SUBADDITIVE PROBABILITIES AND PORTFOLIO INERTIA

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Resumo

Mostra-se que com preferências pela incerteza (no sentido de Knight) dadas pela axiomatização de Schmeidler (1982, 1984) e Gilboa (1987) (e não pela de Savage (1954)) pode-se obter inércia na escolha ótima de carteira com quantidades positivas de todos os ativos. O artigo também apresenta uma resenha unificada da literatura recente sobre a incerteza, com especial ênfase no modelo de probabilidades subaditivas.

Abstract

We show that in the presence of uncertainty (in the sense of Knight), as axiomatized by Schmeidler (1982, 1984) and Gilboa (1987) (as opposed to the classical view of Savage (1954)) one may obtain portfolio inertia with positive quantities held of all assets. We also present a comprehensive survey of the recent literature on uncertainty, with special emphasis on the subadditive probabilities model.

1. Introduction.

Why small changes in relative asset prices do not necessarily lead to an immediate change in individual portfolios is an intriguing question in the theory of financial decisions. Portfolio rigidities may obviously be caused by transaction costs. Corner equilibria, which may emerge when short sales are ruled out can be another reason for inertia. Yet, in many empirical cases, a more powerful source of friction appears to be responsible for the lack of response of quantities to prices.

Subadditive probabilities provide the proper analytical tool to describe this stronger source of portfolio inertia. Formally, a subadditive probability space $(\Omega; \mathcal{A}; \pi)$ is a non-void set Ω (universe), an algebra $\mathcal{A}$ of subsets of Ω (measurable events), and a real valued function π defined for every element of $\mathcal{A}$, such that:

i) $\pi(\phi) = 0$

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ii) $\pi(\Omega) = 1$

iii) $\pi(A \cup B) + \pi(A \cap B) \geq \pi(A) + \pi(B)$ for all measurable events A and B.

Subadditive probabilities have been studied from the probabilistic viewpoint by Dempster (1967) and Shafer (1976).

A random variable X in a subadditive probability space is a real valued measurable function defined for every point $w \varepsilon \Omega$.

From the point of view of decision theory, Schmeidler (1982, 1984), Gilboa(1987) and Gilboa and Schmeidler (1989) analyse subadditive probabilities. They describe a mix of pure risk and pure uncertainty according to Frank Knight's classical distinction (Knight (1921)). An alternative formalization of the notion of uncertainty is due to Bewley(1986). It also leads to inertia in a sense similar to the one defined here. Bewley's model, however, has much less predictive power than Schmeidler and Gilboa's, framework.

Uncertainty means, in fact, incomplete information about the true probabilities. All that is known about the true probability $\pi'(B)$ of event B is that it is greater than or equal to $\pi(B)$. Since $B \bigcup B^c = \Omega$, where B^c stands for the complement of B, $\pi(B) + \pi(B^c) \leq 1$, because of the subadditivity. As a consequence, all available information on the true probability of B is that $\pi(B) \leq \pi'(B) \leq 1 - \pi(B^c)$. The degree of uncertainty of event B can be measured by $1 - \pi(B) - \pi(B^c)$. From this point of view, Dow and Werlang(1988) define and examine a measure of uncertainty aversion (or degree of uncertainty).

An easy way to create a subadditive probability space is to take a family $(\Omega, \mathcal{A}, \pi_j)$ of probability spaces, for the same universe and the same algebra of measurable events, but different probability functions; then take:

$$\pi(B) = \inf_j \{\pi_j(B)\}$$

Conversely, one may easily prove that every subadditive probability space can be generated by a family of probability functions as above, as shown by Gilboa and Schmeidler(1989).

The attractiveness of the concept of subadditive probabilities is that it might provide the best possible description for what is behind the widespread notion of subjective probabilities in the theory of financial decisions. Probability theory gained its scientific prestige among other reasons because physicists, insurance companies and card players realized that the law of large numbers works in the real world. Now, the whole power of the theory was based on the existence of objective probabilities. Subjective probabilities, where invented to translate uncertainty into a language where the theory of choice of involving risk could be easily used, and this was the base for the development of the theory of financial decisions. Except that it was probably a bad translation. The translation of uncertainty into risk by the use of additive subjective probabilities, was obtained by Savage(1954), and Anscombe and Aumann (1963), under very stringent conditions. Subjective probabilities suggest a lack of empirical information that, at best, allows the individual to estimate the probability of an event within a certain interval, and not as a precise real number. Now, this means describing subjective probabilities not by a conventional, but by a subadditive probability space, where the true probability of an event B lies in the interval $[\pi(B), 1 - \pi(B^c)]$.

Incidentally, the extreme cases of pure risk and pure uncertainty can also be described by particular subadditive probability spaces. The case of pure risk, of course, corresponds to a conventional additive probability. Total uncertainty by $\pi(B) = 0$ for every measurable event such that $B^c \neq \emptyset$. This is the message that except for the universe and the void set, the probability of any event B may lie in any point of the closed $[0\ ;\ 1]$ interval.

The crucial point is that optimization under pure risk and under pure uncertainty follows different criteria both incidentally explored in the von Neumann-Morgenstern seminal work on game theory (Von Neumann and Morgenstern (1947)). Under pure risk, the individual should maximize his expected utility given his budget constraint. Under pure uncertainty, there is no other possible advice but prudence, that leads to a maxmin choice: the portfolio should be selected so as to maximize the individual's wealth in the worst possible state of nature.

A convenient definition of expected value EX of a random variable X in a subadditive probability space unifies the two theories of choice, one involving risk the other involving uncertainty. The trick is to define EX as the lowest possible expected value of X in an additive probability space compatible with the available information on probabilities:

$$EX = \inf_{\pi_j} \quad E_j X \tag{1}$$

where $E_j X$ is the expected value of X for an additive probability function consistent with π_j. Equation (1) means solving the uncertainty issue by the most prudent rule: since the true expected value of X is uncertain, let us take its infimum.

As an example, let us assume a subadditive probability space with three states of nature. Observe that, since the subadditive probability of the union of disjoint sets is, in principle, different from the sum of the probabilities of the sets, then one has to define the probability for all events. In this case, we have, for example, $\pi_{12} =$ probability of $\{1, 2\}$, and so forth. Thus, let us consider:

$\pi_1 = 0.2$
$\pi_2 = 0.2$
$\pi_3 = 0.2$
$\pi_{12} = 0.5$
$\pi_{13} = 0.6$
$\pi_{23} = 0.6$
$\pi_{123} = 1$

and that the values of the random variable in the three states are $X_1 = 1$, $X_2 = 5$, $X_3 = 3$.

Indicating by π_1', π_2', π_3' the true probabilities of the state of nature ($\pi_1' + \pi_2' + \pi_3' = 1$), the true expected value of X is $\pi_1' X_1 + \pi_2' X_2 + \pi_3' X_3 = \pi_1' + 5\pi_2' + 3\pi_3'$. Since π_1', π_2', π_3' are uncertain, the lowest possible value for EX is obtained by taking π_2' as low as possible, namely, $\pi_2' = 0.2$; then, π_3' as low as possible, given π_2', which yields $\pi_3' = \pi_{23} - \pi_2' = 0.4$. It remains only one possibility for π_1', namely, $\pi_1' = 0.4$, and as a consequence:

$$EX = 0.4 \times 1 + 0.2 \times 5 + 0.4 \times 3 = 2.6$$

The example above indicates how the expected value of a random variable in a subadditive probability space should be computed by the safest level criterium: given the available information, states of nature with high realizations of the random variable should be weighted as little as possible. A general formula can be provided in terms of distribution functions. Given a random variable X, for every real number μ let us define:

$$F(\mu) = 1 - \pi(X > \mu)$$

$F(\mu)$ is the maximum possible probability of the event $(X > \mu)$. Then, compute EX according to the usual formula:

$$EX = -\int_{-\infty}^{0} F(\mu)d\mu + \int_{0}^{\infty} (1 - F(\mu))d\mu \qquad (2)$$

This the formula of Choquet's (1955) integral. The general relation between the two ways of defining the expected value of a random variable is described in Gilboa and Schmeidler(1989).

Some important propositions about expected values in subadditive probability spaces are listed below:

Proposition 1:
probability space, then:

$$E(X+Y) \geq EX + EY \qquad (3)$$

Proof: According to (1),

$$E(X+Y) = \inf_{\pi_j} E_j(X+Y) = \inf_{\pi_j} (E_jX + E_jY)$$
$$\geq \inf_{\pi_j} E_jX + \inf_{\pi_j} E_jY = EX + EY$$

□

Proposition 2: If X is a random variable in a subadditive probability space, c a positive constant, then:

$$E(cX) = cEX$$

Proof: $E(cX) = \inf_{\pi_j}\{E_jcX\} = \inf_{\pi_j}\{cE_jX\} = c \inf_{\pi_j} \{E_jX\} = cEX$ □

Proposition 3: If X and Y are random variables in a subadditive probability space and if X dominates stochastically Y, i.e, $X(w) \geq Y(w)$ for every $w \varepsilon \Omega$, then $EX \geq EY$.

Proof: We first note that $E_j X \geq E_j Y$ for any additive probability function π_j. Then use (1). □

Proposition 4: If X is a random variable in a subadditive probability space, then:

$$E(X) + E(-X) \leq 0$$

Proof: Since $E0 = 0$, the result follows immediately from proposition 1. □

The importance of this proposition is that it sets both, the upper lower bound and the lower upper bound of the true expected value of X, $E_{\pi_j}X$, that must be such that $E_{\pi_j}X + E_{\pi_j}(-x) = 0 \cdot$ EX, by definition, is the upper lower bound of the true expected value $E_{\pi_j}X$;

$$-E(-X) = -\inf_{\pi_j} E_{\pi_j}(-X) = -\inf_{\pi_j} -(E_{\pi_j}X) = +\sup_{\pi_j} E_{\pi_j}X$$

That is to say that the true expected value of X lies in the uncertainty interval $EX \leq E_{\pi_j}X \leq -E(-X)$.

Proposition 5: Let X be a random variable in a subadditive probability space; $H(x)$ a real valued increasing and differentiable function of the real variable x; then, if:

$$F(\mu) = E\ H(\mu X)$$

μ indicating a real variable, $F(\mu)$ is differentiable for every $\mu \neq 0$. For $\mu = 0$ the right side derivative is $F'_+(0) = H'(0)EX$, the left side derivative $F'_-(0) = -H'(0)E(-X)$.

Proof: Let us first assume $\mu > 0$. In this case $H(\mu X)$ is an increasing function of X, except for $\mu = 0$, where it becomes a constant. Hence, if for two additive probability functions $E_{\pi_i}X$

$E_{\pi_i} H(\mu X)$
cording to equation 2 in the same additive probability space. In his case E can be treated as a linear operator and, as a consequence:

$$F'(\mu) = E\ XH'\ (\mu X)$$

For $\mu < 0$ it suffices to make $\mu X = \lambda Y$, where $\lambda = -\mu$ and $Y = -X$.

Proposition 6: Let $P' = (P'_1, P'_2, \ldots, P'_n)$ be an n-dimensional random vector in a subadditive probability space; $y = (y_1, y_2, \ldots, y_n)$ a point in R^n; $P'.y = y_1 P'_1 + y_2 P'_2 + \ldots + y_n P'_n$; and $U(W)$ a real valued concave function of the real variable W. Then:

$$G(y) = EU(P' \cdot y)$$

is a concave function of y.

Proof: Let y' and y'' indicate two points in R^n and a real number such that $0 \leq \mu \leq 1$, since $U(W)$ is concave and $P' \cdot y$ linear:

$$U(P' \cdot ((1-\mu)y' + \mu y'')) \geq (1-\mu)U(P' \cdot y') + \mu U(P' \cdot y'')$$

Hence, by proposition 3 (stochastic dominance):

$$G((1-\mu)y' + \mu y'') = EU(P' \cdot ((1-\mu)y' + \mu y'') \geq E((1-\mu)\ U\ (P' \cdot y') + \mu U(P' \cdot y''))$$

Now, as a result os propositions 2 and 3:

$$E((1-\mu)U(P' \cdot y') + \mu U(P' \cdot y'')) \geq (1-\mu)\ E\ U\ (P' \cdot y') + \mu\ E\ U\ (P' \cdot y'') = (1-\mu)\ G\ (y') + \mu G(y'')$$

□

2. How subadditivity may lead to inertia.

Before we proceed with the general discussion of portfolio choice, let us clarify why subadditive probabilities may lead to inertia. The key point is that the expected value operator E is no longer Gateaux differentiable. As a result, even if the von Neumann-Morgenstern utility function is differentiable, the portfolio indifference curves may be kinked.

The new point of this paper, is to provide an example where portfolio inertia may occur with positive quantities of all assets held. The fact that inertia could happen in such models was first noted in Dow and Werlang (1988). However, in their example, one of the assets always had to have zero demand at the inertia region.

To see that, let us assume that a risk neutral individual is to distribute his wealth between two assets, whose unit present prices are P_1 and P_2. His initial endowment are quantities x_1, x_2 of these assets, which is to say that his initial wealth is measured by $W = P_1x_1 + P_2x_2$. These assets can be purchased or sold without transaction costs, implying that the individual can choose any portfolio with quantities (y_1, y_2) as long as:

$$P_1y_1 + P_2y_2 \leq W = P_1x_1 + P_2x_2 \tag{4}$$

To guarantee the existence of an equilibrium, we shall prohibit short sales, adding to the budget constraint (4) the inequalities:

$$y_1 \geq 0; \ y_2 \geq 0 \tag{5}$$

Given these constraints the individual, who is non satiable and risk neutral, will select his portfolio so as to maximize the expected value $E(y_1P_1' + y_2P_2')$ of his future wealth, P_1', P_2' indicating the random future asset prices.

We now assume that the individual is uncertain about the probability function in the event space. It may be either π_1, π_2 or a convex combination of both. If it is π_1, the expected future asset prices are E_1P_1', E_1P_2'. If it is π_2, these expected values change to E_2P_1', E_2P_2'. To make the discussion interesting we shall assume that:

$$E_1P_1' > E_2P_1' \tag{6a}$$

$$E_2P_2' > E_1P_2' \tag{6b}$$

What is the expected value of the individual's future wealth with a portfolio (y_1, y_2) ? All available information is that it may be either $y_1E_1P_1' + y_2E_1P_2'$, $y_1E_2P_1' + y_2E_2P_2'$ or some convex combination of these values. The maxmin rule is to choose (y_1, y_2) in the budget constraint area so as to maximize:

$$E(y_1P_1' + y_2P_2') = \min \{y_1E_1P_1' + y_2E_1P_2';\ y_1E_2P_1' + y_2E_2P_2'\}$$

The portfolio indifference curves are very easy to draw since they are homothetical, as shown in figure 1. In region I, $y_1E_1P_1' + y_2E_1P_2' > y_1E_2P_1' + y_2E_2P_2'$, which implies:

$$\frac{y_1}{y_2} > \frac{E_2P_2' - E_1P_2'}{E_1P_1' - E_2P_1'}$$

and $E(y_1P_1' + y_2P_2') = y_1E_2P_1' + y_2E_2P_2'$. The absolute value of the slope of any indifference curve in this region is given by:

$$-\frac{dy_2}{dy_1} = \frac{E_2P_1'}{E_2P_2'}$$

In region II, $y_1E_2P_1' + y_2E_2P_2' > y_1E_1P_1' + y_2E_1P_2'$, which is to say that:

$$\frac{y_1}{y_2} < \frac{E_2P_2' - E_1P_2'}{E_1P_1' - E_2P_1'}$$

with $E(y_1P_1' + y_2P_2') = y_1E_1P_1' + y_2E_1P_2'$ and with indifference curve slopes such that:

$$-\frac{dy_2}{dy_1} = \frac{E_1P_1'}{E_1P_2'}$$

Inequalities (6a) and (6b) imply, since expected asset prices should be non negative:

$$\frac{E_1P_1'}{E_1P_2'} > \frac{E_2P_1'}{E_2P_2'}$$

which is to say that indifference curves are steeper in region II compared to region I, as in figure 1.

Now let us discuss the problem of portfolio inertia. Given the budget constraint and the present asset prices, the conclusion is that as long as the present asset prices P_1 and P_2 are such that:

$$\frac{E_1 P_1'}{E_1 P_2'} > \frac{P_1}{P_2} > \frac{E_2 P_1'}{E_2 P_2'} \tag{7}$$

the optimum portfolio will be the point of the budget frontier AB. $P_1(y_1 - x_1) + P_2(y_2 - x_2) = 0$ in the borderline of regions I and II, namely, where:

$$\frac{y_1}{y_2} = \frac{E_2 P_2' - E_1 P_2'}{E_1 P_1' - E_2 P_1'}$$

If the initial endowment of the individual is already on this ON straight line, a small change in present asset prices, which means a rotation of his budget constraint with the fixed point N, will not change the individual optimum portfolio, as long as inequalities (7) are fulfilled. Out of these inequalities, the individual will concentrate his portfolio in a single asset, moving to a corner equilibrium, since short sales are prohibited.

Why a portfolio along the straight line ON is inertial, as opposed to chose in region I or region II, deserves both a mathematical and an economically intuitive explanation. The first is obvious: along the ON straight line, expected utility is not a differentiable function of the portfolio quantities. The second seems harder to generalize, but is obviously more inspiring: an ON portfolio clears uncertainties. In fact, with such a portfolio proportions, $y_1 E_1 P_1' + y_2 E_1 P_2' = y_1 E_2 P_1' + y_2 E_2 P_2'$, namely, its future expected value is no longer uncertain. The general case is much more complicated. Section 4 explores the case of pure uncertainty.

How subadditive probabilities can lead to portfolio inertia was shown for a particular case by Dow and Werlang(1989). An individual must distribute his wealth W between two assets: money, which is certain, riskless but that has no yield; and a risky and uncertain stock the present price of which is indicated by P. The individual is non satiable, and risk averse. Short sales are unrestricted, so that his budget constraint is given by:

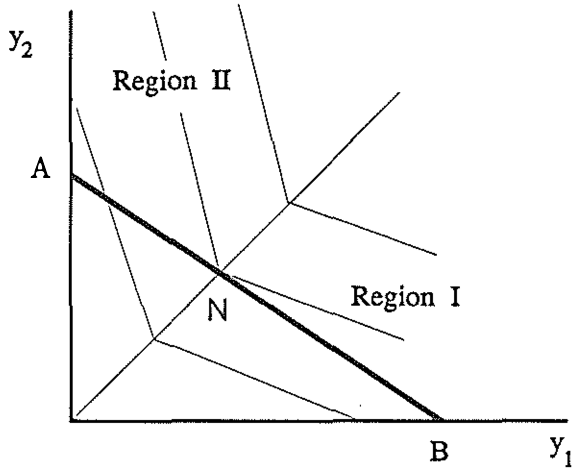

Figure 1.

$$y_1 + Py_2 \leq W$$

y_1 and y_2 indicating the quantities of money and of the risky asset. Since utility is an increasing function of wealth, the individual will operate on his budget frontier, where $y_1 = W - Py_2$, choosing y_2 so as to maximize:

$$F(y_2) = EU(y_1 + P'y_2) = EU(W + y_2(P' - P))$$

where P' is the random future price of the stock. Now, $U(W + y_2(P' - P))$ is an increasing function of $P' - P$. Hence, according to proposition 5, $F(y_2)$ is differentiable for every $y_2 \neq 0$, but may be non differentiable for $y_2 = 0$. For $y_2 = 0$, the right and left side derivatives are:

$$F'_+(0) = U'(W)\ (EP' - P)$$
$$F'_-(0) = -U'(W)\ E\ (P' - P) = -U'(W)\ (P + E(-P'))$$

Espected utility will be maximum for $y_2 = 0$ if and only if

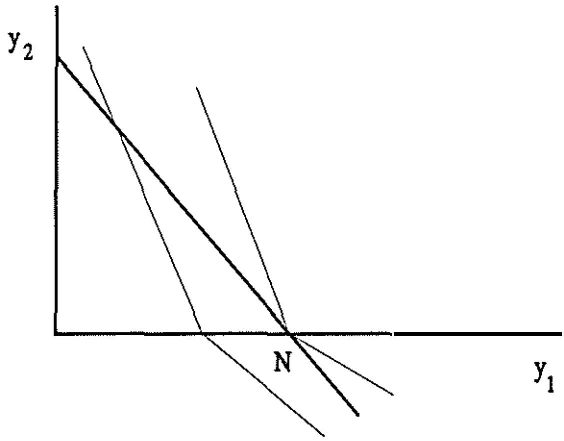

Figure 2.

$$F'_+(0) \leq 0 \leq F'_-(0)$$

namely, if and only if:

$$EP' \leq P \leq -E(-P') \tag{8}$$

Inertia occurs if the present stock price P lies in this interval. The individual will neither buy nor sell short the risky asset, concentrating all his wealth on money. The reason is easy to understand: the individual is uncertain whether the true expected price of the stock in the future will be higher or lower than its present price.

Outside the interval (8) the individual will either buy or sell short some quantity of the stock. Now inertia is no longer possible, since $F(y_2)$ is differentiable for $y_2 \neq 0$. In the example due to Dow and Werlang all expected indifference curve kinks lie in the horizontal axis, as in figure 2.

3. Portfolio optimization.

We can now present the general portfolio selection problem with subadditive probabilities. A non satiable and risk averse individual has a von Neumann-Morgenstern utility function $U(W)$, namely U is an increasing and strictly concave function of wealth W. There are n assets, the initial prices of which are described by the n-dimensional vector P. The individual is initially endowed with quantities $x \varepsilon R^n$ of these assets, so that his initial wealth is given by $W = P \cdot x \cdot$ Assets can be purchased and sold without transaction costs. Moreover, short sales are unrestricted. This is to say that any $y \varepsilon R^n$ represents a feasible portfolio as long as:

$$P \cdot y \leq P \cdot x \tag{9}$$

Future asset prices are indicated by the n-dimensional random vector P'. The problem of the individual is to choose a feasible portfolio y so as to maximize the expected utility of his future wealth:

$$F(y) = EU(P'.y)$$

Formally, the problem appears the same of portfolio choice with additive probabilities. This is due to the definition of expected value of a random variable with subadditive probabilities. The difference now is that even if the utility function is differentiable, the expected utility $F(y)$ may not be. The hypothesis of risk aversion, however, avoids most of the complications of non differentiability. In fact, according to proposition 6, $F(y)$ is a concave function of y, defined in all R^n, since shorts sales are unrestricted. And a concave function defined in an open set has a derivative in every possible direction.

Let us describe the geometrical solution to the problem. Since the individual is non satiable, the portfolio equilibrium y_0, assuming it exists, will lie in the budget frontier $P \cdot y = P \cdot x$. Let us indicate by:

$$B(y_0) = \{y \varepsilon R^n | F(y) > F(y_0)\}$$

$B(y_0)$ is the set of portfolio combinations with expected utility higher than y_0. Obviously y_0 is an equilibrium portfolio if and only if any point of $B(y_0)$ violates the individual's budget constraint. Moreover, since $F(y)$ is concave, $B(y_0)$ is a convex subset of R^n. These observations are summarized in:

Proposition 7: A feasible portfolio y_0 is optimal if and only if the budget frontier $P \cdot (y - x) = 0 \cdot$ is a supporting hyperplane of $B(y_0)$ at y_0. Now, the proposition above admits two subcases:

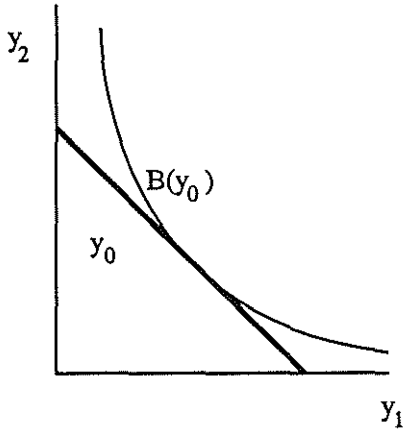

Figure 3.

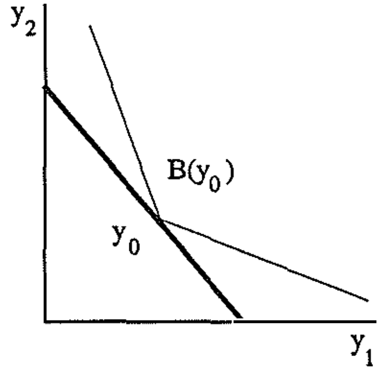

Figure 4.

a) The supporting hyperplane of $B(y_0)$ at y_0 is unique (figure 3). In this case, any normal of the supporting hyperplane must be proportional to the present asset price vector P. This is the case where $F(y)$ is differentiable at y_0. This is the non inertial case: any change in relative present asset prices must displace the portfolio equilibrium;

b) There are multiple supporting hyperplanes of $B(y_0)$ at y_0 (figure 4). This is the case where $F(y)$ is non differentiable at y_0. The budget frontier can rotate around the fixed point y_0 as a result of a change in relative asset prices, as long as P remains as a normal to one of these multiple supporting hyperplanes. This is the inertial case: within certain limits: a change in the prices of the assets will leave unchanged the portfolio composition.

To provide an analytical solution to the problem it suffices to recall the definition of directional derivative and supergradient of a concave function and to use the theorem of Kuhn and Tucker.

Let $G(y)$ be a real concave function defined in an open neighborhood of $y \varepsilon R^n$; u a vector of $R^n (u \neq 0)$. The directional derivative

$G'(y,u)$, namely, the derivative of G at point y in the direction u is defined as:

$$G'(y,u) = \lim_{\lambda \to 0+} \frac{G(y+\lambda u) - G(y)}{\lambda}$$

being a positive real variable.

One can prove that:

i) $G'(y,u)$ exists for every $u \neq 0$ and is a superlinear function of u, i.e., $G'(y,\mu u) = \mu G'(y,u)$ for every positive real number μ and $G'(y,u_1+u_2) \geq G'(y,u_1) + G(y,u_2)$;

ii) $G(y)$ has its absolute maximum at y_0 if and only if $G'(y_0,u) \leq 0$ for every $u \neq 0$ in R^n;

iii) there exists at least one vector $v = v(y) \varepsilon R^n$ such that $G'(y,u) \leq v.u$ for every $0 \neq u \varepsilon R^n$, v.u indicating the euclidean inner product of v and u. Any vector v fulfilling this condition is called a supergradient of G at point y:

iv) $G(y)$ is differentiable at y_0 if and only if $G'(y,u)+G'(y;-u) = 0$ for every $u \neq 0$ in R^n. In this case, the supergradient of G at y_0 is unique and $G'(y_0,u) = v.u$. Conversely, if the supergradient of G at y_0 is unique, G is differentiable at y_0.

Let us now turn back to the portfolio optimization problem. According to Kuhn and Tucker's theorem, a feasible portfolio y_0 will be optimal if and only if, for some non negative Lagrange multiplier λ:

$F(y_0) + \lambda\ P.\ (x-y) = F(y_0) \geq F(y) + \lambda P \cdot (x-y)$, for every $y\ \varepsilon\ R^n$

The maximum of the Lagrangean, in terms of directional derivatives, can be expressed by the necessary and sufficient condition:

$$F'(y_0,u) - P \cdot u \leq 0 \text{ for every } u \neq 0 \text{ in } R^n.$$

since the directional derivative of $P \cdot (x-y)$ in the direction u is equal to $-P \cdot u$.

Summing up, a feasible portfolio y_0, located at the individual's budget frontier $P \cdot y = P \cdot x$ is optimal if and only if the present asset vector price P is proportional to a supergradient of $F(y)$ at

y_0. Inertia will occur if and only if this supergradient is not unique, namely, if $F(y)$ is not differentiable at y_0.

4. Pure uncertainty.

A particular case to be discussed is that portfolio choice under pure uncertainty in the sense of Frank Knight. Let $(\Omega; \mathcal{A}, \pi)$ be the subadditive probability space of future events, $P' = (P'_w)_{w \epsilon \Omega}$ the n-dimensional random vector describing future asset prices. For any event $B \neq \emptyset$ such that $B^c \neq \emptyset$, $\pi(B) = 0$, which is to say that its true probability may lie at any point of the closed [0, 1] interval. Accordingly, the expected value and the expected utility of a portfolio $y \epsilon R^n$ are given by:

$$H(y) = E(P' \cdot y) = \inf_{w \epsilon \Omega} \{(P'_w \cdot y)\}$$
$$EU(P' \cdot y) = \inf_{w \epsilon \Omega} U(P'_w \cdot y)$$

or, since $U(W)$ is an increasing continuous function of W:

$$EU(P' \cdot y) = U(H(y))$$

Summing up, an optimal portfolio is one that maximizes $H(y)$ given the individual's budget constraint. (The existence of both, $H(y)$ and of an equilibrium portfolio can be assured if short sales are restricted, namely, $y + y_0 \geq 0$ for some non negative vector (y_0)). The von Neumann-Morgenstern utility function is irrelevant, as long as it is increasing. This should be no surprise, since the role of the utility function is to order preferences involving risk. Is the case under discussion there is no risk but only pure uncertainty.

The interesting point is that pure uncertainty can lead to anything in terms of portfolio inertia, depending on how future asset prices vary along the possible states of nature. Two extreme possibilities are shown below. In both cases Ω is the open interval (0,1), and the individual must distribute his wealth between two assets, namely, $P'_w = (P'_{1w}; P'_{2w})$. Short sales are ruled out, meaning that $y = (y_1, y_2) \geq 0$.

Example A:

$$P'_{1w} = w;\ P'_{2w} = 1 - w$$
$$\text{In this case, } H(y) = \inf_{0<w<1}\{wy_1 + (1-w)y_2\}$$
$$= \min\{y_1; y_2\}$$

This is case of total inertia. The optimum portfolio combines equal quantities of the two assets, independently of the present asset prices. Portfolio indifference curves are straight angles.

Example B:

$$P'_{1w} = -\log w$$
$$P'_{2w} = -\log(1-w)$$

$$\text{Now, } H(y) = \inf_{0<w<1}\{-y_1 \log w - y_2 \log(1-w)\}$$
$$= (y_1 + y_2)\log(y_1 + y_2) - y_1 \log y_1 - y_2 \log y_2.$$

a differentiable function of y. This example shows that uncertainty can lead to portfolio inertia but not necessarily creates inertia.

5. Conclusions.

This paper intended to address the following question: can an investor stay put under small variations of the prices of assets? The very general answer is yes: as long as there is uncertainty, as opposed to risk. It should be clear, however, that uncertainty alone is **not** sufficient to generate inertia. In fact, the example in section four shows it: not even in the case of pure uncertainty can one assure that inertia will occur. It is only the possibility of the existence of such inertia that the paper addresses. The theory of uncertainty, as viewed by Savage (1954), was a simple reduction to subjective risk. Alternatively, we used here Schmeidler-Gilboa's uncertainty theory: the consumer maximizes expected utility with respect to subadditive probabilities. This theory can be applied to several apparently paradoxical phenomena:

(1) Why do some large creditor banks do not sell or buy debt in the secondary market?

(2) The theory of inflationary inertia assumes that people are not Nash- equilibrium players. (See Simonsen (1987)). In a certain sense, this view is that inertia is always an out of equilibrium phenomenon. By having Nash equilibria with non additive probabilities, we find out that inflationary inertia may arise in equilibrium.

(3) Keynes' theory of investment, where "animal spirits" play a rôle, may be viewed as a reflex of uncertainty in the sense described here.

(4) The investigation of Bayesian learning of independent trials of a random variable (as, for example, tossing the same coin over and over), in connection with a subadditive prior, may lead to the intuitive conclusion that uncertainty decreases with experience: the more the uncertain phenomenon is observed, the less uncertain it becomes. We expect to pursue some of these problems in the future.

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