Bounds for the probability distribution function of the linear ACD process

Marcelo Fernandes

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Abstract: This paper derives both lower and upper bounds for the probability distribution function of stationary ACD($p,q$) processes. For the purpose of illustration, I specialize the results to the main parent distributions in duration analysis. Simulations show that the lower bound is much tighter than the upper bound.

JEL Classification: C22, C41.

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1 Introduction

The statistical properties of the autoregressive conditional duration model of first order are by now quite well known. Engle and Russell (1998) derive not only the first two moments of the ACD(1,1) model, but also the conditions under which it is stationary and $\beta$-mixing. Carrasco and Chen (2002) extend the latter results so as to consider the more general ACD($p, q$) model. Bauwens and Giot (2000) provide a recursive formula to compute the autocorrelation function of the ACD(1,1) process with exponential errors. Fernandes and Grammig (2002) establish conditions for the existence of higher-order moments, strict stationarity and $\beta$-mixing property as well as moment recursion relations and autocovariance function of a richer class of nonlinear ACD(1,1) processes that encompasses most autoregressive conditional duration models in the literature.

This note aims at deriving nonasymptotic characterizations of the tail behavior of unconditional distribution function of the ACD($p, q$) process. More precisely, I derive both lower and upper bounds for the probability density function of the duration process. These bounds are quite relevant to the analysis of liquidity risk in that trade and volume durations are intimately related to market activity and liquidity (see Gouriéroux, Jasiak and Le Fol, 1999). Another interesting application relates to actuarial models of credit risk contagion as in Focardi (2001).

The remainder of this paper is organized as follows. Section 2 bounds the probability density function of the ACD($p, q$) process without assuming a particular distribution for the error term. Section 3 sharpens the result by considering the usual specifications of the density of the error term. Section 4 investigate the precision of these bounds through a simple simulation study.

2 Bounds

Let $x_t = \tau_t - \tau_{t-1}$ denote the time spell between two events occurring at times $\tau_t$ and $\tau_{t-1}$. Engle and Russell (1998) propose to account for the serial dependence of financial duration data by formulating an accelerated time process $x_t = \psi_t \epsilon_t$, where $\psi_t$
where \( \psi_t \equiv E(x_t \mid \Omega_{t-1}) \), \( \epsilon_t \) is iid with unity mean, and \( \Omega_{t-1} \) is the set including all information available at time \( \tau_{t-1} \). The ACD\((p, q)\) then assumes that \( \psi_t \) and \( \epsilon_t \) are stochastically independent, and

\[
\psi_t = \omega + \sum_{i=1}^{p} \alpha_i x_{t-i} + \sum_{j=1}^{q} \beta_j \psi_{t-j},
\]

where \( \omega > 0 \), \( \alpha \equiv (\alpha_1, \ldots, \alpha_p) \geq 0 \), and \( \beta \equiv (\beta_1, \ldots, \beta_q) \geq 0 \). This parameter restrictions ensure the nonnegativeness of the duration process, whereas imposing \( \gamma \equiv \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1 \) guarantees stationarity.

I take benefit from the close parallel between ACD and GARCH models in order to establish both lower and upper bounds of the probability distribution function. More precisely, I start with a trivial upper bound and then derive a nontrivial lower bound using the techniques put forth by Pawlak and Schmid (2001). Denoting by \( F_\epsilon \) the probability distribution function of the error term, it follows that

\[
Pr(x_t \leq z) = E\left[Pr\left(\psi_t \epsilon_t < z \mid I_{t-1}\right)\right]
= E\left[F_\epsilon\left(\frac{z}{\omega + \sum_{i=1}^{p} \alpha_i x_{t-i} + \sum_{j=1}^{q} \beta_j \psi_{t-j}}\right)\right].
\]

It is readily seen that

\[
Pr(x_t \leq z) \leq F_\epsilon(z/\omega)
\]

given that durations are nonnegative as well as \( \alpha \) and \( \beta \).

As in Pawlak and Schmid (2001), the nontrivial lower bound for the probability distribution function of the duration process holds only for \( z \in [0, \bar{z}] \), where \( \bar{z} \) varies according to the distribution of the error term. The reason is that, to apply Jensen’s inequality to bound (2) from below, one must find the conditions under which the function

\[
H(u, v) = F_\epsilon\left(\frac{z}{\omega + \sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j v_j}\right)
\]

is convex for all \( u \equiv (u_1, \ldots, u_p) \geq 0 \) and \( v \equiv (v_1, \ldots, v_q) \geq 0 \). It is straightforward to show that the Hessian of (4) is convex if

\[
2f_\epsilon(z/M) + \frac{z}{M} f'_\epsilon(z/M) \geq 0
\]
where $M = \omega + \sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j v_j$, given that

$$
\frac{\partial^2 H(u,v)}{\partial u_s \partial u_a} = \alpha_r \alpha_s \frac{z^2}{M^4} f_c'(z/M) + 2 \alpha_r \alpha_s \frac{z}{M^3} f_c(z/M) \quad (6)
$$

$$
\frac{\partial^2 H(u,v)}{\partial u_s \partial v_s} = \alpha_r \beta_s \frac{z^2}{M^4} f_c'(z/M) + 2 \alpha_r \beta_s \frac{z}{M^3} f_c(z/M) \quad (7)
$$

$$
\frac{\partial^2 H(u,v)}{\partial v_s \partial u_a} = \beta_r \beta_s \frac{z^2}{M^4} f_c'(z/M) + 2 \beta_r \beta_s \frac{z}{M^3} f_c(z/M) \quad (8)
$$

for $r \neq s$. I am now ready to state the main result.

**Theorem.** Let $x_t \sim \text{ACD}(p,q)$ satisfying the nonnegativeness and stationarity conditions. Assuming that the density $f_c$ of the error term is differentiable then yields that

$$
F_c(x) \leq \Pr \left( x_t \leq \frac{\omega}{1 - \gamma} \right) \leq F_c \left( \frac{x}{1 - \gamma} \right), \quad (9)
$$

where the lower bound holds only for $x \in \left[ 0, (1 - \gamma)c \right]$ with

$$
c \equiv \sup_{\tau > 0} \left\{ 2f_c(x) + xf_c'(x) \geq 0 \text{ for all } 0 < x < \tau \right\}. \quad (10)
$$

**Proof.** It ensues from condition (5) that

$$
\sup_{\tau > 0} \left\{ 2f_c(z/M) + \frac{z}{M} f_c'(z/M) \geq 0 \text{ for all } 0 < x < \tau \right\} = M \sup_{\tau > 0} \left\{ 2f_c(x) + xf_c'(x) \geq 0 \text{ for all } 0 < x < \tau \right\} = Mc. \quad (11)
$$

The Hessian of (4) is therefore convex if

$$
z \leq c \left( \omega + \sum_{i=1}^{p} \alpha_i u_i + \sum_{j=1}^{q} \beta_j v_j \right). \quad (12)
$$

This is true for any $\alpha_i, \beta_i, \gamma, \nu_j \geq 0$ if $z \leq c \omega$. The result then follows by applying the Jensen’s inequality to (2) with $z = \frac{\omega}{1 - \gamma} x$.

As is apparent, the applicability of the lower bound depends essentially on the constant $c$, and hence it is interesting to evaluate (10) for the usual distributions in the ACD literature. I therefore consider in the next section five particular cases, namely exponential, Weibull, Burr, generalized gamma, and uniform. In all instances I normalize the distribution so as to impose unity mean.
3 Examples

The ACD modeling aims to match two stylized features in financial duration data, namely serial correlation and overdispersion. Engle and Russell (1998) show indeed that the simple ACD model with exponential errors produce overdispersion. Further, quasi maximum likelihood methods provide consistent estimates for ACD process only if based on the exponential distribution (see Drost and Werker, 2001). It seems therefore natural to start with the exponential assumption for the error term, which yields $f_E(x) = e^{-x}$ and $f_E^0(x) = -e^{-x}$.

By (5), this implies that $c = 2$.

Engle and Russell (1998) argue that the nature of financial durations are more in line with a decreasing baseline hazard rate function. The exponential assumption implies however that the baseline hazard rate function is flat. A natural candidate then is the Weibull distribution with parameter $\theta$, which gives rise to either monotonically decreasing ($0 < \theta < 1$) or increasing ($\theta > 1$) baseline hazard rate functions. The density function of the Weibull distribution and its first derivative are

$$f_W(x) = \frac{\theta}{\Gamma(1 + 1/\theta)} x^{\theta - 1} \exp \left[-\frac{x^\theta}{\Gamma(1 + 1/\theta)}\right]$$

and

$$f_W^0(x) = -f_W(x) \left[\frac{1 - \theta}{x} + \frac{\theta}{\Gamma(1 + 1/\theta)} x^{\theta - 1}\right],$$

respectively. Condition (5) then becomes

$$f_W(x) \left[2 - 1 + \theta - \frac{\theta}{\Gamma(1 + 1/\theta)} x^\theta\right] \geq 0,$$

which implies that $c = \left[\Gamma(2 + 1/\theta)\right]^{1/\theta}$. As a sanity check, observe that in the exponential case $\theta = 1$, recovering $c = \Gamma(3) = 2$. If one assumes that $0 < \theta < 1$ as in Engle and Russell (1998), then the sup condition in (10) becomes less stringent than in the exponential context, viz. $c > 2$.

Grammig and Maurer (2000) advocate the use of the Burr distribution in order to accommodate more flexible hazard rate functions. Indeed the Burr density

$$f_B(x) = \frac{\theta \xi_B^\theta x^{\theta - 1}}{(1 + \kappa \xi_B^\theta x^\theta)^{1+1/\kappa}},$$

5
where $\theta > \kappa > 0$ and
\[
\xi_B = \frac{\Gamma(1 + 1/\theta) \Gamma(1/\kappa - 1/\theta)}{\kappa^{1+1/\theta} \Gamma(1 + 1/\kappa)},
\] (17)
entails a nonmonotonic baseline hazard rate function if $\theta > 1$. It is easy to show that
\[
f_B'(x) = f_B(x) \left[ \frac{\theta - 1}{x} - \left( 1 + \frac{1}{\kappa} \right) \frac{\kappa \theta \xi_B^\theta}{1 + \kappa \xi_B^\theta x^{\theta-1}} \right],
\] (18)
and hence
\[
c = \left( \frac{\theta + 1}{\theta - \kappa} \right)^{1/\theta} \xi_B^{-1} = \left( \frac{\theta + 1}{\theta - \kappa} \right)^{1/\theta} \frac{\kappa^{1+1/\theta} \Gamma(1 + 1/\kappa)}{\Gamma(1 + 1/\theta) \Gamma(1/\kappa - 1/\theta)}.
\] (19)
Special cases of the Burr family of distributions are the Weibull ($\kappa \to 0$), the exponential ($\kappa \to 0, \theta = 1$), and the log-logistic ($\theta > \kappa = 1$) distributions.

As an alternative to the Burr model, Lunde (1999) puts forward the generalized gamma ACD process, where $\epsilon_t$ is iid with density
\[
f_G(x) = \frac{\xi_G^\theta \theta x^{\theta - 1}}{\Gamma(\kappa)} \exp\left(-\xi_G^\theta x^\theta\right)
\] (20)
where $\xi_G \equiv \Gamma(\kappa + 1/\theta)/\Gamma(\kappa)$. The generalized gamma distribution nests the standard gamma ($\theta = 1$), log-normal ($\kappa \to \infty$), Weibull ($\kappa = 1$), exponential ($\theta = \kappa = 1$), and half-normal ($\kappa = 1/2, \theta = 2$) distributions.

Although the baseline hazard rate has no closed-form solution, it is possible to derive its shape properties according to the parameter values (Glaser, 1980). If $\theta \kappa < 1$, the hazard rate is decreasing for $\theta \leq 1$, and U-shaped for $\theta > 1$. Conversely, if $\theta \kappa > 1$, the hazard rate is increasing for $\theta \geq 1$, and inverted U-shaped for $\theta < 1$. Lastly, if $\theta \kappa = 1$, the hazard rate is decreasing for $\theta < 1$, constant for $\theta = 1$ (exponential case), and increasing for $\theta > 1$. The derivative of the density is
\[
f_G'(x) = f_G(x) \left[ \frac{\kappa \theta - 1}{x} - \theta \xi_G^\theta x^{\theta-1} \right],
\] (21)
yielding
\[
c = (\kappa + 1/\theta)^{1/\theta} \frac{\Gamma(\kappa)}{\Gamma(\kappa + 1/\theta)}.
\] (22)
It is easy to show that the lower bound is valid for $c = \sqrt{\pi}$ and $c = 1 + 1/\kappa$ in the particular cases of the half-normal and standard gamma distributions, respectively.
Finally, albeit the ACD process with $\epsilon_t \sim U(0, 2)$ does not have much appeal in practice, it entails a very interesting result. Indeed, it turns out that (5) holds for every value of $x$, and so the lower bound is always valid. Figure 1 summarizes these results by displaying the constant $c$ as a function of the distributional parameters. There are no plots for the exponential, half-normal and uniform distributions in view that $c$ does not vary for them.

4 Sharpness of the bounds

To investigate how tight these bounds are, I perform a simple simulation study using an ACD(2,2) process with exponential errors. I set $\alpha = (0.10, 0.05)$ and $\beta = (0.45, 0.25)$ and normalize the unconditional expected duration to one by imposing $\omega = 1 - \gamma$. Next, I initialize (1) with $\psi_0 = 1$ and simulate 10,000 realizations of the process and then estimate the unconditional cumulative distribution of the duration process using the empirical distribution of the last 8,000 observations of the sample. Figure 2 illustrates the fact that, despite the slackness of the trivial upper limit, the nontrivial lower bound is extremely sharp and informative. Further simulations show that this result is quite robust to the specification of the linear ACD process. The simulations also indicate that by substituting maximum likelihood estimates for the true values of the parameters, the 95% confidence interval of the lower bound provides a tight confidence band to the true probability distribution function of the process.

References


Grammig, J., Maurer, K.-O., 2000, Non-monotonic hazard functions and the autoregressive conditional duration model, Econometrics Journal 3, 16–38.


Figure 1: The constant $c$ as a function of the distributional parameters

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Figure 2: Bounds for the linear ACD probability distribution function

This example refers to an ACD(2,2) process with exponential errors. The parameters are set to $\alpha = (0.10, 0.05)$, $\beta = (0.45, 0.25)$, $\omega = 1 - \gamma$, and $\psi_0 = 1$. The bounds are based on 10,000 realizations of the process and on the empirical distribution function of the last 8,000 observations of the sample.

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