SUSTAINED GROWTH, GOVERNMENT EXPENDITURE AND INFLATION

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Abstract: We construct and simulate a theoretical model in order to explain particular historical experiences in which inflation acceleration apparently helped to spur a period of economic growth. Government financed expenditures affect positively the productivity growth in this model so that the distortionary effect of inflation tax is compensated by the productive effect of public expenditures. We show that for some interval of money creation rates there is an equilibrium where money is valued and where steady state physical capital grows with inflation. It is also shown that zero inflation and growth maximization are never the optimal policies.

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1 Introduction

During the fifties and sixties, inflation and growth was a recurrent theme in growth theory as shown in the influential articles by Tobin (1965) and Mundell (1965). It was also part of the political discussion and in Latin America it was perhaps the most important economic topic and source of controversy between "monetarists" and "structuralists". This debate took place not only in the economic journals but also in the newspapers and among political parties. However, in much of the recent literature of growth this phenomenon is ignored, as most economists believe that, in general, distortion, and specially inflation, hurt rather then promote economic growth. The purpose of this paper is to develop a theoretical model of sustained growth to study these facts.

In models like Lucas (1988) and Azariadis and Drazen (1990) sustained growth is achieved through investment in human capital. By its replacement of physical labor in the production function, and by advocating a linear function for human capital investment, there will be no fixed factor in these models. As human capital grows it raises the marginal productivity of physical capital, stimulating firms to increase investment. The final result (assuming enough homotheticity) is a constant rate of growth in income and physical capital.

In our model we use this basic framework but we change some important features. In these models the equation of human capital accumulation is linear in its current stock and grows with the fraction of total human capital devoted to training and learning. The investment decision and the path of human capital depend thus on the decision of the individuals as to how much training they are willing to undergo. As everything else in the model depends on the path of human capital, the dynamics of these economies is completely determined by the way individuals decide to allocate their time.

This hypothesis ignores the fact that in most countries education is provided, at least up to the high school, by the government and in general is mandatory. Moreover it also does not deal with the fact that the effective productivity of labor does not depend only on the amount of investment in education. It is also related, among other factors, to investments in health and infrastructure, which do not depend on individual choices. Finally, it ignores problems of credit rationing, due to moral hazard factors, that individuals may face when deciding how much to invest in education. We postulate in our article that the quality of labor in a given economy depends on government expenditures so that labor productivity varies with increases in public investments. We dot not provide the individuals with an investment function in education. By adopting

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1 See Kaldor (1974) and the comments following his article for a good presentation of this debate and the position of both groups in it.
the opposite view of what is generally accepted, we hope to stress that differences in
growth rates in per capita income among countries may be explained by government
intervention in the form of investment in education, health, sanitation, transportation, etc.

The second departure from the literature is that we will assume that the
government finances its expenditures through seignorage. In doing so we are thinking of
particular economic experiences in which government involvement was essential for the
promotion of growth but also prompted an inflationary process.

The idea is not that the government resorts to inflation taxes rather than other
taxes because it is less distortionary (which is probably not true), but because it is the
easiest and least costly way to overcome restrictions and rigidities in the tax structure.
What we mean is that once the (political) decision for the government intervention in the
economy through investment in education and infrastructure is taken, it is necessary to
establish how to finance these expenditures. More often than not the fiscal system is
obsolete and too weak for this task - the tax base is too small, there is widespread tax
evasion and, what maybe the most important reason, the underground economy is
disproportionate large - and it would take a long time to modernize it. Moreover,
political pressures from different groups concerning which of them will pay for the
increase in taxes can postpone a fiscal reform for years and even threaten the survival of
the government. Given that even today there are a large number of countries without an
independent central bank so that monetary policy will be conducted, directly or
indirectly, by the central government it is easy to see that inflationary finance is the
fastest way for the government to achieve its aims and avoid the problems associated
with broad tax reforms.

We could of course resort to a more complete set-up where the government
could finance its investments through taxes, bonds and money creation. For instance, in
a model close to Alessina and Drazen (1989) seignorage would be used until the cost
distribution of the new tax system is decided among different players (or until the cost of
being taxed through inflation is lower than the alternatives). Another possibility is to
suppose that, even if the cost distribution is settled quickly, there would be a time gap
between the decision to tax and the actual taxation, because of the time needed to build
the new tax system. In this case, an impatient government would use money expansion
while the new system is not working, or even after if the government expenditures grow
at a fast rate. A third alternative is to suppose the existence of a cost to tax which would
increase with the amount taxed. In this case a certain amount of inflation would always
exist in equilibrium, and one would expect that the higher the public expenditures and its
requirements for funds, the higher the use of money expansion and thus a higher inflation
rate.
In all these cases our model would be part of a more complete framework, but we would still get seignorage as an important form of government finance. We chose, however, to ignore these more complex set-ups. We think that inflationary financing of public investment is an important topic by itself and the introduction of more variables would add unnecessary complications to the treatment of the subject. What we would gain in realism, would be more than offset by the loss in clarity and emphasis. Furthermore, it would not address the previously mentioned limits that large underground economies put on tax collection.

Moreover, this is not an unfamiliar idea for economists around the world. In Latin America, for instance, from the fifties to the seventies a large number of countries experienced high rates of growth sustained in part by government investments. In many of these cases - Brazil and Argentina, for instance - prices rose uninterruptedly during the period at rates greater than ten per cent per year. Let us take as an example the tenure of Juscelino Kubitscheck as president of Brazil.

President Kubitscheck took office in 1955 and set an ambitious plan for fast growth. There would be not only tax breaks to attract international capital but also a huge amount of public investment in infrastructure, specially in transportation and energy (as well as the construction of the new capital, Brasilia). There were no broad tax or financial reforms. The existence of an antiquated usury law limiting nominal interest rate to a bound that was, in most years, below the expected inflation rate restricted the borrowing ability of the government. Nonetheless the projects were completed to a point were the government participation in capital formation grew from 41% in 1950 to 45% in the earlier sixties, while the GNP grew at rates between six and seven per cent, reaching 7.7% in 1961. There is no question among Brazilian economists that much of the government expenditures were financed through money creation. There was no central bank in Brazil at this time and the agency in charge of monetary policy was a department of the same government-owned bank, Banco do Brasil, which financed part of public investments and programs. The commercial branch of Banco do Brasil was a de facto second money authority with unchecked liberty to expand money and credit. The rate of money expansion jumped from 19.1% in 1955 to 44.7% in 1961 and inflation followed this trend: it was 16.4% in 1955 and went to 45% in 1961.

In summary, we think there are plenty of historical examples and theoretical arguments that can be used to justify this model. Moreover, unlike the already cited papers by Tobin and Mundell, as well as articles by Weiss (1980) and Summers (1985), in our model government expenditures cause not only inflation and growth but inflation and sustained growth, because they introduce an externality by raising the quality of labor services. This is not a linear effect and we will see in the paper that too much inflation can be detrimental to the economy, as the flight from money can destroy the
inflation tax base. In other words, money can affect the rate of growth of real variables while the inflation rate is not too high (in a sense that we will explain later), after this point we only have nominal effects over real variables, and the economy halts at some level of per capita income.

Another important point is that in our model we show that even employing a distortionary tax like inflation, it may be the case that government expenditures, within limits, can improve the welfare of the economy because of the spillover effect of the public investment on education and production. We prove in the paper that, unless government expenditures cannot influence productivity at all as in the conventional models, it is always optimal to introduce some inflation in the economy.

2 The Model

Consider an overlapping generations economy with no population growth. Each generation is composed of a large number of individuals who live for two periods, except the first generation that only lives for one period. In the first period of their lives, "youth", the individuals are endowed with one unit of labor. We will assume that they do not value leisure, so that they supply their labor inelastically. When young the individuals work, receive a wage, consume the only good of this economy and save. In the second and last period of their lives, "old" people do not work, but consume the proceedings of their savings.

There are two different assets in the economy competing for the savings of the young generation. One is fiat money issued by the government and the other is a capital asset issued by the firms. Money may or may not be valued in equilibrium. In this case individuals will hold only capital in their portfolios (and the equilibrium will be called non-monetary). The capital and money levels in the first period (time zero) are given by history.

The consumers' utility function follows standard assumptions: it is concave, increasing and (at least) twice differentiable in all of its components. We will also assume that it is homothetic. The problem of the consumer is to maximize his utility by choosing the saving level as well as its distribution between capital and money. The budget constraint of a young person born at time t is given by:

\[ C_t^f = W_t - S_t \]

where \( W_t \) are the wages and \( S_t \) is the total saving.
Letting $R_{t+1}$ be the gross return on capital and $\Pi_t$ the gross return on money (the inverse of inflation factor), the consumer's budget constraint when he is old is:

\[(2)\quad C_{t+1}^0 = [S_t - m_t]R_{t+1} + m_t\Pi_t\]

Applying (1) in (2), we have the lifetime budget constraint, so that the consumer's problem can be expressed as:

\[(3)\quad \max_{c_t^y, c_{t+1}^0} U(C_t^y, C_{t+1}^0)\]

\[(4)\quad \text{s.t. } C_{t+1}^0 + R_{t+1}C_t^y + [\Pi_t - R_{t+1}]m_t = w_tR_{t+1}\]

It is easily seen that the solution of this problem is given by:

\[(5)\quad U_1(C_t^y, C_{t+1}^0) = U_2(C_t^y, C_{t+1}^0)R_{t+1}\]

\[(6)\quad R_{t+1} \geq \Pi_t, \quad \text{if } m_t > 0\]

Expression (5) equates the marginal rate of substitution with the marginal rate of transformation between consumption when young and capital. It says that the loss in utility by giving up one unit of consumption when young has to be equal, in equilibrium, to the gross yield of capital (in utility terms) in the next period. Equation (6) is a no-arbitrage condition expressing the fact that in equilibrium the returns on the assets must be the same in order for consumers to be willing to hold both of them in their portfolios, otherwise it will hold only capital.

As we have already commented in the introduction we did not introduce taxes in this economy. As we stated, this is not necessary. Although we are interested in this paper in analyzing the effects of money creation and inflation on the rate of growth of capital and income, we also could introduce tax (and loans) as a source of government revenue. This would not change the behavior of this economy as long as we introduced at the same time an upper bound on tax collection - or any of the alternatives we already discussed - such that if the government wanted to spend more than this bound it would be less costly to use money creation. This would mean that, only for small levels of public expenditures seignorage would not be necessary, a fact not unfamiliar for most of Latin
America countries or countries with high inflation experiences For the time being we opt to ignore taxes as their introduction would not affect in any significant way our results.

Government expenditures perform a very particular task in this economy: they enhance the productivity of labor. The idea is that by investing in public education, health services, sanitation, transport system and so on, the government can increase the quality of the labor force. As work services are measured in efficiency units in the model, public investment increases the flow of labor services per unit of time. In particular, calling $L_t$ the flow of efficiency units of labor of a worker born at time $t$, we assume that

$$L_{t+1} = \lambda(g_t)L_t$$

The function $\lambda(g_t)$ is the government expenditure function. It transforms each unit of public investment in infra-structure, by a relative increase of $L_{t+1}/L_t$ in labor productivity. We will assume that $\lambda$ is positive, increasing and differentiable. $\lambda(0)$ is assumed to be one so that if the public infra-structure remains the same the labor productivity does not change. We also take $\lambda$ as bounded so that when government expenditures are very large the marginal gain in labor productivity is close to zero. However, we do not assume that $\lambda$ is everywhere concave.

In this economy, the government budget constraint is $P_tG_t = M_t - M_{t-1}$, where $P_t$ is the price level at time $t$, $G_t$ is real government expenditures and $M_t$ is nominal money holdings at time $t$. If we suppose a constant and preannounced rate of money creation ($\mu$), we get $M_t = (1 + \mu)M_{t-1}$, which implies

$$g_t \left( \frac{\mu}{1 + \mu} \right) m_t$$

where $m_t$ and $g_t$ are real money holdings and real government expenditures per efficient unit of labor, respectively.

The production side of the economy is represented by competitive firms with a technology that is homogeneous of degree one in capital and efficiency labor. They use capital, that fully depreciate with use, and labor services to produce a homogeneous good. Both inputs are rented at market prices. These hypotheses imply that output per efficiency unit of labor, $y_t$, depends only on the capital-labor (in efficiency units) ratio $k_t$:

$$y_t = f(k_t)$$
We will endow the production function \( f \) with the standard assumptions:

\[
(10) \quad f(0) = 0, f' > 0, f'' < 0, f \text{ concave}
\]

From the solution of the problem of the firms, given the homogeneity and competitiveness assumptions, we can derive both the labor and the capital demand schedules:

\[
(11) \quad r_t = f'(k_t) \\
(12) \quad w_t = f(k_t) - k_t \cdot f'(k_t)
\]

where \( w_t \) in this model is wage rate per efficient unit of labor.

In this economy the labor market is always in equilibrium, as we assumed that workers supply inelastically their services. So, there are two markets that need to be checked for equilibrium, the goods market and the asset market. By Walras' law we need only to examine the equilibrium in one of them to determine the equilibrium conditions for the whole economy. We will study the asset market. It will clear when the total demand for assets, represented by the total savings of the young, equates the total supply. The supply of assets is composed by the stock of money held by the old plus money created in this period and the debt issued by firms to finance capital investment.

The saving function is derived from the first order conditions of the consumer problem, more precisely from equation (5). Using the implicit function theorem we can write it as:

\[
S = S(w_t, r_{t+1})
\]

We will assume that consumption in the first and second period of life are both normal goods, so that savings are increasing in wages. We are able now to write the equilibrium condition in the asset market:

\[
S(w_t, r_{t+1}) = K_{t+1} + M_t / P_t,
\]

which can be rewritten, after some manipulations, as

\[
(13) \quad s(w_t, r_{t+1}) = \lambda(g_t)k_{t+1} + m_t
\]
An equilibrium in this economy is a capital sequence \( \{k_t\} \) and a money holdings sequence \( \{m_t\} \) such that, for a given initial \( k_0 \), in every period the dynamic system given by equations (6), (8), (11), (12) and (13) is satisfied.

This system can be reduced to two equations. First, note that the return to fiat money, which is the inverse of the inflation rate \( (P_t/P_{t-1})_t \), is equal to \( \lambda(k_t)/(1+\mu)[(m_{t+1})/m_t] \) in this economy. Applying this expression in equation (6) and then, applying (8), (11) and (12) in the remaining two, we get the following two-dimensional first order dynamical system:

\[
\begin{align*}
(14') \quad f'(k_{t+1}) &= \frac{\lambda \left( \frac{\mu - m_t}{1 + \mu} \right) m_{t+1}}{m_t} \\
(15') \quad s(w(k_t), R(k_{t+1})) &= \lambda \left( \frac{-\mu}{1 + \mu} m_t \right) k_{t+1} + m_t
\end{align*}
\]

In what follows we will only study stationary equilibria for this economy, as there can be several capital and money sequences satisfying these two equations. However, it is worth noting that steady states in this economy are not constant sequences, but a balanced growth path in which capital, consumption and income grow at the same rate \( \lambda(g) \).

Given the homotheticity property of the utility function\(^2\), the above dynamical system becomes:

\[
\begin{align*}
(14') \quad f'(k_{t+1}) &= \frac{\lambda \left( \frac{\mu - m_t}{1 + \mu} \right) m_{t+1}}{m_t} \\
(15') \quad s(w(k_t), R(k_{t+1})) &= \lambda \left( \frac{-\mu}{1 + \mu} m_t \right) k_{t+1} + m_t
\end{align*}
\]

For a given money growth rate there are three types of stationary equilibrium satisfying the above system. The first one is a trivial one given by \( (k_t, m_t) = (0,0) \): with zero capital, there is no production and so no savings and money demand in any period. The

\(^2\) It is shown in a previous version of this article that this hypothesis does not change the dynamic behavior of the monetary steady state, the possible numbers of stationary equilibria and the general structure of the dynamic system.
second type of steady state is such that individuals hold only capital in their portfolio, and is given by \((k_t, m_t) = (k_d, 0)\), where \(k_d > 0\) solves \(sw(k_t) = k_t\), for all \(t\).

The third type of stationary equilibrium is a monetary equilibrium in which both assets are held by the individuals: \((k_t, m_t) = (k^*, m^*)\), such that \(k^*\) and \(m^*\) solve, for all periods

\[
m^* = sw(k^*) - \lambda \left( \frac{\mu}{1 + \mu} m^* \right) k^*
\]

(16)

\[
f' \left( \frac{sw(k^*) - m^*}{\lambda \left( \frac{\mu}{1 + \mu} m^* \right)} \right) = -\frac{\lambda \left( \frac{\mu}{1 + \mu} m^* \right)}{1 + \mu}
\]

(17)

Equation (16) represents the locus in the economy where \(k\) is constant while equation (17) is the locus where money balances per efficiency units of labor is constant.

Note that for the first two types of equilibrium the no-arbitrage condition holds as an inequality, it does not bind. With zero money holdings the above planar system turns into the standard Diamond (1965) one-sector-growth-model with only one asset. It is clear thus that these two types of steady states correspond to the steady states of the Diamond model. In what follows we will call \((k_t, m_t) = (k_d, 0)\) the "Diamond equilibrium".

Under mild and standard assumptions equation sixteen can be represented by a parabola that grows initially with money, reaches a single peak and then falls, cutting the capital axis at zero and \(k_d\). If we rewrite it as

\[
\frac{m_t}{k_t} = \frac{sw(k_t)}{k_t} - \lambda \left( \frac{\mu}{1 + \mu} m_t \right),
\]

(18)

we can see that as capital goes to infinity the limit of the money-capital ratio goes to minus \(\lambda(g)\) that is negative number\(^3\). On the other hand when capital approaches zero there can be no production and savings, so that \(m_t\) also approaches zero, and we have

\(^3\) Note that
In order for this limit to be positive we have to assume that the limit of the saving-capital ratio is greater than one close to the origin. This is not a strong assumption. In fact it is the same one needed for the existence of a stationary equilibrium in the standard one sector OLG model.

Finally, given the assumption of monotonicity for the function $\lambda$ and of normality of consumption (which imply a monotone saving function), the condition for a maximum in this curve\(^4\), $sw'(k) = \lambda$, is met only once - for a given $m$ - at some capital level $k_m$.

For equation 17 we will need slightly stronger assumptions. It is shown in appendix A that for $\mu$ not large enough ( $\mu$ such that $1/\mu > \lambda' k [\sigma/(1-\alpha) - 1]$ - 1, where $\sigma$ is the elasticity of substitution between capital and labor inputs and $\alpha$ is the capital share of output) this curve has a positive slope. Also, it will cross the capital line at $k_\mu$ defined by

$$f'(sw(k_\mu)) = \frac{1}{1+\mu}$$

while its intercept with the money axis will be negative. This can be easily seen because when capital is zero, the money phase line becomes

$$f\left(\frac{-m}{\lambda\left(\frac{\mu}{1+\mu}\right)}\right) = \frac{\lambda\left(\frac{\mu}{1+\mu}\right)}{1+\mu}$$

that is equal to zero by L'Hôpital rule.

\(^4\) If we rewrite equation (17) as $m_t = G(k_t, \mu)$, it is easy to see that for a given $\mu$ the maximum is reached at a $(m_\mu, k_\mu)$ point such that

$$G_k = \frac{d m_t}{d k_t} = \frac{-\lambda'}{1+\lambda' k \mu} = 0 \Leftrightarrow sw' = \lambda$$

we can also see that to the left of $k_\mu$, $sw' > \lambda$ (for small $k$ and $m$, $\lambda$ is close to zero while $sw'$ is large) so that $G_k$ is positive. On the other hand, to the right of $k_\mu$, $G_k$ is negative as $sw'$ decreases to zero as $k$ increases, while $\lambda$ is bounded away from zero.
Note that the derivative of the production function is only defined in $\mathbb{R}^+$, so that $m$ has to be negative for $\{-m/\lambda\}$ be positive. Putting the two curves together we have, for $\mu$ not very large, the following diagram:

**Figure 1:**
The Money-Capital System

Figure 1 above also displays the phase diagram of the system. By visual inspection, the three steady states of the system, $(k_L,m_L) = (0,0)$, $(k_d,0)$ and $(k^*,m^*)$, seen to be a source point, an attractor and a saddle point, respectively. In appendix B we study the dynamic properties of the system composed by equations 16-17 and prove that this is in fact the case under mild assumptions.
3 The Monetary Equilibrium

In this section we prove the existence of a monetary equilibrium and we study the effect of the rate of money creation on the steady state levels of capital and money holdings.

**Proposition 1**: There exists a monetary equilibrium for $\mu$ sufficiently small. Furthermore, for $\mu$ such that $1/\mu > \{ \lambda' \ k \ [ \sigma/(1-\alpha) - 1 ] - 1 \}$ there can also exist a monetary steady state as long as $k_\mu$ is to the left of $k = k_d$ which solves $sw(k_d) = k_d$.

**Proof**: For $\mu = 0$ the dynamical system given by equations 16-17 becomes

$$k_{t+1} = sw(k_t) - m_t,$$
$$m_{t+1} = f(k_{t+1})m_t. \tag{16'} \tag{17'}$$

This is the standard OLG model with national debt, bubbles or fiat money that is studied by Azariadis (1990), Tirole (1986) and Wallace (1975). We know from these authors that a monetary steady state exists and is a saddle point.

Now, we can use a continuity argument to prove existence of equilibrium for $\mu$ different from zero. As $\mu$ increases away from zero the money phase line will start to shift down and to the right while its slope will eventually become negative for large $\mu$. First, note that by expression (19) above, as $\mu$ increases, the capital intercept of the money phase line also increases. And by equation (20) the intercept on the money axis eventually shifts down. On the other hand the slope of this curve, given by expression (19), will be positive for $\mu$ not very large ($\mu$ such that $1/\mu > \{ \lambda' \ k \ [ \sigma/(1-\alpha) - 1 ] - 1 \}$) while for intermediate values of $\mu$, $k$ and $m$ it can even be negative (for large $\mu$, given the boundness of $\lambda$, the slope will be $sw'$ again). But the important point is that the slope does not jump but changes smoothly.

On the other hand, the intercepts of equation 16 at the capital axis will not change as $m$ departs from zero, because as $m_t$ is zero $g_t$ does not change with $\mu$. However the level of money holdings will be lower for a given capital stock, and the parabola shift down keeping the same intercepts: for any $m_t$ positive, given that $\lambda(g_t) > \lambda(0) = 0$, for $g_t > 0$, the curve given by $m_t = sw - \lambda(g_t)k_t$ is always below $m_t = sw - k_t$.

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Note that for large $\mu$ equation (20) becomes $f'(-m/\lambda(m)) = \lambda(m)/(1 + \square) = 0$, which implies that $\mu$ goes to minus infinity. For $\mu$ close to zero, given the assumption that $\lambda'$ is not large at $g_t = 0$, we can expect that $\lambda(m)$ increases less than $(1+\mu)$ initially. This implies that $f'$ also decrease, so that $-m$ increases and $m$ shifts down.
In other words, as \( \mu \) increases from zero there will be no jumps in either curve. This fact implies that as the economy departs from a situation depicted by equations (16')-(17') to the more general model of equations (16)-(17), there still exists a positive intercept between both phase lines, and the economy will have (at least) one stationary monetary equilibrium \( (k^*(\mu), m^*(\mu)) \) for each \( \mu \). For large money growth rates (\( \mu \) such that \( 1/\mu < \{ \lambda'k \left[ \sigma/(1-\alpha) - 1 \right] - 1 \} \)), the slope of equation (17) becomes negative. It may also be the case that for \( \mu \) large enough, equation (17) shifts "too much" to the right, so that \( k_\mu \) is to the right of \( k_d \). In both cases there will be no intersection between both curves in the positive quadrant.

The fact that for large \( \mu \) there will be no monetary equilibrium makes economic sense: as the amount of money creation becomes very large, as does the rate of inflation, individuals will flee from money to alternative assets to avoid losses.

The same reasoning allows us to do a graphic and heuristic analysis of the possible scenarios of the effects of changes in \( \mu \) over capital and money demand. Suppose the economy is initially at the point \( E_0 \) in the figure 2 below:

As \( \mu \) rises, both curves shift down. The economy goes to successively lower levels of money demand and higher levels of capital: \( E_1, E_2, \) etc. For a certain value of the money growth rate, the money phase line will become negatively sloped, or it will be still positively sloped but it will cross the capital axis at the right of \( k_d \). At this point there will
be no monetary equilibrium and the economy will stay forever at \((kd, 0)\). Here the steady state is no longer a balanced growth path but a constant value: because of the flight from money the government cannot use inflationary financing to improve labor productivity\(^6\).

If \(\mu = 0\) the monetary equilibrium is at a point to the left of the maximum of the curve given by equation (16) it is possible that for some levels of money creation both asset demands increase. Here, for a "well behaved" economy (where movements in both curves are approximately of the same size), money creation by the government has a Laffer-effect in money demand and a positive effect on the rate of growth of capital up to a level of \(k = kd\). Through money and inflation taxes the government can finance the investments needed to raise labor productivity and consequently to increase the growth rate of the income and capital. However, as inflation reaches higher levels, the government capacity to stimulate the real side of the economy declines, as the (inflation) tax base shrinks.

This graphic reasoning is supported by analytical results that we present as a proposition below:

**Proposition 2**: The level of capital per efficiency units of labor in the steady state grows with the rate of money creation, for \(\mu\) sufficiently small (and very large). Real money balances grow (fall) with \(\mu\) if \(sw' > \lambda\) (\(sw' < \lambda\)) and \(\mu\) being either very small or very large.

**Proof**: see appendix C

In the appendix it is shown that the derivative of \(m\) with respect to \(\mu\), in the steady state, for small values of the money growth rate is

\[
\frac{dm}{d\mu} = -\frac{\lambda}{(1 + \mu)^{2f''}}[sw' - \lambda],
\]

(21a)

while for large \(\mu\) it is given by

\[
\frac{dm}{d\mu} = -\frac{1}{f''}[sw' - \lambda].
\]

(21b)

In both cases the sign of the derivative will depend upon whether the monetary equilibrium is to the left or to the right of the peak of the curve given by equation (17). To

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\(^6\)This is only one of the possible scenarios, although for small and large values of the parameter \(\mu\) this will always be the case. We do not know the relative size of the shifts of both curves, unless we make some very unreasonable assumptions. If the shifts in the phase line of capital are, for some levels of \(\mu\), much larger than the ones in the money phase line, it may be the case that both the demand for money and for capital will decrease for some time.
the left, \( sw' \) is greater than the function \( \lambda(.) \), so that money demand grows with \( \mu \). On the other hand, to the right of the peak of the parabola, \( \lambda(.) \) is larger than \( sw' \) so that individuals are willing to hold less money for larger rates of money creation by the government. However, for intermediate values anything can happen, a result that coincides with what we just saw through graphical analysis.

The interesting result here is that it is possible to have inflation and money demand growing at the same time in the case where \( sw' \) is greater than \( \lambda \). This is apparently a non-intuitive result but can be easily explained because government expenditures are financed by inflation tax, and \( g_t \) has a spillover effect on the variables of the model, so that when it increases (within certain bounds) national income increases. For this interval, where saving is still growing fast (\( sw' > \lambda \)), this income effect on the demand for money is stronger than the substitution effect caused by the decrease in the return on money holdings.

On the other hand, the derivative of the steady state capital with respect to \( \mu \) is positive for both large and small rates of money creation. In both cases we can reduce this derivative to

\[
\frac{dk}{d\mu} = \frac{\lambda}{(1 + \mu)^2}.
\]  

(22)

that is always positive. Here money creation by the government is not only non-neutral but can raise the growth rate of the whole economy. This is the so-called Tobin effect: as inflation increases, the return on money declines boosting the demand for the alternative asset, capital. However, in concordance with the graphical analysis, for intermediate values of \( \mu \), the stock of capital can go to any direction and even be crowded out by government expenditures. For this reason we cannot say that in this model inflationary finance has everywhere a positive and monotone effect on the growth rates of capital and product. It might be the case that for large intervals the inflation rate capital falls with increases in \( \mu \). In the simulations we ran in section 5, however, capital always grows with inflation.

4 Optimal Monetary Policy

In this section we study the optimal monetary policy for a government that wants to maximize the utility of its subjects and take their actions as given. In this model, money creation has two opposite effects on the well-being of consumers. The first one is that, creating inflation, the government distorts the optimal allocation of the economy. This is a general "by-product" of inflation finance, and it only exists here because the government
cannot use lump sum taxes in this economy. The second effect is that money creation finances public investment, and thus increases the growth rate of output and consumption and therefore improves consumer utility. We would expect that the best policy for the government has the fine tuning property of maximizing growth while introducing a minimum distortion in the economy. We will see below that this may not be the case.

The government goal is to choose the \( \mu \) that maximizes the flow of utility from consumption in the economy. First we will use the well established fact that along the balanced path all the variables grow at a common rate, \( \lambda \). In this case, we can rewrite the consumption of young and old at time \( t \) as \( \lambda^t c_1 \) and \( \lambda^t c_2 \), respectively. The government thus solves

\[
\max_{\mu} \sum_{t=0}^{\infty} \beta^t U(w - S, RS)
\]

where we took \( \lambda^t \) outside the utility function using its homotheticity property. If we assume that the term \( (\beta \lambda)^t \) is always between zero and one (i.e., \( \lambda \) is bounded by a number smaller than \( 1/\beta \)), we can rewrite the government problem as

\[
\max_{\mu} \frac{1}{1 - \beta \lambda} U(w - S, RS)
\]

We basically want to study the relationship between the optimal \( \mu^* \) which solves the above problem and the money growth rate that maximizes growth (and seignorage). We would expect that a government which does not discount the future very heavily would equalize both rates, maximizing growth while not creating much distortion. The following proposition says that for a large number of economies this maybe not the case.

**Proposition 3:** For economies where \( (s + s / S) > \alpha \) (respectively \( < \alpha \), \( \alpha \) being the capital share of output, and \( s \) the saving rate, a government which wants to maximize the welfare of the present and all future generations, should set \( \mu \) at a smaller (respectively, higher) level than the one which maximizes growth and it should operate on the upward (respectively, downward) sloped side of the Laffer curve.

Proof: The first order condition for the government problem is:

\[
\frac{1}{1 - \beta \lambda} U_1 \left( \left[ \frac{\partial w}{\partial \mu} - \frac{\partial S}{\partial \mu} \right] \frac{\partial \lambda}{\partial \mu} + U_2 \left( \left[ \frac{\partial R}{\partial \lambda} S - \frac{\partial R}{\partial \mu} \right] \frac{\partial \mu}{\partial \mu} \right) \right) + \frac{\beta}{(1 - \beta \lambda)^2} \left( \frac{m}{(1 + \mu)^2} + \frac{\mu \partial m}{1 + \mu \partial \mu} \right) U = 0
\]

Using the envelope theorem and collecting terms we can rewrite this expression as
Introducing into the above equation the expressions for the derivative of wages and interest rate with respect to capital, we have

\[ \frac{U_1}{\partial k} \frac{\partial w}{\partial k} + U_2 \frac{\partial R}{\partial k} S \frac{\partial k}{\partial \mu} + \frac{\beta}{(1 - \beta \lambda)(1 + \mu)} \left( \frac{m}{(1 + \mu)} + \mu \frac{\partial m}{\partial \mu} \right) U = 0 \]

Introducing into the above equation the expressions for the derivative of wages and interest rate with respect to capital, we have

\[ (U_2 S - U_1 k)^n \frac{\partial k}{\partial \mu} + \frac{\beta}{(1 - \beta \lambda)(1 + \mu)} \left( \frac{m}{(1 + \mu)} + \mu \frac{\partial m}{\partial \mu} \right) U = 0 \]

Using the fact that \( U_1 = RU_2 \), we obtain

\[ (S - Rk)U_2 f^n \frac{\partial k}{\partial \mu} + \frac{\beta}{(1 - \beta \lambda)(1 + \mu)} \left( \frac{m}{(1 + \mu)} + \mu \frac{\partial m}{\partial \mu} \right) U = 0 \]

Finally, introducing in the left most bracket the expression for savings and interest rate, we have

\[ (s - (1 + s) f' k)U_2 f^n \frac{\partial k}{\partial \mu} + \frac{\beta}{(1 - \beta \lambda)(1 + \mu)} \left( \frac{m}{(1 + \mu)} + \mu \frac{\partial m}{\partial \mu} \right) U = 0 \]

(23)

Given the concavity assumption of the production function the first expression on the left is negative for \((s/(1+s)) > \alpha\). This implies that in order for equation (23) to be zero, the term in brackets in the second expression has to be positive. But this term is the expression for the slope of the Laffer curve, so that the optimal money growth rate is to the left of the maximum seignorage and the government is not maximizing revenues from money creation. Given that in this model maximization of seignorage coincides with maximization of growth, it follows that the optimal policy is not maximization of growth.

Remark 1: Note that in this model \( s \) is actually a propensity to save for old age out of wage income and does not correspond to the observed saving rate of the economy. If this was the case the only relevant scenarios would be when \( s/(1+s) \) is smaller than \( \alpha \), because the observed saving rate is around 0.2 and \( \alpha \) is around one quarter or one third for most economies. The optimal policy would be always excessively distortionary. In this model, however, given the definition of \( s \), \( \alpha < s/(1+s) \) looks more plausible, because it implies \( S \) larger than \( Rk \).

The above result has interesting implications. The first one is that in a model where government expenditures are not "wasted" or lump sum transferred but directly affect the productivity growth of the economy, proposition 3 says that the best policy implies always some inflation and that it is usually optimal to introduce some distortion. Note that if \( \lambda' \) is always equal to zero (government expenditures does not affect labor productivity) equation (23) cannot vanish, and the best policy is to set \( \mu^* \) equal to zero. This is the
standard result: in a world where money introduces only distortion, the best thing for the government to do is to avoid any inflation. For all other cases some inflation is always optimal because the benefits of money creation over accumulation are larger than the distortion costs, up to a certain level. In this way we can explain endemic inflation cases, because when a government is unable to enlarge tax collection (for social or political reasons), the importance for the economy of government expenses and its external effect over capital, force the authorities to resort to money creation.

The second implication is that, under the more plausible scenario \( s/(1+s) > \alpha \), it is never optimal for the authorities to maximize growth and they should keep inflation below the level that maximizes seignorage. As the rate of money creation rises, the increase in seignorage becomes progressively small, and so do the welfare gains from economic growth. At a certain point, the loss due to the distortionary effect of inflation tax overcome the gains from growth. In section five, we obtain this result in our simulations for a large number of combinations of parameters. For instance, in the case where \( \alpha \) is one quarter and \( s \) equals 0.65, the optimal inflation rate is 10 points below the maximand of growth when the discount rate is 0.9. If, however, the opposite case ever holds and the ratio \( s/(1+s) \) is smaller than \( \alpha \), we have an intriguing result: a government which wants to maximize the welfare of the individuals should always introduce excessive distortion in the economy. the rate of growth, however, will never be maximized.

5 An Example

In this section we parameterize the main functions of the model in order to simulate it. We do not claim that the results we obtain with this example are a definite confirmation of what we proved in the model, they are illustrations. However, as we will see below, the excellent fit of these simulations with respect to the analytical results are without a doubt an indication that at least for this stylized economy, our model not only displays a positive relationship between inflation and growth but also reproduces all the analytical results we previously obtained.

We will assume that the production function is Cobb-Douglas given by \( f(k) = k^\alpha \), while the utility function is given by \( U(c_1,c_2) = \beta \ln c_1 + (1-\beta) \ln c_2 \), where \( \alpha \) and \( \beta \) are positive real numbers between zero and one. The government expenditure function \( \lambda \) will be of the form \( \lambda(g_t) = 2 - \exp(-g_t/\phi) \), where \( \phi \) is a real number greater than one. This last function has two properties we assumed in the text, it is bounded above (by 2) and has a small derivative at zero if \( \phi \) is large enough. The fact that it is not bounded below is of little importance as we will not work with values of government expenditures below zero.
With these functions the model at the steady state (equations (16)-(17) in section II) becomes:

\[ m = (1 - \alpha)(1 - \beta)k^\alpha \left[ 2 - \exp\left( -\frac{\mu m}{1 + \mu \phi} \right) \right]^k, \]

(24)

\[ k = \frac{\left( 2 - \exp\left( -\frac{\mu m}{1 + \mu \phi} \right) \right)^{\frac{1}{\alpha - 1}}}{(1 + \mu \alpha)} \]

(25)

We will not investigate the behavior of the model outside the steady-state nor try to find the saddle-path for different parameters. Although these exercises would be of some importance, the technicalities involved are too complicated for what is supposed to be a simple illustration of the model. In this sense we will only investigate what occurs with the monetary steady state, for different combinations of the parameters \( \alpha, \beta \) and \( \phi \), when the government changes the rate of money creation.

Figure 3 below shows, for different values of the rate of money creation, the behavior of equilibrium levels of capital, money holdings and savings, together with the inflation rate\(^7\). For this particular figure we used a capital share of one quarter, \( \beta \) equal 0.35 and \( \phi \) equal to 10.

**Figure 3**

Steady State Equilibria Levels

Although the corresponding values are different, the behavior of money and capital vis-a-vis the inflation rate for a wide range of parameters coincides with the one displayed

---

\(^7\) Note that in this model the inflation rate at the steady state is given by \(((1+\mu)\lambda(g) - 1)\), which is equal to \(\mu\) only at \(g = 0\). However, it follows \(\mu\) very close.
in the above figure. For $\mu$ equal or close to zero the agents hold similar quantities of capital and money on their portfolios. For successively higher rates of inflation, the steady state level of money decreases until it reaches zero while the capital per efficiency unit increases until the economy reaches the Diamond steady state. The positive correlation between inflation and capital reproduces the so called Tobin effect. Overall, savings are higher for higher rates of inflation.

These are quite satisfactory results. They say that through an inflationary financing scheme the government can stimulate the economy and that higher levels of steady state inflation correspond to higher levels of capital growth rates. At higher stationary inflation rates, money demand is very small or null. Thus, there is a bound on the ability of the government to use inflation to finance growth. These outcomes very closely reproduce those obtained in parts 2 and 3 of the paper.

Tables I and II below present the steady state level of money and capital, respectively, for different values of money growth rates, using six combinations of parameters: $\alpha$ equal to one third and one quarter, and $\beta$ equal to 0.5, 0.45 and 0.35. For all combinations we set $\phi$ equal to 10, as changes in this parameter did not significantly affect the results.
The steady state value of money holdings falls monotonically with \( \mu \) for all the combinations of parameters. However the level at which it will reach zero depends crucially on the value of the \( \alpha \) and \( \beta \) coefficients. The smaller their values, the higher the inflation rates for which monetary steady state will exist. For the lowest combination (capital share of one quarter and \( \beta \) equal 0.35) there will be a monetary steady state for rates of money creation up to 0.95, which corresponds roughly to a inflation rate of 94% per period in the model. On the other hand, for \( \alpha \) and \( \beta \) equal to one third and one half, respectively, there is no monetary steady state for \( \mu \) larger than 0.02. This is not unexpected because the higher the \( \beta \) the higher the importance for the individual utility function of the consumption in the first period of his life, and so, the lower the propensity to save. Furthermore, as we can see from Table II below, capital increases with \( \alpha \) so that the participation of money in the total savings, everything being the same, decreases with this parameter.

### Table I

#### Steady State Money Stock Levels

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \alpha = 1/4 )</th>
<th>( \alpha = 1/3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \beta = 0.35 )</td>
<td>( \beta = 0.45 )</td>
</tr>
<tr>
<td>0</td>
<td>0.1496</td>
<td>0.102</td>
</tr>
<tr>
<td>0.01</td>
<td>0.1485</td>
<td>0.101</td>
</tr>
<tr>
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<td>0.1475</td>
<td>0.1</td>
</tr>
<tr>
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<td>0.1463</td>
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</tr>
<tr>
<td>0.04</td>
<td>0.1452</td>
<td>0.097</td>
</tr>
<tr>
<td>0.11</td>
<td>0.1369</td>
<td>0.088</td>
</tr>
<tr>
<td>0.12</td>
<td>0.1357</td>
<td>0.087</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1294</td>
<td>0.075</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1117</td>
<td>0.06</td>
</tr>
<tr>
<td>0.32</td>
<td>0.1087</td>
<td>0.057</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0968</td>
<td>0.044</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0811</td>
<td>0.027</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0644</td>
<td>0.009</td>
</tr>
<tr>
<td>0.64</td>
<td>0.0575</td>
<td>0.001</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0469</td>
<td></td>
</tr>
<tr>
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<td>0.0097</td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>1E-07</td>
<td></td>
</tr>
</tbody>
</table>
**TABLE II:**

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \alpha = 1/4 )</th>
<th>( \alpha = 1/3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( \beta = 0.35 )</td>
<td>( \beta = 0.45 )</td>
</tr>
<tr>
<td>0.0</td>
<td>0.1575</td>
<td>0.1575</td>
</tr>
<tr>
<td>0.01</td>
<td>0.1596</td>
<td>0.1596</td>
</tr>
<tr>
<td>0.02</td>
<td>0.1617</td>
<td>0.1616</td>
</tr>
<tr>
<td>0.03</td>
<td>0.1637</td>
<td>0.1638</td>
</tr>
<tr>
<td>0.04</td>
<td>0.1658</td>
<td>0.1658</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1785</td>
<td>0.1787</td>
</tr>
<tr>
<td>0.11</td>
<td>0.1807</td>
<td>0.1808</td>
</tr>
<tr>
<td>0.12</td>
<td>0.1828</td>
<td>0.1829</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2227</td>
<td>0.2230</td>
</tr>
<tr>
<td>0.32</td>
<td>0.2273</td>
<td>0.2276</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2458</td>
<td>0.2463</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2694</td>
<td>0.2701</td>
</tr>
<tr>
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<td>0.2938</td>
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</tr>
<tr>
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<td>0.3046</td>
</tr>
<tr>
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<td>0.3046</td>
</tr>
<tr>
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<td>0.3046</td>
</tr>
<tr>
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<td>0.3704</td>
<td>0.3046</td>
</tr>
<tr>
<td>0.95</td>
<td>0.3837</td>
<td>0.3046</td>
</tr>
</tbody>
</table>

The behavior of capital is, in certain sense, a mirror image of the behavior of money: its level on the steady state is higher for higher values of the coefficients \( \alpha \) and \( \beta \), at a given \( \mu \). It increases monotonically with \( \mu \) for given capital share and \( \beta \). The later fact, as we have already commented, reproduces the Tobin effect. Another result easily observed from table II is that the effect of \( \alpha \) is much more strong and dominant over the value of capital in the steady state than variations in \( \beta \). This is not entirely unexpected because, from equations (27) and (28) above, we can see that \( \beta \) affects capital only indirectly through its effects over money holdings, while \( \alpha \) is a parameter of the production function.

Additionally, due to the fact that as \( \alpha \) and \( \beta \) fall there will exist monetary steady states for increasingly higher inflation rates, the Diamond steady state will have higher levels of capital per capita the smaller are these coefficients: it is 0.386 for \( \alpha \) equal to 0.25 and \( \beta \) equal to 0.35, but only 0.197 for \( \alpha \) equal to one third and beta equal to one half.

The outcome of one last exercise is shown in figure 5 below, where we plot the steady state levels of the utility\(^8\) of one generation as well as growth rates against rates of money creation for \( \alpha \) equal to one quarter and \( \beta \) equal to 0.35.

---

\(^8\) For a matter of scale we subtracted 0.15 from the utility level, otherwise the utility line would be out of scale. Utility* means utility minus 0.15.
The important result of this exercise is that the inflation rate which maximizes the utility of the consumer is lower than that which maximizes growth (0.33 and 0.5, respectively). So, if the government wants to maximize the welfare of the present generation it should, consequently, operate on the left side of the Laffer curve, below the maximum revenue it can get from money creation, and he should not maximize economic growth. This result is consistent with proposition 3, as in this case $s/(1+s)$ equals 0.4, which is greater than the assumed capital share of output. Also, it holds for a large number of combinations of $\beta$ and $\alpha$, only that when $s/(1+s)$ approaches $\alpha$ the optimal $\mu$ and the $\mu$ which maximizes growth get closer and closer. They are, for instance, respectively 0.09 and 0.1 when $\alpha$ is 0.2 and $\beta$ is 0.72, which corresponds to a $s/(1+s)$ ratio of 0.21.

For discount rates other than zero, this result has to be qualified. It is still true, unless the present government gives equal weight to all present and future generations, that the optimal money growth rate will not be superior to the one that maximizes growth. However, as the discount rate increases, the optimal rate approaches 0.5. It is still 0.33 for discount rates equal to one half and it goes to 0.37, 0.44, 0.49 for discount rates equal to 0.8, 0.9, and 0.99 respectively. The intuition behind this is that if the government cares about the future generations it will be willing to accept more distortion and inflation in order to achieve higher rates of growth with the larger expenditures he will be able to finance.
7 Conclusions

We have developed in this paper an overlapping generations model where government expenses positively affects the growth rate of human capital and consequently the rate of growth in productivity. Endemic inflation is thus explained because of, in one hand, the inability of the government to enlarge the tax base (which we did not model here) and, in the other hand, the essential role that infrastructure and education, mostly public financed, play in the economy. The lack of alternative financial sources forces the government to use money to pay for its projects, which are instrumental for economic growth.

We proved that monetary equilibria exist in this economy for large intervals of parameters and we also showed that steady state capital (per efficiency units of labor) increases with increases in the rate of money creation. However, for large enough rates, the authorities cannot affect real variables and all we are left with are nominal effects. We also proved that for reasonable values of parameters, the optimal rate of money creation is never above the maximum seignorage rate. Those results are also displayed by the simulations we run in the end of the paper.

There are some extensions of the present framework that we think are worth pursuing. The first and more immediate one is to include taxes in the model. Although there are plenty of historical experiences to justify the present model, such additions would add more realism to it. A second extension would be to endow agents with a more sophisticated human capital investment decision, in such a way that human capital growth would depend not only on public investment but also on the number of hours individuals spend on training. This would give individuals a more important role in the determination of the rate of growth of the economy.
Appendix A

In this appendix we show the necessary conditions for equation (17) in section 2 to be positively sloped and to cut equation (16) to the left of $k_d$. Equation (17) is given by

$$
\frac{f'(sw(k*) - m*)}{\lambda(\frac{\mu}{1+\mu} - m*)} = \frac{\lambda(\frac{\mu}{1+\mu} - m*)}{1 + \mu}
$$

First let us rewrite it taking $m_t$ as an implicit function of $k_t$ and $\mu$, in which we will take $\mu$ as a given parameter:

$$
m_t = H(kt, \mu)
$$

To obtain the slope of $H$, we implicitly differentiate $m_t$ with respect to $k_t$, which gives us, after some simplification and substitution

$$
h_k = \frac{dm}{dk} = \frac{sw'}{1 + \lambda'k \frac{\mu}{1+\mu} \frac{1-\alpha}{1-\alpha}}
$$

(A1)

where $\sigma$ is the elasticity of substitution. We can easily verify that when the parameter $\mu$ is zero the slope of this curve is equal to $sw'$ for any value of $\sigma/(1-\alpha)$. This slope will be still positive (as it will always be the case for production functions with small elasticity of substitution) for values of $\mu$ close to zero or not large enough$^9$.

For large values of $\mu$, the denominator becomes $\{1 + \lambda'k[1 - (\sigma/(1-\alpha))]\}$. Clearly it can be negative if the expression $\lambda'k[.]$ is greater than one in absolute value as, for many production functions, the Cobb-Douglas among them, the expression in brackets is negative. However, for small and large values of $m_t$ and $k_t$ we can suppose without problems that the numerator is positive. For large values, as $\lambda$ is a bounded function we only have to imagine that $\lambda'$ approaches zero at a faster pace than that the capital goes to infinity. On the other hand, for small values of $m_t$ and $k_t$, we only need that the derivative $\lambda'(0)$ be bounded. $^{10}$ In both cases the denominator is reduced to one and the slope of the money phase line is again $sw'$.

$^9$ For "not large enough" I mean $\mu$ such that $1/\mu > \lambda'k[\sigma/(1-\alpha) - 1] - 1$. Note that for values of $\sigma/(1-\alpha)$ lower than one this inequality always holds.

$^{10}$ This assumption makes economic sense. It says that when the government starts its investments, the marginal gain in labor productivity is not unbounded. This maybe be the case because there are some scale problems with respect to the size of schools, hospitals, etc. However, in order to avoid jumps in the $\lambda$ function we assume a slow but smooth increase in the labor productivity for small values of $g_t$. 
For intermediate values of the money growth rate, everything depends on the value of the ratio of the elasticity of substitution of the labor share of production. Again, it can be the case that the value of the slope is negative everywhere except for very large and for very small values of capital and money holdings, if this ratio is greater than one. On the other hand, it will cross the capital line (if it crosses at all) at \( k\mu \) defined by

\[
(A2)\quad f'(sw(k\mu)) = \frac{1}{1 + \mu}
\]

while its intercept with the money axis will be negative. This can be easily seen because when capital is zero, the money phase line becomes

\[
(A3)\quad f'\left(\frac{-m}{\lambda} \right) = \frac{\lambda \left( \frac{\mu}{1 + \mu} m \right)}{1 + \mu}
\]

Note that the derivative of the production function is only defined in \( \mathbb{R}^+ \), so that \( m \) has to be negative for \( \{-m/\lambda\} \) be positive.

Appendix B

This section deals with the dynamic properties of the system composed of equations (16)-(17). Graph IV below depicts the phase diagram of this system. We will briefly comment on the determination of the arrows of motion of the system. Let us first study the capital phase line, equation 16.

If \( k_{t+1} > k_t \), then we have the following:

\[
m_t = sw(k_t) - \lambda(g_t)k_{t+1} < sw(k_t) - \lambda(g_t)k_t
\]

so that capital is increasing below the phase line.

For the money phase line, suppose that \( m_{t+1} > m_t \), so that

\[
f'(k_{t+1}) = \frac{\lambda \left( \frac{\mu}{1 + \mu} m_t \right) m_{t+1}}{1 + \mu} > f' \left( \frac{sw(k_t) - m_t}{\lambda \left( \frac{\mu}{1 + \mu} m_t \right)} \right) = \frac{\lambda \left( \frac{\mu}{1 + \mu} m_t \right)}{1 + \mu}
\]

which implies that
This result, in turn, implies that the arrows of motion are upward to the left of the money phase line.

By visual inspection, the tree steady states of the system, \((k_1, m_1) = (0, 0), (k_d, m_d) = (0, 0)\) and \((k^*, m^*)\), seen to be a source point, an attractor point and a saddle point, respectively. In what follows, while examining the Jacobian of the linearized system, we will see that this is case under mild assumptions.

For ease in computing the stability conditions it will be useful to rewrite the system as

\[
\begin{align*}
\lambda &\frac{\mu}{1 + \mu} m, \\
sw(k, m) - m, \\
\lambda &= \lambda, \\
\end{align*}
\]

We can now state the following proposition:

**Proposition 4:** If \((sw'/\lambda) < 1\), the monetary steady state is saddle path so that this equilibrium is determinate. Otherwise, a sufficient condition for a saddle is given by expression (B3) below. The trivial steady state is source point, while the Diamond equilibrium is an attractor if the sufficient condition \(sw'f < (1 + \mu)\) holds.

Proof: The Jacobian of the system (B1)-(B2) at the steady state is given by

\[
J = \begin{bmatrix}
h_k \\
1 + \frac{\mu}{\lambda} mf''h_k \\
1 + \frac{\mu}{\lambda} mf''h_m + f\left(1 - \frac{\lambda'}{\lambda} \frac{m\mu}{1 + \mu}\right)
\end{bmatrix}
\]

where \(h_k = (sw')/\lambda \geq 0\)

\(h_m = -(1/\lambda) [1 + (\lambda'k\mu/(1 + \mu))] \leq 0\)

We can see immediately that the trivial steady state \((k, m) = (0, 0)\) is a source point. The trace of the Jacobian evaluated at this point reduces to \(sw' + (1 + \mu)f\).
which is positive and greater than one for any \( \mu \). The determinant becomes 
\[ sw'(1+\mu)f' \] 
which is also positive and greater than one. We can conclude that both eigenvalues are positive (it is also easily checked that they are real). Thus, we need only to check the characteristic polynomial at one. It is given by:

\[ P(1) = (1 - sw')(1 - f(1+\mu)) \]

As \( P(1) \) is positive and the determinant is greater than one, both eigenvalues have to be greater than one, so that the trivial steady state is in fact a source.

For the Diamond steady state \((k,d,0)\), both the trace and the determinant have the same expression as the trivial case only now the \( f \) and \( sw' \) evaluated at \( kd \) are relatively small. The trace and the determinant are still positive, but the determinant may be less than one depending on the relative magnitude of \( fsw' \) vis-a-vis \( 1+\mu \). If this is the case, the Diamond equilibrium is an attractor point: the characteristic polynomial evaluated at one for \((kd,0)\) is positive, which, together with a determinant between zero and one, implies that both eigenvalues are positive and smaller than one.

For the monetary steady state \((k,m) = (k^*,m^*)\) the Jacobian of the system can be rewritten as

\[ J = \begin{bmatrix} h_k & h_m \\ m'f'h_k & 1 - \eta + m\frac{f'}{f}h_m \end{bmatrix} \]

where \( \eta \) is the elasticity of the government expenditure function with respect to \( g_t \).

It is easily checked that the eigenvalues are always real. The characteristic polynomial evaluated at one and minus one is, respectively, after collecting terms

\[ p(-1) = (2-\eta)(1+(sw'/\lambda)) + m.h_m(f'/f), \]
\[ p(1) = \eta(1-(sw'/\lambda)) - m.h_m(f'/f). \]

We know that the condition for a stationary equilibrium to be a saddle point is that the value of these characteristic polynomials have opposite signs. The first expression is positive if we assume \( \eta \) smaller than \( \) or close to \( \) two. The second polynomial is always negative if \((k^*,m^*)\) are at the upward part of equation \((B1)\) where \( (sw'/\lambda) \) is greater than one. Otherwise a sufficient condition for negativity is

\[ 1 - \frac{sw'}{\lambda} < \frac{1 - \frac{\eta + \frac{1+\mu}{\mu} \cdot 1}{\sigma}}{\frac{1+\mu}{k\lambda}} \]

\((B3)\)

We see no problem with condition \((B3)\) except perhaps for large values of \( k \). In this case, the ratio \( sw'/\lambda \) is very small. But when capital is very large, money demand is close to zero in this economy, so that the derivative \( \lambda' \) is also near zero. This fact would imply that \( k\lambda' \) is very small if \( \lambda' \) goes faster to zero, so that the condition would hold. For a small stock, of capital the ratio \( sw'/\lambda \) is greater than one so that the left hand side of the inequality is negative, while the right hand side is always positive.
The condition that the elasticity of government expenditures be less than two, which is only sufficient, is not a strong one. It places a reasonable bound on the magnitude of the relative response of labor productivity to increases in the government investments in infrastructure. In summary, if condition B3 (together with a \( \eta \) not much bigger than two) holds, the monetary stationary equilibrium is a saddle point.

Appendix C

In this section we will study the derivative of the steady state money demand and capital with respect to the rate of money creation. We want to prove that for \( \mu \) small and large these derivatives can be reduced to equations (21) and (22), respectively, of section II. To simplify calculations and notation we will rewrite the system as

\[
\begin{align*}
    m_t + \lambda \left( \frac{\mu}{1 + \mu} m_t \right) k_t - sw(k_t) &= 0 \\
    m_t + \lambda \left( \frac{\mu}{1 + \mu} m_t \right) \Omega(k_t, \mu) - sw(k_t) &= 0
\end{align*}
\]

(C1)

(C2)

Where the function \( \Omega(k_t, \mu) \) solves for \( k_{t+1} \) implicit as the solution of

\[
    f'(k_{t+1}) = \frac{\lambda \left( \frac{\mu}{1 + \mu} m_t \right)}{1 + \mu}
\]

Note that this time we construct the system plugging equation (14) into equation (15) to define the system in the steady state. In the paper we worked in the reversed order. This does not change the results, but only simplify things.

The derivative of money demand with respect to \( \mu \) is given by

\[
    \frac{dm}{d\mu} = \frac{-\Omega_s [sw' - \lambda] + \frac{\lambda'}{\Omega_m}{\Omega_m}}{\Omega_m [sw' - \lambda] - \left[ 1 + \frac{\lambda'}{\mu} \right] \frac{\Omega_m}{1 + \mu}},
\]

(C3)

where \( \Omega_m \) is given by \( \{(\lambda/\mu)/(1+\mu)^2\} \), which is non-positive, while \( \Omega_{\mu} \) is given by \( \{[\lambda/(1+\mu)^2] f''[\eta/\mu] - 1\} \), which can be either positive or negative, depending on \( \eta/\mu \) being smaller or greater than one.
As \( \mu \) becomes larger it can be immediately seen, given previous hypothesis, that \( \Omega_{m} \) approach zero because \( \lambda' \) goes to zero and \( \mu/(1+\mu)^2 \) becomes small. On the other hand, \( \lambda'\Omega \) also approaches zero, as \( \Omega \) is bounded by \( kd \), and the denominator goes to minus one. This all results that \( dm/d\mu \) is reduced to

\[
\frac{dm}{d\mu} = -\Omega_{\mu} [sw'-\lambda]
\]

This expression can be reduced immediately to equation (21a).

For \( \mu \) small we have that \( \Omega_{m} \) goes to zero, because \( \lambda' \mu \) goes to zero, while the second term in the numerator reduces to minus one, again because of the behavior of \( \lambda'\mu \). Given that \( \eta(0) \) is zero, \( \Omega_{\mu} \) is reduced to \((-\lambda/(1+\mu)f')\). Collecting terms we get

\[
\frac{dm}{d\mu} = - \frac{\lambda}{(1+\mu)f''} [sw'-\lambda],
\]

which is equation (21b).

Differentiating the system (C1)-(C2) with respect to capital we get

\[
\frac{dk}{d\mu} = \frac{-\Omega_{m} [1 + \lambda' \mu \Omega] + \lambda' \Psi m \Omega \Omega_{m}}{\Omega_{m} [sw'-\lambda] [1 + \lambda' \mu \Omega]} - \frac{1}{1 + \mu}
\]

Once again there is no definite sign for this expression, unless we make some unreasonable assumptions. However is not difficult to define its sign for limit rates of money supply.

For \( \mu \) close to zero, the second term in the numerator goes to zero (as both \( \Omega_{m} \) and \( \lambda' \) are zero), while the denominator reduces to minus one, as we have seen in the previous case. The whole derivative reduces to \( \Omega_{\mu} \) that approaches equation (22) close to zero. For large \( \mu \), we already know that the denominator goes to minus one. As for the numerator, the second term goes to zero as both \( \Omega_{m} \), \( \lambda' \) and \( (1/(1+\mu)^2) \) go to zero, while \( \lambda \) is bounded. The second term reduces again to \( \Omega_{\mu} \), and we complete the proof that the derivative of capital stock in the steady state can be reduced to equation 22 for both small and large rates of money creation.

Note that we did not need any further assumptions to get these results. We only used assumptions we made in the body of the paper.
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