Nº 252

ENTREPRENEURIAL RISK AND LABOUR'S SHARE IN OUTPUT

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Janeiro de 1995
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ABSTRACT

This paper presents a theoretical model which discusses the role played by the entrepreneurial risk on the distribution of the national income. In a two-period general equilibrium framework with competitive risk-averse entrepreneurs it is shown that the highest the risk borne by firms, the lower will be real wages, employment and the labour's share in output. The model helps explain the fall of the labour's share in the Brazilian output during the 1980s.
INTRODUCTION

Since the first contributions to the theory of implicit labour contracts by Baily (1974), Gordon (1974) and Azariadis (1975) economists began to understand some important characteristics of wages. When markets are not complete in the sense of Arrow (1964) or Debreu (1959), firms and workers may find it Pareto improving to assign to wages an insurance role. If this is the case, then gross salaries will differ from the marginal productivity of labour even in a competitive labour market. If workers could obtain insurance against fluctuations in wages or employment then they would not find any incentive to include an insurance premium in wages.

Incompleteness of insurance markets arise due to many different sources; information asymmetries, moral hazard and adverse selection are sufficient to undermine such kind of markets. The literature on implicit labour contracts puts risk averse workers as the main source of implicit contracts. Though managers may behave in a risk-averse way, large companies tend to follow a risk-neutral behaviour since they can diversify away non-systematic risks for which there are no insurance markets. These issues are discussed in Azariadis (1983), Cooper (1983) and (1985), Fethke & Policano (1984), Grossman & Hart (1981), Grossman (1983), Hart (1983). Enforcement of firm's fulfilment of labour contracts may be impossible when bankruptcy is allowed for as discussed in Farmer (1985), Kahn & Scheinkman (1985). Rosen (1985) is a good survey on implicit contracts.

The model presented in this paper is an attempt to explain the stylized fact shown in Figure I which were extracted from Oliveira (1993). It shows the ratio of worker's payroll to output in the Brazilian formal economy. Although the very low figures may well be the result of under reporting due to various sources, the downward trend suggests that the worker's share in the Brazilian output fell during the 1980s.

![Payroll/GDP Chart](chart.png)

**Source:** RAIS and INSS
The risks faced by Brazilian entrepreneurs in the 1980s were higher than in previous decades. During the 1980s there were spells of very dear money for most Brazilian firms, since the government borrowed massively internally, thus guzzling almost all private savings. There were also spells of very negative real interest rates when the government relented to financing its deficits with monetary expansion. Inflation rates were unpredictable, at times plummeting to -0.5% as well as rocketing to 80% per month. Basic constitutional principles as the one which assures that income (a flux), but never financial assets (a stock), could be source of taxation, were flouted. Price controls were caustically used. Pernickety labour laws were made ever more complicated to apply, leaving almost all firms liable to high fines from the Labour Ministry. These distortions increased the uncertainty faced by firms.

It is important to realize that most of the above adversities do not impact all companies at the same time. They reach different firms randomly. A company flush with cash during a period of hefty real interest rates could rake in huge gains, while another firm facing a transitory negative cash flow could go bust. Price controls could reach suppliers of raw materials but not producers of final goods or just the other way around. This is why I use in the theoretical model presented below a stochastic production shock for each firm which is non-correlated across firms. In the aggregate, the shocks cancel out, but for each firm it varies significantly.

The model is based on the assumption that entrepreneurs cannot diversify away the risk they run. In a competitive labour market, each entrepreneur cannot hire workers for a wage lower than the market's. Given the market's wage and the level of risk involved in the production process, the entrepreneur optimally chooses the amount of resources he is willing to allocate to his risky business. The higher the wage he has to pay and the higher the risk he has to run, the lower will be his willingness to hire workers. In partial equilibrium each entrepreneur cannot change the market's wage but he can reduce the number of hours of work employed.

An increase in the risk borne by firms is followed by a reduction in the demand for labour. Each entrepreneur hires less workers, leading to an aggregate reduction in the demand for labour, which leads to a fall in the market's wage. At the new (lower) market wage, entrepreneurs tend to increase their demand for labour, (partially) offsetting the initial reduction due to the higher level of risk. The reduction in the market wage is interpreted as an increase in the risk premium that workers implicitly pay to employers who are the agents running the risk of producing goods.

The fall of the labour's share in the aggregate output is due to the fact that this share is calculated in an ex-post basis. Since the ex-ante profits increases due to the fall of wages, and since the number of firms in the economy is large, the Law of Large Numbers apply and the ex-post and ex-ante worker's share in output are the same.

The paper is presented in three sections. Section II describes the basic model in which there are only two markets, the market of goods and the market of labour. In this abridged model there is only one market clearing variable, namely the wage. Section III extends the basic model, introducing a third market, the market of loans, and consequently a second market clearing variable: the interest rate. Section IV concludes
THE BASIC MODEL

Consider a competitive two-period economy with one good and two kinds of agents, entrepreneurs and workers. The former own the technology of production and hire the latter for a real wage $W$ during the last period.

II.1. WORKERS' BEHAVIOUR

There are $H$ identical workers who work and consume at period 1. They supply labour competitively according to a consumption-leisure decision: $L'$ hours of work during period 1 allows consumption of $WL'$ units of the good. It is assumed that both leisure and consumption are normal goods so that there is a real-valued function $L': \mathbb{R} \to \mathbb{R}$ with positive derivative which ascribes to each wage $W$ the number of hours of work each worker supplies at that wage.

II.2. ENTREPRENEURS' BEHAVIOUR

There are $I$ identical entrepreneurs with an amount of real resources of $A$ units of the good at period 1. This amount can be either (1) lent or borrowed at a real riskless rate $r$ in, say, the international bonds market or (2) used to hire workers who will produce the good according to the risky technology described below.

The production technology is described by a real valued twice differentiable production function $f: \mathbb{R} \to \mathbb{R}$ and a random variable $\varepsilon$. An entrepreneur who contracts $L$ hours of work during period 1 will have an output of $\tilde{\varepsilon} f(L)$ units of the good at period 2, where the tilde over $\varepsilon$ stands for the realization of the random variable. The function $f$ is assumed to be increasing ($f' > 0$), to exhibit decreasing marginal productivity, i.e., to be strictly concave ($f'' < 0$) and such that $f(0) = 0$. The random variable $\varepsilon$ is normally distributed with mean $\mu$ and variance $\sigma^2$, where $\mu$ is sufficiently larger than $\sigma$ so that a negative realization of $\varepsilon$ can be ruled out. The random variables $\varepsilon_i, i = 1, 2, ... , I$ are identical and independently distributed across entrepreneurs.

Each entrepreneur is non-satiable, risk averse and becomes less risk averse the richer he gets. His preferences are described by a real valued twice differentiable utility function $U: \mathbb{R} \to \mathbb{R}$ with positive first derivative ($U' > 0$), negative second derivative and decreasing absolute risk aversion.

Given the market real wage $W$, entrepreneur $i$ chooses how many hours of work $L^0_i$ he will demand during period 1 in order to maximize the expected utility of his period 2 consumption. Since at period 1 he spends $WL^0_i$ with his payroll, he will have $A - WL^0_i$ left to invest at the riskless rate $r$. At period 2 the amount of goods he will possess will be the sum of his firm's production $\tilde{\varepsilon}_i f(L^0_i)$ and the accrued value of his riskless investment $(A - WL^0_i)(1 + r)$. His choice $L^0_i$ is, therefore the solution of the maximization below:
The properties of the choice \( L^D \) are presented in the proposition I.

**PROPOSITION I:** The solution to (1) is a continuously differentiable function \( L^D : \mathbb{R}^7 \rightarrow \mathbb{R} \), \( L^D = L^D(W, \mu, \sigma, A, r) \) with the following partial derivatives:

a) \( \partial L^D / \partial W < 0 \), \( \partial L^D / \partial \mu > 0 \), \( \partial L^D / \partial \sigma < 0 \), \( \partial L^D / \partial A > 0 \);  
b) If entrepreneurs borrow resources at the riskless rate, i.e., if \( A - WL^D < 0 \), then \( \partial L^D / \partial r < 0 \)

**Proof:** See the appendix.

The partial derivatives above show that a non satiable risk averse entrepreneur will demand more labour, i.e., allocate a higher share of his available resources \( A \) to the risky investment opportunity, the lower the cost of labour (wage), the higher the expected productivity of labour, the lower the underlying risk. Since he becomes less risk averse the richer he gets, the higher the amount of available resources he has, the higher his demand for labour.

The partial derivative with respect to the riskless rate \( r \) is indeterminate. On the one hand the higher this rate, the higher the opportunity cost of investing in the risky business, which tends to reduce the demand for labour. On the other hand, the higher the riskless rate, the higher the (future) value of the available resources, which tends to increase the demand for labour. If his relative risk aversion were constant, this income effect would be null and an increase in the riskless rate would reduce the demand for labour. If the solution of (1) is such that the entrepreneur finances part of his risky business with borrowed money, than the higher the riskless rate the lower his demand for labour, since an increase in \( r \) would increase his exposure to risk.

The risk aversion hypothesis also implies that given the market wage \( W \), each entrepreneur will hire labour at an amount sufficiently low such that:

a) the expected present value of the marginal productivity of labour be greater than the real wage;  
b) the expected rate of return on each unit of resource ploughed into his business be greater than the riskless rate;

These results are presented in proposition II.
PROPOSITION II: Given the market real wage $W$, the demand for labour $L^D$ is such that:

\[ \frac{\mu f'(L^D)}{1+r} > W; \]  

(2.a)

\[ \frac{\mu f(L^D) - WL^D}{WL^D} > r. \]  

(2.b)

Proof: Appendix.

II.3. GENERAL EQUILIBRIUM

The market real wage $W^*$ is competitively determined in the labour market such that the aggregate supply of labour $HL'(W^*)$ be equal to the aggregate demand $IL^D(W^*, \mu, \sigma, A, r)$:

\[ \lambda L'(W^*) = L^D(W^*, \mu, \sigma, A, r) \]  

(3)

where $\lambda = H: I$ is the number of workers hired by each entrepreneur. The properties of the equilibrium real wage $W^*$ are presented in proposition III.

PROPOSITION III: The equilibrium real wage $W^*$ is a continuously differentiable function $W^*: \mathbb{R}^4 \rightarrow \mathbb{R}$, $W^* = W^*(\mu, \sigma, A, r, \lambda)$ with the following partial derivatives:

a) $\partial W^*/\partial \mu > 0$, $\partial W^*/\partial \sigma < 0$, $\partial W^*/\partial A > 0$, $\partial W^*/\partial \lambda < 0$.

b) If entrepreneurs borrow resources at the riskless rate, i.e., if $A - W^* L^D < 0$, then $\partial W^*/\partial \lambda < 0$.

Proof: Appendix

The above partial derivatives are straightforward consequences of the demand side of the market. The equilibrium real wage increases (decreases) when the demand for labour also increases (decreases).

II.4. LABOUR'S SHARE IN OUTPUT

At period 2 the i-th entrepreneur will have a level of production of $\bar{e}_i f(L^D_i)$ units of the good. The national product $Y$ is the sum over $i$ of each firm's output:
\[ Y = \sum_{i=1}^{I} \tilde{e}_i f(L^D_i) = f(L^D) \sum_{i=1}^{I} \tilde{e}_i \]

where the second equality follows from the fact that all entrepreneurs employ the same amount of labour. The hypothesis that the random variables are independently distributed across entrepreneurs and that the number of entrepreneurs is sufficiently large gives, by the Law of Large Numbers:

\[ \frac{1}{I} \sum_{i=1}^{I} \tilde{e}_i = \mu \]

These two last expressions give the national product:

\[ Y = I \mu f(L^D) \]

The labour income is the aggregate payroll \( IL^D W^* \). Thus the labour's share in output \( \theta \) is given by:

\[ \theta = \frac{IL^D W^*}{I \mu f(L^D)} = \frac{W^*(\mu, \sigma, A, r, \lambda) L^D W^*(\mu, \sigma, A, r, \lambda)}{\mu f(L^D)[W^*(\mu, \sigma, A, r, \lambda), \mu, \sigma, A, r]} \]

The properties of the labour's share in output are given in proposition IV.

**PROPOSITION IV:** The labour's share in output \( \theta \) is a continuously differentiable function \( \theta: \mathbb{R}^5 \rightarrow \mathbb{R} \), \( \theta = \theta(\mu, \sigma, A, r, \lambda) \), with the following partial derivatives:

a) \( \frac{\partial \theta}{\partial \sigma} < 0, \quad \frac{\partial \theta}{\partial A} > 0, \quad \frac{\partial \theta}{\partial r} < 0; \)

b) If entrepreneurs borrow resources at the riskless rate, i.e., if \( A - WL^D < 0 \), then \( \frac{\partial \theta}{\partial r} < 0 \).

Proof: Appendix.

The partial derivatives above show that the labour's share in output decreases when the risk borne by entrepreneurs increases.

The results above are illustrated in graph 1. The vertical axis represents the wages and the horizontal axis the labour supply and demand for each firm. If there were no risk in the production technology, i.e., if \( \sigma \) were 0 (zero), then the demand for labour would be given by the usual first order condition in which the wage is equal to the (present value of the) marginal productivity of labour:

\[ W^* = \mu f'(L^D) (1 + r). \]
According to proposition II, the demand for labour for a positive $\sigma$, i.e., $L^D$ is a curve below the previous one. The labour supply $L'$ is an upward sloping curve and the equilibrium wage $W^*$ is determined at the point where $L'$ crosses $L^D$.

**GRAPH 1**

The (average) output per firm $y$ is given by

$$y = \frac{Y}{L} = \mu f(L^*) = \mu \int_0^{L^*} f'(L) dL$$

This is represented by the area below the curve $\mu f'(L) = W(1+r)$ and the horizontal axis up to the point where $L = L^*$ (points GEFD). The payroll per firm is given by the area of the rectangle BEFC. The area of the rectangle ABCD represents the risk premium implicitly paid by workers to entrepreneurs who bear the risk of producing goods. The area GAD represents the pure profits received by entrepreneurs due to the ownership of the technology. An increase of $\sigma$ shifts the demand $L(W, \sigma, \mu, A, r)$ downwards. This reduces ratio of the area rectangle BEFC to GEFD.
III. THE EXTENDED MODEL

The basic model of section II has a unique market clearing variable, namely the real wage. The risk premium implicitly paid to the risk bearers was given from workers to entrepreneurs solely through this variable. In this section, the riskless rate \( r \) is endogenized so that a second variable clearing a savings market is introduced. The goal is to ascertain whether the impact of the entrepreneurial risk on the workers' share in output is changed when a more complete setting is allowed for.

III.1. WORKER'S BEHAVIOUR

In this section each worker works at period 1 and consumes at periods 1 and 2. He supplies labour competitively according to a consumption - savings - leisure decision: \( L' \) hours of work during period 1 gives \( WL' \) units of the good. If he consumes \( C_i \) units at period 1 he will have \( (WL' - C_i)(1+r) \) units for consumption at period 2. Each worker's endowment of time as of period 1 is \( L \). Thus he has to decide how many hours of leisure \( \ell^B = L - L' \) he will demand at period 1, how many units \( C_i \) and \( C_j \) of the good he will consume at periods 1 and 2, subject to the budget constraint below:

\[
L(1+r)W = \ell^B W(1+r) + C_i(1+r) + C_j
\]

Increases of the real wage has both income and substitution effects. The income effect is the fact that the worker feels richer and, therefore, tends to increase his demand for leisure and consumption in both periods. The substitution effect, on the other hand, makes leisure relatively more expensive vis-à-vis consumption, which tends to reduce leisure.

Likewise, increases of interest rate has both income and substitution effects. The income effect, which tends to increase his demand for leisure and consumption in both periods, is the fact that the higher the interest rate, the richer a worker feels since his consumption possibilities increase. On the other hand, a higher interest rate makes leisure and period 1 consumption more expensive relative to period 2 consumption, which tends to reduce the latter.

It is assumed that leisure and periods 1 and 2 consumption are all normal goods so that the substitution effects are larger than the income effects. This means that an increase in the wage increases period 1 consumption and reduces the demand for, leisure, i.e., increases the supply of labour. An increase in the interest rate reduces period 1 consumption and leisure, i.e., increases the supply of labour:

\[
L' = L'(W, r), \quad L'_w \geq 0 \quad \text{and} \quad L'_l \geq 0
\]

\[
C_1 = C_1(W, r), \quad C'_1 \geq 0, \quad \text{and} \quad C'_1 \leq 0
\]
III.2. ENTREPRENEURS' BEHAVIOUR

Entrepreneurs face the same choices of the basic model. The only difference is that instead of borrowing or lending at the international (exogenous) riskless rate $r$, they will not have access to foreign markets and $r$ will represent the home interest rate. While making his choice of $L^D$ each entrepreneur takes the interest rate $r$ as given. But $r$ will be determined endogenously in general equilibrium. Propositions I and II still hold but for this reinterpretation of the riskless rate $r$.

III.3. GENERAL EQUILIBRIUM

The market real wage $W^*$ and the interest rate $r^*$ are competitively determined in the labour and loans markets. The workers' supply of loans is $H(WL' - C_i)$ and the entrepreneur's supply is $I(A - WL^D)$:

$$H(WL^D - C_i) + I(A - WL^D) = 0$$

The labour market equilibrium requires $HL' = IL^D$, the equilibrium real wage $W^*$ and riskless rate $r^*$ in both labour and loans markets are determined by the following two equations:

$$\lambda L'(W^*, r^*) = L^D(W^*, r^*, \mu, \sigma, A) \quad (4)$$

$$\lambda C_i(W^*, r^*) = A \quad (5)$$

Since workers must save something in order to consume at period 2, it follows that their savings will be lent to entrepreneurs. Therefore condition (b) of proposition I will hold and the demand for labour will be lower the higher the interest rate. Workers are now partially financing their own wages. Proposition V substitutes for proposition III of the basic model.

**PROPOSITION V:** The equilibrium real wage $W^*$ and interest rate $r^*$ are continuously differentiable functions $W^*: \mathbb{R} \to \mathbb{R}$, $r^*: \mathbb{R} \to \mathbb{R}$, $W^*(\mu, \sigma, A, \lambda)$, $r^*(\mu, \sigma, A, \lambda)$, with the following partial derivatives:

$$\partial W^*/\partial \mu > 0, \quad \partial W^*/\partial \sigma < 0, \quad \partial W^*/\partial A > 0, \quad \partial W^*/\partial \lambda < 0$$

$$\partial r^*/\partial \mu > 0, \quad \partial r^*/\partial \sigma < 0, \quad \partial r^*/\partial A \leq or \geq 0, \quad \partial r^*/\partial \lambda \leq or \geq 0$$

**Proof:** Appendix.

The results above show that the higher the expected productivity of labour, the higher will be the equilibrium values of factor prices. On the other hand, the higher the risk born by firms, the lower will be those prices, for the risk averse entrepreneurs will require a premium in order to run the risk of employing workers as well as of borrowing resources. The impact of the entrepreneurs' available resources on
the wage is positive because the entrepreneurs become less risk averse, and hence less loath to borrow, the richer they are. Its impact on the interest rate is ambiguous because the higher those resources, the higher the aggregate supply of loans.

III.4. LABOUR'S SHARE IN OUTPUT

As in section II.4 the labour's share in output is the aggregate payroll divided by the aggregate output:

\[ \theta^* = \frac{L^D W}{L^D f(L^D)} = \frac{W^*(\mu, \sigma, A, \lambda) L^D [W^*(\mu, \sigma, A, \lambda), r^*(\mu, \sigma, A, \lambda), \mu, \sigma, A]}{\mu f(L^D [W^*(\mu, \sigma, A, \lambda), r^*(\mu, \sigma, A, \lambda), \mu, \sigma, A])} \]  

(6)

The effect of the exogenous parameters on \( \theta \) are qualitatively the same as those found in the basic model, the results are summarized in proposition VI

**PROPOSITION VI**: The labour share in output in the extended model is a continuously differentiable function \( \theta^*: \mathcal{R}^4 \rightarrow \mathcal{R} \). \( \theta^* = \theta^*(\mu, \sigma, A, \lambda) \), with the following partial derivatives

\[ \frac{\partial \theta^*}{\partial \sigma} < 0 \quad \frac{\partial \theta^*}{\partial A} < 0 \]

\[ \frac{\partial \theta^*}{\partial A} \geq 0 \text{ or } \leq 0 \quad \frac{\partial \theta^*}{\partial \lambda} \geq 0 \text{ or } \leq 0 \]

The results above show that the introduction of a second endogenous variable in the general equilibrium, namely the interest rate, does not modify the impact of the risk born by firms on the workers' share in output. Actually, it just provides the economy with a second means of conveying the risk premium from the workers to the entrepreneurs.

IV. CONCLUSION

This paper presented a two-period theoretical model which provides a possible explanation for one of the causes of the fall of the labour's share in the output of Brazil during the 1980's. The model suggests that the higher risk faced by firms in the unfriendly economic environment of the 1980s might have been one of the causes of that fall. Needless to say, there might have been other causes such as the imperfect indexation of wages, the real devaluation of the Brazilian currency intent on generating trade balance surpluses to service the external debt, the decrease in labour qualification (average human capital) that were not focused by the paper.
APPENDIX

Proposition I

Proof: The maximization (1) can be rewritten with $x = \frac{E - \mu}{\sigma}$ to yield:

$$\max_L \int_{\sigma}^{\infty} e^{-x^2/2} U\left( (\sigma x + \mu) f(L) + (A - WL)(1 + r) \right) dx$$

The first order condition is

$$\int_{\sigma}^{\infty} e^{-x^2/2} U'\left( (\sigma x + \mu) f'(L) - W(1 + r) \right) dx = 0$$

where $$(x) = (\sigma x + \mu) f(L) + (K - WL)(1 + r)$$

The second order condition is

$$I(L) = \int_{\sigma}^{\infty} e^{-x^2/2} \left\{ U''(x) \left[ (\sigma x + \mu) f''(L) - W(1 + r) \right] + U'(x) (\sigma x + \mu) f''(L) \right\} dx < 0$$

From $U'' < 0, f'' < 0, W(1 + r) > 0$ and the first order condition it follows that $I(L) < 0$.

The Implicit Function Theorem applied to the first order condition yields:

$$-I(L) dL = I(\mu) d\mu + I(W) dW + I(\sigma) d\sigma + I(A) dA + I(r) dr$$

where:

$$I(\mu) = \int_{\sigma}^{\infty} e^{-x^2/2} \left\{ U''(x) \left[ (\sigma x + \mu) f''(L) - W(1 + r) \right] + U'(x) (\sigma x + \mu) f''(L) \right\} dx$$

$$I(W) = \int_{\sigma}^{\infty} e^{-x^2/2} \left\{ U''(x) \left[ (\sigma x + \mu) f''(L) - W(1 + r) \right] + U'(x) (1 + r) \right\} dx$$

$$I(\sigma) = \int_{\sigma}^{\infty} e^{-x^2/2} x \left\{ U''(x) \left[ (\sigma x + \mu) f''(L) - W(1 + r) \right] + U'(x) f'(L) \right\} dx$$

$$I(A) = (1 + r) \int_{\sigma}^{\infty} e^{-x^2/2} \left\{ U''(x) \left[ (\sigma x + \mu) f''(L) - W(1 + r) \right] + U'(x) f'(L) \right\} dx$$

$$I(r) = \int_{\sigma}^{\infty} e^{-x^2/2} \left\{ (A - WL) U''(x) \left[ (\sigma x + \mu) f''(L) - W(1 + r) \right] - U'(x) \right\} dx$$

Lemmas 4, 5, 6 and 7 prove that $I(\mu) > 0, I(W) < 0, I(\sigma) < 0, I(A) > 0$. Lemma 8 proves that if $A - WL < 0$ then $I(r) < 0$. Lemmas 1, 2 and 3 are used to prove the other Lemmas.

Lemma 1: Define the constant $\bar{x} \in \mathbb{R}$ such that $(\sigma \bar{x} + \mu) f'(L) - W(1 + r) = 0$

Then $\bar{x} < 0$:

Proof: From $\int_{-\infty}^{\infty} e^{-x^2/2} x dx = 0$, it follows that:

$$-\bar{x} \int_{-\infty}^{\infty} e^{-x^2/2} dx = \int_{-\infty}^{\infty} e^{-x^2/2} (x - \bar{x}) dx$$

It will be shown that the integral on the right hand side of the above equality is positive.

In order to avoid cluttering with too many symbols, define:
\[
U'(\bar{x}) = U'((\sigma \bar{x} + \mu)f'(L) + (A - WL)/(A - WL))
\]

Taking any \( x < \bar{x} \) and recalling that \( U'' < 0 \), one finds that \( U'(x) > U'(\bar{x}) \). From \( (x - \bar{x}) < 0 \) it follows that: \( U'(x)(x - \bar{x}) < U'(\bar{x})(x - \bar{x}) \), for all \( x < \bar{x} \). By the same token, taking any \( x > \bar{x} \) and recalling that \( U'' < 0 \), \( \bar{x} \) has that: \( U'(x) < U'(\bar{x}) \). From \( (x - \bar{x}) > 0 \) it follows that: \( U'(x)(x - \bar{x}) < U'(\bar{x})(x - \bar{x}) \) for all \( x > \bar{x} \). This proves that for any \( x \in \mathbb{R} \) the inequality below holds:

\[
U'(x)(x - \bar{x}) \geq U'(x)(x - \bar{x})
\]

with equality only when \( x = \bar{x} \).

Multiplying (*) by \( \sigma f'(L)e^{-x^2/2} \) and integrating

\[
\int_{-\infty}^{\infty} e^{-x^2/2} U'(x)(x - \bar{x})dx > \int_{-\infty}^{\infty} e^{-x^2/2} U''(x)(x - \bar{x})dx = \int_{-\infty}^{\infty} e^{-x^2/2} U'(x)((\sigmax + \mu)f'(L) - W(1 + r))dx = 0
\]

where the quality to zero is given by the first order condition. Dividing through by \( \sigma f'(L)U'(\bar{x}) \) gives:

\[
\int_{-\infty}^{\infty} e^{-x^2/2}(x - \bar{x})dx > 0 \]

which completes the proof.

**Corollary:** The definition of \( \bar{x} \) implies that:

\[
\int_{-\infty}^{\infty} e^{-x^2/2}(x - \bar{x})U'(x)dx = 0
\]

**Lemma 2:** Define \( Z = \lim_{x \to +\infty} e^{-x^2/2}(x - \bar{x})U''(x) = \lim_{x \to +\infty} e^{-x^2/2}(x - \bar{x})U'(x) \). Then \( Z \geq 0 \)

**Proof:** Since \( \lim_{x \to -\infty} e^{-x^2/2} = \lim_{x \to +\infty} e^{-x^2/2} \), \( Z \) can be rewritten as

\[
Z = \lim_{x \to +\infty} e^{-x^2/2}\left[\lim_{x \to +\infty} (x - \bar{x})U''(x) - \lim_{x \to +\infty} (x - \bar{x})U'(x)\right]
\]

The same argument used in the proof of Lemma 1 can be repeated in order to find that for any \( x_1 \) and \( x_2 \) such that \( x_1 < \bar{x} < x_2 \). The following inequality holds:

\[
(x_1 - \bar{x})U''(x_1) < 0 < (x_2 - \bar{x})U''(x_2)
\]

letting \( x_1 \to -\infty \) and \( x_2 \to +\infty \) the proof is completed.

**Lemma 3:** The following inequality holds:

\[
\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U'(x)dx > 0
\]

**Proof:** Since \( x = (x - \bar{x}) + \bar{x} \), the integral above is equal to:

\[
\int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U'(x)dx + \bar{x} \int_{-\infty}^{+\infty} e^{-x^2/2}(x - \bar{x})U'(x)dx
\]

The first integral above is positive and the second one is zero according to the corollary. This completes the proof.
Lemma 4: \( I(\mu) > 0 \)

**Proof:** Noting that
\[
\frac{d}{dx} [U'((x)][(ax + \mu)f'(L) - W(1 + r))] =
= \sigma[U''((x)]f(L)[(ax + \mu)f'(L) - W(1 + r)] + U'((x)]f'(L)]
\]
one can write:
\[
I(\mu) = \int \frac{1}{\sigma} e^{-x^2/2} \frac{d}{dx} \left[ U'((x)][(ax + \mu)f'(L) - W(1 + r)]\right] dx
\]
Integration by parts gives:
\[
I(\mu) = \int \frac{1}{\sigma} e^{-x^2/2} U''((x)] + \int e^{-x^2/2} x U'((x)] dx
\]
where \([..] = (ax + \mu)f'(L) - W(1 + r) = \sigma f''(L)(x - \bar{x}).\)

From lemmas 2 and 3, it follows that \( I(\mu) > 0. \)

Lemma 5: \( I(W) < 0 \)

**Proof:** \( I(W) = -(1 + r) \int \frac{1}{\sigma} e^{-x^2/2} \left[ U''((x)]L\sigma f'(L)(x - \bar{x}) + U'((x)]\right] dx \)

Integrating by parts the first integral one finds:
\[
\int e^{-x^2/2} (x - \bar{x}) U''((x)] dx = \frac{1}{\sigma f'(L)} \int e^{-x^2/2} (x - \bar{x}) \frac{d}{dx} U'((x)] dx =
\]
\[
\frac{1}{\sigma f'(L)} \left[ e^{-x^2/2} (x - \bar{x}) U'((x)] \right]^{+\infty}_{-\infty} - \int e^{-x^2/2} \left[ 1 - x(x - \bar{x}) \right] U''((x)] dx
\]
therefore,
\[
I(W) = -(1 + r) \left[ \frac{f''(L)}{f'(L)} \right] \int e^{-x^2/2} U'((x)] dx
\]
\[
+ \left( \frac{f(L)}{L f'(L)} - 1 \right) \int e^{-x^2/2} x U'(x)] dx
\]
The strict concavity of \( f, f(0) = 0 \) and the Mean Value Theorem assures the existence of \( \tilde{L}, 0 < \tilde{L} < L \) such that
\[
\frac{f(L)}{L} - \frac{f(0)}{L - 0} = f'(\tilde{L}) \geq f'(L)
\]
therefore \( f(L)/ L f'(L) \geq 1 \). Thus, from Lemmas 2 and 3 it follows that \( I(W) < 0. \)

Lemma 6: \( I(\sigma) < 0 \)

**Proof:** \( I(\sigma) = f'(L) \int \frac{1}{\sigma} e^{-x^2/2} x U''((x)] f(L) \sigma(x - \bar{x}) + U'((x)] dx = \)
Integrating by parts the second integral:

\[ I(\sigma) = f'(L)\left[ \int_{-\infty}^{\infty} e^{-x^2/2}(x - \bar{x})^2 U''[(x)] dx + \int_{-\infty}^{\infty} e^{-x^2/2} x U'[(x)] dx \right] + \int_{-\infty}^{\infty} e^{-x^2/2} x U'[(x)] dx \]

where the inequality is assured by Lemmas 1, 2 and 3 and the corollary.

**Lemma 7:** \( I(A) > 0 \)

**Proof:** Define \( g: \mathbb{R} \to \mathbb{R} \), \( g(x) > 0 \), \( g'(x) < 0 \) such that \( g(x) = -\frac{U''[(x)]}{U''[(x)]} \)

then, the mathematical expression of \( I(A) \) is given by:

\[ -I(A) = (1 + r) \int_{-\infty}^{\infty} e^{-x^2/2} U''[(x)] (x - \bar{x}) g(\bar{x}) dx \]

Since \( g' < 0 \), the same argument used in Lemma 1 can be used to conclude that for any \( x \in \mathbb{R} \), \((x - \bar{x}) g(x) \leq (x - \bar{x}) g(\bar{x})\), where the equality holds only if \( x = \bar{x} \).

Therefore:

\[ -I(A) < (1 + r) \int_{-\infty}^{\infty} e^{-x^2/2} (x - \bar{x}) U'[(x)] dx = 0 \]

where the equality to zero is assured by the corollary.

**Lemma 8:** If \( A - WL < 0 \) then \( I(r) < 0 \)

**Proof:** \( I(r) = \frac{A - WL}{1 + r} I(A) - W \int_{-\infty}^{\infty} e^{-x^2/2} U'[(x)] dx < 0 \)

where the inequality is assured by Lemma 8.

**Proposition II**

**Proof of (a):** From Lemma 1:

\[ \mu f''(L) - W(1 + r) = -\sigma \bar{x} f''(L) > 0 \]

**Proof of (b):** In order for maximization (1) to have a positive solution \( L^D > 0 \) it is necessary that the expected period 2 amount of goods be higher with \( L^D > 0 \) than with \( L^D = 0 \). That is:

\[ \mu f(L^D) + (1 + r)(A - WL) > (1 + r)A \]

which implies inequality (b).
Proposition III

Proof: Apply the Implicit Function Theorem to the labour market equilibrium equation (3) and use the partial derivatives given by proposition I.

Proposition IV

Proof: The proof given below for $\sigma$ can easily be repeated for $\mu, \lambda$ and $r$:

$$\frac{d\theta}{d\sigma} = \frac{1}{L^D f(L^D)} \left[ L^D f(L^D) \frac{dW}{d\sigma} + W(f(L^D) - L^D f'(L^D)) \right] \frac{dL^D}{d\sigma}$$

From proposition III, $dL^D / d\sigma$ is given by:

$$\frac{dL^D}{d\sigma} = L_\sigma + L_w \frac{dW}{d\sigma} = L^D + L^D \frac{L^D}{\lambda L_w^S - L_w^S} = \frac{\lambda L_w^S L^D}{\lambda L_w^S - L_w^S} < 0$$

Since $dW / d\sigma < 0$ and from Lemma 5 $f(L^D) > L^D f'(L^D)$, it follows that $d\theta / d\sigma < 0$.

Proposition V

Proof: Apply the Implicit Function Theorem to the system of equations (4) and (5).

Proposition VI

Proof: Apply the Implicit Function Theorem to equations (4), (5) and (6).
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