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The Constant Amortization Scheme With Multiple Contracts

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CLOVIS DE FARO

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The Constant Amortization Scheme With Multiple Contracts

1 - Introduction

Since its inception by the Brazilian System of Home Financing (Sistema Financeiro de Habitação – SFH) in 1971, the constant amortization scheme has become a very popular method of debt financing, even competing, particularly for home financing, with the traditional method of constant payments.

For the case of the constant payments scheme of debt amortization, De-Losso, Giovannetti and Rangel (2013), proposed a multiple contracts variation that may imply, in terms of present values, substantial income tax reduction for the financial institutions granting the loans.

The purpose of the present paper is to extend their analysis to the case of the constant amortization method of debt financing. It will be shown that if a single contract is split into multiple contracts, one for each of the payments of the single contract, the tax reductions for the financial institution can be even greater.

2 - The case of a single contract

Denoting by F the loan amount, and by i the periodic interest rate, suppose that, with a single contract, it has been stipulated that the debt has to be reimbursed by n periodic payments in accordance with the constant amortization scheme.

As it is well known (cf. de Faro, 2014, p. 262), it follows that the outstanding loan balance S_k , immediately after the k -th payment has been made, decreases linearly in such a way that:

$$S_k = F(1 - k/n), \quad k = 1, 2, \dots, n \quad (1)$$

Taking into account that the interest parcel J_k of the k -th payment is equal to $i.S_k$, and that the constant amortization parcel is equal to F/n , it follows that the k -th payment, denoted by p_k , decreases also linearly, in such a way that:

$$p_k = (F/n)\{1 + (n+1-k)i\}, \quad k = 1, 2, \dots, n \quad (2)$$

That is, the periodic payments follow an arithmetic progression which ratio is $-i.F/n$.

At this point, it should be noted that the contract implies that the borrower will have to pay, from a strict accounting point of view, a total of $(n+1)i.F/2$ as interest.

3 - The case of n subcontracts

Alternatively, suppose now that, instead of a single contract, the borrower is required to sign n subcontracts: one for each of the n payments that would be associated with the case of a single contract. With the principal of the k -th subcontract being the present value, at the same interest rate i , of the k -th payment of the single contract.

That is, the principal of the k -th subcontract, denoted by F_k , is:

$$F_k = p_k (1+i)^{-k} = (F/n) \{1 + (n+1-k)i\} (1+i)^{-k}, \quad k = 1, 2, \dots, n \quad (3)$$

In this case, the parcel of amortization associated with the k -th payment, which will be denoted by \hat{A}_k , will be:

$$\hat{A}_k = F_k = (F/n) \{1 + (n+1-k)i\} (1+i)^{-k}, \quad k = 1, 2, \dots, n \quad (4)$$

Namely, the parcel of amortization associated with the k -th subcontract is exactly equal to the value of the corresponding principal.

On the other hand, from an accounting point of view, it follows that the parcel of interest associated with the k -th subcontract, which will be denoted by \hat{J}_k , is.

$$\hat{J}_k = (F/n) \{1 + (n+1-k)i\} \{1 - (1+i)^{-k}\}, \quad k = 1, 2, \dots, n \quad (5)$$

It should be noted that:

a) For the set of the n subcontracts, the consolidated outstanding loan balance, just after the k -th payment, which will be denoted by \hat{S}_k , is:

$$\hat{S}_k = F(1 - k/n), \quad k = 1, 2, \dots, n \quad (6)$$

That is, $\hat{S}_k = S_k$, for all k . Hence, we will also have the linear decrease of the consolidated loan balance.

In other words, for the set of the n subcontracts, we have the basic characteristic of the constant amortization scheme as well.

b) From the strict accounting point of view, the total of interest payments is the same both in the case of a single contract as well as in the case of the n subcontracts. However, the present value of the difference of the respective sequences of the parcels of interest is, for any positive interest ρ , always positive. That is:

$$\sum_{k=1}^n J_k (1+\rho)^{-k} - \sum_{k=1}^n \hat{J}_k (1+\rho)^{-k} > 0 \quad (7)$$

assuming that the interest rate ρ has the same period as the rate i .

The implication is, therefore, that similar to the case that was addressed by De-Losso, Giovannetti and Rangel (2013), which refers to the case of constant payments, the financial institutions may derive substantial fiscal gains.

4 – The fiscal gain

Before proceeding with a formal analysis, it is convenient to present a numerical example.

In Table 1 we have the evolution of the numerical values associated with the case of a loan of R\$ 1.200.000,00, which has to be paid in 12 monthly payments, with the interest rate of 2% per month, considering the constant amortization scheme.

That is, in Table 1 we have the corresponding values of $S_k, p_k, \hat{A}_k, J_k, \hat{J}_k$, as well as the difference $d_k = J_k - \hat{J}_k$, for $k = 1, 2, \dots, 12$.

Table 1
Evolution of the Numerical Values

k	S_k	p_k	\hat{A}_k	J_k	\hat{J}_k	d_k
0	1.200.000,00	–	–	–	–	–
1	1.100.000,00	124.000,00	121.568,63	24.000,00	2.431,37	21.568,63
2	1.000.000,00	122.000,00	117.262,59	22.000,00	4.737,41	17.262,59
3	900.000,00	120.000,00	113.078,68	20.000,00	6.921,32	13.078,68
4	800.000,00	118.000,00	109.013,76	18.000,00	8.986,24	9.013,76
5	700.000,00	116.000,00	105.064,77	16.000,00	10.935,23	5.064,77
6	600.000,00	114.000,00	101.228,74	14.000,00	12.771,26	1.228,74
7	500.000,00	112.000,00	97.502,74	12.000,00	14.497,26	-2.497,26
8	400.000,00	110.000,00	93.883,94	10.000,00	16.116,06	-6.116,06
9	300.000,00	108.000,00	90.369,57	8.000,00	17.630,43	-9.630,43
10	200.000,00	106.000,00	86.956,92	6.000,00	19.043,08	-13.043,08
11	100.000,00	104.000,00	83.643,36	4.000,00	20.356,64	-16.356,64
12	0,00	102.000,00	80.426,30	2.000,00	21.573,70	-19.573,70
Σ	–	1.356.000,00	1.200.000,00	156.000,00	156.000,00	0,00

(Values in R\$)

The numerical values in Table 1 present a basic feature. The sequence of differences $\{d_1, d_2, \dots, d_n\}$ is increasing and with only one change of sign. The implication being, as it will be shown, that this sequence characterizes a conventional financing project to which, as it is well known (cf. de Faro, 1974), is associated a unique internal rate of return. Which is null, as

$$\sum_{k=1}^n d_k = \sum_{k=1}^n (J_k - \hat{J}_k) = 0.$$

Let us now proceed with the proof that the sequence $\{d_1, d_2, \dots, d_n\}$ characterizes a conventional financing project in which the initial components of the cash flow are positive, followed by negative components.

To this end, we will make use of the following results.

$$a) J_{k+1} - J_k < 0$$

Trivially, from relation (1), we have:

$$J_{k+1} - J_k = -i.F / n \quad (8)$$

As with the sequence of payments, the sequence of the parcels of interest also follows an arithmetic progression with ratio $-i.F / n$.

$$b) \hat{J}_{k+1} - \hat{J}_k < 0$$

From relation (5), we have:

$$\hat{J}_{k+1} - \hat{J}_k = (i.F / n) \left\{ [2 + (n+1-k)i] (1+i)^{-k-1} - 1 \right\} \quad (9)$$

Therefore, for $k = 1, 2, \dots, n-1$, it follows that:

$$d_{k+1} - d_k = -(i.F / n) \left\{ 2 + (n+1-k)i \right\} (1+i)^{-k-1} < 0, \text{ if } i > 0 \quad (10)$$

Consequently, the sequence $\{d_1, d_2, \dots, d_n\}$ is increasing, and in such a way that:

$$d_1 = i.F(n-1)(1+i)^{-1} / n > 0, \text{ if } i > 0 \text{ and } n \geq 2 \quad (11)$$

and

$$d_n = (F / n) \left\{ (1+i)^{-n} - 1 \right\} < 0, \text{ if } i > 0 \text{ and } n \geq 2 \quad (12)$$

Therefore, the sequence $\{d_1, d_2, \dots, d_n\}$ can be characterized as a conventional financing project with its unique internal rate of return being null.

Thus, for any positive interest ρ , which can be interpreted as representing the opportunity cost for the financial institution, we have that:

$$\sum_{k=1}^n d_k (1+\rho)^{-k} = \sum_{k=1}^n J_k (1+\rho)^{-k} - \sum_{k=1}^n \hat{J}_k (1+\rho)^{-k} > 0 \quad (13)$$

In other words, in terms of present values, the financial institution has a fiscal gain if a single contract is substituted by n subcontracts, one for each of the n payments.

5 – Relevance of the fiscal gain

For the case of a single contract, the present value at the positive interest rate ρ of the sequence of the parcels of interest, denoted by $V_1(\rho)$, is:

$$\begin{aligned}
V_1(\rho) &= \sum_{k=1}^n i.F(1-k/n)(1+\rho)^{-k} \\
&= (i.F) \left\{ n.\rho + (1+\rho)^{-n} - 1 \right\} / (n.\rho^2)
\end{aligned} \tag{14}$$

On the other hand, for the case of the n subcontracts, the present value of the sequence of the parcels of interest, denoted by $V_2(\rho)$, is:

$$\begin{aligned}
V_2(\rho) &= \sum_{k=1}^n (F/n) \left\{ 1 + (n+1-k)i \right\} \left\{ 1 - (1+i)^{-k} \right\} (1+\rho)^{-k} \\
&= (F/n) \left\{ \frac{(i-\rho) \left[(1+\rho)^{-n} - 1 \right] + n.i.\rho}{\rho^2} - \frac{(i-\hat{\rho}) \left[(1+\hat{\rho})^{-n} - 1 \right] + n.i.\hat{\rho}}{\hat{\rho}^2} \right\}
\end{aligned} \tag{15}$$

where $\hat{\rho} = \rho + i + \rho.i$

With $V_1(0) = V_2(0) = i.F(n+1)/2$, if $\rho = 0$

In Tables 2 and 3, which refer to the cases where the contractual interest rate $i = 1\%$ p.m. and $i = 2\%$ p.m., respectively, we present, as a function of the number n_a of years of contract, the percentual increase of the fiscal gain δ , given by $V_1(\rho_a)/V_2(\rho_a) - 1$, for some values of the anual interest rate ρ_a , which express the opportunity cost of the financial institution.

Table 2
Fiscal Gain When $i = 1\%$ p.m.

n_a	$\rho_a(\%)$						
	0	5	10	15	20	25	30
1	0	1,5	2,9	4,2	5,5	6,8	8,0
5	0	7,1	14,3	21,6	28,9	36,3	43,4
10	0	12,2	26,4	40,3	54,6	69,0	83,4
15	0	17,4	35,9	55,1	74,7	94,2	113,1
20	0	29,9	43,4	66,4	89,6	112,3	134,5
25	0	23,8	49,1	74,9	100,4	125,2	149,1
30	0	26,4	53,6	81,3	108,3	139,4	159,4

Table 3
Fiscal Gain When $i = 2\%$ p.m.

n_a	$\rho_a(\%)$						
	0	5	10	15	20	25	30
1	0	1,4	2,8	4,1	5,3	6,6	7,8
5	0	6,2	12,4	18,6	24,8	30,9	37,0
10	0	10,0	20,3	30,6	40,9	51,2	61,3
15	0	12,5	25,2	38,1	50,7	63,2	75,3
20	0	14,1	28,5	42,8	56,8	70,4	83,5
25	0	15,3	30,7	45,9	60,7	74,9	88,6
30	0	16,1	32,3	48,1	63,3	78,0	92,0

The results presented in Tables 2 and 3 are sufficient to support the conclusion that the fiscal gains may be very significant. Therefore, disregarding administrative costs, the financial institution may derive expressive fiscal gains if a single contract is substituted by n subcontracts.

6 – Comparison with the case of constant payments

The results presented in the previous section, coupled with those presented by De-Losso, Giovannetti and Rangel (2013), indicate that the financial institutions may derive substantial fiscal gains, if a single contract is substituted by n subcontracts, one for each payment, both in the case of the constant amortization scheme and in the case of the constant payments scheme.

As the financial institution may have the possibility of choosing between the two amortization schemes, it is pertinent to extend the analysis to include a comparison of the corresponding fiscal gains.

Considering the same parameters of section 2, it is well known (cf. de Faro (2014, p. 248)), that the constant payment, denoted by p , is such that:

$$p = i.F / \left\{ 1 - (1+i)^{-n} \right\} \quad (16)$$

With the k -th parcel of interest, denoted by J_k' , as given by De-Losso, Giovannetti and Rangel (2013, expression 6), being equal to:

$$J_k' = p \left\{ 1 - (1+i)^{k-n-1} \right\}, \quad k = 1, 2, \dots, n \quad (17)$$

For our purpose, it should be noted that:

$$J'_{k+1} - J'_k = -i.p(1+i)^{-k-n-1} < 0, \text{ if } i > 0 \quad (18)$$

That is, the constant payments scheme implies that the parcels of interest form a decreasing sequence.

On the other hand, in the case of subdivision in n subcontracts, it follows that parcel of interest associated with the k -th contract, denoted by \hat{J}'_k , is:

$$\hat{J}'_k = p\{1 - (1+i)^{-k}\}, \quad k = 1, 2, \dots, n \quad (19)$$

As

$$\hat{J}'_{k+1} - \hat{J}'_k = i.p(1+i)^{-k-1} > 0, \text{ if } i > 0 \quad (20)$$

it follows that the parcels of interest, in the case of subdivision in n subcontracts, form an increasing sequence.

At this point, it seems to be pertinent to present a numerical comparison with what was presented in Table 1.

In Table 4, which also refers to a loan of R\$ 1.200.000,00, with 2% p.m. interest rate, now with 12 constant monthly payments of R\$ 113.471,52, we present the values of J'_k , \hat{J}'_k , of the difference $d'_k = J'_k - \hat{J}'_k$, and of \hat{J}_k , as given in Table 1, as well as of the difference $d''_k = \hat{J}_k - \hat{J}'_k$. for $k = 1, 2, \dots, 12$.

Table 4
Comparison of the Two Schemes

k	J'_k	\hat{J}'_k	d'_k	\hat{J}_k	d''_k
1	24.000,00	2.224,93	21.775,07	2.431,37	206,44
2	22.210,57	4.406,24	17.804,33	4.737,41	331,17
3	20.385,35	6.544,77	13.840,58	6.921,32	376,55
4	18.523,63	8.641,37	9.882,26	8.986,24	344,87
5	16.624,67	10.696,87	5.927,80	10.935,23	238,36
6	14.687,73	12.712,06	1.975,67	12.771,26	59,20
7	12.712,06	14.687,73	-1.975,67	14.497,26	-190,47
8	10.696,87	16.624,67	-5.927,80	16.116,06	-508,61
9	8.641,37	18.523,63	-9.882,26	17.630,43	-893,20
10	6.544,77	20.385,35	-13.840,58	19.043,08	-1.342,27
11	4.406,24	22.210,57	-17.804,33	20.356,64	-1.853,93
12	2.224,93	24.000,00	-21.775,07	21.573,70	-2.426,30
Σ	161.658,19	161.658,19	0,00	156.000,00	-5.689,19

The following points should be stressed:

- a) From the strict accounting point of view, the total amount of interest in the case of the constant payment scheme, which is equal to $n \cdot p - F$, is greater than the corresponding one in the case of the constant amortization scheme; which amounts to $i \cdot F (n+1)/2$.

The difference can be substantial. For instance, for a contract of 30 years with monthly payments, if the effective annual interest rate is 10%, the debtor may have to pay over 42% more, if he chooses the constant payment scheme instead of the constant amortization scheme.

b) As

$$d'_{k+1} - d'_k = -i \cdot p \left\{ (1+i)^{-k-1} + (1+i)^{k-n-1} \right\} < 0, \text{ if } i > 0 \quad (21)$$

it follows that the sequence $\{d'_1, d'_2, \dots, d'_n\}$ is decreasing, and with a unique change of sign since

$$d'_1 = p(1+i)^{-1} \left\{ 1 - (1+i)^{1-n} \right\} > 0, \text{ if } i > 0 \text{ and } n \geq 2, \text{ and } d'_n = p \left\{ (1+i)^{-n} - (1+i)^{-1} \right\} < 0, \text{ if } i > 0 \text{ and } n \geq 2$$

Consequently, the sequence $\{d'_1, d'_2, \dots, d'_n\}$ also characterizes a conventional financing project with its unique internal rate of return being null.

In other words, for any positive interest rate ρ , it follows that $V_3(\rho) > V_4(\rho)$, where:

$$V_3(\rho) = p \left\{ \frac{1 - (1+\rho)^{-n}}{\rho} - \left[\frac{(1+i)^{-n} - (1+\rho)^{-n}}{\rho - i} \right] \right\} \quad (22)$$

an expression which is analogous of relation (8) in De-Losso, Giovannetti and Rangel (2013), which is valid only if $\rho \neq i$, or

$$V_3(\rho) = p \left\{ \frac{1 - (1+\rho)^{-n}}{\rho} - n(1+i)^{-n-1} \right\}, \text{ if } \rho = i \quad (22')$$

with

$$V_4(\rho) = p \left\{ \frac{1 - (1+\rho)^{-n}}{\rho} - \frac{1 - (1+\hat{\rho})^{-n}}{\hat{\rho}} \right\} \quad (23)$$

an expression which is analogous of relation (13) in De-Losso, Giovannetti and Rangel (2013), if the income tax rate λ is equal to one.

- c) In the case of our numerical example, it should be noted that the sequence $\{d''_1, d''_2, \dots, d''_n\}$ is initially increasing, and subsequently decreasing.

As a consequence, it is not possible to assure that this sequence characterizes a conventional financing project. Thus, in principle, observing that $V_4(0) > V_2(0)$, it is not possible to assure that we will have $V_4(\rho) > V_2(\rho)$, for any positive interest rate ρ .

As a numerical illustration, Tables 5 and 6, which consider the monthly interest rates $i = 1\%$ and $i = 2\%$, respectively, present the percentual values of the expression.

$$\delta' = V_4(\rho)/V_2(\rho) - 1, \quad \rho > 0 \quad (24)$$

as well of the expression

$$\delta'' = V_4(0)/V_2(0) - 1 = \frac{2\left\{n \cdot i / \left[1 - (1+i)^{-n}\right]\right\} - 1}{(n+1)i} - 1 \quad (25)$$

which refers to the case where the rate ρ is null.

Tabela 5
Comparison of the Fiscal Gains if $i = 1\%$ a.m.

n (years)	$\rho_a(\%)$						
	0	5	10	15	20	25	30
1	1,82	1,79	1,76	1,73	1,69	1,67	1,64
5	9,73	8,95	8,20	7,48	6,78	6,11	5,48
10	19,28	16,39	13,66	11,12	8,78	6,64	4,69
15	28,21	22,19	16,79	12,05	7,99	4,54	1,63
20	36,32	26,48	18,21	11,52	6,24	2,13	-1,08
25	43,50	29,43	18,49	10,39	4,54	0,31	-2,79
30	49,75	31,26	18,05	9,14	3,21	-0,81	-3,62

Tabela 6
Comparison of the Fiscal Gains if $i = 2\%$ a.m.

n (years)	$\rho_a(\%)$						
	0	5	10	15	20	25	30
1	2,33	3,56	3,50	3,43	3,27	3,32	3,26
5	19,03	17,60	16,20	14,89	13,65	12,47	11,35
10	36,04	30,98	26,41	22,30	18,62	15,36	12,46
15	49,44	39,65	31,37	24,48	18,82	14,18	10,40
20	59,41	44,39	32,68	23,79	17,10	12,07	8,25
25	66,64	46,32	31,85	21,84	14,94	10,11	6,34
30	71,91	46,45	29,95	19,55	12,92	8,23	5,49

In Table 6, which refers to the case where $i = 2\%$, we have a situation where $V_4(\rho)$ is always greater than $V_2(\rho)$. That is, at least when the opportunity cost ρ of the financial institution is not greater than 30% per year, the option for the constant amortization method should be the preferred one.

On the other hand, in Table 5, which refers to the case where $i = 1\%$, we see that we may have cases where the option for the constant payment method should be the preferred one. For instance, this occurs whenever ρ is 30% and n is 20 years or more.

As we do not have an unequivocal dominance, it is suggested that, in a concrete situation, numerical comparisons, making use of relations (15) and (23), should be performed.

7 – Conclusion

Strictly from an accounting point of view, financial institutions may derive significant fiscal gains, in terms of tax reductions, if the policy of substituting a unique contract by n subcontracts, one for each of the periodic payments, is adopted.

Obviously, this holds true taking into due consideration the costs that may be associated with bookkeeping and registration of the subcontracts.

Once the associated costs and the fiscal gains are netted, it is possible that, as previously pointed out by De-Losso, Giovannetti and Rangel (2013), the implementation of the policy of multiple contracts may even imply in a reduction of the interest rates that are currently charged by the financial institutions.

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