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VICTOR BLUHU DA ANNUNCIAÇÃO

MONEY MINING: TECHNOLOGY DIFFUSION AND TRANSACTIONS VALIDATION

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Dissertação apresentada à Escola de Economia de São Paulo como pré-requisito à obtenção de título de mestre em Economia de Empresas.

Orientador: Bráz Ministério de Camargo.

Coorientador: Luis Fernando Oliveira de Araújo.

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Resumo

Nós desenvolvemos um modelo a la Lagos-Wright em tempo contínuo para estudar a produção privada e decentralizada de dinheiro por meio de uma tecnologia de mineração consumidora de tempo. Incluímos ainda um processo de difusão para essa tecnologia. Existe um único equilíbrio com *perfect-foresight* em que a economia atinge um estado estacionário monetário e existe um contínuo de equilíbrios com *perfect-foresight* indexados pelo valor inicial do dinheiro e em que o dinheiro desvaloriza gradualmente mesmo sem ter períodos de apreciação. Em oposição a resultados anteriores na literatura, o dinheiro privado é utilizado para transações ao longo do caminho de equilíbrio. Estes resultados ainda se mantêm quando os agentes escolhem um portfolio com um *fiat money* competidor ofertado pelo governo. Adicionalmente, demonstramos que a política monetária pode tornar desinteressante a atividade mineradora. Por fim, no atrelamos transações com uma estrutura de validação e criamos um contínuo de equilíbrios com *boom and bust* por meio de um imposto inflacionário no portador de dinheiro.

Palavras-chave: Bitcoin, Criptomoeda, Moeda Digital, Mineração, Dinâmica de Preço, Validação de Transações

Abstract

We develop a continuous-time Lagos-Wright model to study private and decentralised money production with a time-consuming mining technology and with a diffusion process for this technology. There exists a unique equilibrium where the value of money reaches a monetary steady state and a continuum of perfect-foresight equilibria indexed by the starting value of the currency where the price of money vanishes gradually even without appreciation periods. In opposition to previous results in the literature, private money is used for transactions along the equilibrium path. These results still hold when agents choose a portfolio with a competing government fiat money. We additionally demonstrate that the monetary policy can prevent mining activity. Finally we entangle transactions and their validation and create a continuum of boom and bust equilibria via inflationary taxation on mined money holders.

Keywords: Bitcoin, Cryptocurrency, Digital Currency, Mining, Price Dynamics, Validation

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1 Introduction

As of December 19th 2019, there are 4,950 cryptocurrencies with total market capitalisation of U$190bi being tracked by CoinMarketCap, a leading provider on cryptoassets data.[1,2] While there is a long going discussion on whether these assets could or should be classified as currencies or not, the fact is that currently many of those assets play a role as a medium of exchange and not only as a speculative asset. The novelty of these cryptocurrencies is that none of them were introduced, backed, produced or distributed by any sovereign state as the predominant currencies were and, therefore, are arguably the purest form of fiat money ever known. It is only natural then to investigate typical money-related questions such as why people value and carry any of those assets, how can their initial and long-run prices be determined and when or why will their bubbles burst. Yet, there is more to cryptocurrencies than that and we focus on building a model that tackles two relevant dimensions: the mining technology diffusion and the validation of transactions.

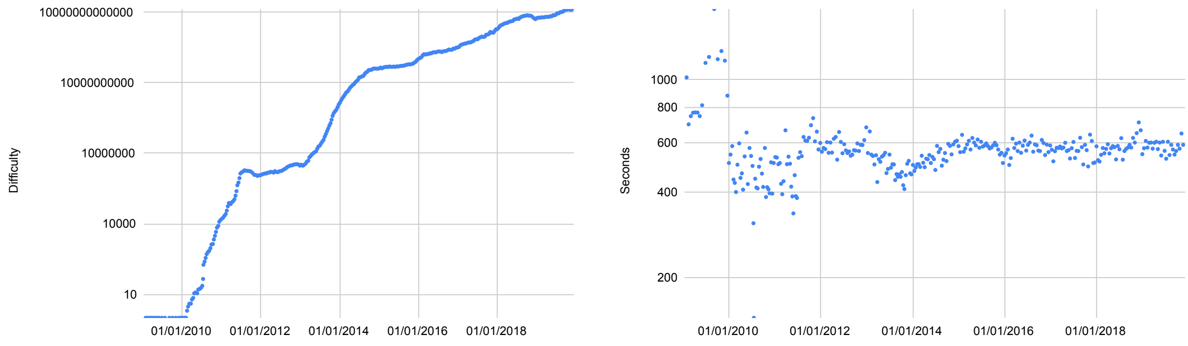
While (fiat) money is and old, known and widely used technology, cryptocurrencies are not. Bitcoin's introduction in January 2009 was not associated with immediate and widespread interest and usage, but it currently corresponds to 68.3% of cryptocurrencies' market capitalisation and to 31.3% of their total daily volume. From the supply side, there is evidence that the amount of resources and agents dedicated to entering new blocks in the blockchain has increased through time. Figure 1 depicts how the difficulty per block in Bitcoin's blockchain has evolved through time in order to maintain the average time per block around 10min.[3] Figure 2 depicts how the revenue-computational power ratio has declined over time. These factors suggest that the entrance of new players, i.e., mining technology diffusion might have played a significant role in the bitcoin mining sector. If this is true, can that have affected the Bitcoin's market capitalisation or its appreciation rate? Can we say anything about the welfare consequences of having people/resources being diverted to the production of payment instruments?

We build on these thoughts to create a continuous-time version of the Lagos & Wright (2005) model in Choi & Rocheteau (2019a), Choi & Rocheteau (2019b) fashion.

[1] CoinMarketCap (2019) methodology for choosing the list of tracked cryptoassets is available in their website: https://coinmarketcap.com.

[2] This market capitalisation is equivalent to 5% of US M1.

[3] The process by which miners include blocks in the blockchain can be approximated by sequentially drawing a number from a uniform distribution from 0 to 1 until a number lower than a threshold $\varepsilon > 0$ is drawn. The difficulty is a measure of how low this threshold is and also of the probability of having success. The higher the computational power/the amount of miners, more draws can happen per unit of time. In order to make the average time per block constant, the difficulty and computational power must move in the same direction.



(a) Difficulty per Block (b) Average Time per Block

Figure 1: Historical Data for Bitcoin Mining Technology

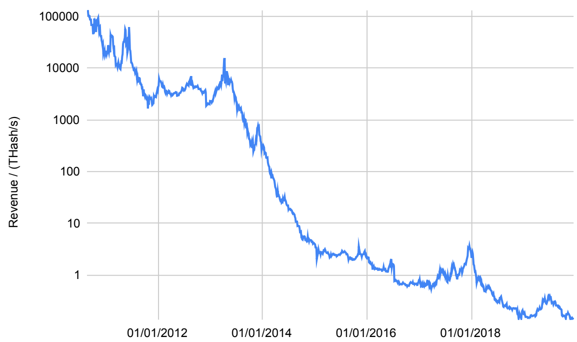


Figure 2: Return over Computational Power

There is a time-consuming mining technology that agents may possess and with which they may produce an asset that can be used as a medium of exchange. A key feature is that there is a maximum supply of the mineable asset and the technology exhibits diminishing returns over the stock of mined assets. A centralised market for this mined asset is always open and all agents are capable of adjusting the real balance they carry at every moment, except when they encounter other agents from whom they may want to purchase a differentiated good (as in Shi (1995) and Trejos & Wright (1995)). Agents with the mining technology will have an occupational choice: they either choose to mine the asset or choose to be available to produce the differentiated good for agents they might meet. The occupational choice endogenously determines the mass of producers and thus affects the marginal benefit of carrying real balances (via liquidity premium) and the equilibrium appreciation rate. In a world where all agents possess the technology (as in Choi & Rocheteau (2019a)), all agents engage in mining when its profitable to do so, no one produces, the liquidity premium collapses to zero (because no transactions will ever happen) and money must appreciate at the agents' impatience rate[4]. The diffusion process for the mining technology dampens the appreciation rate because it induces a diminishing lower bound on the mass of producers, thus reducing the equilibrium appreciation rate

[4] Otherwise the market for real balances would not clear.

(and possibly turning it negative). By doing so, we produce equilibrium trajectories that are more realistic, because transactions happen all the time, and more flexible and adherent to data, because money is allowed to appreciate at lower rates or even depreciate while agents mine.

We improve on the baseline model by allowing agents to hold a diverse portfolio and to make transactions with different assets. From the demand side there will be no arbitrage balancing each assets' real rate of return. From the miners side the opportunity cost goes up because agents can pay with more assets and trade in larger quantities. Even with the technology diffusion process, the model is flexible enough to incorporate different encounters a la Lester, Postlewaite & Wright (2012). As is traditional, a no arbitrage condition imposes that all assets carried in positive quantities must have a liquidity premium that is equal to the difference between the agents' discount rate and the real rate of return of the considered asset. We focus on the coexistence of cryptocurrency and fiat (government) money and show that the monetary authority can follow a policy that prevents agents from mining. If the monetary authority does not follow such a policy, then there will be a unique monetary equilibrium leading to a monetary steady-state and there will be infinite equilibria with boom phases and infinite without them; a cutoff value for the initial price will separate the equilibria.

Following this analysis, we embed a transaction validation structure to our model. We impose that any payment order from buyers to sellers of the differentiated good must be validated, i.e., registered in a public ledger by active miners. The rate with which transactions can be validated is increasing in the number of active miners. If the validation rate is lower than the rate with which encounters between buyers and sellers happen, only some transactions will be randomly selected to be validated and the other will be discarded. If few agents are mining, there is a low probability that transactions will be validated and increasing the mass of miners will raise the liquidity premium even in spite of reducing the mass of producers. If enough agents are mining, all transactions can be validated and increasing the mass of miners will reduce the liquidity premium because there will be fewer available producers. We are the first to entangle transactions and their validation in a new monetarist model with a maximum supply of the mineable asset. Previous attempts such as Kang (2019) involved departing from the maximum supply hypothesis, a distinctive feature of some dominant cryptocurrencies (e.g., Bitcoin). We are able to produce perfect-foresight equilibria with boom and bust phases and transactions in both phases.

As Choi & Rocheteau (2019b) points out, the continuous-time assumption has a few advantages over the usual discrete-time framework in the new monetarist literature. It allows markets to be concurrently open (which is arguably more realistic than a rigid timing structure), provides tractability via a system of differential equations and acts as an equilibrium refinement ruling out exotic dynamics (as in Oberfield & Trachter (2012)).

Cryptocurrencies price dynamics have been analysed in many different frameworks. A first branch of the literature comprises more traditional macroeconomic models that assume away some technological peculiarities cryptocurrencies usually share. Athey et al. (2016) develop a remittance model with a fixed cryptocurrency supply well suited to analyse adoption amongst potential users. Schilling & Uhlig (2018) use an Euler-equation approach in an stochastic endownment economy with two types of infinitely-lived agents that alternate periods in which they consume goods and produce bitcoins. The authors obtain a martingale for the bitcoin price process. Fernández-Villaverde & Sanches (2019) build a discrete-time new monetarist model with private money producers that only interact with the other agents by selling money in the centralised market. They analyse the introduction of private fiat money in the economy, but focus on the effects of private currencies competition on price stability and long run prospects.

The coexistence between cryptocurrencies and other (private or governmental) media of exchange are an important question for these models so they focus on the agents' portfolio choice whenever they study monetary policy. For instance, in Fernández-Villaverde & Sanches (2019) there is room for welfare improvement via government money, but any successful constant rate of return policy on government money drives private money out of the economy. In this aspect, our model also has room for a government pegging a positive rate of return on its money, but our framework is more flexible in the sense that private-money can coexist with government money in such a scenario and can even improve the economy's liquidity.

Another branch of the literature is one that embeds some blockchain technological aspects in the new monetarist workhorse. Chiu & Koeppl (2019) and Kang & Lee (2019) analyse the transaction settlement algorithm in bitcoin and estimate that there is a substantial welfare loss to avoid a double-spending problem.[5] [6] Kang (2019) analyses the double-spending problem and points that digital wallets reputation can substantially reduce and even eliminate the welfare loss without settlement delays.

Some papers outside the new monetarist framework that focus on miners services and are worth mentioning notably include Pagnotta (2018) and Pagnotta & Buraschi (2018) who relate Bitcoin price and mining services in decentralised networks. On the introduction of a new cryptocurrency one can also identify an Initial Coin Offering (ICO) literature with Li & Mann (2018) pointing how ICOs may solve coordination issues, and Sockin & Xiong (2018) that grounds cryptocurrencies value both as a membership token

[5]Chiu & Koeppl (2019) estimate that Bitcoin generates a welfare loss about 500 times as large as a monetary economy with 2% inflation. Optimal design of the incentive mechanism lowers the welfare loss to the equivalent of a monetary economy with "moderate inflation" of about 45%.

[6]Kang & Lee (2019) estimate that the coexistence of money and Bitcoins generate a welfare loss of 0.048% of consumption, in terms of a consumption-equivalent measure under an inflation rate of 2%.

for users and as a compensation for miners services. This literature can bring insights on equilibrium refinement for the multiple equilibria we typically find in new monetarist models.

The rest of the paper is organised as follows. Section 2 presents the baseline model with technology diffusion, the (perfect-foresight) equilibrium definition and the steady-state properties. Section 3 investigates the equilibria set and its characteristics. Section 4 extends the baseline model definitions and allow agents to carry multiple assets and make a richer portfolio decision. Section 5 introduces the validation structure and analyse the equilibria set with an inflationary tax compensating miners for their services. Section 6 concludes the paper.

2 Setup

There is a $[0,1]$-continuum of agents and at $t=0$ a new technology is discovered. The technology allows agents to dedicate time to produce a unit of a divisible and durable asset that can be recognised by other agents. This asset is a Lucas tree with dividend flow $d \geq 0$. We refer to this asset as money and we say the technology mines money. There is a maximum amount $\bar{A}$ of money that can be mined and an agent mines a unit of money according to a Poisson process with rate $\lambda(\bar{A}-A_t)$ where A_t is the mass of money mined up to period t.

All agents have a technology that allows them to produce instantaneously x units of a numéraire good that grants them a consumption utility x and a production disutility $-x$. Except for $M_0 \in (0,1)$ agents, all agents are given the mining technology according to a Poisson process with arrival rate δ. The mining technology has no depreciation, so an agent never loses it after acquiring it. Let "miner" denote an agent that has acquired the mining technology and let M_t denote the fraction of miners in period t. We will use m and n as superscript for miners' and non miners' functions and variables.

Agents are also endowed with a technology to produce instantaneously q units of a differentiated good at a cost q. Agents cannot consume the differentiated goods they produce. Agents meet randomly in pairwise meetings according to a Poisson process with rate α and all meetings are equally likely. With probability $\sigma \in (0,1)$, one agent likes the differentiated good the other agent produces. We also assume there is no room for barter. When an agent consumes q units of this differentiated good, he enjoys utility $u(q)$, in which u is increasing, strictly concave, $u'(0)=\infty$ and $u'(\infty)=0$.

The agents' lifetime expected discounted utility is given by

$$\mathfrak{U} = \mathrm{E}\left[\sum_{n=1}^{\infty} e^{-rT_n^b} u(q(T_n^b)) - \sum_{n=1}^{\infty} e^{-rT_n^s} q(T_n^s) + \int_0^{\infty} e^{-rt}\,\mathrm{d}X_t\right] \tag{2.1}$$

where the time indices $\{T_n^b, T_n^s\}_{n=1}^{\infty}$ in the first and second summations refer to the periods in which the agent meets someone and is, respectively, the buyer and the seller; r is the agents' discount factor and X_t is the measure of the cumulative net consumption of numeráire up to period t. X_t is restricted to the set of functions of bounded variation on any finite interval such that the Lebesgue-Stieltjes integral is defined. This cumulative net consumption function is allowed to have discrete jumps only in a countable set and we further restrict it to happen only at $\{T_n^b, T_n^s\}_{n=1}^{\infty}$.[7,8]

A centralised market for money and the numéraire is open in every period. Agents have full access to it except when they meet another agent. In this market agents decide how much money they want to carry with them. We denote by a^m and a^n the real balance the agents carry (in terms of the numéraire) and $\phi > 0$ is the price of money. We only consider ϕ_t such that $\frac{\dot{\phi}}{\phi}$ is continuous. Market clearing imposes that

$$MA^m + (1 - M)a^n = A. \tag{2.2}$$

Because agents can always rebalance their portfolio by buying or selling money/producing or consuming numéraire goods, we have that

$$V^i(a) = \max_{h \in [-a, \infty)} \{-h + V^i(a + h)\} = a + \max_{a^* \in [0, \infty)} \{-a^* + V^i(a^*)\} \tag{2.3}$$

for $i \in \{m, n\}$. The optimal portfolio is such that $V'(a^*) = 1$.

When two agents meet and one likes the differentiated good the other agent produces, they bargain over the terms of trade. We denote the traded amount by q and the price by p. The buyer and the seller enter the negotiation with real balances a^b, a^s and the surplus of the transaction is given by

$$\begin{aligned} S(a^b, a^s) &= [u(q) + V^i(a^b - p) - V^i(a^b)] + [V^j(a^s + p) - q - V^j(a^s)] \\ &= [u(q) - p] + [p - q]. \end{aligned}$$

We impose a Kalai bargain with θ as the buyer's bargain power, so $p = (1 - \theta)u(q) + \theta q \coloneqq w(q)$.[9] Therefore, the terms of trade only depend on the buyer's portfolio, $p = w(q)$ and the traded amount is such that $w(q) = \min\{w(q^*), a^b\}$, in which q^* is the efficient quantity ($u'(q^*) = 1$). Buyers have utility $\theta[u(q(a^b)) - q(a^b)]$ from transactions.

Let $\rho \coloneqq \frac{d + \dot{\phi}}{\phi}$, $m \in [0, M]$ be the fraction of miners dedicating their time to mine and $W(a) \coloneqq V^m(a) - V^n(a)$. We impose that an agent cannot produce the differentiated good when she meets another agent if she is dedicating her time to mine money. On the other hand, miners have no restriction on being on the buyer side. The optimal portfolio for miners and non miners maximises the agents utility flows by solving the following

[7] The latter restriction implies that any discrete jump will be associated with a pairwise meeting. This is a non-binding restriction but helps with the intuition that agents cash in/out frictionlessly after a transaction. Whenever a seller produces q units of differentiated good in a meeting, he will obtain disutility $-q < 0$ and will receive money that he will immediately exchange for the numeráire, so $X(t^+) - X(t^-) > 0$. The buyer will immediately replenish his real balance and will have $X(t^+) - X(t^-) < 0$.

[8] The probability distribution that agents use for their expected utility is consistent with the mining technology acquisition random process and their decision between mining and being available to sell differentiated goods.

[9] Kalai bargain is chosen in opposition to Nash bargain because it is monotonous in a and generates concave value functions in the pairwise meeting problem. Moreover, it renders a much more tractable model.

Hamilton-Jacobi-Bellman (HJB) equations

$$rV^m(a^m) = \alpha\sigma(1-m)\theta[u(q(a^m)) - q(a^m)] + \max\{\alpha\sigma(1-\theta)[M[u(q(\bar{a}^m)) - q(\bar{a}^m)] + (1-M)[u(q(\bar{a}^n)) - q(\bar{a}^n)]], \lambda(\bar{A} - A)\phi\} + \max_h[-h + V'(a^m)(h + \rho a^m)] + \dot{V}^m(a^m) \quad (2.4)$$

and

$$rV^b(a^n) = \alpha\sigma(1-m)\theta[u(q(a^n)) - q(a^n)] + \alpha\sigma(1-\theta)[M[u(q(\bar{a}^m)) - q(\bar{a}^m)] + (1-M)[u(q(\bar{a}^n)) - q(\bar{a}^n)]] + \max_h[-h + V'(a^n)(h + \rho a^n)] + \delta W(a^n) + \dot{V}^n(a^n). \quad (2.5)$$

For miners, the RHS has utility flows for (1) the expected surplus from being a buyer, (2) the maximum between the expected surplus from producing and from mining, (3) the portfolio appreciation and dividend payment and (4) the variation in the expected utility of having portfolio a^m. For non miners, the only differences arise from the fact that they cannot mine and from the expected utility flow from the mining technology acquisition.

Equation 2.3 implies that $\delta W(a^n) = \delta W(0)$, so a^n is irrelevant for the technology acquisition benefit. Also, because the terms of trade care not about the sellers portfolio, the only difference in miners and non miners portfolio choice could come from $\dot{V}^m(a^m)$ and $\dot{V}^n(a^n)$. Note that both agents choose a^i such that $V'(a^i) = 1$, so

$$\max_h[-h + V'(a^i)(h + \rho a^i)] = \rho a^i$$

and from the envelope condition and the HJB equations,

$$\begin{aligned} r \cdot 1 &= \alpha\sigma(1-m)\theta\left[u'(q(a^i))q'(a^i) - q'(a^i)\right] + \rho \\ r - \frac{d + \dot{\phi}}{\phi} &= \alpha\sigma(1-m)\theta\left[\frac{u'(q(a^i)) - 1}{(1-\theta)u'(q(a^i)) + \theta}\right]. \end{aligned} \quad (2.6)$$

Equation 2.6 implies all agents choose the same real balance a and care only about the money value trajectory. This allows us to further simplify Equations 2.4 and 2.5 to

$$rV^m(a) = \alpha\sigma(1-m)\theta[u(q(a)) - q(a)] + \max\{\alpha\sigma(1-\theta)[u(q(\phi A)) - q(\phi A)], \lambda(\bar{A} - A)\phi\} + \rho a + \dot{V}^m(a^m) \quad (2.7)$$

and

$$rV^b(a) = \alpha\sigma(1-m)\theta[u(q(a)) - q(a)] + \alpha\sigma(1-\theta)[u(q(\phi A)) - q(\phi A)] + \rho a + \delta W(0) + \dot{V}^b(a). \quad (2.8)$$

The opportunity cost of mining is the benefit from being a seller in pairwise meetings. The profit flow of mining is given by

$$\Delta(A, \phi) := \lambda(\bar{A} - A)\phi - \alpha\sigma(1-\theta)[u(q(\phi A)) - q(\phi A)] \quad (2.9)$$

so

$$\begin{aligned} &\text{if } \Delta(A,\phi) > 0, \quad m = M, \\ &\text{if } \Delta(A,\phi) = 0, \quad m \in [0, M], \\ &\text{if } \Delta(A,\phi) < 0, \quad m = 0. \end{aligned} \tag{2.10}$$

The mass of miners dictates how money grows:

$$\dot{A} = m\lambda(\bar{A} - A). \tag{2.11}$$

Agents are given the mining technology according to a Poisson process, so the mass of miners evolution is such that

$$\dot{M} = (1 - M)\delta. \tag{2.12}$$

Equilibrium

An equilibrium is a list $\{V_t^b, V_t^m, a^t, M_t, m_t, A_t, \phi_t\}$ such that

1. The centralised market is always cleared (Equation 2.2)
2. Portfolio is optimally chosen (Equation 2.6)
3. Agents become miners according to Equation 2.12
4. Miners optimally decide to mine (Equations 2.9 and 2.10)
5. Money grows in accordance with the technology and the mass of miners (Equation 2.11)
6. The HJB equations are solved (Equations 2.7 and 2.8)

2.1 Steady State

We investigate steady-states (SS) in which $\dot{M} = \dot{A} = \dot{\phi} = \dot{W} = m = 0$. We focus our attention on monetary SS, i.e., SS with $\phi^s > 0$.

If no agents are mining, it need be that

$$\Delta(A^s, \phi^s) \leq 0. \tag{2.13}$$

Because $\dot{M} = 0$, we have that $M = 1$ and $W(0) = 0$. From the fact that $\dot{\phi} = m = 0$, we have that

$$r = \frac{d}{\phi^s} + \alpha\sigma\theta \left[\frac{u'(q(\phi^s A^s)) - 1}{(1-\theta)u'(q(\phi^s A^s)) + \theta} \right]. \tag{2.14}$$

The SS is obtained by the intersection of two different loci: the locus of pairs (A, ϕ) such that $\dot{\phi} = 0$ when $m = 0$ and the locus such that $\Delta(A, \phi) = 0$. The following proposition establishes conditions for the existence of the SS and describes its characteristics. If $q^s < q^*$, we shall say there is scarce liquidity in the SS; otherwise, there is be abundant liquidity.

Proposition 2.1 (Steady-State Existence and Characterisation)**.** *Besides the trivial $\phi = 0$ SS, four scenarios are possible:*

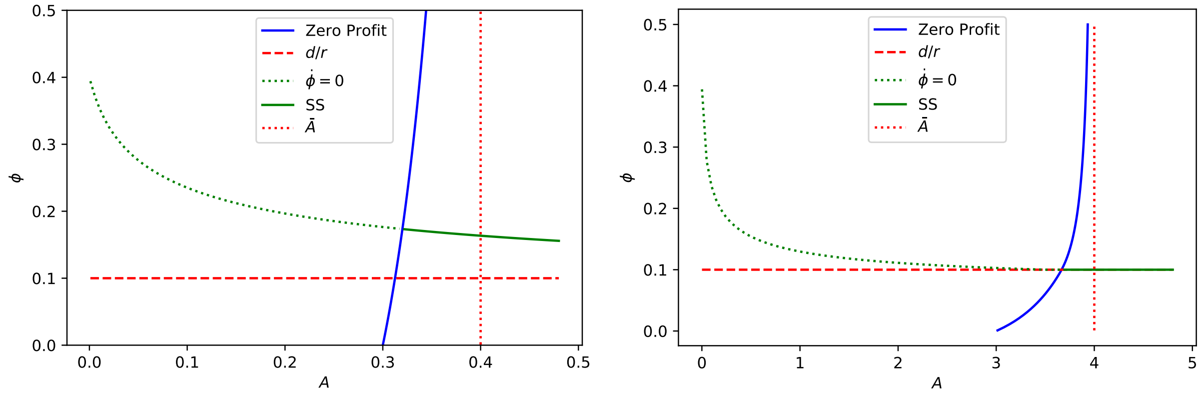
1. *$d = 0$ and $r \geq \alpha\sigma\frac{\theta}{1-\theta}$: there is no monetary SS;*
2. *$d = 0$ and $r < \alpha\sigma\frac{\theta}{1-\theta}$: there is scarce liquidity in the SS;*
3. *$d > 0$ and $\bar{A}\frac{d}{r} \geq \frac{\alpha\sigma(1-\theta)[u(q^*)-q^*]}{\lambda} + w(q^*)$: there is abundant liquidity in the SS; and*
4. *$d > 0$ and $\bar{A}\frac{d}{r} < \frac{\alpha\sigma(1-\theta)[u(q^*)-q^*]}{\lambda} + w(q^*)$: there is scarce liquidity in the SS.*

As usual, when money bears no dividends, agents must be sufficiently patient in order to value it in a monetary SS. If $r \geq \alpha\sigma\frac{\theta}{1-\theta}$, the highest possible liquidity premium (the one obtained with vanishing real balance) is not enough to compensate agents impatience. Therefore, $\rho = \frac{\dot{\phi}}{\phi} > 0$ for any $q > 0$ and no SS is possible.

If $d > 0$ or $r < \alpha\sigma\frac{\theta}{1-\theta}$ there is a monetary SS because the system of equations established by the two loci impose a decreasing and an increasing relation between A and ϕ. The pairs $(A, \phi) > 0$ that satisfy $\Delta(A, \phi) = 0$ form an increasing curve in the space $A \times \phi$ with an asymptote. On the other hand, regardless of the scenario, the $\rho = 0$ locus is a downward sloping curve with a plateau on $\phi = \frac{d}{r}$ if $d > 0$. Graphically, the abundant liquidity will be characterised only when the $\Delta(A, \phi) = 0$ curve intersects the $\phi = 0$ curve in $\phi = \frac{d}{r}$ plateau.

The characteristics of both loci are derived in the Appendix A. Figure 3 exhibits the aforementioned curves for two economies in which money bears dividend: one with scarce liquidity and the other with abundant. The two economies differ solely by the maximum amount of money $\bar{A}$. The reported figures were made under the assumption that $r > \alpha\sigma\frac{\theta}{1-\theta}$.

Figure 4 exhibits the same curves but for a money that bears no dividend. In accordance to Proposition 2.1, we varied r to ensure $r < \alpha\sigma\frac{\theta}{1-\theta}$.



(a) Scarce Liquidity (b) Abundant Liquidity

Figure 3: Steady-States — $d > 0$

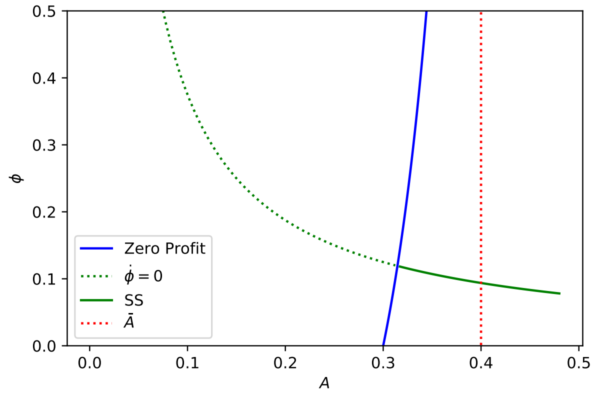


Figure 4: Steady-State — $d = 0$

3 Dynamics

All agents' decisions depend solely on the amount of money in the economy and the current price of money. In this environment, the stock of money is non decreasing, but the price could a priori have any trajectory. To understand the equilibrium trajectories we need to understand how the money value varies through time in a potential equilibrium. Our first goal in this section is to separate the appreciation and depreciation regions.

After separating the aforementioned regions, we identify the conditions an equilibrium trajectory must meet in order to reach a monetary SS. We first analyse equilibria in a world with scarce liquidity and positive dividends ($d > 0$) and argue there will be only one equilibrium trajectory and that the economy reaches the monetary SS. We then proceed to fiat money ($d = 0$) to allow a richer set of equilibrium trajectories. In this latter case, there will be only one equilibrium leading to the monetary SS and infinite equilibria with vanishing price. Later we argue how an equilibrium with abundant liquidity must be.

3.1 Appreciation vs. Depreciation

We want to separate the regions in which money appreciates from the ones in which it depreciates. Because the mining technology is spreading in the economy, we will not have Choi & Rocheteau (2019a)'s case in which money is always appreciating until the SS. We will use the fact that all miners engage in mining until the economy has reached the zero profit region and this implies that there is a bijection between the amount of miners and the total amount of money in any equilibrium path. This bijection will allow us to know the mass of producers and the liquidity premium in any point of the economy's trajectory, so we will be able to determine the appreciation and depreciation regions.

We start by understanding the link between the mass of miners and the amount of money in any equilibrium path.[10] The mass of miners evolves exogenously and

$$M_t = M_0 + (1 - M_0)(1 - e^{-\delta t}). \tag{3.1}$$

If $\Delta(A_t, \phi_t) > 0$ for $t \in [0, b)$, then

$$\dot{A} = M_t \lambda(\bar{A} - A) \implies A_t = \bar{A} - \bar{A} e^{-\lambda t} e^{\frac{\lambda}{\delta}(1-M_0)(1-e^{-\delta t})}. \tag{3.2}$$

The case in which the technology is available to every agent is the limit case in which $\delta \to \infty$, so $A_t = \bar{A} - \bar{A}e^{-\lambda t}$.[11]

[10] For the sake of simplicity we illustrate the equations with $M_0 = 0$.

[11] The Poisson process for the mining technology acquisition is very convenient for the closed expressions we have here and for the parametrisation we use for the graphical examples throughout the paper. It must be noted that the results we obtain though are only dependent on an exogenous and strictly increasing mass of miners.

The rate δ impacts the dynamics because the price trajectory depends on how many agents are mining at any given moment. Recall that agents buy money in order to make transactions with people that are not mining. If all agents can mine ($M = 1$) and mining is more profitable than producing ($\Delta(A, \phi) > 0$), all agents will be mining ($m = 1$) and no transactions will be made. This situation implies that the price of money evolves according to

$$\frac{\dot{\phi}}{\phi} = r - \frac{d}{\phi}, \tag{3.3}$$

so

$$\phi > \frac{d}{r} \implies \dot{\phi} > 0. \tag{3.4}$$

Furthermore, if agents agree that $\phi_0 \geq \frac{d}{r}$, then the price of money evolves according to

$$\phi = \left(\phi_0 - \frac{d}{r}\right) e^{rt} + \frac{d}{r} \tag{3.5}$$

while all agents mine. This implies that money has a non decreasing value as its stock in the economy is growing. Note that agents want to hold money in this situation because the appreciation rate compensates them for their impatience (r) and the dividend they accrue ($-d/\phi$). In this limit case, if the SS has abundant liquidity (so $\phi^s = d/r$), then $\phi_t = d/r$ in every moment. If there is scarce liquidity in the SS, then money is always appreciating until the SS.

In contrast to the previous limit case with $\delta \to \infty$, if the total amount of miners in the economy is lower than 1 ($M < 1$) and it is profitable to mine (so $m = M$),

$$\frac{\dot{\phi}}{\phi} = r - \frac{d}{\phi} - \alpha\sigma(1 - M)\theta \left[\frac{u'(q(\phi A)) - 1}{(1-\theta)u'(q(\phi A)) + \theta}\right] \tag{3.6}$$

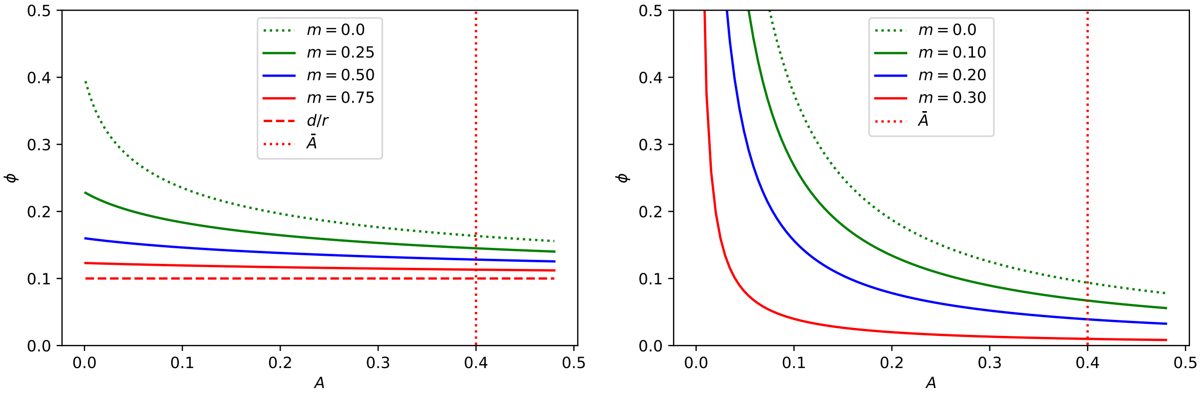
which in our parametrisation corresponds to

$$\frac{\dot{\phi}}{\phi} = r - \frac{d}{\phi} - \alpha\sigma e^{-\delta t}\theta \left[\frac{u'(q(\phi A)) - 1}{(1-\theta)u'(q(\phi A)) + \theta}\right].$$

In our environment, money will always have an appreciation rate that is lower than in Choi & Rocheteau (2019a) and can even have a decreasing value while all miners are engaged in mining. To illustrate this, we exhibit different threshold levels for $\dot{\phi} = 0$ for economies with scarce liquidity in Figure 5. Given a current M_t, if the pair (A_t, ϕ_t) lies above the respective threshold level, then $\dot{\phi} > 0$. The converse holds if it lies below that level.

The existence of a bijection between A_t and M_t along the equilibrium paths allows us to map the effective thresholds the agents would consider in any given moment. By knowing A_t, we can retrieve $M(A_t)$, the mass of miners the economy must have accumulated up to that point considering that all miners have engaged in mining up to A_t. With that information, we can solve the threshold ϕ such that $\dot{\phi} = 0$ with the equation

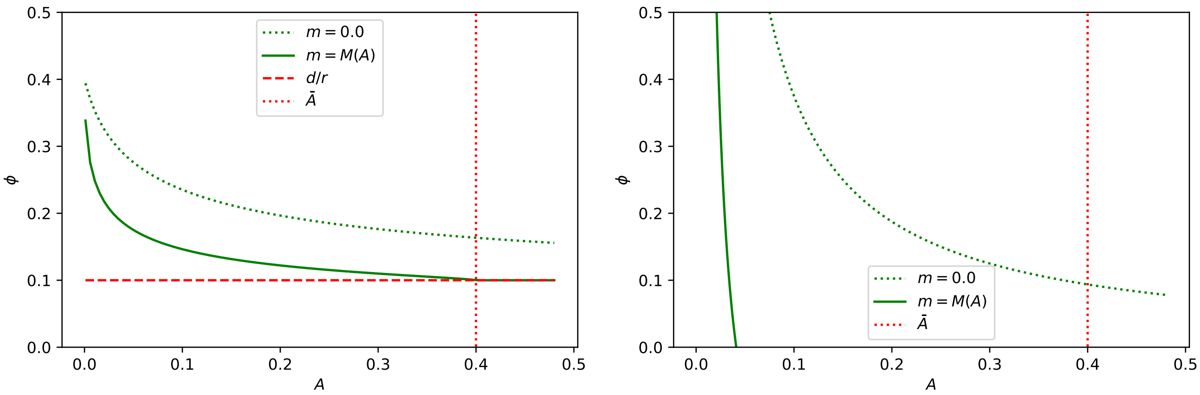
$$0 = r - \frac{d}{\phi} - \alpha\sigma(1 - M(A))\theta \left[\frac{u'(q(\phi A)) - 1}{(1-\theta)u'(q(\phi A)) + \theta}\right]. \tag{3.7}$$



(a) $d > 0$ (b) $d = 0$

Figure 5: Thresholds for $\dot{\phi} = 0$ – Scarce Liquidity

The solution is unique for any A. We report the effective threshold for economies with scarce liquidity in Figure 6. In any equilibrium path, if the economy is currently below the effective threshold, the traded amount is too low for the associated mass of producers $(1 - M(A))$, so the liquidity premium is higher than the agent's impatience rate (discounted by the accrued dividends) and the economy must have $\dot{\phi} < 0$. The converse holds above the effective threshold.



(a) $d > 0$ (b) $d = 0$

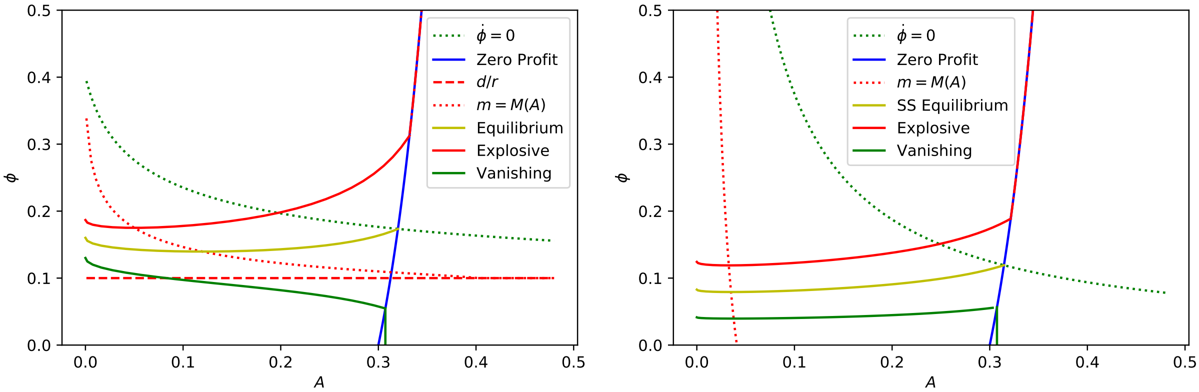
Figure 6: Effective Thresholds for $\dot{\phi} = 0$ – Scarce Liquidity

3.2 Equilibria with Scarce Liquidity

To understand the economy's trajectory, we must understand how the price of money evolves. This can be done by comparing the appreciation rate for a given stock of money and prices $\phi > \tilde{\phi}$. Note that the liquidity premium in (A, ϕ) is not as large as in $(A, \tilde{\phi})$ and that $d/\phi < d/\tilde{\phi}$, so it must be that $\frac{\dot{\phi}}{\phi} > \frac{\dot{\tilde{\phi}}}{\tilde{\phi}}$ (from Equation 3.6) and thus $\dot{\phi} > \dot{\tilde{\phi}}$. Note also that, as long as agents are willing to mine with both prices, $\dot{A}$ will be the same in both cases, so the trajectory of an economy at (A,ϕ) is steeper than of one at $(A,\tilde{\phi})$:

$$\left.\frac{\partial\phi}{\partial A}\right|_{(A,\phi)} > \left.\frac{\partial\phi}{\partial A}\right|_{(A,\tilde{\phi})}. \tag{3.8}$$

Therefore, two trajectories with different initial prices will never cross each other as long as these trajectories are still in the region in which mining has always been profitable. Furthermore, if money starts at a higher initial value, it will always sustain a higher value. To illustrate this feature, we plot 3 different trajectories for 3 different initial prices in Figure 7.[12]



(a) $d > 0$ (b) $d = 0$

Figure 7: Trajectories with Scarce Liquidity

For all trajectories, ϕ initially declines until it reaches the $\dot{\phi} = 0$ threshold. After that point, money appreciates and has an increasing value in the rest of the trajectory because the liquidity premium and the mass of producers have become too low, so agents demand an appreciation rate to hold money. As we anticipated, two trajectories never cross each other in the region where $\Delta(A,\phi) > 0$. The absence of intersections in this case implies that there can be only one trajectory such that agents mine all their way until the SS ($m_t = M_t$ until $A_t = A^s$) and stop mining all together ($m_t = 0$ after that). If such a trajectory is an equilibrium, we will say there is full mining until the SS. Otherwise, we will say there is partial mining. As we argue next, the comparison between the slope of a full mining trajectory and of the zero profit curve at the SS is necessary to determine whether a full mining equilibrium exists or not.

The slope of the trajectory is given by

$$\begin{aligned}\frac{\partial\phi}{\partial A} = \frac{\dot{\phi}}{\dot{A}} &= \frac{\phi\left\{r - \frac{d}{\phi} - \alpha\sigma\theta(1-m)\left[\frac{u'(q(\phi A))-1}{(1-\theta)u'(q(\phi A))+\theta}\right]\right\}}{m\lambda(\bar{A}-A)} \\ &= \frac{\frac{\phi}{m}\left\{r - \frac{d}{\phi} - \alpha\sigma\theta\left[\frac{u'(q(\phi A))-1}{(1-\theta)u'(q(\phi A))+\theta}\right]\right\} + \phi\alpha\sigma\theta\left[\frac{u'(q(\phi A))-1}{(1-\theta)u'(q(\phi A))+\theta}\right]}{\lambda(\bar{A}-A)}.\end{aligned} \tag{3.9}$$

[12]It must be noted that not all depicted trajectories are equilibrium trajectories.

If the trajectory of the economy reaches the indifference curve, then we know that $\lambda(\bar{A} - A)\phi = \alpha\sigma(1-\theta)[u(q(\phi A) - q(\phi A)]$, so

$$\frac{\partial \phi}{\partial A} = \frac{\frac{\phi}{m}\left\{r - \frac{d}{\phi} - \alpha\sigma\theta\left[\frac{u'(q(\phi A))-1}{(1-\theta)u'(q(\phi A))+\theta}\right]\right\} + \phi\alpha\sigma\theta\left[\frac{u'(q(\phi A))-1}{(1-\theta)u'(q(\phi A))+\theta}\right]}{\alpha\sigma(1-\theta)\frac{[u(q(\phi A))-q(\phi A)]}{\phi}}. \tag{3.10}$$

Let $f(A, \phi; m)$ denote the ratio between the rates with which money appreciates and the stock of money grows when the economy has m active miners and is in the zero profit region, i.e.,

$$f(A, \phi; m) := \frac{\partial \phi/\phi}{\partial A/A} = \frac{\frac{1}{m}\left\{r - \frac{d}{\phi} - \alpha\sigma\theta\left[\frac{u'(q(\phi A))-1}{w'(q(\phi A))}\right]\right\} + \alpha\sigma\theta\left[\frac{u'(q(\phi A))-1}{w'(q(\phi A))}\right]}{\alpha\sigma(1-\theta)\frac{[u(q(\phi A))-q(\phi A)]}{w(q)}}. \tag{3.11}$$

The slope of the indifference curve can be obtained by differentiating the zero profit condition. Let $g(A, \phi)$ denote the elasticity of the price of money relative to its stock in the economy along the indifference curve:

$$\begin{aligned} g(A, \phi) := \left.\frac{\partial \phi/\phi}{\partial A/A}\right|_{\Delta(A,\phi)=0} &= \frac{\lambda + \alpha\sigma(1-\theta)\left[\frac{u'(q(\phi A))-1}{w'(q(\phi A))}\right]}{\alpha\sigma(1-\theta)\left[\frac{u(q(\phi A))-q(\phi A)}{w(q(\phi A))} - \frac{u'(q(\phi A))-1}{w'(q(\phi A))}\right]} \\ &= \frac{\lambda + \alpha\sigma(1-\theta)\left[\frac{u'(q(\phi A))-1}{w'(q(\phi A))}\right]}{\alpha\sigma(1-\theta)\frac{[u(q(\phi A)-q(\phi A)]}{w(q)}\left[1 - \frac{w(q(\phi A))[u'(q(\phi A))-1]}{w'(q(\phi A))[u(q(\phi A))-q(\phi A)]}\right]}. \end{aligned} \tag{3.12}$$

Proposition 3.1 (Equilibrium with a Monetary SS — Existence and Characterization)**.** *There is a unique equilibrium leading to the monetary SS. If $f(A^s, \phi^s; M(A^s)) \leq g(A^s, \phi^s)$ and the backwards integration from the SS does not enter the negative profit region, this equilibrium is a full mining equilibrium. Else, this equilibrium is a partial mining equilibrium.*

The intuition of the proposition is as follows. We can always use Equation 3.9 with $m = M(A)$ to calculate a trajectory with full mining that would lead to the SS with scarce liquidity by backwards integration. Note that

$$\phi_t = \phi^s - \int_{A_t}^{A^s} \left.\frac{\dot{\phi}}{\dot{A}}\right|_{m=M(A)} \mathrm{d}A \tag{3.13}$$

Let ϕ_0^* be the price we obtain by this method for $t = 0$.[13] If the trajectory we obtain never crosses the zero profit curve, it means that we have successfully found an equilibrium and that the equilibrium is with full mining. Any other trajectory that started with $\phi_0 > \phi_0^*$ would always have a higher appreciation rate than the equilibrium trajectory and would have $\phi > \phi^s$ at A^s, so money would be ever appreciating and would violate $\lim e^{-rt}\phi_t = 0$. On the other hand, a trajectory that had $\phi_0 < \phi_0^*$ would have a lower appreciation rate

[13] $M_0 > 0$ ensures this integral converges for $t = 0$.

and would reach the zero profit curve at $(A, \phi) < (A^s, \phi^s)$. Any reduction in the number of miners in this case would make the economy cross the zero profit curve and lead to $\lim \phi_t = 0$. Figure 7 illustrates these 3 cases.

Another possibility is that the full mining trajectory that started with the aforementioned ϕ_0^* crosses the zero profit curve and would only be consistent with mining activity where mining is not as profitable as producing. If that is the case, there is a higher initial price $\tilde{\phi}_0$ whose full mining trajectory will lead the economy to a point where the slope of the full mining trajectory is tangent to the zero profit curve. The equilibrium will then be that all miners are engaged with mining until the economy reaches this tangency point and after that the equilibrium trajectory coincides with the zero profit curve. This is possible because the mass of active miners will be lower than the mass of miners in the economy and both the appreciation rate and the mining rate will be reduced. Because $m_t < M_t$ over the zero profit curve, this will be a partial mining equilibrium.

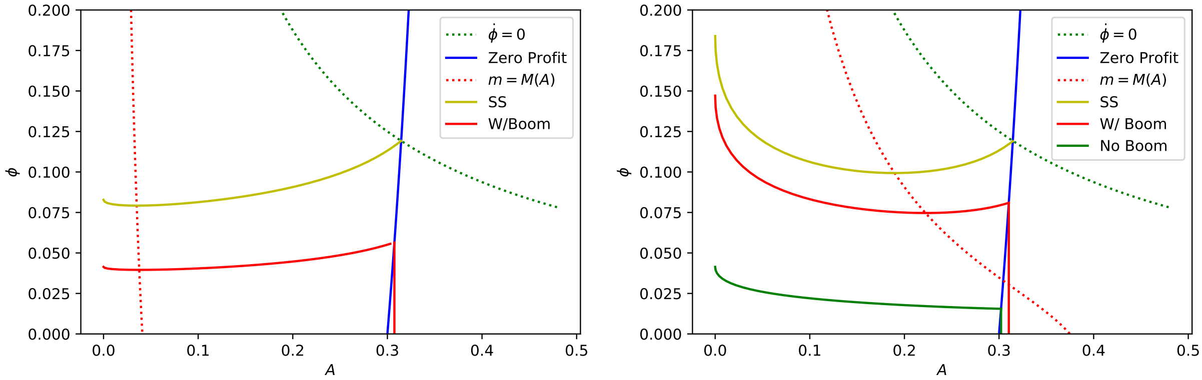
As with the full mining equilibrium, any trajectory that starts with a higher initial price would violate $\lim e^{-rt}\phi_t = 0$ and any trajectory with a lower initial price would have $\lim \phi_t = 0$. In addition to these, there are infinite trajectories starting with the same initial price of the partial mining equilibrium, but in which agents stop mining along the zero profit curve. These trajectories can be indexed by the moment in which agents stop mining and all will have $\lim \phi_t = 0$.

Proposition 3.2 (Monetary Equilibria)**.** *If money bears dividends, then there is a unique monetary equilibrium. If money bears no dividends, there are infinite monetary equilibria that can be indexed as follows:*

1. *Let ϕ_0^* be the initial price of the equilibrium leading to the monetary SS.*
2. *Each trajectory starting with $\phi_0 \in (0, \phi_0^*)$ is an equilibrium.*
3. *If there is a partial mining equilibrium, then let T be the period in which the economy reaches the zero profit region. Each trajectory starting with $\phi_0 = \phi_0^*$ can be indexed by the period $\tilde{t} \in [T, \infty)$ in which all agents stop mining. All these trajectories will also be equilibria.*

Fiat money in this environment has infinite equilibria in which there is a bust phase with no mining and vanishing price. In opposition to Choi & Rocheteau (2019a), it need not be the case that there is a boom phase prior to this bust phase because, for a low enough δ, there are also infinite equilibria in which money has ever decreasing value until the economy reaches the zero profit region. This is the case because a lower δ implies there will be fewer miners (lower $M(A)$) in the region where miners have zero profit, thus leading to higher liquidity premia and lower (and possibly negative) appreciation rate at the indifference curve. Figure 8 illustrates how equilibria with boom phases and equilibria without them are possible if δ is low enough, while only equilibria with boom phases are possible for high enough δ. Proposition 3.3 formalizes this statement.

Proposition 3.3 (Boom Phases with Fiat Money)**.** *Let* $A_H := \frac{\lambda}{\lambda+\alpha\sigma}\bar{A}$ *and* $A_L := M^{-1}\left(\frac{\alpha\sigma\frac{\theta}{1-\theta}-r}{\alpha\sigma\frac{\theta}{1-\theta}}\right)$*. There is a unique* δ^b *such that* $A_L = A_H$*. If* $\delta < \delta^b$*, there are infinite equilibria in which money is ever depreciating, thus with no boom phase. If* $\delta \geq \delta^b$*, all equilibria have a boom phase in which money appreciates.*



(a) High δ: $\delta > \delta_b$

(b) Low δ: $\delta < \delta_b$

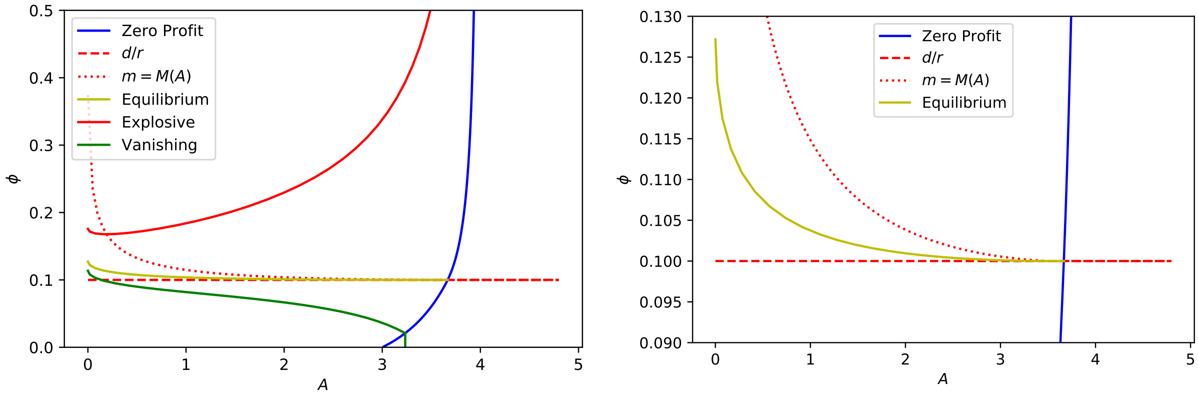
Figure 8: Equilibria with Fiat Money and Different δs

A fuller discussion of Propositions 3.1, 3.2 and 3.3 can be found in Appendix B.

3.3 Equilibrium with Abundant Liquidity

As in the case of scarce liquidity, when money bears dividends but the maximum amount of mined assets $\bar{A}$ is large enough to have abundant liquidity, the economy will have at most one equilibrium. The difference from the previous cases is that this equilibrium must have continuous depreciation until the point where $A_t\phi_t = w(q^*)$. After that, the economy will have $\phi_t = d/r$ and will reach the SS in finite time. As for the scarce liquidity economies, we report 3 trajectories starting from different prices in Figure 9.

With abundant liquidity, any trajectory that has $\phi_t = d/r$ with $q_t < q^*$ will have an ever decreasing price for money and $\lim \phi_t = 0$. Because money bears dividends, such trajectories cannot be equilibria. In addition to that, any trajectory that has $\rho = 0$ with $\phi > d/r$ will have $\lim e^{-rt}\phi_t \neq 0$, so cannot be an equilibrium either. Even though these potential trajectories are not equilibria, they provide respectively lower and upper bounds for the equilibrium trajectory and an initial equilibrium price. These bounds are used to demonstrate equilibrium existence and uniqueness subsequently. A fuller discussion of the following proposition can be found in Appendix C.



(a) $d > 0$

(b) $d > 0$ — Zoomed Plot

Figure 9: Trajectories with Abundant Liquidity

Proposition 3.4 (Monetary Equilibrium)**.** *If money bears dividends and there is abundant liquidity, then there is a unique monetary equilibrium.*

4 Multiple Assets

We now allow agents to carry assets that cannot be mined. An asset i may bear dividends d_i, will have price ϕ_i and may or may not be used for transactions. Let $\rho_i = \frac{d_i + \dot{\phi}_i}{\phi_i}$.[14] We know that, from the adapted HJB equation, all agents choose to carry $a_i = \phi_i A_i$ independently of having already acquired the mining technology. Furthermore, there is a no arbitrage condition in this economy:

$$\begin{aligned} r &= \alpha\sigma(1-m)\theta\left[\frac{u'(q)-1}{(1-\theta)u'(q)+\theta}\right] + \rho \\ r &= \mathbf{1}_{\{i\in\Omega\}}\alpha\sigma(1-m)\theta\left[\frac{u'(q)-1}{(1-\theta)u'(q)+\theta}\right] + \rho_i, \forall i \\ \implies \rho - \rho_i &= \alpha\sigma(1-m)\theta\left[\frac{u'(q)-1}{(1-\theta)u'(q)+\theta}\right]\left[\mathbf{1}_{\{i\in\Omega\}} - 1\right], \forall i \end{aligned} \tag{4.1}$$

where Ω is the set of indices of assets that can be used in transactions. If there are different kinds of transactions and the set of assets that can be used in each transaction is different, then the portfolio decision takes into account the benefit of bringing an additional unit of $\phi_i A_i$ to each transaction j:

$$\begin{aligned} r &= \sum_j \mathbf{1}_{\{0\in\Omega_j\}}\alpha_j\sigma_j(1-m)\left[\theta\frac{u'(q_j)-1}{(1-\theta)u'(q_j)+\theta}\right] + \rho \\ r &= \sum_j \mathbf{1}_{\{i\in\Omega_j\}}\alpha_j\sigma_j(1-m)\theta\left[\frac{u'(q_j)-1}{(1-\theta)u'(q_j)+\theta}\right] + \rho_i, \forall i \\ \implies \rho - \rho_i &= (1-m)\theta\sum_j \alpha_j\sigma_j\left[\frac{u'(q_j)-1}{(1-\theta)u'(q_j)+\theta}\right]\left[\mathbf{1}_{\{i\in\Omega_j\}} - \mathbf{1}_{\{0\in\Omega_j\}}\right], \forall i. \end{aligned} \tag{4.2}$$

With multiple assets, the decision between producing or mining is also altered. The profit from mining relative to being a producer is now

$$\Delta(A, \phi, \{A_i, \phi_i\}_{i=1}^{I}) = \lambda(\bar{A} - A)\phi - \sum_j \alpha_j\sigma_j(1-\theta)\left[u(q_j) - q_j\right]. \tag{4.3}$$

When there were no other assets, if $\phi_0 > 0$ it was profitable to mine when $A = 0$. Now there are assets that may be used in transactions, so the initial traded amount might already be high enough to discourage any mining if ϕ_0 is too low.

4.1 One Asset (dollars) and One Transaction

We initially consider the special case in which agents can use dollars, an asset that does not bear dividends and whose rate of return can be controlled by a central authority.

[14] We will not use subscripts for the asset we considered in the previous sections.

Constant Growth Rate

The simplest case to analyse is the scenario in which the central authority injects (or subtracts) dollars with a fixed rate $\gamma > -r$ and in which there is only one type of transaction being made. Prior to the mining technology being discovered, we consider that the economy was in a SS with constant real balance $\phi_d A_d$. In this case

$$\frac{(\dot{\phi_d A_d})}{\phi_d A_d} = \frac{\dot{\phi}_d}{\phi_d} + \frac{\dot{A}_d}{A_d} = 0 \implies \frac{\dot{\phi}_d}{\phi_d} = -\gamma \tag{4.4}$$

and then, from the agents portfolio decision,

$$\left[\frac{u'(q) - 1}{(1-\theta)u'(q) + \theta}\right] = \frac{r+\gamma}{\alpha\sigma\theta} > 0 \tag{4.5}$$

and $w(q) = \phi_d A_d < w(q^*)$.

We consider a post-technology discovery SS in which each real balance is constant. There is no mining in the SS, so $\dot{A} = 0$ and

$$\frac{(\dot{\phi A + \phi_d A_d})}{\phi A + \phi_d A_d} = \frac{\phi A}{\phi A + \phi_d A_d}\left(\frac{\dot{\phi}}{\phi} + 0\right) + \frac{\phi_d A_d}{\phi A + \phi_d A_d}\left(\frac{\dot{\phi}_d}{\phi_d} + \gamma\right) = 0. \tag{4.6}$$

From the portfolio choice and the fact that there is only one kind of transaction, we know that $\rho = \rho_d$, so $\frac{\dot{\phi}_d}{\phi_d} = \frac{d + \dot{\phi}}{\phi}$ and

$$\frac{\dot{\phi}}{\phi} + \frac{\phi_d A_d}{\phi A + \phi_d A_d}\left(\frac{d}{\phi} + \gamma\right) = 0. \tag{4.7}$$

Proposition 4.1 (Steady-State Characterization)**.** *If money bears dividends, then a constant rate deflationary policy for dollars will lead to a SS in which the real balance is the same as before the mined money introduction. The mined money will provide no liquidity gains in the SS.*

If money bears no dividends and there is a constant emission policy, the mined money and the dollar cannot be both valued in the SS unless $\gamma = 0$, i.e., the dollar is supply is fixed. In the SS where dollars have value, the real balance is the same as before the mined money introduction. If the mined money has value, the SS is the same as in the baseline model.

In an environment with constant emission policies, the mined money does not increase the liquidity in the economy's SS, except if it is a fiat money and it dominates the other asset.

4.2 One Asset (dollars) and Three Transactions

We focus on the case in which $d = 0$ and the central authority aims to hold $\phi_d A_d =: C$ constant. Similarly to Choi & Rocheteau (2019a) and Lester, Postlewaite &

Wright (2012), we consider an economy in which agents can have three types of encounters: one in which only dollars are accepted, one in which only the mined asset is accepted and another in which both assets are accepted. In this case, there is more flexibility in the rate with which each asset appreciates or depreciates. Let d be the subscript for transactions in which only dollars are accepted and 2 be the subscript for transactions in which both assets are accepted. In this case,

$$\begin{aligned} r &= \alpha_2\sigma_2(1-m)\theta\left[\frac{u'(q_2)-1}{(1-\theta)u'(q_2)+\theta}\right] + \alpha\sigma(1-m)\theta\left[\frac{u'(q)-1}{(1-\theta)u'(q)+\theta}\right] + \rho \\ r &= \alpha_2\sigma_2(1-m)\theta\left[\frac{u'(q_2)-1}{(1-\theta)u'(q_2)+\theta}\right] + \alpha_d\sigma_d(1-m)\theta\left[\frac{u'(q_d)-1}{(1-\theta)u'(q_d)+\theta}\right] + \rho_d \\ &\implies \rho - \rho_d = (1-m)\theta\left[\alpha_d\sigma_d\left[\frac{u'(q_d)-1}{(1-\theta)u'(q_d)+\theta}\right] - \alpha\sigma\left[\frac{u'(q)-1}{(1-\theta)u'(q)+\theta}\right]\right]. \end{aligned} \tag{4.8}$$

The central authority holds the dollar real balance $\phi_d A_d$ constant, so there is a lower bound for how much is traded in d- and 2-type transactions. If q_d is high enough and ϕ is low enough, agents will not have any profit with mining relative to being a producer. This is the case if

$$\Delta(A,\phi,A_d,\phi_d) = \lambda(\bar{A}-A)\phi - \sum_j \alpha_j\sigma_j(1-\theta)\left[u(q_j)-q_j\right] \leq 0. \tag{4.9}$$

The indifference equation for the mining activity in this economy is different from the one we previously analysed. It is still an upward-sloping curve, but it starts in the vertical axis, i.e., as $A \to 0$, $\phi \to \phi_{min} := \frac{(\alpha_d\sigma_d+\alpha_2\sigma_2)(1-\theta)[u(q_d)-q_d]}{\lambda\bar{A}} > 0$ and ϕ_{min} only depends on the dollar real balance.

4.2.1 Steady-State

Proposition 4.2 (Steady-State Existence)**.** *If*

$$r \leq \theta\left[\alpha_2\sigma_2\frac{u'(q_2(C))-1}{(1-\theta)u'(q_2(C))+\theta} + \alpha\sigma\frac{1}{1-\theta}\right]$$

there are infinite steady-states. The SS with the least amount of money is a SS with scarce liquidity in encounters where only money is accepted, i.e., $q^{SS} < q^*$.

As before, agents need to be sufficiently patient to value fiat money. The necessary condition is different from the baseline model, but it is still related to the liquidity premium of the fiat money when no agent mines. It is different because the marginal benefit of acquiring money at the CM also considers the traded amount in the encounters where dollars are also accepted.

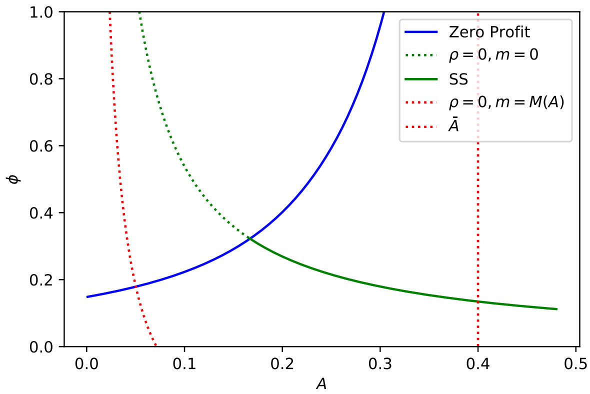


Figure 10: Steady-State and Effective Threshold — Multiple Assets

In the SS, there is no mining, so Equation 4.9 holds. We want a SS with $\dot{\phi} = 0$, so

$$r - \alpha_2\sigma_2\theta \left[\frac{u'(q_2) - 1}{(1-\theta)u'(q_2) + \theta}\right] = \alpha\sigma\theta \left[\frac{u'(q) - 1}{(1-\theta)u'(q) + \theta}\right] \quad (4.10)$$

$$r - \alpha_2\sigma_2\theta \left[\frac{u'(q_2) - 1}{(1-\theta)u'(q_2) + \theta}\right] = \alpha_d\sigma_d\theta \left[\frac{u'(q_d) - 1}{(1-\theta)u'(q_d) + \theta}\right] + \rho_d. \quad (4.11)$$

Given the chosen value for $\phi_d A_d$, Equation 4.10 pins the SS value for ϕA: its LHS is increasing in ϕA and its RHS is decreasing. ρ_d is then given by Equation 4.11 and Equation 4.9 with equality pins ϕ (and A). We know $q < q^*$ because $q \leq q_2$ and $q = q^*$ would imply $q_2 = q^*$ and $r = 0$.

Figure 10 depicts the steady-states in this economy for sufficiently patient agents. As before, we will focus on equilibria that lead to the SS with the least amount of money. We discuss the isocline with $m = M(A)$ in the next subsection.

4.2.2 Dynamics

As before, the equilibrium trajectories in this economy are governed by the agents decision to mine and to acquire the available real balance in the economy. To understand the economy's evolution is to understand the joint evolution of A_t and ϕ_t. We initially separate the monetary equilibria in which no agent mines from the other equilibria.

Proposition 4.3 (Existence of No Mining Monetary Equilibria). *There are infinite monetary equilibria in which no agent mines. Each equilibrium can be indexed by their initial price $\phi_0 \in (0, \phi_{min}]$ and has ever decreasing price.*

In any equilibrium, the market for money must clear with $A = 0$. To ensure no agent wants to carry money, the rate of return ρ must be such that

$$\rho = \frac{\dot{\phi}}{\phi} \leq r - \alpha_2\sigma_2\theta \left[\frac{u'(q_d) - 1}{(1-\theta)u'(q_d) + \theta}\right] - \alpha\sigma\frac{\theta}{1-\theta} < 0, \quad (4.12)$$

so money has an ever decreasing price. The no mining condition from Equation 4.9 calculated at $A = 0$ is equivalent to

$$\phi_t \leq \frac{(\alpha_d \sigma_d + \alpha_2 \alpha_2)(1-\theta)[u(q_d) - q_d]}{\lambda \bar{A}} = \phi_{min}. \tag{4.13}$$

The condition for having no active miners is then that $\phi_0 \leq \phi_{min}$. Finally, Equation 4.11 gives us the dollar rate of return in such equilibria.

Any equilibrium with mining must have $\phi_0 > \phi_{min}$. In order to investigate how the economy will evolve in an equilibrium with mining, we must understand how the price of the mined asset will evolve if $m = M(A)$. Similarly to the previous section, we need to analyse the $\dot{\phi} = 0$ curve with $m = M(A)$:

$$r = (1 - M(A))\theta \left[\alpha_2 \sigma_2 \frac{u'(q_2) - 1}{(1-\theta)u'(q_2) + \theta} + \alpha\sigma \frac{u'(q) - 1}{(1-\theta)u'(q) + \theta} \right]. \tag{4.14}$$

As before, this is a downward-sloping curve that separates the region in which money depreciates from the region in which money appreciates, conditional on all miners being engaged in mining[15]. Figure 10 depicts this curve in an $A \times \phi$ plot.

The same analysis we performed for the baseline model can be done for the case in which there are multiple assets. Any equilibrium with mining activity, as before, will have declining price prior to reaching the $\dot{\phi}|_{m=M(A)} = 0$ curve and there is only one equilibrium leading to the SS with $\phi A > 0$. The condition for full mining until the SS is still the same, but the slope of the trajectory is now different because miners may also produce in exchange for dollars and buyers benefit from carrying the mined asset in type-2 encounters as well.

If the trajectory of the economy reaches the indifference curve, then we know that $\lambda(\bar{A} - A)\phi = (1-\theta)\sum_j \alpha_j \sigma_j [u(q_j) - q_j]$, so let $f(A, \phi; m)$ be the ratio of the appreciation

[15] As A increases, $(1 - M(A))$ (the mass of producers) decreases and the term in squared brackets must increase; this last term increases as q_2 and q decreases, i.e., as ϕA decreases; finally, for ϕA to decrease, ϕ must decrease. Additionally, as $A \to 0$, $1 - M(A) \to 1$. The necessary condition to have an equilibrium in which money is valued in the long run is now

$$r \leq \theta \left[\alpha_2 \sigma_2 \frac{u'(q_2(C)) - 1}{(1-\theta)u'(q_2(C)) + \theta} + \alpha\sigma \frac{1}{1-\theta} \right]. \tag{4.15}$$

The latter condition implies $\exists L \geq 0$ such that $\phi A \to L$ as $A \to 0$ and

$$r = \theta \left[\alpha_2 \sigma_2 \frac{u'(q_2(C+L)) - 1}{(1-\theta)u'(q_2(C+L)) + \theta} + \alpha\sigma \frac{u'(q(L)) - 1}{(1-\theta)u'(q(L)) + \theta} \right].$$

In addition, there is $A_L > 0$ such that

$$r = (1 - M(A_L))\theta \left[\alpha_2 \sigma_2 \frac{u'(q_2(C)) - 1}{(1-\theta)u'(q_2(C)) + \theta} + \alpha\sigma \frac{1}{1-\theta} \right],$$

i.e., there is a minimum A_L for which it is not possible to have $\dot{\phi} = 0$ with $m = M(A)$.

rate and the money growth rate at the indifference curve, i.e.,

$$f(A,\phi;m) := \frac{\partial\phi/\phi}{\partial A/A} = \frac{\frac{1}{m}\left\{r - \theta\left[\alpha\sigma\frac{u'(q)-1}{w'(q)} + \alpha_2\sigma_2\frac{u'(q_2)-1}{w'(q_2)}\right]\right\} + \theta\left[\alpha\sigma\frac{u'(q)-1}{w'(q)} + \alpha_2\sigma_2\frac{u'(q_2)-1}{w'(q_2)}\right]}{(1-\theta)\frac{\sum_j \alpha_j\sigma_j[u(q_j)-q_j]}{w(q)}}. \tag{4.16}$$

As before, for any $\phi A < \phi^s A^s$ we have that term in the curly brackets is negative, so reducing the amount of active miners will make the trajectory flatter (reduces f). The differentiation of $\Delta(A,\phi,A_d,\phi_d) = 0$ gives us the slope of the indifference curve and by performing similar operations we have:

$$g(A,\phi) := \left.\frac{\partial\phi/\phi}{\partial A/A}\right|_{\Delta(A,\phi,A_d,\phi_d)=0} = \frac{\lambda + (1-\theta)\sum_j \alpha_j\sigma_j\left[\frac{u'(q_j)-1}{w'(q_j)}\right]}{(1-\theta)\sum_j\left\{\alpha_j\sigma_j\frac{w(q_j)}{w(q)}\left[\frac{u(q_j)-q_j}{w(q_j)} - \frac{u'(q_j)-1}{w'(q_j)}\right]\right\}}. \tag{4.17}$$

Proposition 4.4 (Equilibrium with a Monetary SS and Multiple Assets)**.** *There is a unique equilibrium leading to the monetary SS. If $f(A^s,\phi^s;M(A^s)) \le g(A^s,\phi^s)$ and the backwards integration from the SS does not enter the negative profit region, this equilibrium is a full mining equilibrium. Else, this equilibrium is a partial mining equilibrium.*

Proposition 4.5 (Monetary Equilibria with Multiple Assets)**.** *There are infinite equilibria that can be indexed as follows:*

1. *Let ϕ_0^* be the initial price of the equilibrium leading to the monetary SS.*
2. *Each trajectory starting with $\phi_0 \in (\phi_{min}, \phi_0^*)$ is also a monetary equilibrium.*
3. *If there is a partial mining equilibrium, then let T be the period in which the economy reaches the zero profit region. Each trajectory starting with $\phi_0 = \phi_0^*$ can be indexed by the period $\tilde{t} \in [T,\infty)$ in which all agents stop mining. All these trajectories will also be equilibria.*

Proposition 4.6 (Boom Phases with Multiple Assets and Fiat Money)**.** $\exists\tilde{\phi} \in (\phi_{min}, \phi_0^*)$ *such that $\phi_0 \in (\phi_{min}, \tilde{\phi}]$ implies the equilibrium has no boom phase and $\phi_0 \in (\tilde{\phi}, \phi_0^*]$ implies the equilibrium has a boom phase.*

In the baseline model there was a cutoff value δ_b that allowed for equilibria without boom phases, i.e., equilibria with never increasing money price. With multiple assets, the existence of encounters in which agents can use different assets increases the active miners opportunity cost for every (A,ϕ) and, in particular, can make producing more profitable than mining for any $A \ge 0$. The additional encounters increase the equilibrium liquidity premium for the mined asset and reduce the appreciation rate in comparison with the baseline model. Because the furthest point the economy reaches in vanishing equilibria is increasing in the initial price, there is a maximum initial price for which money does not appreciate in equilibrium. This maximum price would lead the economy to reach the

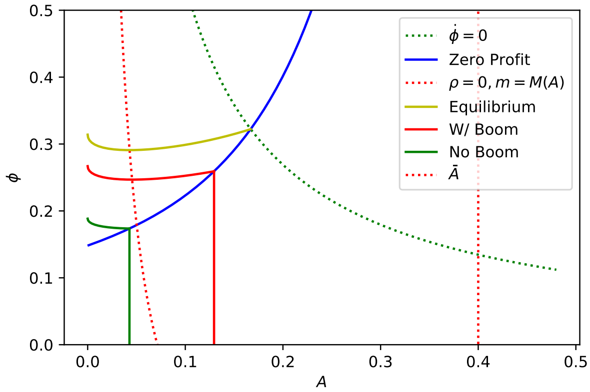


Figure 11: Equilibria — Multiple Assets

zero profit curve where $\dot{\phi}|_{m=M(A)} = 0$. Figure 11 depicts the possible equilibria in this environment.

5 Validating Transactions

We now entangle the mined asset production and its usage in transactions. To keep the analysis closer to real cryptocurrencies, we focus on the case where money bears no dividends. Mining will still be a time-consuming activity, so the trade-off between mining and producing will still exist. We modify the baseline model to make the mined asset production dependent on the transactions agents make with it. More specifically, transactions can only happen if they are entered in a ledger and miners are rewarded for the services they provide.

Potential transactions arrive for agents with rate $\alpha\sigma(1-m_t)$, but only at most λm_t transactions can enter the ledger at any given moment. The probability with which an encounter will enter the ledger is given by

$$\beta = \min\left\{\frac{\lambda m_t}{\alpha\sigma(1-m_t)}, 1\right\} \tag{5.1}$$

and all potential transactions are validated ($\beta = 1$) as long as

$$m_t \geq m_1 := \frac{\alpha\sigma}{\lambda + \alpha\sigma}. \tag{5.2}$$

As long as there are agents dedicated to mining the asset, its amount increases according to $\dot{A} = \pi(\bar{A} - A)$. The active miners are given the newly mined assets and an additional amount that can either come from a transaction fee or from the confiscation scheme. The mined assets and the additional amount are given to the m_t active miners who are rewarded with n_t units of the mined asset according to a Poisson process that arrives with rate λ. The specific reward structure is fundamental to understand how many miners will be active in the SS and in the transition path.

Miners will engage in the mining activity as long as that is more profitable than producing, so

$$m_t \begin{cases} = 0 & \text{, if } \Delta(A,\phi,n_t) < 0 \\ \in [0, M_t] & \text{, if } \Delta(A,\phi,n_t) = 0 \\ = M_t & \text{, if } \Delta(A,\phi,n_t) > 0 \end{cases} \tag{5.3}$$

in which

$$\Delta(A,\phi,n_t) = \lambda n_t \phi_t - \beta\alpha\sigma(1-\theta)[u(q) - q] \tag{5.4}$$

The reward n_t is detailed in the following Subsection.

5.1 Confiscation Scheme

We assume that all agents that carry the mined asset are subject to a confiscation scheme. In this economy, agents lose a fraction χ of their mined asset with rate λ_c. In any period, the expected rewards are such that

$$\lambda m_t n_t = \pi(\bar{A} - A) + \lambda_c \chi A. \tag{5.5}$$

In this environment, agents have less incentive to acquire the mined asset because there is an expected disutility flow $\lambda_c \chi a$ associated with having a fraction χ of the real balance a confiscated to reward the miners. Due to that, agents choose their portfolio such that

$$r_c := r + \lambda_c \chi = \rho + \beta\alpha\sigma(1 - m_t)\theta \left[\frac{u'(q) - 1}{(1-\theta)u'(q) + \theta}\right]. \tag{5.6}$$

The previous equation implies that the agents' portfolio choice with the confiscation scheme is similar to the one in the baseline model. The confiscation scheme has the same effect as an increase in the agents impatience would have. An agent in the baseline model with discount rate $r_c = r + \lambda_c \chi$ and an agent facing the confiscation scheme would carry the same portfolio for all (A, ϕ) given $\beta = 1$ and there were the same amount of active miners m_t in both economies.

Note that the confiscation scheme does not alter the traded amount with respect to the traded amount in the baseline model conditional on both having the same real balance $A\phi$. Each economy's trajectory will differ because the equilibrium ρ will be different, but the traded amount is not changed, so the confiscation scheme could be seen as non distortionary, at least in a direct way, of the DM.

In the SS, $A = \bar{A}$, so the miners' compensation comes solely from the confiscated assets: $\lambda mn = \lambda_c \chi \bar{A} > 0$, so $m > 0$ and $n = \frac{\lambda_c \chi \bar{A}}{\lambda m}$. A SS with $\phi_t = \phi$ requires $m < 1$ and $q < q^*$, otherwise $r + \lambda\chi = 0$. $m \in (0, 1)$ implies that $\Delta(A, \phi, n) = 0$. m and ϕ are such that

$$r + \lambda_c \chi = \beta\alpha\sigma(1 - m)\theta \left[\frac{u'(q) - 1}{(1-\theta)u'(q) + \theta}\right] \tag{5.7}$$

$$\lambda_c \chi \bar{A}\phi = m\beta\alpha\sigma(1-\theta)[u(q) - q] \tag{5.8}$$

Figure 12 depicts the SS conditions $\rho = 0$ (Equation 5.7) and $\Delta = 0$ (Equation 5.8) in our enviroment. The Zero Profit condition imposes a lower bound m_{min} on the number of agents that can be mining in the SS. More specifically, m_{min} is the solution to

$$m_{min}\beta(m_{min}) = \frac{\lambda_c \chi}{\alpha\sigma}. \tag{5.9}$$

The agents' portfolio decision in the SS is associated with a range of values for the number of active miners in the SS. These values are bounded by the solutions to

$$\beta(1 - m_\rho) = \frac{r + \lambda_c \chi}{\alpha\sigma} \frac{1-\theta}{\theta}. \tag{5.10}$$

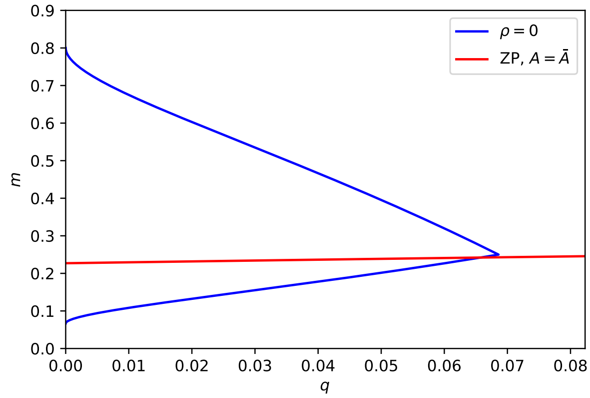


Figure 12: Confiscation Scheme — Steady-State

Note that the m_ρ solutions become more extreme as r decreases, i.e., as agents become more patient. Let r_1 be such that

$$\beta(m_{min})(1 - m_{min}) = \frac{r_1 + \lambda_c \chi}{\alpha\sigma} \frac{1-\theta}{\theta}. \tag{5.11}$$

The condition for the existence of a SS is that m_{min} is within the range of values for the mass of active miners in the SS, i.e., there is a monetary SS as long as $\lambda_c \chi < \alpha\sigma$ (the confiscation scheme is not too intense) and $r \leq r_1$ (agents are patient enough)

Proposition 5.1 (Steady-State Existence)**.** *If $\lambda_c \chi < \alpha\sigma$ and $r \leq r_1$, then there is a SS with scarce liquidity.*

5.1.1 Dynamics

In the individual miner's view, it is profitable to mine as long as

$$\begin{aligned} \Delta(A_t, \phi_t, n_t) &= \frac{1}{m_t}[\pi(\bar{A} - A_t) + \lambda_c \chi A_t]\phi_t - \beta\alpha\sigma(1-\theta)[u(q) - q] \\ &= \frac{1}{m_t}\Big[[\pi \bar{A} + (\lambda_c \chi - \pi)A_t]\phi_t - m_t \beta\alpha\sigma(1-\theta)[u(q) - q]\Big] \\ &= \frac{\phi_t}{m_t}\left[\pi \bar{A} + \left(\lambda_c \chi - \pi - m_t \beta\alpha\sigma(1-\theta)\frac{u(q) - q}{A_t \phi_t}\right) A_t\right] \geq 0. \end{aligned} \tag{5.12}$$

Note that the profit with a given pair (A, ϕ) is strictly decreasing in m, so there is a unique solution m for every pair (A, ϕ). In addition, the profit is increasing in ϕ_t, so there is a minimum ϕ such that there is zero profit with the maximum amount of active miners $(m_t = M_t)$.

5.1.2 Dynamics with $M_t = 1$

If all agents are endowed with the mining technology from $t = 0$ onwards, then whenever mining is more profitable than being a producer, $m_t = 1$, there will be no trade

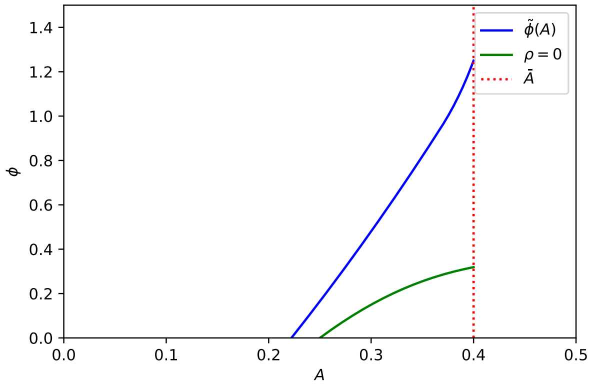


Figure 13: Confiscation Scheme — $\tilde{\phi}$ and $\rho = 0$

even though $\beta = 1$ and $\rho_t = r + \lambda\chi > 0$. This situation happens as long as

$$\Delta(A_t, \phi_t, n_t) = \phi_t \left[\pi(\bar{A} - A_t) + \left(\lambda_c \chi - \alpha\sigma(1-\theta) \frac{u(q) - q}{A_t \phi_t} \right) A_t \right] \geq 0. \tag{5.13}$$

Note that $\Delta(A_t, \phi_t, n_t) = 0$ implies m_t solves

$$\frac{1}{\bar{A}} [\pi(\bar{A} - A_t) + \lambda_c \chi A_t] = m_t \beta\alpha\sigma(1-\theta) \frac{u(q) - q}{A_t \phi_t} \frac{A_t}{\bar{A}}. \tag{5.14}$$

Given A_t, the LHS is constant and can be seen as a weighted average of π and $\lambda_c\chi$ whose weights are the fraction of the assets that is yet to be mined and the fraction that was already mined. The Zero Profit condition shows that as the stock of money grows in the economy, the confiscation scheme becomes more important for the miners compensation vis-à-vis the newly mined money.

As can be seen in Equation 5.14, the miners profit is increasing in the price of money, so the equilibrium mass of active miners that induces zero profit is higher for higher prices. For every amount of money for which Zero Profit is possible, we can find a maximum price $\tilde{\phi}(A)$ with which miners have no profit. If $\phi > \tilde{\phi}$, the profit from mining is positive even when all agents are dedicated to mining. We depict $\tilde{\phi}(A)$ in Figure 13. To the left and above the $\tilde{\phi}(A)$ curve, all agents mine and money appreciates with $\rho = r + \lambda_c\chi$. To the right and below $\tilde{\phi}(A)$, there is zero profit from mining and the amount of active miners, as well as the appreciation rate, will depend on the price of money.

Let $\underline{A}$ be the greatest amount of money for which there is profit from mining regardless of the price of money.[16] Because the economy starts with $A_0 = 0 < \underline{A}$, as long as the economy has $\phi_t > \tilde{\phi}(A_t)$, all agents will mine and $\rho = r + \lambda_c\chi > 0$. If the economy

[16] $\underline{A}$ is the solution to the limit equation

$$\pi\bar{A} + (\lambda_c\chi - \pi - \alpha\sigma)\,\underline{A} = 0.$$

never has $\phi_t < \tilde{\phi}(A_t)$, then money is ever appreciating with a rate greater than agents impatience, so this trajectory cannot be an equilibrium because.

Considering now only trajectories in which there is a t^* after which $\phi_t < \tilde{\phi}(A_t)$, we can obtain the mass of active miners $m_t \in (0, 1)$ from the zero profit equation and we later obtain ϕ_t's growth rate from the agents' portfolio decision:

$$\rho = r + \lambda_c \chi - \beta\alpha\sigma(1-m)\theta \left[\frac{u'(q)-1}{(1-\theta)u'(q)+\theta}\right]. \tag{5.15}$$

We depict the locus $\rho = 0$ in Figure 13 and discuss how ρ responds to changes in ϕ in the following paragraph.

Given A, the previous equation indicates that the money appreciation rate is increasing in ϕ if $m \geq m_1$ (so $\beta = 1$): An increase in ϕ increases q and induces more agents to be active miners, but this increment only predates encounters with producers without increasing the probability of validating a transaction, thus reducing the liquidity premium. For $m < m_1$, $\beta\alpha\sigma(1-m) = \lambda m$ and q, m and β increase as ϕ increases, so the net effect on ρ is at first sight ambiguous. The net effect on ρ is positive even with $m < m_1$ because the increase in the expected rate of valid encounters is less than the necessary increase to compensate for the reduction in the liquidity premium. These effect is sufficient to establish that money depreciates to the right and below the $\rho = 0$ curve.

Proposition 5.2 (Monetary Equilibria Characterization)**.** *There is at most one equilibrium leading to the monetary SS. If this equilibrium exists, it is with ever increasing price (no bust phase). Let ϕ_0^* denote the initial price of money in such equilibrium.*

There are infinite equilibria with vanishing price and all have a boom and a bust phase. If the equilibrium leading to the monetary SS exists, all vanishing equilibria can be indexed by their initial price $\phi_0 \in (0, \phi_0^*)$.

We exhibit two equilibrium trajectories for the case in which $m^{SS} < m_1$ in Figure 14. As we anticipated in Proposition 5.2, the equilibrium leading to the monetary SS is with an ever increasing price of money. The vanishing equilibria, on the other hand and in spite of having an initial increasing value for money, has a bust phase with $\lim \phi_t = 0$. We are able to replicate Choi & Rocheteau (2019a) boom-and-bust equilibria for mined fiat money, but in an environment that entangles transactions and their validation.

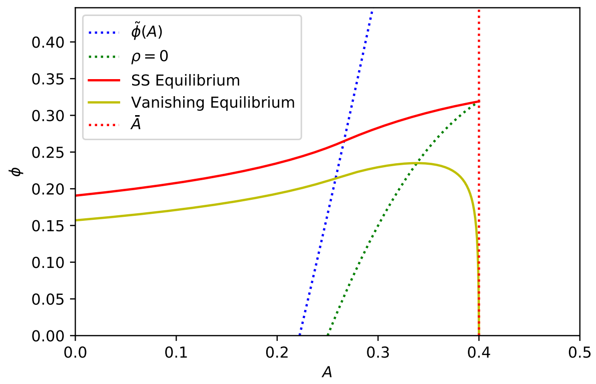


Figure 14: Confiscation Scheme — Equilibria

6 Conclusion

In this paper we study how a mining diffusion process and a validation structure for transactions can affect a cryptocurrency price dynamics in a world where agents have almost no (timing) frictions on their real balance portfolio choice. The continuous-time assumption allows us to refine the equilibria set in many different setups. In common, they all share the property that the higher the initial price in any equilibrium, the more inflated the bubble on the fiat money will be.

Via the occupational choice, we are able to include a channel through which the mining activity has welfare consequences in the decentralised market. Via the diffusion process and the validation structure, we are able to adhere more to data on cryptocurrencies real return rates and to the fact that they are used in real life as media of exchange in transactions even when it is profitable to mine.

Unlike many papers, we are capable of building a scenario where cryptocurrencies have an essential role and may be valued even when the economy has different assets or a government-pegged fiat money. We believe this is a neat way of addressing the fact that there might be privacy gains on the cryptocurrency technology and that may be positively valued by society.

Our model is also flexible enough to approximate the diminishing returns on the mining technology and the supply upper bound many cryptocurrencies share. The introduction of a validation structure to the baseline model allows us to analyse equilibria in a world where miners are providing a service to the cryptocurrency community and there is no seignoriage revenue.

Interesting extensions include, but are not limited to, an entanglement of the random selection of potential miners and an entry decision to the mining sector; heterogeneity on the agents' mining skills; a validation structure and a compensation scheme with an endogenous transaction fee; and a model that can generate heterogeneity in the agents' transaction fee and equilibrium sequences of distributions over transactions waiting for validation.

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Appendix

APPENDIX A – Steady-State Characterisation

For $d > 0$, the locus of pairs (A, ϕ) is such that $\dot{\phi} = 0$ forms a downward sloping curve with a horizontal plateau at $\phi = \frac{d}{r}$:

- If (A, ϕ) is in the locus, $\phi A \geq w(q^*)$ if, and only if, there is abundant liquidity and $\phi = \frac{d}{r}$;
- If $\left(A, \frac{d}{r}\right)$ is in the locus, then $A \geq \frac{rw(q^*)}{d}$;
- If $\phi A < w(q^*)$, (A_0, ϕ_0) and (A, ϕ) are in the locus, and $\phi_0 > \phi > \frac{d}{r}$, then $A_0 < A < \frac{rw(q^*)}{d}$, and no other pair (A_0, ϕ) can be in the locus;
- For any sequence (A_n, ϕ_n) in the locus such that $A_n \to 0$, if $r > \alpha\sigma\frac{\theta}{1-\theta}$, then $\phi_n \to \frac{d}{r-\alpha\sigma\frac{\theta}{1-\theta}}$; if not, then $\phi_n \to +\infty$.

For $d = 0$, the only equilibrium is with scarce liquidity and the locus of pairs (A, ϕ) is a hyperbole with constant real balance lower than $w(q^*)$:

$$r = \alpha\sigma\theta\left[\frac{u'(q(\phi^s A^s)) - 1}{(1-\theta)u'(q(\phi^s A^s)) + \theta}\right]. \tag{A.1}$$

Note that the only way a monetary steady-state can exist is if $r < \alpha\sigma\frac{\theta}{1-\theta}$, otherwise $\dot{\phi} > 0$ for all scenarios in which money is valued. In this environment, agents must be sufficiently patient to value a fiat currency.

In the SS, agents are not mining, so we also investigate the locus of points such that agents are indifferent between mining and producing. This locus is given by $\Delta(A, \phi) = 0$ or

$$\lambda(\bar{A} - A)\phi = \alpha\sigma(1-\theta)[u(q(\phi A)) - q(\phi A)]. \tag{A.2}$$

This equation is trivially satisfied by $\phi = 0$. Considering now only $\phi > 0$, the solution forms an upward sloping curve with vertical asymptote in $\bar{A}$:

- If (A_0, ϕ_0) is in the locus, then no other pair (A, ϕ_0) can be in that locus;
- If (A_0, ϕ_0) and (A, ϕ) are in the locus with $A_0 < A$, then $\phi_0 < \phi$;
- For sufficiently high values of A and ϕ, $\phi A \geq w(q^*)$ and the right hand side of the equation is constant, so $\phi \to \infty$ as $A \to \bar{A}$; and
- As $\phi \to 0$, $A(\phi) \to \frac{\lambda}{\lambda+\alpha\sigma}\bar{A}$.

APPENDIX B – Equilibria in Scarce Liquidity Economies

We initially discuss the equilibria in an economy where money bears dividends and later proceed to the case of fiat money ($d = 0$).

B.1 $d > 0$

If money bears dividends, any trajectory in which, at a given period t, $\phi_t < d/r$ or $\phi_t = d/r$ and money has a positive liquidity premium will lead to $\lim \phi_t = 0$. Of course, any trajectory with one of these properties cannot be an equilibrium for dividend-bearing money.

B.1.1 Full mining until the SS

As we previously discussed, one condition for having a trajectory that reaches the SS with $m_t = M_t$ until $A_t = A^s$ is that $f(A^s, \phi^s; M(A^s)) \leq g(A^s, \phi^s)$ when $m = M(A^s)$, or

$$\left[1 - \frac{w(q^s)[u'(q^s) - 1]}{w'(q^s)[u(q^s) - q^s]}\right] \cdot \\ \cdot \left\{\frac{1}{M(A^s)}\left\{r - \frac{d}{\phi} - \alpha\sigma\theta\left[\frac{u'(q^s) - 1}{w'(q^s)}\right]\right\} + \alpha\sigma\theta\left[\frac{u'(q^s) - 1}{w'(q^s)}\right]\right\} \\ \leq \lambda + \alpha\sigma(1 - \theta)\left[\frac{u'(q^s) - 1}{w'(q^s)}\right], \quad \text{(B.1)}$$

If Inequality B.1 holds, we can integrate the trajectory backwards and retrieve the unique initial price ϕ_0^* such that the trajectory has the aforementioned properties. This initial price will give us the unique monetary equilibrium trajectory provided there is a unique intersection between the trajectory and the indifference curve. Any trajectory starting with $\phi_0 > \phi_0^*$ would have an ever increasing ϕ in the long run and would violate $\lim_{t\to\infty} e^{-rt}\phi_t = 0$, so it would not be an equilibrium. Any trajectory starting with $\phi_0 < \phi_0^*$ would have $\lim_{t\to\infty} \phi_t = 0 < d/r$ and would also not be an equilibrium.

B.1.2 Partial mining until the SS

If Inequality B.1 does not hold and were we to retrieve the initial price of the trajectory that leads to the SS assuming all miners are engaged with mining in every period, we would have a trajectory that would cross the indifference curve at least twice and there would be mining in the region where $\dot{A} = 0$. If Inequality B.1 holds, but the intersection with the indifference curve is not unique, we would also have mining in the region where $\dot{A} = 0$ and there would be at least three intersections with the indifference curve. In spite of imposing mining in regions where it is not optimal for the agents do so, let $\tilde{\phi}_0$ denote the initial price of the trajectory obtained this way. This initial price is too low to make for an equilibrium trajectory and the economy reaches the indifference curve with a slope that is lower than the latter curve's slope. Because $m = M(A_t)$ at that point, if some agents stop mining ($m < M(A_t)$) the slope of the trajectory would become even lower, the economy would have an ever decreasing price, and $\lim \phi_t = 0 < d/r$. As we argued previously, this cannot be an equilibrium.

The only trajectory that can reach the SS is a trajectory in which the economy has $m_t = M_t$ up to the intersection with the indifference curve and m_t is such that $f(A_t, \phi_t; m_t) = g(A_t, \phi_t)$ after that, i.e., the second part of the trajectory coincides with the indifference curve. Because Inequality B.1 does not hold, it must be that there is only one pair $(\tilde{A}, \tilde{\phi})$ such that $f(\tilde{A}, \tilde{\phi}; M(\tilde{A})) = g(\tilde{A}, \tilde{\phi})$. Let $\tilde{M} = M(\tilde{A})$.

Just as we had for the equilibrium with full mining, we may retrieve the initial price ϕ_0^* that leads the economy to that pair $(\tilde{A}, \tilde{\phi})$ and any price different from that would either lead to explosive trajectories for the price of money ($\lim e^{-rt}\phi_t > 0$) or to a vanishing price for money ($\lim \phi_t = 0 < d/r$). If an equilibrium exists, it must have $\phi_0 = \phi_0^*$. We now need to prove that we can produce a function m_t such that the economy converges to the SS.

The condition for the coincidence of both curves (and hence, for an equilibrium to be defined) is that $m = m_I := m(q, \phi)$ such that

$$\left[1 - \frac{[u'(q) - 1]/w'(q)}{[u(q) - q]/w(q)}\right] \cdot \\ \cdot \left\{ \frac{1}{m_I} \left\{ r - \frac{d}{\phi} - \alpha\sigma\theta \left[\frac{u'(q) - 1}{w'(q)}\right] \right\} + \alpha\sigma\theta \left[\frac{u'(q) - 1}{w'(q)}\right] \right\} \\ = \lambda + \alpha\sigma(1 - \theta) \left[\frac{u'(q) - 1}{w'(q)}\right]. \quad \text{(B.2)}$$

Additionally, note that ϕ, A, and ϕA increase along the indifference curve, so $q = q(\phi A)$ increases. As q increases, $\frac{u'(q)-1}{w'(q)} / \frac{u(q)-q}{w(q)}$ decreases. The RHS decreases and the first term of the LHS increases as q increases. Note also that $-\frac{1-m}{m}\alpha\sigma\theta\frac{u'(q)-1}{w'(q)}$ is increasing in q and that $-\frac{d}{\phi}$ is increasing in ϕ, so the LHS is increasing in q and in ϕ. Finally, $\frac{1}{m_I}\left\{r - \frac{d}{\phi} - \alpha\sigma\theta\left[\frac{u'(q)-1}{w'(q)}\right]\right\} < 0$[17], so the LHS is decreasing in m_I. We must conclude that as ϕ and q (A) increase, m_I must decrease. Additionally, we have that $m_I \leq \tilde{M}$, so an equilibrium with convergence to the SS exists.

[17]Recall that the term in curly brackets is only equal to zero if $\dot{\phi} = 0$ and that can only happens in the SS if the pair (A, ϕ) is in the indifference curve.

It should be pointed that the reduction in the mass of active miners only makes the trajectory flatter (reduces $f(A, \phi; m)$) because in the area we were considering ($q < q^s$) was under the curve $\dot{\phi}|_{m=0} = 0$, so $\frac{1}{m_I}\left\{r - \frac{d}{\phi} - \alpha\sigma\theta\left[\frac{u'(q)-1}{w'(q)}\right]\right\} < 0$. Were the economy to reach the indifference curve above the curve $\dot{\phi}|_{m=0} = 0$, then the sign of the former expression would be positive and a reduction in m would make the trajectory steeper (increase $f(A, \phi; m)$). That is why, were the economy to reach the indifference curve with $q > q^s$, the trajectory after that would also coincide with the indifference curve.

B.2 $d = 0$

As we mentioned previously, if $d = 0$, then the SS existence condition is $r < \alpha\sigma\frac{\theta}{1-\theta}$, so we will impose this. Because $d = 0$, the curve $\frac{\dot{\phi}}{\phi}\Big|_{m=M(A)} = 0$ is the solution to

$$r = \alpha\sigma(1 - M(A))\theta\left[\frac{u'(q(\phi A)) - 1}{(1-\theta)u'(q(\phi A)) + \theta}\right]. \tag{B.3}$$

The solution forms a downward sloping curve with vertical asymptote in $A = 0$:

- If (A, ϕ) and (A', ϕ') are in the locus and $A < A'$, then $\phi' < \phi$;
- If $A_n \to 0$, then $M(A_n) \to 0$ and $\left[\frac{u'(q(\phi A))-1}{(1-\theta)u'(q(\phi A))+\theta}\right] \to \frac{r}{\alpha\sigma\theta} < \frac{1}{1-\theta}$. This implies $\phi_n A_n \to L \in (0, w(q^*))$ and $\phi_n \to \infty$;
- $\left[\frac{u'(q(\phi A))-1}{(1-\theta)u'(q(\phi A))+\theta}\right] \leq \frac{1}{1-\theta}$, so any solution to Equation B.3 must have $r \leq \alpha\sigma\frac{\theta}{1-\theta}(1 - M(A))$ or $M(A) \leq \frac{\alpha\sigma\frac{\theta}{1-\theta} - r}{\alpha\sigma\frac{\theta}{1-\theta}} < 1$; and
- A is bounded above, so $\phi_n A_n \to 0$ and $M(A_n) \to M(A_L) := \frac{\alpha\sigma\frac{\theta}{1-\theta} - r}{\alpha\sigma\frac{\theta}{1-\theta}}$ as $\phi_n \to 0$.

The mining indifference curve properties we studied in the previous section are not altered by the assumption that $r < \alpha\sigma\frac{\theta}{1-\theta}$, so we know that as $\phi_n \to 0$ along the indifference curve, $A_n \to A_H := \frac{\lambda}{\lambda+\alpha\sigma}\bar{A}$. If $A_H < A_L$, then there is a non empty intersection between the area in which $\dot{\phi}|_{m=M(A)} < 0$ and the area in which $\dot{A} = 0$. If this is the case, then any trajectory that leads the economy to a monetary SS will have the same properties as the trajectory in an economy with scarce liquidity:

- If the slope of the trajectory is less than or equal to the slope of the indifference curve at the SS, then there exists a monetary equilibrium with full mining until the SS. The condition is the same as before:

$$\left[1 - \frac{w(q^s)[u'(q^s) - 1]}{w'(q^s)[u(q^s) - q^s]}\right] \cdot \left\{\alpha\sigma\theta\left[\frac{u'(q^s) - 1}{w'(q^s)}\right]\right\} \leq \lambda + \alpha\sigma(1-\theta)\left[\frac{u'(q^s) - 1}{w'(q^s)}\right]; \tag{B.4}$$

- If Inequality B.4 does not hold, then there is a monetary equilibrium in which the economy reaches the indifference curve and in which a part of the trajectory coincides with the indifference curve. The amount of miners in this equilibrium is obtained using similar equations as before.
  - Note that $f(A^s, \phi^s; M(A^s)) > g(A^s, \phi^s)$ and the slope of the trajectory will be zero at the intersection of the indifference curve and the $\dot{\phi}\big|_{m=M(A)} = 0$. Therefore, there must be an $A < A^s$ such that $f(A, \phi; M(A)) = g(A, \phi)$ in the indifference curve.

Because money now bears no dividends, trajectories with $\lim \phi_t = 0$ cannot be ruled out. $d = 0$ implies that there is a unique equilibrium leading to a monetary SS, but there are infinite equilibria with vanishing price. If the initial price is lower than the initial price corresponding to the monetary equilibrium, then the economy will have a vanishing price in the long run, but it will be valued along the trajectory. Also, if Inequality B.4 does not hold, then any trajectory in which the agents deviate from m_{It} to a lower value will lead to a vanishing price.

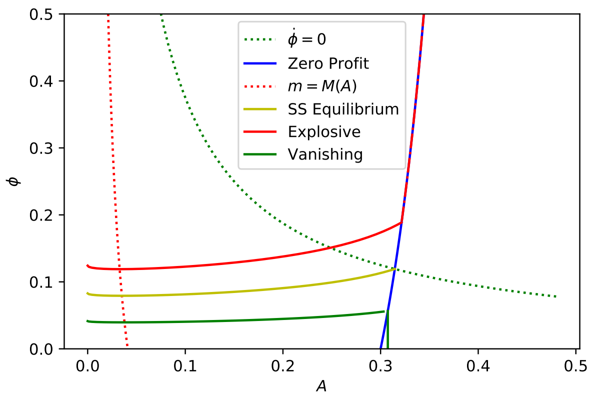

Figure 15: Trajectories with no Dividends

If $A_L < A_H$, it could seem that the economy could have no trajectory leading to a monetary SS. That would be the case if, going backwards from the first point of intersection with the indifference curve, the trajectory would lead to $\phi_t = 0$ with $A_t > A_L$. We argue no trajectory can lead to $\phi_t = 0$ because the slope of the trajectory would become too flat as it approaches $\phi = 0$.

The inclination of the trajectory is given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}A} = \frac{\phi\left[r - \alpha\sigma\theta\left[\frac{u'(q)-1}{w'(q)}\right] + \alpha\sigma\theta M(A)\left[\frac{u'(q)-1}{w'(q)}\right]\right]}{M(A)\lambda(\bar{A} - A)}. \tag{B.5}$$

To prove our statement, we will consider the case in which the economy follows a steeper trajectory given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}A} = \frac{\phi\left[r - \alpha\sigma\theta\left[\frac{u'(q^s)-1}{w'(q^s)}\right] + \alpha\sigma\theta M(A^s)\frac{1}{1-\theta}\right]}{M(A_L)\lambda(\bar{A} - A^s)} = \phi\frac{\alpha\sigma\theta M(A^s)\frac{1}{1-\theta}}{M(A_L)\lambda(\bar{A} - A^s)}. \tag{B.6}$$

The trajectory we obtain with Equation B.6 will always have a price for money that is lower than the price in the effective trajectory. The last equation is a simple differential equation whose solution, given we are integrating backwards from the (A^s, ϕ^s), is

$$\ln(\phi) = \ln(\phi^s) + \frac{\alpha\sigma\theta M(A^s)\frac{1}{1-\theta}}{M(A_L)\lambda(\bar{A} - A^s)}(A - A^s), \tag{B.7}$$

so $\phi > 0, \forall A$ in this steeper trajectory. Therefore, by integrating backwards from the SS we would never have $\phi_t = 0$ and a monetary equilibrium is possible.

APPENDIX C – Equilibrium in Abundant Liquidity Economies

We know $\left(\frac{w(q^*)}{d/r}, d/r\right)$ is in the threshold level for every amount of miners. In this point, the liquidity premium is zero and $\dot{\phi} = r\phi - d = 0, \forall M \in [0,1]$, even though all capable agents are mining.

Any monetary equilibria with abundant liquidity must be such that $\left(\frac{w(q^*)}{d/r}, d/r\right)$ is the first point to be reached of the price $\phi = d/r$

- If the economy starts with $\phi_0 \leq d/r$, then money would have an ever decreasing value even if agents decide to mine.
- If the economy reaches $(A, d/r)$ with $A < \frac{w(q^*)}{d/r}$, then $\dot{\phi} < 0, \forall M_t < 1$ and money would have zero value in the long run. This cannot happen if the money bears dividends.
- The economy cannot reach $(A, d/r)$ with $A > \frac{w(q^*)}{d/r}$ if it has not passed through $\left(\frac{w(q^*)}{d/r}, d/r\right)$. Were it to do so, then at some point it would have crossed the threshold level for $\dot{\phi}|_{m=M(A)} = 0$ at a point with $\phi > d/r$ and money would have an ever increasing value.

Now we obtain lower and upper bounds for the equilibrium initial price.

1. Given a $q_0 < q^*$, we can obtain a unique ϕ_0^- such that an economy starting at $(0, \phi_0^-)$ would achieve $\left(\frac{w(q_0)}{d/r}, d/r\right)$. Let $q_n \to q^*$ with $q_n < q_{n+1}$ and let ϕ_n^- be the associated initial price obtained in the aforementioned way. We know ϕ_n^- is strictly increasing.
2. Given an $A_0 < \frac{w(q^*)}{d/r}$, we can obtain a unique ϕ_0^+ such that an economy starting at $(A, \phi) = (0, \phi_0^+)$ would have $\dot{\phi}|_{m=M(A_0)} = 0$. Let $A_n \to \frac{w(q^*)}{d/r}$ with $A_n < A_{n+1}$ and let ϕ_n^+ be the associated initial price obtained in the aforementioned way. We know ϕ_n^+ is strictly decreasing.
3. Note that $\phi_n^- < \phi_m^+$ for any n, m, ϕ_n^- is strictly increasing and ϕ_m^+ is strictly decreasing, so it must be that $\phi_n^- \to \phi^- \leq \phi^+ \leftarrow \phi_m^+$. Any trajectory starting with $\phi_0 \in [\phi^-, \phi^+]$ is an equilibrium path.

It is easy to conclude that $\phi^- = \phi^+$. Were they different, any trajectory that started from ϕ^- would be steeper than any trajectory starting from $(\phi^-, \phi^+]$ for every A_t, thus reaching a strictly lower ϕ_t for every A_t until the abundant liquidity is reached. Because all equilibria must have $\phi_t = d/r$ at abundant liquidity, it must be that there is a unique equilibrium.