

On the Integrability of Money-Demand Functions by the Sidrauski and the Shopping-Time Models

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Abstract

This paper investigates which properties money-demand functions must satisfy so that they are consistent with Lucas's [Lucas, R.E., Jr., 2000. Inflation and welfare. *Econometrica* 68, 247-274] versions of the Sidrauski and the shopping-time models. We conclude that shopping-time-integrable money-demand functions are necessarily also Sidrauski-integrable, but that the converse is not necessarily true, unless a boundedness assumption on the nominal interest rate is made. Both the log-log with an interest-rate elasticity greater than or equal to one and the semi-log money demands may serve as counterexamples. All the models and results are also extended to the case in which there are several assets in the economy performing monetary functions.

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1.Introduction

Lucas (2000), Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick (2007) are examples of papers using money-demand functions assumed to be consistent with the versions of the Sidrauski (1967) and the shopping-time models employed by Lucas (2000). Other papers in the literature using versions of the Sidrauski and of the shopping-time models which fall into the category we deal with here and which could, therefore, benefit from our results, are Fischer (1979), Siegel (1983), Asako (1983), Weil (1991) and Goodfriend (1997).

Investigating if a certain money-demand specification can be rationalized by such models is a problem in general-equilibrium applied monetary theory which, somehow, parallels the well-known integrability problem in standard partial-equilibrium microeconomics.¹ In microeconomics, given a system of demand functions, one investigates the existence of a utility function from which such demand functions can be derived.

Here, we investigate if a given money-demand function can be rationalized by the particular versions of the Sidrauski and shopping-time models worked out by Lucas (2000). When the answer is in the affirmative, we shall call these demands Sidrauski- and shopping-time-integrable, respectively. We consider Lucas's versions of these models to be particular because utility is assumed to be homothetic (in the Sidrauski model) and because the transacting technology is assumed to be weakly separable (in the shopping-time model). Such versions of these models are theoretically important, among other reasons, because they allow for closed-form calculations of the welfare costs of inflation.

The investigation of integrability arises here as a theoretical complement to Lucas's (2000) analysis regarding the calculation of the welfare costs of inflation in economies with only one type of money. For instance, Lucas provides a numerical solution to the differential equations giving the welfare costs of inflation for each model in the case of a semi-logarithmic money demand. However, he does not investigate if such a functional form is indeed consistent with the versions of the Sidrauski and of the shopping-time model he presents. We take this task

¹See, e.g., Varian (1992, section 8.5).

for ourselves and show that, unless the nominal interest rate must satisfy a certain boundedness property, the semi-logarithmic money demand, although rationalizable by Sidrauski's model, may not be derived from the shopping-time model. Additionally, the log-log money demand, although Sidrauski-integrable, is seen to be shopping-time-integrable if and only if its interest-rate elasticity is less than unity.

Scholnick et al.'s (2006) survey of the literature on credit cards, debit cards and ATMs serves as a suggestion that it may be relevant to broaden our framework and consider the presence of other monetary assets in the economy as well. All our results are extended to this n -dimensional case (the dimension refers to the number of monetary assets available). Lucas (2000) deals exclusively with $n = 1$.

A second contribution of this paper is that of investigating how the Sidrauski model relates to the shopping-time model as far as the integrability problem is concerned. Regarding this second point, our result adds to Feenstra's (1986). This author has demonstrated a functional equivalence between both models. Such an equivalence, though, does not apply to the versions of the Sidrauski and the shopping-time models we deal with here because they do not comply with all of Feenstra's assumptions.

The remainder of the paper is structured as follows. In Section 2 we use the unidimensional case worked out by Lucas and investigate the two main points mentioned above. Section 3 extends the results of Section 2 to a multidimensional setting, including the two aforementioned examples. Section 4 summarizes and concludes the work.

2. The case with only one type of money

In this section we analyze the case of an economy with only one type of money (currency).

2.1. The Sidrauski model

We shall assume, as in Lucas (2000, sec. 3), a forever-living, perfectly-foresighted, representative agent maximizing a time-separable constant-relative-risk-aversion utility function, the arguments of which are the flows of real consumption of a single nonstorable good and

of holdings of real cash balances. For every $t \in [0, +\infty)$, let $B_t \in \mathbb{R}_+$, $M_t \in \mathbb{R}_+$, $H_t \in \mathbb{R}$, $Y_t \in \mathbb{R}_{++}$ and $C_t \in \mathbb{R}_+$ represent the nominal values of, respectively, holdings of government bonds, cash, a lump-sum tax (if negative, a transfer from the government to the individual), the output of the economy and consumption, all measured at instant t of time.

The dynamic budget constraint faced by our representative agent is $\dot{B}_t + \dot{M}_t = Y_t - C_t - H_t + r_t B_t$, where the dots mean time-derivatives and $r_t \in \mathbb{R}_{++}$ stands for the nominal interest rate that bonds yield at time t (cash, by definition, yields a 0 nominal interest rate).

Let $P_t \in \mathbb{R}_{++}$ be the (both expected and realized) price level, $y_t := Y_t/P_t$ real output and $\pi_t := \dot{P}_t/P_t$ the inflation rate at time t . We assume real output grows at constant rate γ ($y_t = y_0 e^{\gamma t}$) and normalize y_0 to 1. Lowercase variables stand for the nominal variables as a share of nominal GDP (that is, $b_t := B_t/Y_t$, $m_t := M_t/Y_t$ etc.).

The utility function is assumed to be homogeneous of degree $1 - \sigma$, in which case we can state our agent's problem (P_{MU}) as:

$$\begin{aligned} \max_{c, b, m \geq 0} \int_0^{+\infty} e^{(-\rho + (1-\sigma)\gamma)t} U(c_t, m_t) dt \text{ subject to} \quad & (P_{MU}) \\ \dot{b}_t + \dot{m}_t = y_t - h_t - c_t + (r_t - \pi_t - \gamma)b_t - (\pi_t + \gamma)m_t, \forall t \in (0, +\infty). \end{aligned}$$

As in Lucas (2000), we make use of a homothetic utility function $U : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \bar{\mathbb{R}}$ given by

$$U(c, m) = \frac{1}{1 - \sigma} \left(c \varphi \left(\frac{m}{c} \right) \right)^{1 - \sigma}, \quad (1)$$

extended by continuity to the ray $\{0\} \times \mathbb{R}_+$, where $\sigma > 0$ and $\sigma \neq 1$, and $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function satisfying:

Assumption φ_0 . $(\varphi/\varphi')(0+) = 0$; and

Assumption φ . There exists an $\bar{m} \in (0, +\infty]$ such that $\varphi|_{[0, \bar{m})}$ is strictly increasing, $\varphi|_{[\bar{m}, +\infty)}$ is constant, $\varphi''|_{(0, \bar{m})} < 0$ and $\varphi'(\bar{m}-) = 0$.

From φ 's continuity, Assumption φ_0 is equivalent to: $\varphi(0) = 0$ or $\varphi'(0+) = +\infty$. The

number \bar{m} in Assumption φ is obviously unique, and equal to infinity if φ is strictly increasing. Note still that Assumption φ says that φ is twice-differentiable with the exception of at most one point: \bar{m} . Example 2 ahead illustrates this point.

Definition. We shall refer to the present version of Sidrauski's model using (1), Assumptions φ_0 and φ , as "Sidrauski's model".

Note that such a class of utility functions, besides the works of Lucas (2000), Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick (2007), includes the one used by Fischer (1979), Siegel (1983), Asako (1983) and Weil (1991). Indeed, these authors use a Cobb-Douglas utility function given by $U(c, m) = (c^\alpha m^\beta)^{1-R} / (1-R)$, where $R \geq 0$, $R \neq 1$, $\alpha, \beta > 0$ and $\alpha + \beta \leq 1$. This is just a particular case of (1) for which $\sigma = 1 - (\alpha + \beta)(1-R)$ and $\varphi(m) = (\alpha + \beta)^{\frac{1}{1-\sigma}} m^{\frac{\beta}{\alpha+\beta}}$. It can be noted from Example 1 ahead that this φ generates only and exactly the unitary-elasticity log-log money-demand function $m = (\beta/\alpha) r^{-1}$.

Remark 1. For positive m 's, the expression $\varphi(m) - m\varphi'(m)$ is positive. Indeed, for $m \geq \bar{m}$, it is simply $\varphi(\bar{m})$, which is greater than $\varphi(0) \geq 0$ from φ 's strict increasingness in $[0, \bar{m}]$. For $m < \bar{m}$, the strict concavity of φ gives $\varphi'(m)(0 - m) > \varphi(0) - \varphi(m) \geq -\varphi(m)$.

Using this remark, it can be seen that U is strictly increasing in each of its variables and strictly concave. Therefore if (P_{MU}) has a solution (which we assume to be true), it will be unique. In equilibrium, since c is taken as a fraction of output, $c = 1$. Euler equations yield

$$r = \frac{U_m}{U_c}, \quad (2)$$

that is,

$$r = \frac{\varphi'(m)}{\varphi(m) - m\varphi'(m)}, \quad (3)$$

which corresponds to equation 3.7 in Lucas (2000). Equation (3) gives us r as a non-negative differentiable function of m , for which we shall write $r = \psi(m)$, where $\psi : (0, \bar{m}) \rightarrow \mathbb{R}_{++}$.

Since

$$\psi'(m) = \frac{\varphi(m)\varphi''(m)}{(\varphi(m) - m\varphi'(m))^2} < 0, \quad (4)$$

ψ is strictly decreasing, and therefore one-to-one. From its continuity (since φ is twice-differentiable over $(0, \overline{m})$, φ' is continuous on this interval), its image is also connected, that is, an interval $(\psi(\overline{m}-), \psi(0+))$. From Assumption φ and (3), we get $\psi(\overline{m}-) = 0$. From Assumption φ_0 and Remark 1, we also have $\psi(0+) = +\infty$.

We shall call its inverse function $m : (0, +\infty) \rightarrow (0, \overline{m})$ a "money-demand function". Since, for any $r > 0$, $m'(r) = 1/\psi'(m(r)) < 0$, it must satisfy a decreasingness property:

Property D. $m' < 0$.

So the money-demand function is differentiable, strictly decreasing and surjective by construction, and the maximum value it attains is \overline{m} .

Definition. We shall call a money-demand function "Sidrauski-integrable" when it may be obtained from Sidrauski's model.

As a practical matter, since the economist does not know φ , he ends up using a money-demand function presented by the econometric practice. This leads us to our first integrability problem in the unidimensional setting.

Proposition 1. *Given an onto money-demand function $m : (0, +\infty) \rightarrow (0, \overline{m})$, it is Sidrauski-integrable if and only if it satisfies Property D.*

Proof. Necessity of Property D has already been proved. To see that it is also sufficient, all we need to do is exhibit a proper φ consistent with such an m and satisfying Assumptions φ_0 and φ . Let $\psi := m^{-1}$, which, from Property D and the Inverse Function Theorem on \mathbb{R} , is also differentiable. Note that equation (3) may be rewritten as

$$\varphi'(m) = \frac{\psi(m)}{1 + m\psi(m)}\varphi(m), \forall m \in (0, \overline{m}), \quad (5)$$

a separable ordinary differential equation with the general solution

$$\varphi(m) = Ce^{\int_1^m \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu}, C > 0. \quad (6)$$

Bearing this in mind, take $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$\varphi(m) = \begin{cases} e^{\int_1^m \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu} & \text{if } m \leq \bar{m} \\ e^{\int_1^{\bar{m}} \frac{\psi(\mu)}{1+\mu\psi(\mu)} d\mu} & \text{if } m > \bar{m} \end{cases}. \quad (7)$$

So $\varphi \geq 0$ and, for $m < \bar{m}$, we get (5), as wanted. Still for $m < \bar{m}$, $\varphi'(m) > 0$ and $\varphi''(m) = \varphi(m)\psi'(m)/(1+m\psi(m))^2 < 0$, so that Assumption φ holds true. Equation (5) gives $(\varphi/\varphi')(m) = m + 1/\psi(m)$, whence Assumption φ_0 is also valid.

In case \bar{m} is finite, we still have to check for φ 's differentiability at \bar{m} . Since $\psi(\bar{m}-) = 0$ (m 's surjectivity and decreasingness), we have $\varphi'(\bar{m}-) = 0$. Since $\varphi'(\bar{m}+) = 0$, we're done.

□

In the Appendix, a few departures from the Sidrauski model just presented are made, in order to assess the robustness of this result.

2.2. The shopping-time model

We start by briefly introducing the unidimensional version of McCallum and Goodfriend's (1987) shopping-time model in the same form as in Lucas (2000). Any variable sharing its symbol with a variable in the previous subsection also shares with it its meaning. Our representative agent's problem is:

$$\max_{c,b,m \geq 0, 0 \leq s \leq 1} \int_0^{+\infty} e^{(-\rho+(1-\sigma)\gamma)t} U(c_t) dt \text{ subject to} \quad (\text{P}_{\text{ST}})$$

$$\dot{b}_t + \dot{m}_t = 1 - h_t - c_t - s_t + (r_t - \pi_t - \gamma)b_t - (\pi_t + \gamma)m_t, \forall t \in (0, +\infty),$$

$$c_t = m_t \phi(s_t), \forall t \in (0, +\infty),$$

where U is of the CRRA kind, $s \in [0, 1]$ is a choice variable such that $s + y = 1$ (it represents the portion of time the individual dedicates to transacting instead of producing), $\phi : [0, 1] \rightarrow \mathbb{R}_+$ is a twice-differentiable function such that $\phi' > 0$ and $\lim_{s \rightarrow 1-} (1 - s) \phi'(s) = 0$. Additionally, the following assumptions on ϕ will be considered:

Assumption ϕ_01 . $(\phi/\phi')(0+) = 0$.

Assumption ϕ_02 . $\phi(0) = 0$.

Assumption $\phi1$. The function $\theta : [0, 1] \rightarrow \mathbb{R}_+$ defined by $\theta(s) := \phi(s)^2 / [(1 - s) \phi'(s)]$ is such that $\theta' > 0$.

Assumption $\phi2$. $(\phi/\phi')' \geq 0$.

As happens with the φ in Sidrauski's model, Assumption ϕ_01 means that if $\phi'(0+) < +\infty$, then $\phi(0) = 0$. That is, Assumption ϕ_01 is weaker than Assumption ϕ_02 . The latter is used, for instance, in Simonsen and Cysne (2001).

Regarding the second pair of assumptions, we shall see ahead Assumption $\phi1$ to be the weakest possible condition in order for the money-demand function arising from this model to satisfy Property D . The rationale for Assumption $\phi2$ will be seen in Remark 2 ahead. In McCallum and Goodfriend (1987), only $\phi' > 0$ is assumed.² In Simonsen and Cysne (2001), a stronger assumption than ours, $\phi'' \leq 0$, is made. We shall see in Section 2.3 that this would make even the standard log-log money-demand specification, irrespective of its elasticity, irrationalizable by this model. Nevertheless, if its absolute-value elasticity were lower than 1, its integrability would be guaranteed as long as ϕ were required to satisfy Assumption $\phi2$ (which, although stronger than Assumption $\phi1$, is weaker than the concavity assumption).

Definition. We shall refer to the present version of the shopping-time model using the transacting technology $c_t = m_t \phi(s_t)$ and Assumptions ϕ_01 and $\phi1$ as "the shopping-time model".

Note that this separable version of the model is important, among other reasons, because

²See eq. 2 on p. 776 in that paper.

it allows for straightforward derivations of measures of the welfare costs of inflation (Lucas, 2000). Another reference in the literature using such a weakly-separable version of the shopping-time model is Goodfriend (1997).

Now let us turn to the solution of (P_{ST}) . Considering interior solutions only, we get the first-order condition (as in Simonsen and Cysne, 2001)

$$rm = \left(\frac{\phi}{\phi'} \right) (s). \quad (8)$$

Cysne (2006) (where Arrow's sufficiency theorem in control theory is applied) and Hiraguchi (2008) (where (P_{ST}) is concavified through a change of variables) show that this is not only necessary but also sufficient to characterize (P_{ST}) 's solution, provided the coefficient of relative risk aversion in U is of at least 0.5.³

The equilibrium equation is given by

$$m\phi(s) = 1 - s, \quad (9)$$

which gives s as a function of m alone. For this purpose, we only need to note that the function $H : (0, 1] \rightarrow \mathbb{R}_+$ defined by $H(s) = (1 - s)/\phi(s)$ is invertible. Since $H'(s) = -[\phi(s) + (1 - s)\phi'(s)]/\phi(s)^2 < 0$, $H(1) = 0$ and $\lim_{s \rightarrow 0+} H(s) = +\infty$, we are finished. Letting τ be its inverse function (therefore also strictly decreasing, with $\tau'(m) = 1/H'(\tau(m)) < 0$), our demand equation may be written as $r = \psi(m)$, where

$$\psi(m) = \frac{1}{m} \left(\frac{\phi}{\phi'} \right) (\tau(m)). \quad (10)$$

As in the last subsection, we shall call ψ 's inverse function a money-demand function and denote it by m . As was the case there, here we also have $\psi(0+) = +\infty$. In fact, note that by using (9), (10) can also be written as $\psi(m) = \theta(\tau(m))$. Then $\psi(0+) = \theta(1) =$

³Specifically for the log-log case considered in Lucas (2000, pp. 265-267), Cysne (2008) shows that this coefficient could be as low as 0.0085.

$\phi(1)^2 / \lim_{s \rightarrow 1-} [(1-s)\phi'(s)] = +\infty$ (using our hypothesis that $\lim_{s \rightarrow 1-} (1-s)\phi'(s) = 0$).

Definition. We shall call a money-demand function "shopping-time-integrable" when it may be obtained from the shopping-time model.

Lemma 1. *Shopping-time-integrable money demands also satisfy Property D.*

Proof. Since $\psi(m) = \theta(\tau(m))$, $\psi'(m) = \theta'(\tau(m))\tau'(m) < 0$ (using Assumption $\phi 1$). \square

Remark 2. From (10), Assumption $\phi 2$ is the necessary and sufficient condition to guarantee that not only ψ , but also $m\psi$, is a decreasing function of m . This yields a positive correlation between rm and r or, in other words, an inelastic money-demand function m .⁴

Here we have a result similar to Proposition 1, although it does not give a test as practical as the one presented there.

Proposition 2. *Given an onto money-demand function $m : (0, +\infty) \rightarrow (0, \overline{m})$, it is shopping-time-integrable if and only if it satisfies Property D and there exists a C^1 function $\phi : [0, 1] \rightarrow \mathbb{R}_+$ solving*

$$\begin{cases} m\left(\frac{\phi(s)^2}{(1-s)\phi'(s)}\right) = \frac{1-s}{\phi(s)} \\ \phi(0) = \frac{1}{\overline{m}} \end{cases} \quad (11)$$

Proof. The necessity of the decreasingness property has been shown in the previous lemma, while the first equation in (11) comes directly from (8) and (9). Making $s \downarrow 0$ in (11), since $(\phi/\phi')(s) \rightarrow 0$, we get \overline{m} on the left-hand side and $1/\phi(0)$ on the right-hand side, ending the necessity part of the demonstration.

For the converse, it suffices to show that the assumptions made for ϕ in the shopping-time model are valid. We start with $\phi_0 1$: again making $s \downarrow 0$ in (11), and using m 's strict decreasingness (Property D), one concludes that $\theta(s) = \phi(s)^2 / [(1-s)\phi'(s)] \rightarrow 0$, whence

⁴The interest-rate-elasticity of the money-demand function m (in absolute value) is smaller, equal or bigger than 1 if rm increases, stays still or decreases with r , respectively.

$(\phi/\phi')(s) \rightarrow 0$. Now rewrite the first equation in (11) as

$$\phi'(s) = \frac{\phi(s)^2}{(1-s)\psi\left(\frac{1-s}{\phi(s)}\right)}, \quad (12)$$

where $\psi := m^{-1}$. Since ψ is also differentiable (from Property D), ϕ is twice differentiable. Since the right-hand side of (12) is non-negative, $\phi' \geq 0$ (and (11) itself implies it cannot be 0 anywhere). Still from (11), $m'(\theta(s))\theta'(s) = H'(s) < 0$, so that $\theta' > 0$ and Assumption $\phi 1$ is also true. Finally, (12) gives $\lim_{s \rightarrow 1-} (1-s)\phi'(s) = \phi(1)^2/\psi(0+) = 0$. \square

The main propositions of this and the last subsections have the following immediate

Corollary 3. *Shopping-time-integrable money-demand functions are also Sidrauski-integrable.*

Note that this is *not* an equivalence result, but an embedding result only. This is the case even if we impose $\bar{m} = +\infty$. We shall see in the next subsection a few examples of demand functions which can be derived from the Sidrauski, but not the shopping-time, model.⁵

Remark 3. The reader may want to note that under Assumption $\phi_0 2$ one would necessarily end up with $\bar{m} = m(0+) = +\infty$, thus forcibly excluding from our consideration the semi-log money demand. In fact, if $r \rightarrow 0$ and $m(r)$ remained limited, then (9) would give $s \rightarrow 0$. But then (8) would represent a contradiction, since its right-hand side would be approaching a finite positive value, whereas its left-hand side would be approaching zero.⁶

One way of making both models equivalent would be to impose

⁵The reader might be wondering if this isn't in disagreement with Feenstra (1986), where it is shown that the Sidrauski and the shopping-time models are equivalent. Actually, the shopping-time model used here does not necessarily satisfy some of his assumptions. In fact, one can see that the three conditions listed under assumption 1(c) there correspond to taking a concave ϕ here. As will be made clear in the following example, this seemingly innocent requirement (although stronger than both Assumptions $\phi 1$ and $\phi 2$) makes, in our present model, even the usual log-log money-demand specification non-integrable, irrespective of its elasticity. The upshot is that the separability of the technology function, imposed by Lucas (2000) to enable the calculation of the welfare cost of inflation, inevitably brings restrictions on the functional form of the money demands emerging from the model.

⁶The only possibility left would be a non-interior solution to (P_{ST}) , such that $c = 0$, $s = 1$ and $m = 0$ (even with $r \rightarrow 0$), consistent only with an identically zero money-demand specification.

Assumption r . There exists a $\underline{r}, \bar{r} \in (0, +\infty)$ such that $r \in [\underline{r}, \bar{r}]$.

Lucas (2000, ft. 8) does something similar, but only for the lower bound, by imposing a strictly positive r . These bounds for r may be taken as low and as high as one wishes. For instance, $\underline{r} = 0.0001$ and $\bar{r} = 10000$ will certainly serve for any practical purposes. Assuming this boundedness property for r doesn't invalidate any of the results so far, but it will imply a minor modification in our original model: there must exist $\underline{s} > 0$ and $\bar{s} < 1$ such that $s \in [\underline{s}, \bar{s}]$.

In fact, putting $s \downarrow 0$ in (8) now implies $m \rightarrow 0$, in contradiction with the fact that $H' < 0$. Thus Assumption $\phi_0 1$ will no longer be asked for. Besides this, putting $s \uparrow 1$ in (9) also gives $m \rightarrow 0$, but then (8) cannot hold (so that $\lim_{s \rightarrow 1-} (1-s)\phi'(s)$ will not be an issue anymore). Additionally, (8) tells us that, under Assumption r , m as well must be bounded in the same fashion: $m \in [\underline{m}, \bar{m}]$, with $\underline{m} > 0$ and $\bar{m} < +\infty$.

Under this new assumption, we get a converse to Corollary 3, making both models equivalent in terms of the money demands they generate.

Proposition 4. *Under Assumption r , Sidrauski-integrable money-demand specifications are also shopping-time-integrable.*

Proof. Given a Sidrauski-integrable money-demand specification m , we know from Proposition 1 that it satisfies Property D . So Proposition 2 says that checking for m 's shopping-time integrability amounts to finding an appropriate ϕ (with an appropriate domain) satisfying (11) (with a corresponding adjustment in the initial condition, $\phi(\underline{s}) = (1 - \underline{s})/\bar{m}$). If we can show the existence of a C^1 -class ϕ satisfying (12), then ϕ will be twice-differentiable, $\phi' > 0$ and Assumption $\phi_0 1$ will be valid, as shown at the end of the proof of Proposition 2.

Treating the right-hand side of (12) as a function of s and ϕ , partial differentiation with respect to ϕ gives

$$\frac{2\phi\psi\left(\frac{1-s}{\phi}\right) + (1-s)\psi'\left(\frac{1-s}{\phi}\right)}{(1-s)\psi\left(\frac{1-s}{\phi}\right)^2}$$

or, more simply,

$$\frac{1}{r} \left(\frac{2\phi}{1-s} + \frac{\psi'(m)}{r} \right).$$

Since $s \in [\underline{s}, \bar{s}]$, ϕ is bounded, and since $m \in [\underline{m}, \bar{m}]$, ψ' is bounded too. Thus the right-hand side of (12), seen as a function of s , is a Lipschitz function, so that, as proved in Simmons (1991, p. 550, Thm. B), (11) admits a global (that is, over $[\underline{s}, \bar{s}]$) solution, and m is shopping-time-integrable. \square

2.3. Examples

With the propositions of the last two subsections at our disposal, we can now investigate the integrability of the usual semi-log and log-log money-demand specifications.

Example 1 (log-log money demand). Consider the money-demand function given by $m(r) = Kr^{-\alpha}$, where $K > 0$ and $\alpha > 0$. In order for us to be able to apply Proposition 1, we must take $\bar{m} = +\infty$. Since $m'(r) = -\alpha Kr^{-\alpha-1} < 0$, it follows from that proposition that the log-log money demand is Sidrauski-integrable.

Let us find the φ underlying this money-demand specification. If $\alpha = 1$, the integral in equation (6) equals

$$\int_1^m \frac{K}{K+1} \frac{1}{\mu} d\mu = \frac{K}{K+1} \log m,$$

and if $\alpha \neq 1$,

$$\int_1^m \frac{(K/\mu)^{1/\alpha}}{1 + \mu(K/\mu)^{1/\alpha}} d\mu = \int_1^m \left(\frac{1}{\mu} - \frac{\mu^{\frac{1}{\alpha}-2}}{\mu^{\frac{1}{\alpha}-1} + K^{\frac{1}{\alpha}}} \right) d\mu = \log \left(m / \left(\frac{K^{\frac{1}{\alpha}} + m^{\frac{1}{\alpha}-1}}{K^{\frac{1}{\alpha}} + 1} \right)^{\frac{\alpha}{1-\alpha}} \right).$$

Therefore that equation gives us

$$\varphi(m) = \begin{cases} Cm^{\frac{K}{K+1}} & \text{if } \alpha = 1 \\ \frac{C'm}{\left(K^{\frac{1}{\alpha}} + m^{\frac{1-\alpha}{\alpha}} \right)^{\frac{\alpha}{1-\alpha}}} & \text{if } \alpha \neq 1 \end{cases}, \quad (13)$$

for positive constants C and C' . (13) generalizes the integrability investigation carried out by Lucas (2000, p. 257) for the particular value $\alpha = 0.5$.

As it concerns the shopping-time model, the system of equations (11) can be written as

$$\begin{cases} \phi'(s) = \frac{(1-s)^{\frac{1}{\alpha}-1}}{K^{\frac{1}{\alpha}}} \phi(s)^{2-\frac{1}{\alpha}} \\ \phi(0) = 0 \end{cases} \quad (14)$$

Thus $\theta(s) = \phi(s)^2 / [(1-s)\phi'(s)] = (K\phi(s)/(1-s))^{\frac{1}{\alpha}}$, so that Assumption ϕ_1 is valid.

If $\alpha = 1$, we get $\phi'(s) = \phi(s)/K$, which together with the initial condition gives $\phi = 0$, a contradiction with $\phi' > 0$. So we now attain to the case $\alpha \neq 1$. The differential equation in (14) is a Bernoulli equation, and we have to find out if its solution satisfying $\phi(0) = 0$ also satisfies the other properties in our version of the shopping-time model.

In order to solve it, write $y = \phi^{\frac{1}{\alpha}-1}$ (since we must have $\phi \geq 0$, this is a legitimate move), so that

$$y'(s) = \left(\frac{1}{\alpha} - 1\right) \phi(s)^{\frac{1}{\alpha}-2} \phi'(s) = \left(\frac{1}{\alpha} - 1\right) \frac{(1-s)^{\frac{1}{\alpha}-1}}{K^{\frac{1}{\alpha}}}.$$

Therefore $y(s) = (\alpha - 1) K^{-\frac{1}{\alpha}} (1-s)^{\frac{1}{\alpha}} + C$ for some constant C . Since $y(0) = 0$, we get $C = (1 - \alpha) K^{-\frac{1}{\alpha}}$, so that $y(s) = \left((1 - \alpha)/K^{\frac{1}{\alpha}}\right) \left(1 - (1-s)^{\frac{1}{\alpha}}\right)$. Now, since ϕ has to be non-negative, y does too. Thus the possibility $\alpha > 1$ is dismissed, and we may attain to the $\alpha < 1$ case, where $\phi = y^{\frac{\alpha}{1-\alpha}}$, that is,

$$\phi(s) = \left[\frac{1-\alpha}{K^{\frac{1}{\alpha}}} \left(1 - (1-s)^{\frac{1}{\alpha}}\right) \right]^{\frac{\alpha}{1-\alpha}}. \quad (15)$$

We conclude that the necessary and sufficient condition for the log-log money-demand specification to be shopping-time-integrable is that $\alpha < 1$.

This conclusion is consistent with the underlying assumptions of our shopping-time model. Assumption ϕ_0 and the $\lim_{s \rightarrow 1-} (1-s)\phi'(s) = 0$ hypothesis are readily attended, by looking at (14). And after some tedious calculations for ϕ' and ϕ'' , one finds that $\phi'^2 - \phi\phi''$

is always non-negative, so that Assumption $\phi 2$ is valid as well (even though ϕ'' has indefinite sign).⁷

Note that (15) yields, for the $\alpha = 0.5$ case analyzed by Lucas (2000, p. 265), a function ϕ of the form $\phi(s) = (2s - s^2) / 2K^2$ (which is approximated by $\phi(s) = s/K^2$ there).

Obviously, under Assumption r , the log-log money-demand function is shopping-time-integrable even if $\alpha \geq 1$. This is because, as commented in the proof of Proposition 4, the relevant initial condition when one considers Assumption r is $\phi(\underline{s}) = (1 - \underline{s}) / \bar{m}$, which generalizes the one used above, $\phi(0) = 0$. In case $\alpha = 1$, as seen above, we get $\phi'(s) = \phi(s) / K$, so that

$$\phi(s) = \frac{1 - \underline{s}}{\bar{m}} e^{\frac{s - \underline{s}}{K}}.$$

In case $\alpha > 1$, defining y and solving the same way as before (but with the appropriate initial condition), we get

$$\phi(s) = \left[\frac{1 - \alpha}{K^{\frac{1}{\alpha}}} \left((1 - \underline{s})^{\frac{1}{\alpha}} - (1 - s)^{\frac{1}{\alpha}} \right) + \left(\frac{1 - \underline{s}}{\bar{m}} \right)^{\frac{1 - \alpha}{\alpha}} \right]^{\frac{\alpha}{1 - \alpha}},$$

of which (15) is the special case $(\bar{m}, \underline{s}) = (+\infty, 0)$. Here, (\bar{m}, \underline{s}) must be chosen as to guarantee that the expression inside brackets above is positive (equivalently, since it decreases with s , $1 - \underline{s} < K^{\frac{1}{\alpha}} \bar{m}^{1 - \frac{1}{\alpha}} / (\alpha - 1)$).⁸ For either one of these two cases, one has, directly from the differential equation in (14), $\phi' > 0$ and $\theta(s) = (K\phi(s) / (1 - s))^{\frac{1}{\alpha}}$, so that

$$\theta'(s) = \frac{K^{\frac{1}{\alpha}}}{\alpha} \left(\frac{\phi(s)}{1 - s} \right)^{\frac{1}{\alpha} - 1} \frac{(1 - s)\phi'(s) + \phi(s)}{(1 - s)^2} > 0.$$

This serves as an illustration of the argument used in Proposition 4.

Example 2 (semi-log money demand). Consider a money-demand function given by $m(r) = De^{-\xi r}$, where $D > 0$ and $\xi > 0$. Let us first investigate if this demand can be obtained

⁷For instance, for $\alpha = 0.6$ and $K = 0.1$, we have $\phi''(0.4) \approx -8.3$, but $\phi''(0.3) \approx 54.2$.

⁸Just as an example, say $K = 0.05$ (as in Lucas, 2000) and $\alpha = 1.06$. Then, if $\bar{m} = 0.2$, an \underline{s} of 0.10 would do, while if $\bar{m} = 0.3$, one could take $\underline{s} = 0.08$.

in the Sidrauski framework. Here we must take $\bar{m} = D$. Since $m'(r) = -\xi D e^{-\xi r} < 0$, it follows from Proposition 1 that the semi-log money demand is Sidrauski-integrable indeed.

Just for the sake of comprehensiveness, we seek the φ (related to our version of the Sidrauski model) underlying this money-demand specification. Since $\psi(m) = (1/\xi) \log(D/m)$, (7) gives us (dividing the result by a constant makes no difference)

$$\varphi(m) = \begin{cases} e^{\int_{m/D}^1 \frac{\log x}{\xi/D - x \log x} dx} & \text{if } m \leq D \\ 1 & \text{if } m > D \end{cases}.$$

The reader may want to note that this is an example of a proper φ which is *not* twice-differentiable at D (as anticipated in Section 2.1). In fact, while $\varphi''(D+) = 0$, we also obtain $\varphi''(D-) = -1/(\xi D)$, using the formula for φ'' derived in the proof of Proposition 1.

As for the shopping-time model, as we've seen in Remark 3, under Assumption $\phi_0 2$ the semi-log demand could not be obtained from this model. We provide no conclusion if only Assumption $\phi_0 1$ is made. Nevertheless, under Assumption r , the shopping-time integrability of the semi-log money-demand function is guaranteed.

3. The case with many types of monies

In this section we extend all results obtained in Section 2 to a framework in which n types of monies are available. Let $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{R}_+^n$ represent the vector of real quantities of each type of money being demanded, as a fraction of nominal GDP (where m_1 is chosen to be m , real currency per output). Each m_i yields a nominal interest rate of r_i , and $\mathbf{r} := (r_1, \dots, r_n) \in \mathbb{R}_+^n$ (with $r_1 = 0$, by definition). We shall write $\mathbf{u} := (r, r - r_2, \dots, r - r_n) \in \mathbb{R}_{++}^n$ for the vector of opportunity costs of holding money instead of government bonds.

3.1. The extended Sidrauski model

Our representative agent's instantaneous utility function will now have the form

$$U(c, \mathbf{m}) = \frac{1}{1-\sigma} \left(c \varphi \left(\frac{G(\mathbf{m})}{c} \right) \right)^{1-\sigma},$$

where σ and φ are exactly as in Section 2.1, and $G : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is a twice-differentiable 1-homogeneous concave function such that $G_{x_i} > 0$ and $G_{x_i x_i} < 0$ for all $i \in \{1, \dots, n\}$.⁹ Her maximization problem will be:

$$\begin{aligned} \max_{c, b, \mathbf{m} \geq 0} \int_0^{+\infty} e^{(-\rho+(1-\sigma)\gamma)t} U(c_t, \mathbf{m}_t) dt \text{ subject to} \quad & (\text{P}_{\text{MU}^n}) \\ \dot{b}_t + \mathbf{1} \cdot \dot{\mathbf{m}}_t = y_t - h_t - c_t + (r_t - \pi_t - \gamma)b_t + (\mathbf{r}_t - (\pi_t + \gamma)\mathbf{1}) \cdot \mathbf{m}_t, \forall t \in (0, +\infty), \end{aligned}$$

where we write $\mathbf{1}$ for the vector $(1, \dots, 1) \in \mathbb{R}^n$ and \cdot for the canonical inner product of \mathbb{R}^n , and all the non-bold letters have the same meaning as in the unidimensional Sidrauski model. Considering only regular solutions, and by U 's concavity, we obtain (as in Cysne and Turchick, 2007) the necessary and sufficient equilibrium relations

$$r - r_i = \frac{U_{x_i}}{U_c}, \forall i \in \{1, \dots, n\}. \quad (16)$$

In equilibrium, we have $c = 1$, so that (16) gives

$$u_i = \frac{\varphi'(G(\mathbf{m}))}{\varphi(G(\mathbf{m})) - G(\mathbf{m}) \varphi'(G(\mathbf{m}))} G_{x_i}(\mathbf{m}), \forall i \in \{1, \dots, n\}. \quad (17)$$

This equation is analogous to (3), giving us a differentiable function $\boldsymbol{\psi} : C_{\overline{m}} \rightarrow \mathbb{R}_{++}^n$ such that $\mathbf{u} = \boldsymbol{\psi}(\mathbf{m})$, just like the function ψ of Section 2.1, where $C_{\overline{m}} := \{\mathbf{m} \in \mathbb{R}_{++}^n : G(\mathbf{m}) < \overline{m}\}$. But, in contrast with what we've done in that section, we will not work with its inverse function, since one is not assured of its existence in the first place.

Note that if we define $F : \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ by $F(G) = \varphi'(G)/(\varphi(G) - G\varphi'(G))$, (17) can be

⁹If $n = 1$, G would have to be linear, whence $G'' = 0$. Therefore, our analysis in this section is restricted to the case $n > 1$. Even so, it yields the same results as the $n = 1$ framework analyzed in the last section.

rewritten, for $\mathbf{m} \in C_{\overline{m}}$, as

$$\boldsymbol{\psi}(\mathbf{m}) = F(G(\mathbf{m})) \nabla G(\mathbf{m}), \quad (18)$$

where, as seen in (4), $F' < 0$. As in the unidimensional setting, here we also have a decreasingness property for $\boldsymbol{\psi}$:

Property D_n . Along rays starting at the origin, each ψ_i is strictly decreasing.

In fact, for $k > 1$ and fixed $\mathbf{m} \in C_{\overline{m}/k}$, (18) gives $\psi_i(k\mathbf{m}) = F(G(k\mathbf{m})) G_{x_i}(k\mathbf{m}) = F(kG(\mathbf{m})) G_{x_i}(\mathbf{m})$, from G 's 1-homogeneity (and subsequent G_{x_i} 's 0-homogeneity). Since $F' < 0$, we get $F(kG(\mathbf{m})) < F(G(\mathbf{m}))$, so that $\psi_i(k\mathbf{m}) < F(G(\mathbf{m})) G_{x_i}(\mathbf{m}) = \psi_i(\mathbf{m})$.

Proposition 5. *Given G as described above, $\overline{m} \in (0, +\infty]$ and a money-demand specification $\psi : C_{\overline{m}} \rightarrow \mathbb{R}_{++}^n$ such that (i) $\psi_i(\mathbf{m}) \rightarrow 0$ for some $i \in \{1, \dots, n\}$ if $G(\mathbf{m}) \rightarrow \overline{m}$ and (ii) $\psi_j(\mathbf{m}) \rightarrow +\infty$ for some $j \in \{1, \dots, n\}$ if $G(\mathbf{m}) \rightarrow 0$, it is extended-Sidrauski-integrable if and only if it can be put in the form (18), with $F' < 0$.¹⁰*

Proof. Necessity of (18) with $F' < 0$ has been shown above. For the converse, we must exhibit a $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying Assumptions φ_0 and φ such that, for $G < \overline{m}$, $\varphi'(G)/(\varphi(G) - G\varphi'(G)) = F(G)$.

Take φ given by

$$\varphi(G) = \begin{cases} e^{\int_1^G \frac{F(\mu)}{1+\mu F(\mu)} d\mu} & \text{if } G \leq \overline{m} \\ e^{\int_1^{\overline{m}} \frac{F(\mu)}{1+\mu F(\mu)} d\mu} & \text{if } G > \overline{m} \end{cases}.$$

So $\varphi \geq 0$ and, for $G < \overline{m}$, $\varphi'(G) = \varphi(G)F(G)/(1 + GF(G))$, which is equivalent to the equality required above. Still for $G < \overline{m}$, $\varphi'(G) > 0$ and $\varphi''(G) = \varphi(G)F'(G)/(1 + GF(G))^2 < 0$, so that Assumption φ holds true. Since $(\varphi/\varphi')(G) = G + 1/F(G)$, Assumption φ_0 will be valid if $F(0+) = +\infty$. Using Euler's formula for homogeneous functions, (18) becomes $\mathbf{m} \cdot \boldsymbol{\psi}(\mathbf{m}) = F(G(\mathbf{m})) G(\mathbf{m})$. Making $G(\mathbf{m}) \rightarrow 0$ in a way that keeps m_j constant, since $\mathbf{m} \cdot \boldsymbol{\psi}(\mathbf{m}) \geq m_j \psi_j(\mathbf{m}) = +\infty$, it is necessary that $F(0+) = +\infty$.

¹⁰In the unidimensional setting, the hypotheses $\psi_i(\mathbf{m}) \rightarrow 0$ if $G(\mathbf{m}) \rightarrow \overline{m}$ and $\psi_j(\mathbf{m}) \rightarrow +\infty$ if $G(\mathbf{m}) \rightarrow 0$ would read $\psi(\overline{m}-) = 0$ and $\psi(0) = +\infty$, that is, $m(0+) = \overline{m}$ and $m(+\infty-) = 0$. This, by Property D , is equivalent to the hypothesis used in Proposition that m be surjective.

We still need to verify if φ is differentiable at \bar{m} in case \bar{m} is finite. Making $G(\mathbf{m}) \rightarrow \bar{m}$, we know that $\psi_i(\mathbf{m}) \rightarrow 0$. From (18), since $G_{x_i}(\mathbf{m}) > 0$, we get $F(\bar{m}-) = 0$. Therefore $\varphi'(\bar{m}-) = 0 = \varphi'(\bar{m}+)$, as wished. \square

3.2. The extended shopping-time model

Here our representative agent's problem is

$$\begin{aligned} \max_{c, b, \mathbf{m} \geq 0, 0 \leq s \leq 1} \int_0^{+\infty} e^{(-\rho + (1-\sigma)\gamma)t} U(c_t) dt \text{ subject to} \quad & (\text{P}_{\text{ST}^n}) \\ \dot{b}_t + \mathbf{1} \cdot \dot{\mathbf{m}}_t = 1 - h_t - c_t - s_t + (r_t - \pi_t - \gamma)b_t + (\mathbf{r}_t - (\pi + \gamma)\mathbf{1}) \cdot \mathbf{m}_t, \forall t \in (0, +\infty), \\ c_t = G(\mathbf{m}_t) \phi(s_t), \forall t \in (0, +\infty), \end{aligned}$$

where c and s stand for the real level of consumption and the portion of time spent transacting rather than producing.¹¹ The money-aggregator function $G : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is a function just like the one in the previous subsection and $\phi : [0, 1] \rightarrow \mathbb{R}_+$ is exactly as the ϕ in Section 2.2, satisfying Assumptions ϕ_01 and $\phi1$. This model's first-order conditions yield the equilibrium relations (Cysne, 2003, p. 224):

$$\begin{cases} u_i = \frac{1}{G(\mathbf{m})} \left(\frac{\phi}{\phi'} \right) (s) G_{x_i}(\mathbf{m}), \forall i \in \{1, \dots, n\} \\ G(\mathbf{m}) \phi(s) = 1 - s \end{cases} \quad (19)$$

As done in Section 2.2, the last of these equilibrium relations enables us to define a function $\tau : \mathbb{R}_+ \rightarrow (0, 1]$ such that $G = (1 - \tau(G)) / \phi(\tau(G))$, and we have $\tau' < 0$. Then (19) becomes a money-demand specification of n equations and n variables:

$$\psi_i(\mathbf{m}) = \frac{\phi(\tau(G(\mathbf{m})))}{G(\mathbf{m})\phi'(\tau(G(\mathbf{m})))} G_{x_i}(\mathbf{m}), \forall i \in \{1, \dots, n\}, \quad (20)$$

and ψ is as in the previous subsection. Note how ψ also fits perfectly into the general form

¹¹Cysne (2002) shows that weak separability of the transacting technology is a necessary and sufficient condition so that the welfare costs of inflation are well defined as functions of \mathbf{m} .

(18), if we take $F : \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ such that $F(G) = (1/G)(\phi/\phi')(\tau(G))$. As shown in Lemma 1, $F' < 0$. Therefore we have, as in the unidimensional framework, the following

Lemma 2. *Extended-shopping-time-integrable money demands also satisfy Property D_n .*

We also get an exact analog of Proposition 2:

Proposition 6. *Given G as described above, $\bar{m} \in (0, +\infty]$ and a money-demand specification $\psi : C_{\bar{m}} \rightarrow \mathbb{R}_{++}^n$ such that (i) $\psi_i(\mathbf{m}) \rightarrow 0$ for some $i \in \{1, \dots, n\}$ if $G(\mathbf{m}) \rightarrow \bar{m}$ and (ii) $\psi_j(\mathbf{m}) \rightarrow +\infty$ for some $j \in \{1, \dots, n\}$ if $G(\mathbf{m}) \rightarrow 0$, it is extended-shopping-time-integrable if and only if it can be put in the form (18), with $F' < 0$, and there exists a C^1 function $\phi : [0, 1] \rightarrow \mathbb{R}_+$ solving*

$$\begin{cases} \phi'(s) = \frac{\phi(s)^2}{(1-s)F\left(\frac{1-s}{\phi(s)}\right)} \\ \phi(0) = \frac{1}{\bar{m}} \end{cases} \quad (21)$$

Proof. Necessity of (18) with $F' < 0$ has been shown above, and that of the solvability of the initial value problem (21) is obtained as in the proof of Proposition 2.

For the converse, rewriting the differential equation in (21) as $\theta(s) = F((1-s)/\phi(s))$ and making $s \downarrow 0$, we get, from the initial condition, $\theta(0+) = F(\bar{m}-) = 0$ (where the latter calculation was done in the proof of Proposition 5, with the aid of hypothesis (i)). So either $\phi(0) = 0$ or $(\phi/\phi')(0+) = 0$, which guarantees the validity of Assumption ϕ_01 . The differentiability of F makes ϕ twice differentiable, through the first equation in (21), which also shows that $\phi' > 0$. Remembering $H(s)$ was defined in Section 2.2 as $(1-s)/\phi(s)$, we can write $\theta(s) = F(H(s))$, so that the Chain Rule yields $\theta'(s) = F'(H(s))H'(s) > 0$ (Assumption ϕ_1). Finally, (21) readily gives $\lim_{s \rightarrow 1-} (1-s)\phi'(s) = \phi(1)^2/F(0+) = 0$ (the last equality comes from the proof of Proposition 5, using hypothesis (ii)). \square

In light of Propositions 5 and 6, we have the following

Corollary 7. Extended-shopping-time-integrable money-demand functions as in the statement of Proposition 6 are also extended-Sidrauski-integrable.

As in the unidimensional case, an extra assumption makes both models equivalent, for practical purposes, in terms of the money demands they generate:

Assumption r . There exist $\underline{u}, \bar{u} \in (0, +\infty)$ such that $r \leq \bar{u}$ and $u_i \geq \underline{u}, \forall i \in \{1, \dots, n\}$.

As in the unidimensional context, this implies $s \in [\underline{s}, \bar{s}]$, where $\underline{s} > 0$ and $\bar{s} < 1$. In fact, making $s \rightarrow 0$ in the last equation in (19) gives $G \rightarrow 0$, so that, for each $i \in \{1, \dots, n\}$, the i^{th} equation in (19) implies $G_{x_i} \rightarrow +\infty$ (using u_i 's boundedness away from 0). Since $G_{x_i x_i} < 0$, this means that $m_i \rightarrow 0, \forall i$. But then, from G 's continuity, $G \rightarrow 0$, contradiction. The argument for $s \rightarrow 1$ follows the same lines: the last equation in (19) would give $G \rightarrow 0$, while the i^{th} equation in (19) would imply $G_{x_i} \rightarrow 0$ (from u_i 's boundedness from above). Since $G_{x_i} > 0$ and $G_{x_i x_i} < 0$, this means $m_i \rightarrow +\infty, \forall i$, so that $G \rightarrow +\infty$, a contradiction. Furthermore, the last equation in (19) implies that G must be limited in the same fashion: $G \in [\underline{m}, \bar{m}]$, where $\underline{m} > 0$ and $\bar{m} < +\infty$. We then get a perfect analog of Proposition 4:

Proposition 8. Under Assumption **r**, extended-Sidrauski-integrable money-demand specifications are also extended-shopping-time-integrable.

Proof. Exactly the same as the proof of Proposition 4. \square

3.3. Examples

In this subsection we extend the examples given in the last section.

Example 3. We now extend the log-log money-demand specification to the multidimensional case. It is natural enough to propose an extension of the form $\psi(\mathbf{m}) = (K/G(\mathbf{m}))^{\frac{1}{\alpha}} \nabla G(\mathbf{m})$, where $K > 0$ and $\alpha > 0$.¹² Taking G as a simple geometric mean, $G(m_1, \dots, m_n) =$

¹²The reader may note that this is the demand that follows from (13) and (17).

$\prod_{i=1}^n m_i^{1/n}$, we obtain the system of equations

$$\begin{cases} \psi_1(\mathbf{m}) = \frac{K^{\frac{1}{\alpha}}}{nm_1} \prod_{j=1}^n m_j^{\frac{(1-\frac{1}{\alpha})}{n}} \\ \vdots \\ \psi_n(\mathbf{m}) = \frac{K^{\frac{1}{\alpha}}}{nm_n} \prod_{j=1}^n m_j^{\frac{(1-\frac{1}{\alpha})}{n}} \end{cases},$$

which, when inverted,¹³ becomes

$$\begin{cases} m_1(\mathbf{u}) = \frac{K}{n^\alpha} u_1^{-1+\frac{1-\alpha}{n}} \prod_{j=2}^n u_j^{\frac{1-\alpha}{n}} \\ \vdots \\ m_n(\mathbf{u}) = \frac{K}{n^\alpha} \left(\prod_{j=1}^{n-1} u_j^{\frac{1-\alpha}{n}} \right) u_n^{-1+\frac{1-\alpha}{n}} \end{cases},$$

extending the $n = 1$ case.

Putting this demand as in (18), we get a $F : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that $F(G) = (K/G)^{\frac{1}{\alpha}}$. So $F'(G) = -\alpha^{-1} K^{\frac{1}{\alpha}} G^{-1-\frac{1}{\alpha}} < 0$, making this type of money demand extended-Sidrauski-integrable. In order for it to be also extended-shopping-time-integrable, Proposition 6 only requires the initial value problem

$$\begin{cases} \phi'(s) = \frac{(1-s)^{\frac{1}{\alpha}-1}}{K^{\frac{1}{\alpha}}} \phi(s)^{2-\frac{1}{\alpha}} \\ \phi(0) = 0 \end{cases}$$

to be solvable. This is exactly the same problem studied in Example 1, so we conclude that this demand can arise from the shopping-time model if and only if $\alpha < 1$.

Example 4. We now do with the semi-log money-demand specification the same as just done for the log-log case. Here the natural extension is $\psi(\mathbf{m}) = (1/\xi) \log(D/G(\mathbf{m})) \nabla G(\mathbf{m})$, where $D > 0$ and $\xi > 0$. Here $F : C_D \rightarrow \mathbb{R}_{++}$ is given by $F(G) = (1/\xi) \log(D/G)$, and we have $F'(G) = -1/(\xi G) < 0$. Therefore, this money demand is extended-Sidrauski-

¹³The very name of this kind of demand gives us a hint of how to invert this system of equations: apply the logarithm function on each side. This yields a linear system in the variables $\log m_1, \dots, \log m_n$.

integrable.

And as far as the shopping-time model is concerned, the conclusion is the same as the one obtained in the unidimensional case. Under Assumption ϕ_02 , the semi-log specification is not extended-shopping-time integrable. Under the broader Assumption ϕ_01 , no conclusion is provided by our analysis. And under Assumption \mathbf{r} , its integrability is guaranteed.

4. Conclusion

In this paper we have investigated the properties an arbitrary money-demand function given by the econometric practice has to obey in order for it to be consistent with the versions of the Sidrauski and the shopping-time models displayed in Lucas (2000). One conclusion of our study was that log-log money demands with an elasticity equal to or greater than unity, as well as the popular semi-log money demand, may not be rationalized by the shopping-time model, unless a boundedness assumption on the nominal interest rate is made. This problem does not extend to the Sidrauski model.

Besides Lucas (2000), Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick (2007), which use exactly the same utility and/or transacting-technology function, Fischer (1979), Siegel (1983), Asako (1983), Weil (1991) and Goodfriend (1997), dealing with particular cases of such functions, are examples of works which can also benefit from our results.

A second contribution of the paper was that of investigating how such classes of the Sidrauski and the shopping-time models relate to each other. Differently from Feenstra (1986), our analysis focused not on the structure of the models themselves, but on the necessary and sufficient conditions arising from them. We have shown that Lucas's version of the Sidrauski model is, in a strict sense, less stringent than that of the shopping-time model.

All our results have been initially developed for the case of unidimensional money demands, and posteriorly extended to an economy where several assets perform monetary functions. They were exemplified with both the unidimensional and multidimensional versions

of the log-log and the semi-log money-demand functions.

Appendix

We have already mentioned that all our results apply *ipsis litteris* to Lucas (2000), Simonsen and Cysne (2001), Cysne (2003) and Cysne and Turchick (2007), since these papers use the same utility function and/or transacting technology we use here. Fischer (1979), Siegel (1983), Asako (1983), Weil (1991) and Goodfriend (1997), on the other hand, are works which can also directly benefit from some of the results of this paper, since the utility function and/or transacting technology used in such works correspond to particular cases of those treated here (it was shown in Section 2.1 how the embedding into the general case can be done).

In this appendix we treat very briefly the case of two other utility functions used in the literature which cannot be written like (1).

The additive case (Brock, 1974). $U(c, m) = u(c) + v(m)$, where u and v are homogeneous of the same degree, strictly increasing and concave.

In this case, (2) becomes, in steady-state equilibrium, $r = v'(m)/u'(1)$. Therefore, we still have that any strictly decreasing money-demand function is rationalizable by Sidrauski's model. In fact, given ψ , take $v(m) = \int_1^m \psi(\mu) d\mu$, and u the identity function, for instance.

The CES case (Holman, 1998). $U(c, m) = [\pi c^\delta + (1 - \pi) m^\delta]^\frac{1}{\delta}$, where $\pi \in (0, 1)$, $\delta \in (0, 1)$.

As in the Cobb-Douglas case, this utility function generates exclusively log-log money-demand functions, but now the ones with higher-than-one elasticities (in absolute value): $m = ((1 - \pi)/\pi)^{1/(1-\delta)} r^{-1/(1-\delta)}$.

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