.

A second contribution of this paper is that of investigating how the Sidrauski model relates to the shopping-time model, as far as the rationalization problem is concerned. This turns out to be a trivial task once we have characterized the money demands which can be obtained from both models.

Regarding this second point, our result adds to Feenstra’s (1986). This author has demonstrated a functional equivalence between both models. Such an equivalence, though, does not apply to the specific versions of the Sidrauski and the shopping-time models we deal with here because they do not comply with Feenstra’s assumptions. We shall show that every money demand rationalizable by the shopping-time model is also rationalizable by the Sidrauski model, but that the converse statement is not true. Two counterexamples are provided.
As before, all results concerning the relationship between these two general-equilibrium monetary models are first presented in a 1-dimensional context and posteriorly generalized to an $n$-dimensional context.

The remainder of the paper is structured as follows. In section 2 we use the unidimensional case worked out by Lucas and investigate the two main points mentioned above. Examples provided are the log-log and the semi-log money-demand functions. Section 3 extends the results of section 2 to a multidimensional setting, including the two examples. Section 4 summarizes and concludes the work.

2 The case of only one type of money

In this section we analyze the case of an economy with only one type of money (currency).

2.1 The Sidrauski model

We shall assume, as in Lucas (2000, sec. 3), a forever-living, perfectly-foresighted, representative agent maximizing a time-separable constant-relative-risk-aversion utility function, the arguments of which are the flows of real consumption of a single nonstorable good and of holdings of real cash balances. For every $t \in [0, +\infty)$, let $B_t \in \mathbb{R}_+$, $M_t \in \mathbb{R}_+$, $H_t \in \mathbb{R}$, $Y_t \in \mathbb{R}_+$ and $C_t \in \mathbb{R}_+$ represent the nominal values of, respectively, holdings of government bonds, cash, a lump-sum tax (if negative, a transfer from the government to the individual), the output of the economy and consumption, all measured at instant $t$ of time.
The dynamic budget constraint faced by our representative agent reads:

\[
\dot{B}_t + \dot{M}_t = Y_t - C_t - H_t + r_t B_t,
\]

where the dots mean time-derivatives and \( r_t \in \mathbb{R}_{++} \) stands for the nominal interest rate that bonds yield at time \( t \) (cash is, by definition, a monetary asset always yielding a nominal interest rate of 0).

Let \( P_t \in \mathbb{R}_{++} \) be the (both expected and realized) price level, \( y_t = \frac{Y_t}{P_t} \) and \( \pi_t := \frac{P_t}{P_{t-1}} \) the inflation rate at time \( t \). We assume real output grows at constant rate \( \gamma \) \( (y_t = y_0 e^{\gamma t}) \) and normalize \( y_0 \) to 1. Lowercase variables stand for the nominal variables as a share of nominal GDP (that is, \( b_t := \frac{B_t}{Y_t}, m_t = \frac{M_t}{Y_t} \) etc.).

The utility function (see below) is assumed to be homogeneous of degree \( 1 - \sigma \), in which case we can formally state our agent’s problem \( (P_S) \) as:

\[
\max_{c_t, m_t \geq 0} \int_0^{+\infty} e^{(-\rho + (1-\sigma)\gamma)t} U(c_t, m_t) dt \quad (P_S)
\]

subject to

\[
\dot{b}_t + \dot{m}_t = y_t - h_t - c_t + (r_t - \pi_t - \gamma)b_t - (\pi_t + \gamma)m_t, \forall t \in (0, +\infty),
\]

\( b_0 > 0 \) and \( m_0 > 0 \) given.

As in Lucas (2000) and in Cysne and Turchick (2007), we make use of a homothetic utility function \( U : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R} \) given by:

\[
U(c, m) = \frac{1}{1 - \sigma} \left( c \varphi \left( \frac{m}{c} \right) \right)^{1-\sigma}, \quad (1)
\]
where \( \sigma > 0 \) and \( \sigma \neq 1 \), extended by continuity to the ray \( \{0\} \times \mathbb{R}_+ \), and \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a differentiable function such that

**Assumption** \( \varphi \). There exists an \( \overline{m} \in (0, +\infty] \) such that \( \varphi \mid_{(0, \overline{m})} \) is strictly increasing, \( \varphi \mid_{[\overline{m}, +\infty)} \) is constant, \( \varphi'' \mid_{(0, \overline{m})} < 0 \) and \( \varphi'(\overline{m}^-) = 0 \).

Note that \( \overline{m} \) can be equal to infinity as well.

To simplify matters, we denote henceforth simply by "Sidrauski’s model" the present version of the Sidrauski model with a utility function given by (1), where \( \varphi \) satisfies Assumption \( \varphi \).

Note that such a class of utility function, besides the works of Lucas (2000), Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick, includes the one used by Fischer (1979), Siegel (1983), Asako (1983) and Weil (1991). Indeed, such authors use a Cobb-Douglas utility function given by \( U(c, m) = \frac{(e^\alpha m^\beta)^{1-R}}{1-R} \), where \( R \geq 0, R \neq 1, \alpha, \beta > 0 \) and \( \alpha + \beta \leq 1 \). This is just a particular case of (1) for which

\[
\sigma = 1 - (\alpha + \beta) (1 - R)
\]

and

\[
\varphi(m) = (\alpha + \beta)^{\frac{1}{1+\sigma}} m^{\frac{\beta}{\alpha+\beta}}.
\]

We shall see in example 1 that this \( \varphi \) generates only and exactly the unitary-elasticity log-log money-demand function \( m = \frac{2}{\alpha} r^{-1} \).

**Remark 1** The expression \( \varphi(m) - m\varphi'(m) \) will appear many times throughout this work, so it’s in place to note right from the start that, for positive \( m \), this expression is positive. Indeed, for \( m < \overline{m} \), the strict concavity of \( \varphi \) gives \( \varphi(0) - \varphi(m) < \)
\( \varphi'(m)(0 - m) \), and since \( \varphi(0) \geq 0 \), we have \( \varphi(m) - m\varphi'(m) > 0 \). For \( m \geq \overline{m} \), \( \varphi(m) - m\varphi'(m) = \varphi(m) > 0 \).

From this remark, it can be seen that \( U \) is strictly increasing in each of its variables and also strictly concave.\(^2\) Therefore if \((P_S)\) has a solution (which we assume to be true), it will be unique. In equilibrium, since \( c \) is taken as a fraction of output, \( c = 1 \). The usual Euler equations give

\[
r = \frac{U_m}{U_c},
\]

or

\[
r = \frac{\varphi'(m)}{\varphi(m) - m\varphi'(m)},
\]

which corresponds to equation 3.7 in Lucas (2000).

Equation (3) gives us \( r \) as a non-negative differentiable function of \( m \), for which we shall write \( r = \psi(m), \) where \( \psi : (0, \overline{m}) \to (0, +\infty) \).

Since

\[
\psi'(m) = \frac{\varphi(m)\varphi''(m)}{(\varphi(m) - m\varphi'(m))^2} < 0,
\]

\( \psi \) is strictly decreasing, therefore one-to-one. From its continuity (since \( \varphi \) is twice-differentiable over \((0, \overline{m})\), \( \varphi' \) is continuous on this interval), its image is also connected, that is, an interval \((\psi(\overline{m}) - , \psi(0+))\). From Assumption \( \varphi \) and (3), we

\(^2\)All we need to check is that, for \((c, z) \in \mathbb{R}^2_+\),

\[
U_c(c, z) = (c\varphi(\frac{z}{k}))^{-\alpha} (\varphi(\frac{z}{k}) - \frac{1}{k}\varphi'(\frac{z}{k})) > 0,
\]

\[
U_z(c, z) = (c\varphi(\frac{z}{k}))^{-\alpha} \varphi'(\frac{z}{k}) > 0,
\]

\[
U_{cc}(c, z) = (c\varphi(\frac{z}{k}))^{-\alpha-1} \left( -\sigma \left( \varphi(\frac{z}{k}) - \frac{1}{k}\varphi'(\frac{z}{k}) \right)^2 + \frac{1}{k^2} \varphi(\frac{z}{k}) \varphi''(\frac{z}{k}) \right) < 0,
\]

\[
U_{zz}(c, z) = (c\varphi(\frac{z}{k}))^{-\alpha-1} \left( -\sigma \varphi'(\frac{z}{k})^2 + \varphi(\frac{z}{k}) \varphi''(\frac{z}{k}) \right) < 0,
\]

\[
U_{cz}(c, z) = (c\varphi(\frac{z}{k}))^{-\alpha-1} \left( \frac{1}{k^2} \varphi'(\frac{z}{k})^2 - \varphi(\frac{z}{k}) \left( \sigma \varphi'(\frac{z}{k}) + \frac{1}{k}\varphi''(\frac{z}{k}) \right) \right), \text{ and}
\]

\[
U_{cc}(c, z)U_{zz}(c, z) - U_{cz}(c, z)^2 = \frac{1}{k^2} \left( c\varphi(\frac{z}{k}) \right)^{1-2\sigma} \varphi''(\frac{z}{k}) > 0.
\]
get $\psi(\overline{m}) = 0$. Let $R := \psi(0+)$. We shall call its inverse function $m : (0, R) \rightarrow (0, \overline{m})$ a "money-demand function".

**Property $S_1 \ \nabla$.** $m' < 0$. In fact, $m'(r) = \frac{1}{\psi'(m(r))} < 0$.

This function is differentiable, strictly decreasing and surjective by construction. This means that $\overline{m}$ ends up being the maximum value that the money-demand specification attains. As a practical matter, since the economist does not know $\varphi$, he ends up using a money-demand function estimated by the econometrician, leading us to our first rationalization problem in the unidimensional setting.

**Proposition 1** Given an onto money-demand function $m : (0, R) \rightarrow (0, \overline{m})$, it is rationalizable by Sidrauski’s model iff $m' < 0$.

**Proof.** Necessity has already been proved. To see that this condition is also sufficient, all we need to do is exhibit a $\varphi$ consistent with such an $m$ that satisfies Assumption $\varphi$. Let $\psi$ denote $m$’s inverse function, which is also differentiable (Inverse Function Theorem on $\mathbb{R}$). Note that equation (3) may be rewritten

$$\varphi'(m) = \frac{\psi(m)}{1 + m\psi(m)} \varphi(m), \quad (5)$$

which is separable, and readily yields the general solution

$$\varphi(m) = Ce^{\int_{s}^{m} \frac{\psi(u)}{1 + m\psi(u)} \, du}, \quad (6)$$
for some constant $C > 0$. Bearing this in mind, take $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$
\varphi(m) = \begin{cases} 
\int_{1}^{m} \frac{\psi(\eta)}{1+m\psi(\eta)} d\eta, & \text{if } m \leq \overline{m} \\
\int_{1}^{\overline{m}} \frac{\psi(\eta)}{1+m\psi(\eta)} d\eta, & \text{if } m > \overline{m}.
\end{cases}
$$

(7)

So $\varphi \geq 0$ and, for $m < \overline{m}$, $\varphi'(m) = \frac{\psi(m)}{1+m\psi(m)} \varphi(m) > 0$ and $\varphi''(m) = \frac{\psi'(m)}{(1+m\psi(m))^2} \varphi(m) < 0$. In case $\overline{m}$ is finite, we still have to check $\varphi$’s differentiability at $\overline{m}$. Since $\psi(\overline{m}^-) = 0$ ($m$’s surjectiveness and decreasingness), we have $\varphi'(\overline{m}^-) = 0$. Since $\varphi'(\overline{m}^+) = 0$, we have $\varphi$ not only continuous, but also differentiable at $\overline{m}$ (where $\varphi'(\overline{m}) = 0$). Finally, since $\frac{\varphi'(m)}{\varphi(m) - m\varphi'(m)} = \frac{\psi(m)}{1+m\psi(m)} = \psi(m)$, we’re done. ■

In appendix A, we shall make a few departures from the Sidrauski model just presented, in order to assess the robustness of this result.

### 2.2 The shopping-time model

Let’s start by briefly introducing the unidimensional version of the McCallum and Goodfriend’s shopping-time model, in an almost identical form to that of Lucas (2000). Any variable sharing its symbol with a variable in the previous subsection also shares with it its meaning. Our representative agent’s problem is

$$
\max_{c_t \geq 0} \int_{0}^{+\infty} e^{(-\rho+(1-\sigma)\gamma)t} U(c_t) dt \quad (P_{MG})
$$

subject to

$$
\dot{b}_t + \dot{m}_t = 1 - h_t - c_t - s_t + (r_t - \pi_t - \gamma)b_t - (\pi_t + \gamma)m_t, \forall t \in (0, +\infty),
$$

$$
b_0 > 0 \text{ and } m_0 > 0 \text{ given,}
$$

$$
c_t = m_t \phi(s_t), \forall t \in (0, +\infty),
$$
where $U$ is concave, $s \in [0, 1]$ is a choice variable such that $s + y = 1$ (it represents the portion of time the individual dedicates to transacting instead of producing), $\phi : [0, 1] \to \mathbb{R}_+$ is a twice-differentiable function such that $\phi(0) = 0$, $\phi' > 0$ and:

**Assumption $\phi$.** $\forall s \in (0, 1), \phi(s) \phi'(s) + (1 - s)(2\phi'(s)^2 - \phi(s) \phi''(s)) > 0$.

We shall see ahead, in Property MG1, this to be the weakest possible condition in order for the money-demand function arising from this model to be strictly decreasing. In McCallum and Goodfriend (1987), only $\phi' > 0$ is assumed (see eq. 2 on p. 776 in that paper), and $\phi(0) = 0$ is a tacit assumption giving money a role in this model. In Simonsen and Cysne (2001), a stronger assumption than ours, $\phi'' \leq 0$, is made. We shall see in section 2.3 that this would make even the standard log-log money-demand specification, irrespective of its elasticity, irrationalizable by this model. Nevertheless, the rationalization of this specification with absolute-value elasticity lower than 1 would be guaranteed if $\phi$ satisfied the following assumption, stronger than Assumption $\phi$, but weaker than $\phi'' \leq 0$:

**Assumption $\phi2$.** The function $\frac{\phi}{\phi'}$ is increasing, that is, $\phi'' - \phi' \phi'' \geq 0$.

Unless explicitly said, $\phi$ will only be required to satisfy Assumption $\phi$.

We shall refer to the present version of the shopping-time model using the transacting technology $c_t = m_t \phi(s_t)$ and Assumption $\phi$ simply as "shopping-time model". Note that this separable version of the model is important, among other reasons, because it allows for straightforward derivations of measures of the welfare costs of inflation [Lucas (2000)]. Another reference in the literature using such a weakly-separable version of the shopping-time model is Goodfriend (1997).
Now let us go back to the solution of \((P_{MG})\). Considering interior solutions only, we get the first-order conditions [as in Simonsen and Cysne (2001)]:

\[
\begin{align*}
rm &= \left(\frac{\phi}{s}\right)(s) \\
m\phi(s) &= 1 - s.
\end{align*}
\]  \hspace{1cm} (8)

This last equation, \(m\phi(s) = 1 - s\), shows that, in equilibrium, we necessarily have \(s > 0\). More than this, it can be seen to give \(s\) as a function of \(m\) alone. For this purpose, we only need to note that the function \(H : (0, 1] \rightarrow \mathbb{R}_+\) defined by \(H(s) = \frac{1 - s}{\phi(s)}\) is invertible. Since \(H'(s) = -\frac{s + (1 - s)\phi'(s)}{\phi(s)^2} < 0\), \(H(1) = 0\) and \(\lim_{s \to 0^+} H(s) = +\infty\), we are finished. Letting \(\tau\) be its inverse function (therefore also strictly decreasing, with \(\tau'(m) = \frac{1}{H'(\tau(m))} = -\frac{\phi(\tau(m))^2}{\phi(\tau(m)) + (1 - \tau(m))\phi'(\tau(m))} < 0\)), our demand equation may be rewritten \(r = \psi(m)\), where

\[
r = \frac{1}{m} \left(\frac{\phi}{\phi'}\right)(\tau(m)).
\] \hspace{1cm} (9)

As in the last subsection, we shall call \(\psi\)'s inverse function a money-demand function and denote it by \(m\). Keeping the notation \(R\) from the previous subsection, we have \(R = \psi(0+) = +\infty\).

**Property MG1 \(\backslash\).** \(m' < 0\). In fact, for \(m > 0\),

\[
\psi'(m) = \frac{\phi'(\tau(m))^2 \tau'(m) m - \phi(\tau(m)) [\phi'(\tau(m)) + m\phi''(\tau(m)) \tau'(m)]}{m^2\phi'(\tau(m))^2}
\]

\[
= \frac{-\phi(s) \phi'(s) - \frac{1 - s}{\phi(s)} \phi(s)^2}{m^2\phi'(s)^2} \left(\phi'(s)^2 - \phi(s)\phi''(s)\right),
\]

where we’ve used the second equation in (8) and the expression for \(\tau'\). There-
fore, $\psi'(m) < 0$ iff

$$
\phi'(s) (\phi(s) + (1-s)\phi'(s)) + (1-s) (\phi'(s)^2 - \phi(s) \phi''(s)) > 0,
$$

which is the same as the expression in Assumption $\phi$.

**Remark 2** Note that, from (9), Assumption $\phi_2$ is the necessary and sufficient condition to guarantee that not only $\psi$, but also $m\psi$, is a decreasing function of $m$. This implies a positive correlation between $rm$ and $r$ or, in other words, an inelastic money-demand function $m$.

Here we have a similar result to Proposition 1, although it does not give a test as practical as the one presented there.

**Proposition 2** Given an onto money-demand function $m : (0, +\infty) \to (0, +\infty)$, it is rationalizable by the shopping-time model iff $m' < 0$ and the system of equations

$$
\begin{align*}
\phi'(s) &= \frac{\phi(s)^2}{(1-s)\psi(\frac{1-r}{s})}, \\
\phi(0) &= 0
\end{align*}
$$

is solvable.

**Proof.** The necessity of the decreasingness property has been shown in the previous remark, while the differential equation comes directly from (8). For the sufficiency, first note that the differential equation would imply $\phi' > 0$ which, together with the initial value for $\phi$, would give $\phi(s) > 0$ for $s > 0$. Moreover, $\phi$

---

3One can see that the interest-rate-elasticity of the money-demand function $m$ (in absolute value) is smaller, equal or bigger than 1 if $rm$ increases, stays still or decreases with $r$, respectively.
could be seen to satisfy Assumption $\phi$, by simply rewriting the given differential equation for $\phi$ as $\phi(s)^2 = (1-s) \psi \left( \frac{1-s}{\phi(s)} \right) \phi'(s)$ and differentiating both sides:

$$2\phi(s) \phi'(s) = -\psi \left( \frac{1-s}{\phi(s)} \right) \phi'(s) - (1-s) \psi' \left( \frac{1-s}{\phi(s)} \right) \phi'(s) \frac{\phi(s) + (1-s)\phi'(s)}{\phi(s)^2}$$

$$+ (1-s) \psi \left( \frac{1-s}{\phi(s)} \right) \phi''(s)$$

$$> -\psi \left( \frac{1-s}{\phi(s)} \right) \phi'(s) + (1-s) \psi \left( \frac{1-s}{\phi(s)} \right) \phi''(s)$$

$$= -\frac{\phi(s)^2}{(1-s) \phi'(s)} (\phi'(s) - (1-s) \phi''(s)),$$

where in the "\(>\)" sign we used the facts that $\psi' < 0$ (Inverse Function Theorem) and $\phi' > 0$, and in the last equality symbol we used the value of $\psi \left( \frac{1-s}{\phi(s)} \right)$ implied by the given differential equation. This directly implies the validity of Assumption $\phi$, as wished. ■

The main propositions of this and the last subsections have the following immediate

**Corollary 3** Money-demand specifications which are rationalizable by the shopping-time model are also rationalizable by Sidrauski’s model.
Note that this is not an equivalence result, but an embedding result only. This is the case even if we impose \( \overline{m} = +\infty \), in an attempt of approximating both models. We shall see in the next subsection a few examples of demand functions which are rationalizable by the Sidrauski, but not the shopping-time, model.

### 2.3 Examples

Having the theorems of the last two subsections at our disposal, we can now investigate the rationalization of the semi-log and the log-log money-demand specifications by both of our models.

**Example 1 (log-log money demand).** Consider now the money-demand function given by:

\[
m(r) = K r^{-\alpha},
\]

where \( K > 0 \) and \( \alpha > 0 \). In order for us to be able to apply Proposition 1, we must take \( \overline{m} = K \) and \( R = +\infty \). Since \( m'(r) = -\alpha K r^{-\alpha-1} < 0 \), it follows from that proposition that the log-log money demand is also rationalizable by our version of the Sidrauski model.\(^5\)

\(^4\)The reader may wonder why Feenstra’s (1986, prop. 1) equivalence result is not applicable to our models. One of the reasons is that our transacting function, separable in \( m \) and \( s \), is somewhat restrictive. Moreover, some assumptions differ. Observe, for instance, that the first assumption 2(c) on page 281 of Feenstra’s paper (\( W_{xx} < 0 \), where \( W \) is defined in his equation 16’) would imply the strict concavity of the function \( \phi \) we use in the shopping-time model, an assumption which we do not make here (it would rule out the log-log money-demand specification). To see that, note that the application of Feenstra’s strategy to our version of the McCallum-Goodfriend model would imply, using our notation, \( 0 > W_{xx}(x, m) = \frac{m \phi''(s)}{(1+m \phi'(s))^2} \), where \( s \) would be such that \( m \phi'(s) + s = x \).

\(^5\)We shall see still in this example that the log-log money demand is also rationalizable by the shopping-time model if and only if the elasticity with respect to the interest rate is below unity. It follows from Corollary 1, therefore, that the log-log demand is also rationalizable by our Sidrauski model in the case \( \alpha < 1 \). We have just generalized this result for any value of \( \alpha \), a property of the Sidrauski model which is not shared by the shopping-time model.
As in the previous example, let us find the $\phi$ underlying this money-demand specification. If $\alpha = 1$, the integral in equation (6) equals

$$ \int_1^{m} \frac{K}{K + 1} \frac{1}{\mu} d\mu = \frac{K}{K + 1} \log m,$$

and if $\alpha \neq 1$,

$$ \int_1^{m} \frac{(\frac{K}{\mu})^{1/\alpha}}{1 + \mu (\frac{K}{\mu})^{1/\alpha}} d\mu = \int_1^{m} \frac{K^{1/\alpha}}{\mu^{1/\alpha} + K^{1/\alpha}} d\mu = \int_1^{m} \frac{d\mu}{\mu} - \int_1^{m} \frac{\mu^{\frac{1}{\alpha} - 2}}{K^{1/\alpha} + \mu^{\frac{1}{\alpha} - 1}} d\mu = \log m - \frac{\alpha}{1 - \alpha} \int_{K^{1/\alpha} + 1}^{K^{1/\alpha + m^{\frac{1}{\alpha} - 1}}} \frac{du}{u} = \log \left( m \left/ \left( \frac{K^{1/\alpha} + m^{\frac{1}{\alpha} - 1}}{K^{1/\alpha} + 1} \right)^{\frac{\alpha}{1 - \alpha}} \right. \right). $$

Therefore that equation gives us

$$ \varphi (m) = \begin{cases} Cm \frac{K}{K + 1}, & \text{if } \alpha = 1 \\ \frac{Cm}{\left( K^{1/\alpha} + m^{\frac{1}{\alpha} - 1} \right)^{\frac{\alpha}{1 - \alpha}}}, & \text{if } \alpha \neq 1 \end{cases} \quad (10) $$

for a positive constant $C$. (10) generalizes the integrability investigation carried out by Lucas (2000, p. 257) for the particular value $\alpha = 0.5$.

As it concerns the shopping-time model, start by noting that, in case Assumption $\phi 2$ were valid, it would imply $\alpha < 1$. But this wouldn’t be enough to show that any $\alpha < 1$ would work. Let’s apply Proposition 2 to conclude that, even under the broader Assumption $\phi$, the necessary and sufficient condition so that the log-log money-demand specification can be rationalized by the shopping-time model is that
Here, the relevant initial value problem to be solved is

\[ \phi'(s) = \frac{(1-s)^{\frac{1}{\alpha}-1}}{K^{1/\alpha}} \phi(s)^{2-\frac{1}{\alpha}}, \quad \phi(0) = 0. \]  

(11)

If \( \alpha = 1 \), we get \( \phi'(s) = \frac{\phi(s)}{K} \), which together with the initial condition implies \( \phi = 0 \), a contradiction with \( \phi' > 0 \). So we now attain to the case \( \alpha \neq 1 \). This is a Bernoulli equation, and we have to find out if its solution satisfying \( \phi(0) = 0 \) also satisfies the other properties in our version of the shopping-time model. In order to solve it, write \( y = \phi^{\frac{1}{\alpha}-1} \) (since \( \phi \) must be nonnegative, this is a legitimate move), so that

\[ y'(s) = \left( \frac{1}{\alpha} - 1 \right) \phi(s)^{\frac{1}{\alpha}-2} \phi'(s) = \left( \frac{1}{\alpha} - 1 \right) \frac{(1-s)^{\frac{1}{\alpha}-1}}{K^{1/\alpha}}. \]

Therefore \( y(s) = \frac{\alpha-1}{K^{1/\alpha}} (1 - s)^{1/\alpha} + C \) for some constant \( C \). Since \( y(0) = 0 \), we get \( C = \frac{1-\alpha}{K^{1/\alpha}} \), so that

\[ y(s) = \frac{1-\alpha}{K^{1/\alpha}} \left( 1 - (1 - s)^{1/\alpha} \right). \]

Now, since \( \phi \) has to be nonnegative, \( y \) does too. Therefore, the possibility \( \alpha > 1 \) is dismissed, and we can attain to the case \( \alpha < 1 \). In this case, we find the \( \phi \) related to the log-log money-demand function:

\[ \phi(s) = \left[ \frac{1-\alpha}{K^{1/\alpha}} \left( 1 - (1 - s)^{1/\alpha} \right) \right]^\frac{\alpha}{1-\alpha}. \]

For \( s \in (0, 1) \), (11) gives \( \phi'(s) > 0 \). After some tedious calculations for \( \phi' \) and \( \phi'' \)
(or, with the aid of a mathematics software), we get

\[ \phi''(s) = -\alpha \xi (1-s)^{-\frac{2\xi}{\alpha}} \times \left[ (1-\alpha) \left( 1 - (1-s)^{1/\alpha} \right) \right]^{\frac{3\alpha-2}{\alpha}} \left[ (1-\alpha)^2 - \alpha^2 (1-s)^{1/\alpha} \right]. \]

Although it is impossible to tell the sign of this expression,\(^6\) we have

\[ \phi'(s)^2 - \phi(s) \phi''(s) = \frac{(1-\alpha)K^{-\frac{2\xi}{\alpha}}}{\alpha} (1-s)^{\frac{2\xi}{\alpha}} \times \left[ (1-\alpha) \left( 1 - (1-s)^{1/\alpha} \right) \right]^{\frac{3\alpha-2}{\alpha}} \left[ 1 - \alpha \left( 1 - (1-s)^{1/\alpha} \right) \right] \geq 0. \]

Therefore \( \phi \) satisfies Assumption \( \phi_2 \), and Assumption \( \phi \) as well.

**Example 2** (semi-log money demand). Consider a money-demand function given by \( m(r) = B e^{-\xi r} \), \( B > 0 \), \( \xi > 0 \). Let us first investigate if this demand can be obtained in the Sidrauski framework. Here we must take \( \overline{m} = B \) and \( R = +\infty \).

Since \( m'(r) = -\xi B e^{-\xi r} < 0 \), it follows from Proposition 1 that the semi-log money demand is rationalizable by our version of the Sidrauski model. With respect to the shopping-time model, note that equation (9) implies \( m(0+) = +\infty \). Unlike the money-in-the-utility-function framework, the shopping-time framework imposes \( \overline{m} = +\infty \). Therefore, it automatically excludes from our consideration the semi-log specification.

Just for the sake of comprehensiveness, we seek the \( \varphi \) (related to our version of the Sidrauski model) underlying this money-demand specification. Since \( \psi(m) = \)

\(^6\)For instance, for \( \alpha = 0.6 \) and \( K = 0.1 \), we have \( \phi''(0.4) \approx -8.3 \), but \( \phi''(0.3) \approx 54.2 \).
\( \frac{1}{\xi} \log \left( \frac{B}{m} \right) \), (7) gives us (dividing the result by a constant makes no difference)

\[
\varphi(m) = \begin{cases} 
    e^{\int_0^1 \frac{log x}{x^\xi \cdot log x} \, dx}, & \text{if } m \leq B \\
    1, & \text{if } m > B 
\end{cases}
\]

3 The case of many types of monies

In this section we extend all results obtained in section 2 to a framework in which \( n \) types of monies are available. Let

\[
m = (m_1, \ldots, m_n) \in \mathbb{R}_+^n
\]

represent the vector of real quantities of each type of money being demanded, as a fraction of nominal GDP (where \( m_1 \) is chosen to be \( m \), real currency per output). Each \( m_i \) yields a nominal interest rate of \( r_i \), and

\[
r := (r_1, \ldots, r_n) \in \mathbb{R}_+^n
\]

(with \( r_1 = 0 \), by definition). We shall write

\[
u := (r, r - r_2, \ldots, r - r_n) \in \mathbb{R}_{++}^n
\]

for the vector of opportunity costs of holding money instead of government bonds.
3.1 The extended Sidrauski model

Our representative agent’s instantaneous utility function will now have the form

\[ U(c, m) = \frac{1}{1 - \sigma} \left( c^\varphi \left( \frac{G(m)}{c} \right) \right)^{1-\sigma}, \]

where \( \sigma \) and \( \varphi \) are exactly as in section 2.1, and \( G : \mathbb{R}_+^n \to \mathbb{R}_+ \) is a twice-differentiable 1-homogeneous concave function such that \( G_{x_i} > 0 \) and \( G_{xix} < 0 \) for all \( i \in \{1, \ldots, n\} \). Her maximization problem will be:

\[
\max_{c_t, m_t \geq 0} \int_0^{+\infty} e^{-(\rho + (1-\sigma)\gamma)t} U(c_t, m_t) \, dt \quad (P_{S^\sigma})
\]

subject to

\[
\dot{b}_t + 1 \cdot \dot{m}_t = y_t - h_t - c_t + (r - \pi - \gamma)b_t + (r - (\pi + \gamma)1) \cdot m_t, \forall t \in (0, +\infty),
\]

\[ b_0 > 0 \text{ and } m_0 > 0 \text{ given,} \]

where we write \( 1 \) for the vector \((1, \ldots, 1) \in \mathbb{R}^n \) and \( \cdot \) for the canonical inner product of \( \mathbb{R}^n \), and all the non-bold letters have the same meaning as in the model introduced in section 2. Considering only regular solutions, and by \( U \)'s concavity,\(^8\) we obtain [as in Cysne and Turchick (2007)] the necessary and sufficient

---

\(^7\)If \( n = 1 \), \( G \) would have to be linear, whence \( G'' = 0 \). Therefore, our analysis in this section is restricted to the case \( n > 1 \). Even so, it yields exactly the same results as the \( n = 1 \) framework analyzed in the last section.

\(^8\)Let \( V \) stand for the function \( U \) of subsection 2.1 (of only two variables), so that \( U(c, m) = V(c, G(m)) \). Given \( c, d \in \mathbb{R}_{++} \) and \( x, y \in \mathbb{R}_+^n \), we have

\[
U(d, y) - U(c, x) = V(d, G(y)) - V(c, G(x)) \leq D_c V_{c,G(x)}(d-c) + D_m V_{c,G(x)}(G(y) - G(x)) \leq D_c V_{c,G(x)}(d-c) + D_m V_{c,G(x)}DG_x(y - x) = DU_{c,x} (d-c, y - x), \]

where we’ve used the concavity of \( G \) and the fact that \( D_m V \geq 0 \).
equilibrium relations
\[ r - r_i = \frac{U_i}{U_c}, \forall i \in \{1, \ldots, n\}. \tag{12} \]

In equilibrium, we have \( c = 1 \), so that (12) gives
\[ u_i = \frac{\varphi'(G(m))}{\varphi(G(m)) - G(m) \varphi'(G(m))} G_{x_i}(m), \forall i \in \{1, \ldots, n\}. \tag{13} \]

This equation is analogous to (3), giving us a differentiable function \( \psi : C_{\overline{m}} \to \mathbb{R}^n_+ \) such that \( u = \psi(m) \), just like the function \( \psi \) of section 2, where \( C_{\overline{m}} := \{m \in \mathbb{R}^n_{++} : G(m) < \overline{m}\} \). But, in contrast with what we’ve done in that section, we will not work with its inverse function, because we’re not assured of its existence in the first place. More on this issue will be treated in a companion paper.

As in the unidimensional setting, here we also have a decreasingness property for \( \psi \).

**Property S_n \_\_.** Along rays starting at the origin, each \( \psi_i \) is strictly decreasing.

In fact, for a fixed \( m \in \mathbb{R}^n_{++} \) and \( k > 1 \), we have
\[ \psi_i(km) = \frac{\varphi'(kG(m))}{\varphi(kG(m)) - (kG(m)) \varphi'(kG(m))} G_{x_i}(m), \]
where we used \( G \)’s 1-homogeneity (and subsequent \( G_{x_i} \)’s 0-homogeneity).

From (4), we have
\[ \frac{\varphi'(kG(m))}{\varphi(kG(m)) - (kG(m)) \varphi'(kG(m))} < \frac{\varphi'(G(m))}{\varphi(G(m)) - G(m) \varphi'(G(m))}, \]
so that \( \psi_i(km) < \psi_i(m) \).
Note that if we define \( F : \mathbb{R}_+ \to \mathbb{R}_+ \) by\[ F(G) = \frac{\varphi'(G)}{\varphi(G) - G\varphi'(G)}, \] we can write
\[
\psi(m) = F(G(m)) \nabla G(m). \tag{14}
\]

From this equation, two other interesting facts about the structure of these demands follow:

**Property** \( \frac{\psi_i}{\psi_j} \). For every \( i, j \in \{1, \ldots, n\} \) and \( m \in \mathbb{R}_+^n \),
\[
\frac{\psi_i(m)}{\psi_j(m)} = \frac{G_{x_i}(m)}{G_{x_j}(m)}.
\]

Since both \( G_{x_i} \) and \( G_{x_j} \) are homogeneous functions of the same degree, \( \frac{G_{x_i}}{G_{x_j}} \) is a 0-homogeneous function – that is, a constant along each ray starting at the origin (but excluding this point). So, given \( u_1 \) for instance, \( u_2, \ldots, u_n \) should be completely determined by the values they take on \( S_{++}^{n-1} \), the intersection of the positive orthant of \( \mathbb{R}^n \) with the \((n - 1)\)-dimensional sphere.

**Property** \((\psi_i)_{x_j}\). For every \( i, j \in \{1, \ldots, n\} \) and \( m \in \mathbb{R}_+^n \),
\[
(\psi_i)_{x_j}(m) = (\psi_j)_{x_i}(m).
\]

In fact, \((\psi_i)_{x_j}(m) = F'(G(m)) G_{x_j}(m) G_{x_i}(m) + F(G(m)) G_{x_i x_j}(m)\), evidently symmetric in \((i, j)\).

**Proposition 4** Given a function \( G \) satisfying the properties mentioned in the beginning of this subsection, \( \bar{m} \in (0, +\infty] \) and a money-demand specification \( \psi : C_{\bar{m}} \to \mathbb{R}_+^n \), it is rationalizable by the extended Sidrauski model iff there exists \( F : (0, \bar{m}) \to \mathbb{R}_+ \) such that \( F' < 0 \) and \( \psi(m) = F(G(m)) \nabla G(m) \).

20
Proof. The proof that $F' < 0$ is formally identical to the proof that $\psi$ from section 2 was such that $\psi' < 0$. Necessity of the proposition has already been proved. In order to see that the conditions above are also sufficient, we proceed as in section 2: we exhibit a $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying Assumption $\varphi$ and such that $F(G) = \frac{\varphi'(G)}{\varphi(G) - G\varphi'(G)}$. Take $\varphi$ given by

$$
\varphi(G) = \begin{cases} 
  e^{\int_G^{G_0} \frac{F(\gamma)}{1+GF(\gamma)} \, d\gamma}, & \text{if } G \leq \overline{m} \\
  e^{\int_G^{\infty} \frac{F(\gamma)}{1+GF(\gamma)} \, d\gamma}, & \text{if } G > \overline{m} 
\end{cases}
$$

So $\varphi \geq 0$ and, for $G < \overline{m}$, $\varphi'(G) = \frac{F(G)}{1+GF(G)} \varphi(G) > 0$ and

$$
\varphi''(G) = \frac{F'(G)(1+GF(G)) - F(G)(F(G) + GF'(G))}{(1+GF(G))^2} \varphi(G) + \left( \frac{F(G)}{1+GF(G)} \right)^2 \varphi(G) < 0.
$$

In case $\overline{m}$ is finite, we still have to check $\varphi$’s differentiability at $\overline{m}$. Since $F(\overline{m}^{-}) = 0$ ($G$’s surjectiveness and decreasingness), we have $\varphi'(\overline{m}^{-}) = 0$. Since $\varphi'(\overline{m}^{+}) = 0$, we have $\varphi$ not only continuous, but also differentiable at $\overline{m}$ (where $\varphi'(\overline{m}) = 0$). Moreover,

$$
\frac{\varphi'(G)}{\varphi(G) - G\varphi'(G)} = \frac{\frac{F(G)}{1+GF(G)} \varphi(G)}{\varphi(G) - \frac{GF(G)}{1+GF(G)} \varphi(G)} = F(G),
$$

as we wanted. ■

The reader might find it interesting to learn that, hadn’t the function $G$ been given to us, it could still be recovered, taking into consideration only equation (14). This is the subject of appendix B.
3.2 The extended shopping-time model

Here our representative agent’s problem is

$$\max_{c_t \geq 0} \int_0^{+\infty} e^{-(\rho+(1-\sigma)\gamma)t} U(c_t) dt \quad (P_{MG^n})$$

subject to

$$\dot{b}_t + b_t \cdot \dot{m}_t = 1 - h_t - c_t - s_t + (r_t - \pi_t - \gamma)b_t + ( r - (\pi + \gamma) ) b_t, \forall t \in (0, +\infty),$$

$$b_0 > 0 \text{ and } m_0 > 0,$$

$$c_t = G(m_t) \phi(s_t), \forall t \in (0, +\infty),$$

where $c$ and $s$ stand for the real level of consumption and the portion of time spent transacting rather than producing.\(^9\) The money-aggregator function $G : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a function just like the one in the previous subsection and $\phi : [0, 1] \to \mathbb{R}_+$ is exactly as the $\phi$ in section 2.2, satisfying Assumption $\phi$. This model’s first-order conditions imply the equilibrium relations (Cysne 2003, p. 224):

$$\begin{align*}
\left\{ \begin{array}{l}
u_i = \frac{1}{G(m)} \left( \frac{\phi}{\phi'} \right) (s) G_x_i(m), \forall i \in \{1, \ldots, n\} \\
G(m) \phi(s) = 1 - s
\end{array} \right. \quad (15)
\end{align*}$$

As done in section 2.2, the last of these equilibrium relations enables us to define a function $\tau : \mathbb{R}_+ \to (0, 1]$ such that $G = \frac{1-\tau(G)}{\phi(\tau(G))}$, and we have $\tau' < 0$. Then, the preceding set of equations becomes a money-demand specification of $n$

\(^9\)Cysne (2002) shows that weak separability of the transacting technology is a necessary and sufficient condition so that the welfare costs of inflation are well defined as functions of $m$. 

22
equations and $n$ variables:

$$\psi_i(m) = \frac{\phi(\tau(G(m)))}{G(m)\phi'(\tau(G(m)))} G_{x_i}(m), \forall i \in \{1, \ldots, n\}, \tag{16}$$

where $\psi$ is as in the previous subsection. Note how $\psi$ also fits perfectly into the general form (14), if we take $F: \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $F(G) = \frac{\phi(\tau(G))}{G'}$. As already shown in Property MG$_1$, $F' < 0$. So we get

**Property MG$_n \sqsubset$.** Along rays starting at the origin, each $\psi_i$ is strictly decreasing.

As in our extended money-in-the-utility-function model, (14) implies that the money-demand functions arising from the present model also obey Properties $\psi_i$ and $(\psi_i)_{x_j}$.

We also get an exact analog of Proposition 2.

**Proposition 5** Given a money-demand specification $\psi: \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$, it is rationalizable by the extended shopping-time model if and only if it can be put in the form (14), with $F' < 0$ and the system of equations

$$\begin{cases}
\phi'(s) = \frac{\phi(s)^2}{(1-s)F(\frac{1}{1-s})} \\
\phi(0) = 0
\end{cases}$$

solvable.

**Proof.** Formally equal to the proof of Proposition 2. $\blacksquare$

In light of the two previous propositions, we have the following

**Corollary 6** Money-demand specifications which are rationalizable by the extended shopping-time model are also rationalizable by the extended Sidrauski model.

In particular, this implies properties 1 and 2 are also satisfied for money-demand functions arising from the shopping-time model.
3.3 Examples

In this subsection we extend the examples given in the last section.

Example 3 We now extend the log-log money-demand specification to the multi-dimensional case. It is natural enough to propose an extension of the form

\[
\psi(m) = \left( \frac{K}{G(m)} \right)^{1/\alpha} \nabla G(m),
\]

where \( K > 0 \) and \( \alpha > 0 \). The reader should note that this is the demand that follows from (10) and (13). Taking \( G \) as a simple geometric mean, \( G(m_1, \ldots, m_n) = \prod_{i=1}^{n} m_i^{1/n} \), we obtain the system of equations

\[
\begin{align*}
\psi_1(m) &= \frac{K^{1/\alpha}}{m_1^{1/n}} \prod_{j=1}^{n} m_j^{(1-\frac{1}{n})}\frac{(1-\frac{1}{n})}{n} \\
\vdots \\
\psi_n(m) &= \frac{K^{1/\alpha}}{m_n^{1/n}} \prod_{j=1}^{n} m_j^{(1-\frac{1}{n})}\frac{(1-\frac{1}{n})}{n}
\end{align*}
\]

which, when inverted,\(^{10}\) becomes

\[
\begin{align*}
m_1(u) &= \frac{K}{n^{\alpha}} u_1^{-1+\frac{1-\alpha}{n}} \prod_{j=2}^{n} u_j^{\frac{1-\alpha}{n}} \\
\vdots \\
m_n(u) &= \frac{K}{n^{\alpha}} \left( \prod_{j=1}^{n-1} u_j^{\frac{1-\alpha}{n}} \right) u_n^{-1+\frac{1-\alpha}{n}}
\end{align*}
\]

extending the \( n = 1 \) case.

Can this money-demand specification originate from the extended shopping-time (and thus also the Sidrauski) model? Putting this demand in the format (14),

\(^{10}\)The very name of this kind of demand gives us a hint of how to invert this system of equations: apply the logarithm function on each side. This yields a linear system in the variables \( \log m_1, \ldots, \log m_n \).
we get a \( F : \mathbb{R}^{++} \rightarrow \mathbb{R}^{++} \) such that \( F(G) = \left( \frac{K}{G} \right)^{1/\alpha} \). So \( F'(G) = -\frac{K^{1/\alpha}}{\alpha} G^{-1-\frac{1}{\alpha}} < 0 \), so that the demand 17 is rationalizable by the extended Sidrauski model.

In order for it to be also rationalizable by the extended shopping-time model, Proposition 5 only requires that the initial value problem

\[
\phi'(s) = \frac{(1 - s)^{\frac{1}{\alpha} - 1}}{K^{1/\alpha}} \phi(s)^{2 - \frac{1}{\alpha}}, \quad \phi(0) = 0
\]

is solvable. This is exactly the same problem studied in Example 1, so we have that this demand can arise from the shopping-time model iff \( \alpha < 1 \).

**Example 4** We now do with the semi-log money-demand specification the same as just done for the log-log case. Here the natural extension is

\[
\psi(m) = \left( \frac{1}{\xi} \log \left( \frac{B}{G(m)} \right) \right) \nabla G(m), \tag{18}
\]

where \( B > 0, \xi > 0 \). Here \( F : C_B \rightarrow \mathbb{R}^{++} \) is given by \( F(G) = \frac{1}{\xi} \log \left( \frac{B}{G} \right) \), and we have \( F'(G) = -\frac{1}{\xi G} < 0 \). Therefore, 18 is rationalizable by the extended Sidrauski model. Nevertheless, just as in the unidimensional framework, it cannot be rationalized by the extended shopping-time model, since \( B \neq +\infty \).

## 4 Conclusion

In this paper we have investigated the properties an arbitrary money-demand function given by the econometrician has to obey in order for it to be rationalizable by the versions of the Sidrauski and the shopping-time models displayed in Lucas (2000). One striking conclusion of our study is that log-log money demands with
an elasticity equal to or greater than unity, as well as the popular semi-log money demand, cannot be rationalizable by Lucas’ version of the shopping-time model, although the same does not happen concerning the Sidrauski model.

Besides Lucas (2000), Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick (2007), which use exactly the same utility and/or transacting-technology function, Fischer (1979), Siegel (1983), Asako (1983), Weil (1991) and Goodfriend (1997), dealing with particular cases of such functions, are examples of works which can also benefit from some of our results.

A second contribution of the paper was that of investigating how such classes of the Sidrauski and shopping-time model relate to each other. Differently from Feenstra (1986), our analysis focused not on the structure of the models themselves, but on the necessary and sufficient conditions arising from them. We have shown that Lucas’ version of the Sidrauski model is, in a strict sense, less stringent than that of the shopping-time model.

All our results have been initially developed for the case of unidimensional money demands, and posteriorly extended to an economy where several assets perform monetary functions. In both cases, all results have been exemplified, respectively, with the unidimensional and the multidimensional versions of the log-log and the semi-log money demand functions.

A Appendix A

We have already mentioned that all our results apply *ipsis litteris* to Lucas (2000), Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick (2007), since these papers use the same utility function and/or transacting technology we use
here. Fischer (1979), Siegel (1983), Asako (1983), Weil (1991) and Goodfriend (1997), on the other hand, are works which can also directly benefit from some of the results of this paper, since the utility function and/or transacting technology used in such works correspond to particular cases of those treated here (sections 2.1 and 2.2 show how the embedding in the general case can be done).

In this appendix we treat very briefly the case of two other utility functions used in the literature which cannot be written like (1).

**Case 1** Additive (Brock 1974) $U(c, m) = u(c) + v(m)$, where $u$ and $v$ are homogeneous of the same degree, strictly increasing and concave.

In this case, (2) becomes, in steady-state equilibrium, $r = \frac{v'(m)}{u'(1)}$. Therefore, we still have that any strictly decreasing money-demand function is rationalizable by Sidrauski’s model. In fact, given $\psi$, take $v(m) = \int_1^m \psi(\mu) d\mu$, and $u$ the identity function, for instance.

**Case 2** Constant Elasticity of Substitution (Holman 1998) $U(c, m) = \left[ \pi e^{\delta} + (1-\pi) m^{\delta} \right]^{\frac{1}{\delta}}$, where $\pi \in (0,1)$, $\delta \in (0,1)$.

As in the Cobb-Douglas case, this utility function generates exclusively log-log money-demand functions, but now the ones with higher-than-one elasticities (in absolute value): $m = \left( \frac{1-\pi}{\pi} \right)^{\frac{1}{1-\delta}} r^{-\frac{1}{1-\delta}}$.

### B Appendix B

One might ask if, given a money-demand specification in the format (14), what would a suitable $G$ be such that $\psi = (F \circ G) \nabla G$, in case this piece of information
wasn’t provided by the econometrician. That is, we can imagine that the only thing we have been told is that \( \psi \) can be put in that form – but we do not know how. Using (13) and Euler’s formula for homogeneous functions, we get

\[
\psi(m) \cdot m = \frac{G(m) \varphi'(G(m))}{\varphi(G(m)) - G(m) \varphi'(G(m))},
\]

which, together with (13), gives the following system of partial differential equations:

\[
\begin{cases}
G_{x_1}(m) = \frac{G(m)}{\psi(m) \cdot m} u_1(m) \\
\vdots \\
G_{x_n}(m) = \frac{G(m)}{\psi(m) \cdot m} u_n(m)
\end{cases}
\]

(20)

Since \( G \) is positive for \( m \in \mathbb{R}^n_{++} \), we can conveniently rewrite this system as

\[
\begin{cases}
(\log G)_{x_1}(m) = \frac{1}{\psi(m) \cdot m} u_1(m) \\
\vdots \\
(\log G)_{x_n}(m) = \frac{1}{\psi(m) \cdot m} u_n(m)
\end{cases}
\]

(21)

which yields the solution

\[
G^*(m) = De^{\int \frac{\psi(m) \cdot du}{\psi(m) \cdot m}}, \quad D > 0,
\]

(22)

where \( \Gamma : [0, 1] \rightarrow \mathbb{R}^n_{++} \) is a piecewise-\( C^1 \) path such that \( \Gamma(0) = 1 \) and \( \Gamma(1) = m \). Evidently, in order for us to write (22), the integral that appears in this expression has to be well-defined – that is, path-independent. Since \( \log \circ G : \mathbb{R}^n_{++} \rightarrow \mathbb{R} \) is an obvious potential function for the vector field taking \( m \in \mathbb{R}^n_{++} \) into \( \frac{1}{\psi(m) \cdot m} \psi(m) \in \mathbb{R}^n \), we have no problem.
Therefore, just like the solution for $\varphi$ in (6), the solution for $G$ is also unique up to a multiplicative constant. You may also want to note how (22) extends the case $n = 1$, where $G$ would be just the identity function.

Also, in the case of a money-demand function that we happen to know can be rationalized by the extended shopping-time model, its money aggregator function $G$ has to be, up to a multiplicative constant, exactly the same as if we would try to rationalize this demand by the extended Sidrauski model. This is because (16) also implies (20), exactly in the same way (13) does.

References


