Monopolistic Insurance and Competitive Financial Markets

Rio de Janeiro
21 de Março de 2016
João Lucas Thereze Ferreira

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Dissertação submetida a Escola de Pós-Graduação em Economia como requisito parcial para a obtenção do grau de Mestre em Economia.

Orientador: Andrés Carvajal

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JOÃO LUCAS THEREZE FERREIRA

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Abstract

This dissertation studies the interaction between insurance and financial markets. Individuals who differ only in risk can save through a competitive market. They also have access to insurance contracts offered by a monopolist firm. We show that an equilibrium always exists in that economy. Fundamentally, we identify an externality imposed on the insurer's decision by the endogeneity of prices in the financial market. We argue that, because of such externality and in contrast to the pure contract theory case, equilibrium may exhibit pooling. This is shown by means of a numerical example in which equilibrium does not differentiate types.

KEYWORDS: Insurance, Screening, Pooling, Finance
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1 Introduction

Insurance economics has been a major theme for Contract Theory at least since Rothschild and Stiglitz (1976) and Wilson (1977) examined competitive screening in insurance markets. Although much of the work on the area has been done under the assumption that insurers compete with each other, that literature is also related to the one of monopolistic screening.\(^1\) In particular, both of these are interested in properties of the optimal contracts that arise from strategic considerations. Knowledge of such properties is relevant both from a normative and from a positive points of view, because it can inform the decisions of policy makers in the regulation of insurance markets, on one hand, and shed light over the key variables determining the actions of insurance companies, on the other.

A relevant limitation of that research program is that it is too partial by nature, in the sense that it abstracts from any other relation occurring in the economy besides the insurer-insuree one. When the insurance industry is considered in isolation, the close relation between the latter and financial markets is overlooked. Therefore, the fact that assets and insurance contracts are imperfect substitutes is neglected by the partial equilibrium approach. The reason why that relation could be relevant is twofold. First, in each state of nature, individuals’ wealth distribution is endogenous as the latter react to different coverages by purchasing different portfolios. Consequently, the insurer should take into account such endogeneity when choosing the menu of contracts offered.

Second, insurers affect asset prices which, in turn, change the attractiveness of a given set of contracts. As a consequence, any equilibrium concept must recognize that a menu should be attractive taking as given the prices it induces in the competitive asset market. These two effects may account for qualitative alterations in the usual prescriptions derived from pure contract theory. For instance, it could be the case that the very existence of equilibrium was undermined. Other usual predictions that could be altered would be the monotonicity of coverages on risk and the fact that the optimal menu of contracts differentiates types.

Even if none of the aforementioned changes occurred, however, this paper would

\(^1\)Stiglitz (1977) being the pioneer work on monopolistic insurance.
still have a role. Except for some works in Public Finance, Mechanism Design problems are regarded to in a completely partial equilibrium approach. Surprisingly, the results derived under such stringent assumption are seldom called into question. As a consequence, whether qualitative results derived in isolated partial equilibrium models still apply in a more general framework is a research question by itself.

We devise a novel approach in order to incorporate financial markets in the standard insurance problem. It consists on coupling a version of the usual contract theory program to an incomplete markets general equilibrium model of a financial economy. A monopolist is considered who is able to insure individuals in an economy that presents aggregate and idiosyncratic risk. A financial market which is complete with respect to aggregate shocks is available for consumers as the only alternative to carry resources through time. Individuals are assumed to be plenty and to be divided in types according only to risk.

In this environment, the monopolist firm charges a premium that is paid in a first period and promises state-contingent coverage in a second one. In order to fulfil her promises, the insurer must carry resources from the first date to the second. Hence, the model is only complete when such transference technology is specified. In this paper, we consider a firm which relies on an external technology for financing its promises. That is the case, for instance, of an insurance company who can access an international market for debt that is not available for its insurees, or of a firm holding a storage technology which is, also, unaccessible for investors.

First of all, we show that an equilibrium always exists in this very general environment. In other words, the non-competitive and the competitive sides of the economy work fine together. In the position of a market maker, it was intuitively possible that the firm manipulated the market so to make unbounded profits. In particular, as outside options are dependent on market prices, the firm indeed has some control over them. The existence of equilibrium, though, demonstrates that this intuition fails.

Then we start considering the qualitative features of such equilibria. In order to accomplish this goal, we impose further structure both for making the model re-

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3For the case in which the firm trades in the same financial market as its insurees, with some added assumptions, we can recover results that closely parallel those in the main text.

4A similar proof applies also if we allow individuals to differ in endowments in addition to risk.
semble the most pure one in contract theory, and for tractability. Specifically, we investigate the case of no aggregate shock, two idiosyncratic states and two types of individuals. Our main result is that an equilibrium menu of contracts can pool agents of different risks together. We show that by means of an example. This is in stark contrast to the basic models of monopolistic insurance in contract theory.

Fundamentally, we identify the driving force behind the pooling result. Before anything else, we argue that it is not incentive constraints that are hindering separation. Such effort differentiates our result from the literature that deals with the absence of the Spence-Mirrlees condition in mechanism design. We conclude, then, that the source of pooling is an externality imposed on the decision of the firm by general equilibrium effects. Such externality operates through the individual rationality constraint, rather than through incentives, but it makes separation more costly for the monopolist by making it more expensive to assure the attractiveness of a contract relative to the outside option.

Also, our is not a simple consequence of countervailing incentives and type-dependent reservation utilities, as in Maggi and Rodriguez-Clare (1995) or Jullien (2000). Rather, in our model outside options do depend on private information, on the specific contract one is offered and, what is new, on the whole set of contracts posted by the firm. The externality we identify is generated exactly by this endogeneity of one’s reservation utility to the whole menu of contracts. This is a consequence of the fact that individual decisions are tied by the financial market and that the latter is affected by the choice of contracts the company offers.

We call that externality the market effect. In canonical insurance theory models, the riskiest individual is undistorted at the optimal decision of the firm, relative to the Pareto Efficient outcome. Namely, his marginal utility equalizes the marginal cost of providing insurance, which usually implies full insurance at the top of the risk distribution. The intuition behind this result is that there is no need to distort the riskiest agent since the agent’s incentives are always to underreport their own risk and never to overstate it.

That result implies no distortion at the top because the source of any suboptimality in the canonic model stems from incentive compatibility. Since the market effects

\[\text{See Araujo and Moreira (2010).}\]
is an individual rationality phenomenon, we end up with distortions for every type. This is our second most important result. For the riskiest type, that allows us to state that financial markets crowd out insurance in the presence of a monopolist.

In the next section we discuss how this work fits in the literature. Next, we describe the model and define what is to be known as equilibrium for the purpose of this paper. We then prove existence. Following, we present the example of pooling equilibria in the restricted setting. The last sections are dedicated to the identification and description of the theoretical force against separation.
2 Related Literature

This dissertation focuses on an insurance problem with adverse selection when the players inhabit a competitive financial economy. It is, then, investigating the relation between contract economics and general equilibrium. A wide literature has been concerned with this issue since Prescott and Townsend (1984), such as Dubey and Geanakoplos (2002), Bisin and Gottardi (2006) and Rustichini and Siconolfi (2008) or, in a different perspective, Bisin et al. (2011). These papers generally study economies which are competitive and subject to informational problems that generate the possibility of adverse selection. In these economies, they model contracts as exclusive relations between a given firm and a consumer.

In all these papers, contracts, which are usually lotteries on future consumption, are either offered by firms that are competitive or are exogenously given. The latter is the case, for instance, in Dubey and Geanakoplos (2002), where individuals buy insurance by choosing among a number of pools to trade in. Contracts are sold in a competitive market in which both firms and agents take prices as given and, in equilibrium, their prices are type-dependent and reflect the profile of types of agents buying them. Further, the two fundamental questions for this literature are whether equilibrium exists and is efficient.

This is in contrast with our model, in which the insurer is a monopolist company exercising power over not only the price of contracts, but also over other prices. The key here is that we assume a market structure that is divided in two. On one hand, there is the contract market, which is strategic, and in the other a financial market that is competitive. None of the papers cited go in the same direction.

Bisin and Gottardi (2006), however, follows a similar path as ours by identifying an externality which is associated to adverse selection in the competitive economy. In their model, competitive firms sell promises of future consumption in separated markets, one for each type of agents who vary only in risk. Risk is private information but, in equilibrium, insurees self-select to the market designed to their own type. Exclusivity of contracts and the fact that markets are competitive and naturally separated precludes risk-sharing between types of agents, and generates inefficiency.

It turns out that the level of net trades an agent can make in a market is linked to
trades in the other one through incentive compatibility. We, reciprocally, identify an externality the existence of competitive markets imposes on the screening problem. And, differently, our externality is reflected directly in the problem of the firm, rather than on another individual’s problem. This is so exactly because we assume a firm that internalizes directly the possible deviations by agents.

Usually, papers on that research program try to recover the type of equilibrium described in Rothschild and Stiglitz (1976) and, more broadly, to extend that by finding equilibrium concepts that always allow for existence. This is always done in an economy that is similar to the one of the original paper. In Dubey and Geanakoplos (2002), for instance, an equilibrium is shown to always exist and to replicate, within the new framework, the results of Rothschild and Stiglitz (1976). This is also true for Bisin and Gottardi (2006). Existence seems to be eased exactly because in this literature firms loose instruments compared to Rothschild and Stiglitz (1976). In Dubey and Geanakoplos (2002), there is no firm at all, and the contracts available are exogenously fixed. In Bisin and Gottardi (2006), contracts are endogenous, but firms take prices as given in a competitive market for contracts. These assumptions amount to a reduction in the space of deviations of the firm, what makes existence more likely.

In a monopolistic framework, such assumptions are meaningless, so we give back the flexible choice space of the firm, allowing not only the insurance company to be free for tailoring any contract it desires, but also to set prices arbitrarily. That leads the firm to internalize price changes that emerge in other markets. It results in a firm that is at the same time freer and more aware of the effects of its own actions in the economy. As a summary, if we see this work as trying to understand contract theory in the realm of general equilibrium, it inverts what is usually done in the mentioned literature - that is, to take competitive markets as the background, and tries to look at the problem in a partial mechanism design framework.

At the same time, this paper does not fit in the existing mechanism design literature. The latter uses to deal with isolated markets in a complete partial equilibrium approach. We are closer, then, to a branch of Public Finance that studies the problem of a big player who has power over other markets, for instance Golosov and Tsyvinski (2007) and da Costa (2009).

Golosov and Tsyvinski (2007) analyses the problem of an economy in which skills
evolve stochastically and in which individuals can trade in a competitive asset mar-
ket unobservably. They show that competitive insurance companies are unable to
internalize the fact that agents have access to hidden trades and, therefore, there is
a role for government intervention in the insurance market. This consists exactly in
the identification of an externality that arises from hidden trades and is not internal-
ized by competitive firms. Reciprocally, the government, being a big rational player,
has the ability to react to the externality and correct the distortions thence generated.

Golosov and Tsyvinski (2007) is close to our model, but the question we have in
mind is radically different, and so are the underlying concepts behind the two pa-
pers. We are asking what is the best policy for an insurance company that offers
contracts to individuals who can hide trades in a financial market, rather than un-
derstanding the limits of a market for competitive insurance. For that, we consider
a monopolistic company in our model that directly internalizes the externality de-
duced from hidden trades, and it is exactly such internalized externality that gives rise
to pooling equilibria.

Last, our paper relates to the literature of pooling in contract theory. It provides
a new justification for pooling that is not solely due to either of the two most usual
ones: the absence of Spence-Mirrlees conditions as, for instance Araujo and Mor-
eira (2010); or countervailing incentives as in Maggi and Rodriguez-Clare (1995). In
our model, we can show that incentive constraints do not pose a problem for sepa-
ration. We show that, by arguing that a condition which plays the role of the single-
crossing property is preserved by the endogenous counterparts of utility functions in
our framework. That excludes the former as an explanation for pooling in our model.

On the other hand, countervailing incentives in the sense of Lewis and Sapping-
ton (1989) and Maggi and Rodriguez-Clare (1995) are indeed present in our work.
Countervailing incentives happen when a given agent may have incentives either to
understate or overstate their private information depending on the actual realiza-
tion of such private variable. In much cases that is associated with type-dependent
outside options, usually when trading with the firm demands the agent to forego an
opportunity outside the contractual relation that varies with his type. In pure con-
tract theoretical models of insurance, foregone opportunities are always present, but
type-dependence of the outside options is easily controlled.\textsuperscript{6}

Because in our model agents can use the financial market to alter their own levels of wealth, countervailing incentives naturally arise and, even more, endogenously. Therefore, we are adding an extra layer of mixed incentives to the countervailing phenomenon. In our model, outside options are not only type-dependent, but also endogenous because two different contracts generate two levels of savings that affect distinctly the relation between the inside outcome and the outside option.

However, the key factor behind pooling in the economies we study is not that individuals endogenously react to offered contracts - that also happens, for instance, in Netzer and Scheuer (2009) - which, by itself, would make endogenous countervailing incentives the core of our paper. Rather, the driving force for pooling here is that the effect of such reaction leads to changes that affect all the constraints in the economy, including other agents'. Namely, individual deviations, when happening for all individuals of a given type, transform prices in the competitive financial market and, therefore, have a widespread effect in the whole feasibility set, and not only in their own constraints.

\textsuperscript{6}In the sense that incentive compatibility implies that the difference between the inside outcome and the outside option is monotonous in types.
3 The General Economy

We consider an economy with two time periods, allowing for both aggregate and idiosyncratic risk. In the first period, there is only one state of nature. We define the set of aggregate states in the second date to be \( S = \{1, ..., S\} \), and assume that the set of idiosyncratic states is the same for all individuals, \( \Sigma = \{1, ..., \Sigma\} \). Then, the complete description of a state of nature for a given individual is \((s, \sigma) \in S \times \Sigma\). The probability that the economy will fall in an aggregate state \( s \) in date 2 is \( \pi_s > 0 \), so that \( \sum_{s \in S} \pi_s = 1 \).

This economy is populated by a continuum of individuals, normalized so that their measure is one. These individuals have types which determine the odds that they fall in a particular idiosyncratic state. Types are private information and are drawn from the set \( I = \{1, ..., I\} \) and the measure of any type \( i \) in the population is \( \lambda_i > 0 \). For any \( i \), the probability of an idiosyncratic state \( \sigma \in \Sigma \) given that the economy is in aggregate state \( s \in S \) is \( \mu_i(s, \sigma) > 0 \). Consequently, for any \( i \), \( \sum_{\sigma \in \Sigma} \mu_i(s, \sigma) = 1 \) for all \( s \in S \).

Endowments are state-dependent: \( W = (w_{s, \sigma}) \in \mathbb{R}^{S \times \Sigma} \) and, for the purpose of our application, we write it as \( w_{s, \sigma} = w - d(s, \sigma) > 0 \). Preferences are indexed by type and given by \( U_i : \mathbb{R}^{S \times \Sigma + 1} \to \mathbb{R} \) defined, for \( x = (x_0, x_{1,1}, ..., x_{S, \Sigma}) \) as:

\[
U_i(x) = x_0 + \sum_{s \in S} \pi_s \sum_{\sigma \in \Sigma} \mu_i(s, \sigma) u(x_{s, \sigma})
\]

where \( u : \mathbb{R}_+ \to \mathbb{R} \) is a three times differentiable, monotonic and strictly concave function, satisfying Inada conditions.

The financial structure of the economy will generally comprise \( S \) Arrow securities, \( A = \{a_s\}_{s \in S} \) one for each aggregate state. Individuals choose portfolios of Ar-
row securities to transfer resources through time. Portfolios are private information. Accounting for all these properties, an economy can be defined as

\[ \Gamma = \{ u, I, S, \Sigma, (\pi_s), (\mu_i(s, \sigma)), A, W \} \]

The insurance firm is a risk-neutral agent who is allowed to offer, for any individual, a list \((p, \eta)\), with \(p \geq 0\) and \(\eta \in [0, 1]^{S \times \Sigma}\), of premium and coverage so that, for a given state \((s, \sigma)\) the wealth guaranteed by initial endowments plus coverage is

\[ \tilde{W} = \{ \tilde{w}_{s\sigma} \} = \{ w - (1 - \eta_{s\sigma})d(s, \sigma) \}. \]

As a consequence, we write the contract \((p, \eta)\) as \((p, \tilde{w})\), where it should be understood that this is only an equivalent representation in terms of premium and wealth, rather than premium and coverage.

The insurance company posts a menu, \(M\), consisting of any number of contracts, among which agents will self-select. Profits are defined as a function of this menu. Specifically, profits depend on the amount earned by collecting the premia, which we shall call \(e(M)\), and the portfolio of securities it must hold in order to honour its commitments in date two, the vector \(z(M) \in \mathbb{R}^S\). They also depend on the technology the firm uses to transfer resources through time. We shall assume that the insurer has an outside source of funding that is not the financial market in question, which is represented by a linear technology \(v \in \mathbb{R}^S_+\). Formally, then, profits are written:

\[ \Pi(M) = e(M) - v \cdot z(M) \]

The meaning of the technology \(v\) is that for each unit of money the firm wants to save for the second period if state \(s\) occurs, it must pay \(v_s\). One could understand such external funding as a storage technology which is not available for individuals, in which case \(v = (1)_{s \in S}\), or as the price of Arrow securities in an external financial market to which insurees have no access. Another option is to consider \(v = (\pi_s)\), that is, the usual state prices for each state \(s\). It will be noted, a bit further, that this would otherwise.

\[ \text{The bounds on } \eta \text{ are natural, but unnecessary. The existence proof is naturally extended for an environment with any bounds.} \]

\[ \text{As long as no confusion is implied, we will be loose on the specific character of a menu. In principle, we could both see it as a set or a (possibly infinite) tuple of vectors, and there is clearly an equivalence in doing one or the other. In the future it will be clear that it is in our best interest to define it as a tuple.} \]
be equivalent to understand the firm as an average profits maximizer.

Notice that, given a menu of contracts, \( \mathcal{M} \), and a price for Arrow securities in the financial market, \( q \in \mathbb{R}^S \), investors maximize their utilities choosing both their asset holdings and which contract they will accept within the menu, if any. We can break down consumer's optimization in two parts, then. Assume an individual of type \( i \) accepts the menu \((p, \tilde{W})\). Then, the problem she faces in the market is:

\[
V_i(p, \tilde{W}|q) = \max_{\theta \in \mathbb{R}^S} \left\{-p - \theta \cdot q + \mathbb{E}_i u(\tilde{W} + \theta)\right\}
\]

where \( \theta \) is the portfolio to be chosen by the agent. Here, the notation \( \mathbb{E}_i \) means the expectation taken using the distribution induced by idiosyncratic probabilities of type \( i \). Besides, we define \( \tilde{W} + \theta = \{\tilde{w}_{s,\sigma} + \theta_s\}_{(s,\sigma)} \).

The problem is straightforward: the investor is choosing savings given that her wealth will be determined by the contract she accepts. We call \((IP_q)\) the optimization problem in the right-hand side of equation (2). When it comes to optimization within the menu, an individual accepts a contract \((p, \tilde{w})\) given \( q \) out of a menu \( \mathcal{M} \) only if:

\[
V_i(p, \tilde{W}|q) \geq V_i(p', W'|q) \quad \forall (p', W') \in \mathcal{M} \cup \{(0, W)\}
\]

The content on the above constraint is clear. Each individual compares her inside outcome, that is, the utility that accrues from accepting the offered policy, versus the utility value of her outside option, that is, of using only the competitive market for future consumption, and the value of accepting any other contract in the menu. It is worth noting that refusing every contract in the menu means accepting \((0, W)\). That is, paying no premium and getting exactly the endowments as wealth.

These constraints are obviously related to the classical incentive compatibility and individual rationality constraints in pure contract theory. This is why we shall call them \((IC_q)\), for those that compare the value of one's own contract to the utility of another one's, and \((IR_q)\) when the comparison is against the outside option. An important fact about these inequalities is that they are written taking \( q \) as given. That means that when one decides whether it is optimal to choose a contract or not, he

\(^{15}\)We are abusing notation slightly in two ways. First, by writing \( \tilde{W} \) both for the vector of wealth and for the random variable it represents. Second, and arguably graver, for writing \( \tilde{W} + \theta \) as a sum of vectors of different lengths. This is made precise if we see \( \theta \) as a random variable that is constant over idiosyncratic states.
takes prices in the financial market as given. This is already the effect of an important assumption on the financial market: individual investors are too small to affect it. That constraint naturally translates into an equilibrium condition. We see that in details in the next section.
4 Equilibrium

Given a menu $\mathcal{M}$, it is natural to believe that the investor side of the economy, which has already been fully described, arranges itself towards a kind of equilibrium, which does not yet take into account the firm’s problem. This is so because, once the menu is fixed, individuals have all the information they need to make their decisions. In order to be consistent with our intuition of what an equilibrium must be, it should, in some sense, comprise the notion of a Walrasian Equilibrium when it comes to portfolio choice. We wish to define an auxiliary notion of equilibrium that will naturally allow us to state, next, what an equilibrium in the model is. Let us do it in two parts.

Definition. Weak Investor-Side Equilibrium  
Fix a menu $\mathcal{M}$. A Weak Investor-Side Equilibrium (WISE) is a list $\{[c_x, \theta_x]_{x \in [0,1]}, q\}$, of contracts, $c_x \in \mathcal{M}$, portfolios $\theta_x$, one for each individual\(^{16}\) and prices $q$, such that:

1. (Optimization within the menu) Given $\mathcal{M}$ and $q$, $c_x$ satisfies $(IC_q)$, $(IR_q)$ for the appropriate type of agent $x$;

2. (Optimization in the market) Given $q$ and $c_x$, $\theta_x$ solves $(IP_q)$ for the appropriate type of agent $x$;

3. (Market Clearing) $\int_0^1 \theta_x dx = 0$

The above definition says that a WISE must follow the two important desiderata of competitive equilibrium: agents should be maximizing their utilities given prices and feasibility should be respected. The only difference is that the condition of optimization is split in two. Notice that, assuming that the first condition is satisfied, the last two items imply that $(\{\theta_x\}, q)$ is a Walrasian Equilibrium in the abstract economy in which initial endowments are $c_x = (p_x, W_x)$. So, all the condition above asks is that, besides composing a Walrasian Equilibrium, individuals should be optimally choosing their wealth possibilities within the menu.

Now, with that definition in mind, we want to restrict attention to type-symmetric WISE. That is, we shall only be interested at WISE such that $\theta_x = \theta'_x$ and $c_x = c'_x$ when-

\(^{16}\)agents are parametrized by the interval $[0, 1]$. 

20
ever the types of \( x \) and \( x' \) are the same.\(^{17}\) Let \( WI(\mathcal{M}) \) be the set of type-symmetric WISE for a given menu. Then, the following holds:

**Proposition 1.** Let \( \mathcal{M} \) be any menu and \( e \in WI(\mathcal{M}) \). There exists \( \hat{\mathcal{M}} \), containing at most one contract for each type, such that \( e \in WI(\hat{\mathcal{M}}) \).\(^{18}\)

This fact is sufficiently obvious. It can be seen as a partial Revelation Principle, in that it says, implicitly, that it suffices to offer one contract for each type of agent to obtain any equilibrium in the class we are interested. Restricting attention to symmetric equilibria is fundamental here. It assures that adding to the menu contracts that create indifference for one type is immaterial when it comes to inducing an equilibrium, once either all individuals of the same type will get one contract or all of them will accept the other. In each case, there is one one redundant contract in the menu, which plays no role at all.

As a direct consequence of that, we can focus on menus that can be written \( \mathcal{M} = (c_i)_{i \in I} \).\(^{19}\) Henceforth, we shall assume that contracts that compose menus are indexed by type or, in the contract theory language, possess type-recommendations. That allows us to, finally, define the auxiliary equilibrium notion as we intended to do:

**Definition. Investor-Side Equilibrium** Fix a menu, \( \mathcal{M} \). An Investor-Side Equilibrium (ISE) is a list \( \{ (c_i, \theta_i)_{i \in I}, q \} \) of contracts \( c_i \in \mathcal{M} \), portfolios \( \theta_i \), one for each type, and a price vector \( q \), such that \( \{ [c_x, \theta_x]_{x \in [0,1]}, q \} \) is a WISE when \( (c_x, \theta_x) = (c_i, \theta_i) \) for all individual \( x \) of type \( i \).

What this definition adds to a type-symmetric WISE is that each individual \( x \) must be optimizing by accepting the contract \( c_i \) devised for his type.\(^{20}\) This is a powerful request. Again, fixing that an agent of type \( i \) is accepting \( c_i \), we have that portfolios and prices compose a Walrasian Equilibrium for the economy with wealth given by such contracts. For a given menu, let us call \( E(\mathcal{M}) \) the set of associated ISE’s. It is also worth noting that the market clearing condition now reads:

\(^{17}\)We have been assuming throughout that "strategies are pure" in the sense that one (and only one) contract is chosen from the menu.

\(^{18}\)All but a few simple and intuitive proofs are left to the appendix.

\(^{19}\)And we will see it as a tuple of contracts, that is, a matrix.

\(^{20}\)Recall that, now, contracts in a menu are indexed with a type recommendation.
\[
\sum_{i \in I} \lambda_i \theta_i = 0 \in \mathbb{R}^S
\]

All we need to define for understanding what is equilibrium for the whole model is the problem of the firm. Now, the firm is a player with market power that should anticipate movements from the part of its insurees. This anticipation, if any rationality is to be imposed, must take into account the fact that investors will reorganize themselves in some particular way, given any choice of menu. This way of organization we assume to be an ISE. Because individuals will necessarily be in an ISE, the firm knows exactly what are its promises for date two and the amount it collects in the first period. So profits are now explicitly described by:

\[
\Pi(M) = \sum_{i \in I} \lambda_i [p_i - \sum_{s \in S} v_s \sum_{\sigma \in \Sigma} \mu_i(s, \sigma)(\tilde{w}_{s,\sigma} - w_{s,\sigma})]
\]

That is, premia to be collected in the first period is exactly the mean when the frequencies of types in the population are correctly taken into account. Besides, the firm faces no idiosyncratic risk, because of the assumption of a continuum of agents. Therefore, its promises for tomorrow are, in each aggregate state, \(\sum_{\sigma \in \Sigma} \mu_i(s, \sigma)(\tilde{w}_{s,\sigma} - w_{s,\sigma})\), that is, the average promises, conditional on \(s\) being the aggregate state. Also see that, as mentioned before, if the technology \(v\) charges state prices for each aggregate state, then \(\Pi(M)\) is exactly the average profits of the firm. At last, the problem of the firm can be stated as:

\[
\max_{M} \left\{ \Pi(M) : E(M) \neq \emptyset \right\} \tag{PF}
\]

The problem above, which we call \((PF)\) is of maximizing profits, subject to a constraint imposed by rationality of the firm, that knows investors will optimize in a feasible financial market. We can then define

**Definition. Equilibrium.** An Equilibrium in this model is a list \(\{(c_i, \theta_i)_{i \in I}, q\}\) such that, defining \(M = (c_i)_{i \in I}:\)

1. \(\{(c_i, \theta_i)_{i \in I}, q\} \in E(M)\)

2. \(M\) solves \((PF)\)
Essentially, then, an equilibrium is an optimal menu of contracts for the firm, and an ISE which is consistent with such menu. It should be clear by now that an equilibrium exists if and only if the firm’s problem has a solution. Indeed, if $\mathcal{M}$ maximizes the firm’s profits over feasible menus, then an equilibrium list can be constructed by taking any element in $E(\mathcal{M})$. Therefore, proving existence of an equilibrium is tantamount to proving that the monopolist’s program has a solution.

**Proposition 2. Existence.** For any economy $\Gamma$, an Equilibrium exists.

As mentioned before, the proof of this proposition is based on showing that the firm’s problem has a solution. For doing that, we rely heavily on the quasilinear structure of the economies under scrutiny to show that the abstract constraint in $(PF)$ translates naturally to a more concrete and tractable one. The above proposition is fundamental in that it shows the competitive and non-competitive sides of the economy work together fine. In particular, the monopolist does not have unlimited market power, in the sense that he cannot construct menus giving himself unbounded profits and that are still acceptable for agents. That is so because his capacity of manipulating outside options is naturally constrained by agent’s ability to use the market in their own favor.

A particular consequence of quasilinearity is that, for any fixed menu, $\mathcal{M}$, $E(\mathcal{M})$ is either a singleton or the null set. In quasilinear economies, fixed an initial set of endowments, under the conditions we imposed on utilities, a Walrasian Equilibrium always exists\(^{21}\) and is unique. Because the set of ISE’s is included in the set of Walrasian Equilibria,\(^{22}\) there must be at most as many ISE as there are Walrasian equilibria, justifying what we suggested above.

This and some more properties that come from quasilinearity are fundamental in establishing the proposition that follows. First, let us establish that $\mathcal{K}$ is the abstract set from which menus can be chosen. So, $\mathcal{K} \subseteq \mathbb{R}^{I \times S\Sigma}$.

**Proposition 3.** There is a function $q : \mathcal{K} \to \mathbb{R}_+$ which is continuous and such that

---

\(^{21}\)See Geanakoplos and Polemarchakis (1985)

\(^{22}\)This claim is not absolutely correct. Actually, the set of ISE’s is included in the Walrasian Equilibrium Manifold of the economy. Then, it projects into a set of portfolios and prices that is a subset of Walrasian Equilibria given the distribution of endowments suggested by the accepted contracts. To circumvent this preciser but convoluted argument, we abuse of meaning in the sentence above.
$(PF)$ is equivalent to:

$$
\max_{\mathcal{M}} \left\{ \Pi(\mathcal{M}) : V_i \left( p_i, \tilde{W}_i | q(\mathcal{M}) \right) \geq V_i \left( p', W' | q(\mathcal{M}) \right) \right\} \\
\forall (p', W') \in \mathcal{M} \cup \{(0, W)\}
$$

(PF')

Proposition (3) above provides us with a characterization of $(PF)$ that closely resembles the problem of a mechanism designer in pure partial equilibrium environments. Notice that the constraints of this new problem $(PF')$ are exactly the incentive and individual rationality constraints as we defined them in our model. It should be clear that the main difference between this problem and a usual contract theoretical one is that any perturbation in the menu affects the whole set of constraints. That complete dependence of the constraint set on all the entries of the menu is the exact consequence of relaxing the assumption of an isolated insurance market.\textsuperscript{23}

The latter characterization of the firm’s problem is, in a sense, expected. The way we defined the insurer, it is a rational player, self-conscious of its own size in the economy. Therefore, it should act as the usual insurer, but with the knowledge that price variations shall impact the acceptability of its menu. What is more surprising about Proposition (3) is that prices can be understood as a function of menus. Even more, such function is continuous. It turns out that $q(\mathcal{M})$ is the price of a Walrasian Equilibrium when type-$i$ individual’s endowment is the wealth guaranteed by the contract recommended for them within $\mathcal{M}$. By quasilinearity, such price is unique and $q$ is a function. By smoothness of the utility functions, $q$ is seen to be continuous.

In this form, $(PF')$ allows us to search for equilibria by solving a problem similar to a canonical contract theory program. What we do in the next sections is to use that form of the firm’s problem to give a partial characterization of equilibria in a much more restricted setting. Our most important result is that pooling may arise as an equilibrium. We show that by means of an example. We then identify the source of this phenomenon.

\textsuperscript{23}It is worth noting the quasilinearity imposes that the function $q$ in Proposition (3) does not depend on premia. Hence, that dependence is not that complete, as a matter of fact.
5 A Simpler Economy

From now on, we assume the following simplified structure for the economy: there is one aggregate state and two idiosyncratic states, \( \Sigma = \{1, 2\} \equiv \{a, n\} \), where \( a \) stands for “accident” and \( n \) for “no accident”. We further assume that there are two types of individuals, \( \mathcal{I} = \{1, 2\} \equiv \{L, H\} \). For agent of type \( i \in \mathcal{I} \), the probability of falling in \( a \) is \( \mu_i \) in which case his endowment is \( w - d \). Otherwise, his endowment is \( w \). Without loss of generality, \( \mu_L < \mu_H \).

The only available asset is savings that cost \( q \), whose value will be determined in competitive equilibrium. By the definition of a contract, because there is no damage in state \( n \), an insurance policy is composed only of premium and coverage on the state \( a \). Consequently, a contract can be seen as a pair \((p, \tilde{w})\), in which \( \tilde{w} \) is wealth guaranteed by the policy.

Using the fact that individuals satisfy their budget constraints with equality, we can write the problem of an \( i \)-type in this environment, given a contract he accepts, \((p, \tilde{w})\) and a price \( q \) as:

\[
V(p, \tilde{w}|q) = \max_{\theta \in \mathbb{R}} \{ -p - \theta q + \mu_i u(\tilde{w} + \theta) + (1 - \mu_i) u(w + \theta) \}
\]

And define \( \theta_i(\tilde{w}, q) \) as the unique argument that attains the maximum for the above problem. We shall call economies \( \Gamma \) that satisfy these restrictions simple economies. Lastly, we shall impose the following Assumption on utilities and endowments - \( v \) is the transference technology of the firm.

**Assumption.**

\[
u'(w - d) > v > u'(w)
\]

The above assumption is justified as follows. If a benevolent central planner, knowing each agent in the economy, were to redistribute resources ignoring the financial market, but using the same technology as the firm does, it would equalize the marginal benefit of receiving insurance (that is, marginal utility), to the marginal cost of providing it, that is \( v \), for each agent. Our assumption, then, is just that the first-best contract is in the support of feasible coverages. That follows from the Assumption by continuity of marginal utilities.
The simple economies we just defined nearly resemble a pure contract theory one,\(^{24}\) except that in canonic models the insurance company is assumed to redistribute the agents wealth across states, rather than through time. The reason why we model it in contrast to part of the literature is twofold. First, we wish to shed light on the relation between insurance and financial markets. In real life, both savings and insurance contracts are paid in advance and, therefore, both have the role of transferring money either from or to the future. Hence, in order to get a better understanding of the relation between the two markets under scrutiny, this modelling strategy seems to be correct.

Second, it helps in the tractability of the problem. It will be noted that the endogeneity of savings prices naturally brings about technical difficulties that are absent in the pure contract theory program. At the same time, functions that are not explicitly defined arise from the ISE and, with greater reason, nonconvexities become pervasive. That imposes a relevant obstacle to understanding the behaviour of the firm’s problem even locally, let alone globally.

However, departing from the canonic model deprives us from having a consolidated benchmark. In order to have a background against which to compare our results, we shall naturally define what will be called the basic model in this paper. We shall state that a standard economy is exactly one that satisfies all the conditions of being a simple economy, except for its financial structure. Specifically, we shall assume that a standard economy is an economy with no financial market. That is:

**Definition.** An standard economy is an economy of the type \( \Gamma = \{ u, \mathcal{I} = \{ L, H \}, \mathcal{S} = \{ 1 \}, \Sigma = \{ a, u \}, (\pi_\sigma) = 1, (\mu_i(s, \sigma)), A = \{ 0 \} \} \).

Standard economies are not a subset of simple economies, but they are very similar to one another, and differ only in some small details. One of them is the definition of the value of a contract. The value of a contract \((p, \tilde{w})\) for an agent of type \(i\) in a standard economy is:

\[
V_i(p, \tilde{w}|q(\mathcal{M})) \equiv B_i(p, \tilde{w}) = -p + \mu_i u(\tilde{w}) + (1 - \mu_i) u(w)
\]

It should be clear that individual rationality and incentive compatibility constraints

\(^{24}\)See Stiglitz (1977)
can be defined analogously. Defining the standard model in this way amounts to simply changing the contract space of the canonical contract theoretical model of a monopolist insurance to account for time in the same we do it in this paper. Assuming that, the firm’s problem in the standard model is:

$$\max_{M} \left\{ \Pi(M : B_i(p_i, \tilde{w}_i) \geq B_i(p', w') \ \forall (p', w') \in M \cup \{(0, w - d)\} \right\}$$

As a consequence, we can characterize upfront what is the best menu in the standard model. That should provide us with a benchmark to compare our results with.

**Proposition 4.** In the standard model, the optimal menu is $M = \{p_i, \tilde{w}_i\}_{i \in \{L, H\}}$ such that:

1. $B_L(p_L, \tilde{w}_L) = B_L(0, w - d)$;
2. $B_H(p_H, \tilde{w}_H) = B_H(p_L, \tilde{w}_L)$; finally,
3. $u'(\tilde{w}_H) = v$ and $u'(\tilde{w}_L) > 1$

The content of Proposition (4) makes it clear that our model closely resembles the canonical monopolistic insurance one. In words, the safest type of agents is left on indifference to their outside option. The riskiest type is indifferent between the two contracts in the menu. The third condition is a bit more interesting. In the canonical case, type H’s would be undistorted in terms of their Pareto optimal allocation by receiving full insurance. Our results parallels that one.

Here, as we mentioned, the first best allocation should be seen as exactly $u'(x) = v$, by the reasons we exposed previously. In that sense, we recover non-distortion for type $H$ as a result in any standard economy. It follows that separation and the absence of distortions at the top are properties of any equilibrium in standard economies, just as in the canonical models. In the next section, we show that when we add competitive markets, that ceases to be the case. We show that by means of a numerical example. In the sections that follow, we identify what is pushing our model towards pooling. We show that what is making separation harder is exactly an externality that

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25In the absence of the assumption on marginal utilities, this proposition only changes in that market exclusion can be optimal. That is, when $u'(w - d) < v$, it makes no sense to provide insurance for any of the agents, then both are excluded from the market. Besides, if $u'(w) > v$, we would have $\tilde{w}_H = w$. 
arises from the endogeneity of prices in financial markets and constrains the firm's problem.

We argue that such externality operates through individual rationality constraints, rather than through incentives. As a consequence, even when types are observable, the presence of a financial market imposes distortions on the optimal policy. Besides, distortions exist even for the riskiest type.
6 Example

In this section we show, by means of an example, that a pooling menu can in fact be an equilibrium insurance policy in the simple economy of the last section. For that, we shall assume that individuals have preferences defined by CARA utilities with absolute risk aversion $\alpha > 0$, that is:

$$u(x) = -e^{-\alpha x}$$

We proceed by first solving the competitive part of the example in closed form. That procedure provides us with all the functions with which the monopolist is concerned. From there on, we use computation to generate numerical results. First, consider that an $i$ type individual accepts the contract $(p_i, \tilde{w}_i)$. Then, his optimization problem at savings price $q$ is:

$$\max_{\theta \in \mathbb{R}} \left\{ -p_i - \theta q - \Phi_i(\tilde{w}_i)e^{-\alpha \theta} \right\}$$

where $\Phi_i(x) = \mu_i e^{-\alpha x} + (1 - \mu_i)e^{-\alpha \tilde{w}}$.

Therefore, first order conditions to this problem, which are necessary and sufficient, can be rearranged and rewritten in logarithmic form so to get:

$$\theta_i = \frac{1}{\alpha} \log \frac{\alpha}{q} \Phi_i(\tilde{w}_i)$$

And by using the market clearing condition, portfolios vanish and we can find out prices as a function of coverages.

Defining $g(x, y) = \Phi_H(x)^{\lambda_H} \Phi_L(y)^{\lambda_L}$, we have $q = \alpha g(\tilde{w}_H, \tilde{w}_L)$.

As a consequence, it is possible to characterize savings for an agent of type $i$ when accepting coverage $w^*$, and the overall profile of coverages in society is $(w_H, w_L)$ as:

$$\theta_i(w^*, \tilde{w}_H, \tilde{w}_L) = \frac{1}{\alpha} \log \frac{\Phi_i(w^*)}{g(\tilde{w}_H, \tilde{w}_L)}$$

And, finally, indirect utilities are:

---

26In order to avoid confusion we shall differentiate pooling from market exclusion. Exclusion happens when any of the individuals is left away from the insurance market, by receiving zero insurance from the firm. Pooling happens when both individuals receive the same contract and this is not the zero contract.
Further, we must notice that indirect utilities are not relevant by themselves for the firm's problem. Rather, terms which are constant between types can be omitted,\(^{27}\) so that the relevant indirect utility function is:

\[
v_i((p^*, w^*) | \tilde{w}_H, \tilde{w}_L) = -p^* - g(\tilde{w}_H, \tilde{w}_L) \log \Phi_i(w^*)
\]

We can understand this value function as representing the composition of an indirect utility function as defined in the previous sections with the price function \(q\) identified in Proposition (3). The expression above is informative in that it separates precisely the two effects our model is interested in. That is, on one hand, there is the direct effect of coverage on utility, expressed in \(- \log \Phi_i(w^*)\);\(^{28}\) on the other hand, changes in coverage for any type have an indirect effect via market changes. The \(g(\tilde{w}_H, \tilde{w}_L)\) term embodies that. If the firm was interested only in individual coverages, it would not take into account the \(g\) function and we would fall into a pretty standard contract problem.

We can, then, solve the principal’s problem numerically. Because our problem lacks concavity, we cannot rely directly on standard local optimization algorithms. So, it was necessary to resort to a number of strategies in order to come up with optima for the problem. Luckily, local optimization provides the correct results for this example. We know this is so by comparing local optimizers with both grid optimization and multistart methods, that provide, respectively, approximations of the global optimum and robustness checks. Also, for this example equilibrium is unique.

Below, there is the parameter specification for which the optimal menus were calculated.

<table>
<thead>
<tr>
<th>(w)</th>
<th>(d)</th>
<th>(\lambda_L)</th>
<th>(\mu_H)</th>
<th>(\mu_L)</th>
<th>(v)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8</td>
<td>0.3</td>
<td>0.6</td>
<td>0.3</td>
<td>1</td>
</tr>
</tbody>
</table>

\(^{27}\)At the end of the day, each constraint is a difference of indirect utilities for an individual of a given type. In this difference, naturally, constants vanish.

\(^{28}\)Simple algebra shows that such function is increasing and concave.
The risk aversion parameter, $\alpha$, is not present in the table above because we let it vary between 0 and 9. We will see that whether the optimal menu of contracts presents pooling or separation depends on such parameter. Below, we present the profile of coverages that was computationally found.

The graphic shows that, for very small risk aversion, insurance is absent in this economy. This is compatible with Proposition (4) characterization of the standard case. At very low risk aversion, marginal utilities for CARA are below $\nu$ which, in our case, is taken to be 1. As $\alpha$ increases, coverage is first improved for the riskier type but, when individuals get more risk averse, pooling appears, which is the novelty of our model.

Something that should also be noticed is that distortions to the first best are present even for the riskiest type. Just as pooling, these distortions are not explained in the standard model. We argue that they are direct consequences of the externality imposed on the monopolist’s problem by price endogeneity. The rest of this dissertation is dedicated to identifying and understanding the source of these results.
7 Incentive Separation

Having shown that a pooling equilibrium is possible in our model, it is natural to inquire what is the intuition and what are the mechanics giving rise to this result. In classical contract theory, whenever pooling equilibria appear, it is likely that the Spence-Mirrlees condition is violated. If a canonical contract theoretical model satisfies single-crossing, then it is always possible for the mechanism designer to dominate each pooling contract by a separating one that is arbitrarily close to the original menu.

As it turns out, Spence-Mirrlees condition implies separation in the canonical case, as well as in our standard model, because of a specific feature of these models: reservation utilities are exogenous. Therefore, for any pooling contract that is individually rational, it suffices to exhibit an incentive compatible menu that is more profitable and that makes each individual better off to guarantee that such menu is feasible and dominates the original pooling.

Albeit the above logics seems to be straightforward, that is just not true when outside options are endogenous. In that case, building a new menu will per se alter reservation utilities, and it is not immediate that individual rationality constraints continue to hold. This effect shows that separation is not as simple a feat to be accomplished by the principal as the previous literature seems to imply. Specifically, it suggests that when trying to find a separating menu that dominates a given pooling contract, the firm should take into account effects its new menu imposes both over incentives and over outside options, that is, that individual rationality constraints must not be forgotten when separation is under scrutiny.

Aiming to make the understanding of the mechanics behind our model as clear as possible, we shall use a concept that is weaker than separation. We will say that incentive separation is possible whenever there is, for any individually rational pool-

\[ \mu_H > \mu_L \]

We shall use, interchangeably, the expressions Spence-Mirrlees condition an single-crossing property.

Notice that in any insurance problem reservation utilities are type-dependent. However, in our model, they will also be endogenous functions, that are affected by each choice of the firm. Type-dependence and endogeneity are two important and sharply distinct phenomena. Although a large literature has been devoted to the former (see Maggi and Rodriguez-Clare (1995), Jullien (2000)), not much has been said about the latter.
ing contract, an incentive compatible menu that dominates it and such that either both individuals prefer their contracts under this new menu to the pooling one, or the new menu is individually rational. In the language of our model, we can formally define incentive separation:

**Definition (Incentive Separability).** Let $\Gamma$ be either a standard or a simple economy. It is said that incentive separation is possible in $\Gamma$ if, for any pooling menu $M = \{(p, \bar{w})\}$ such that $w - d < \bar{w} < w$, there exists a menu $\tilde{M} = \{(p_i, \tilde{w}_i)\}_{i \in I}$ such that $\tilde{M}$ is incentive compatible and, for $i \in \{L, H\}$

$$V_i(p_i, \tilde{w}_i|q(\tilde{M})) \geq \min \{V_i(p, \bar{w}|q(\tilde{M})), V_i(0, w)|q(\tilde{M})\}$$

It is said that separation is possible if the same holds and $\tilde{M}$ is individually rational.

As we argued before, an economy being incentive separable does not imply any equilibrium being separating. However, our next proposition states that these concepts are equivalent in the set of standard economies. That allows us to disentangle the two aforementioned effects of separation.

**Proposition 5.** Let $\Gamma$ be a standard economy. $\Gamma$ is incentive separable if and only if it is separable.

The argument behind this equivalence is direct. Let us assume there is a pooling menu $M = \{(p, \bar{w})\}$, $w - d > \bar{w} < w$ that is individually rational and such that $u'(\bar{w}) > v$. Because we must have $\mu_H > \mu_L$, the riskiest agent is always willing to pay more for a marginal unit of coverage, which can be promised because $\bar{w} < w$. So it is always possible to offer him a new insurance plan, $\tilde{w}$, and ask for a new premium $\tilde{p}$ such that the low risk individual is not willing to pay for this new contract. The menu $\{(p, \bar{w}), (\tilde{p}, \tilde{w})\}$ is incentive compatible and each agent weakly prefers their own contract to the original pooling. All that is left to prove is that such menu improves profits. That easily follows from $u'(\bar{w}) > v$. A similar argument is true for the case in which $u'(\bar{w}) \leq v$.

The real importance of Proposition (5) is that it shows that the separation witnessed in standard economies is exactly incentive separation. So, if we are able to
show that any simple economy allows for incentive separation, there will be established that the source of pooling in our model is 1) engendered by the addition of a financial market and 2) is not a sole consequence of incentive constraints. It is easily seen, and it is widely used, that the Spence-Mirrlees condition implies incentive separability (and, therefore, separability) in the canonical model. It is, furthermore, straightforward to see that it remains true in our standard economies. The proposition that follows shows that any simple economy allows for incentive separation. For that, we use the fact that any simple economy satisfies a property that we call ordering, that is akin to single-crossing except that it is defined over indirect utility functions. Such property implies incentive separability.

**Definition. Ordering Property.** We say that two indirect utility functions, $V_H, V_L$ satisfy the ordering property at the point $(\tilde{w}, \tilde{w})$, when $q = q(\tilde{w}, \tilde{w})$, if:

$$\frac{\partial V_H}{\partial \tilde{w}}(p, \tilde{w}|q) + \frac{\partial V_H}{\partial \tilde{w}\partial q}(p, \tilde{w}|q) > \frac{\partial V_L}{\partial \tilde{w}}(p, \tilde{w}|q) + \frac{\partial V_L}{\partial \tilde{w}\partial q}(p, \tilde{w}|q)$$

That any simple economy with such property satisfies incentive separability goes as follows. Assume that the economy is at a given pooling menu that presents coverage $\tilde{w}$. If the firm proposes a new menu which is separating, gives more coverage to the riskier individuals and keeps the low-risk type accepting $\tilde{w}$, then price is changed. If, at this new price, marginal utilities are ordered, then these coverages can be incentive compatible for appropriate choices of premia. The ordering property is simply the expression for ordering marginal utilities at new prices when the new contract is arbitrarily close to $\tilde{w}$.

Using this definition, we can prove:

**Proposition 6.** Any simple economy $\Gamma$ allows for incentive separation.

The proof of this proposition relies on showing that marginal utilities can be ordered at any prices, so we can prove any simple economy satisfies the ordering property. That is, fundamentally, a consequence of the fact that, even though savings approximate the marginal utilities in the accident state between types, they are not enough to make high-risks more satiated than low-risks. Otherwise, the low-risk type agents would be willing to pay more than the high-risk ones for one unit of the riskless bond, contradicting equilibrium in the savings market. As a consequence of that
proposition, it should be clear that any obstacle for separation should be a result of some distortion in individual rationality constraints. We develop this argument in the next section.
8 The Market Effect

In this section we identify the externality that prevents separation from being ubiquitous in our model. Such externality is related to the fact that the firm, incorporating its market power, understands that different choices of menus result in different prices for saving in the financial market. That change in prices implies a loss of attractiveness for the menu and, hence, an additional layer of costs for the monopolist.

We shall start with the theoretical result and then proceed to the intuition through a heuristic rationale. Let us say that a menu \( M \) is feasible if \( E(M) \neq \emptyset \), that is, if it is within the set of menus that can be chosen by the monopolist in its problem. Let us consider \( N \) a solution for the firm’s problem of choosing among menus with the added constraint that menus are pooling. \( N \) is called a best pooling menu.

Let us assume that, under the pooling menu, consumption allocations are such that \( x_i \) is the amount consumed by an \( i \)-type in state \( a \).

Lemma. \( \frac{\partial q}{\partial \tilde{w}_i} < 0 \), and \( \theta_i(\tilde{w}, q) \) is decreasing in \( \tilde{w} \) for any \( i \in \mathcal{I} \).

Proposition 7. Assume the best pooling menu is \( \tilde{M} \) and presents coverage \( \tilde{w} \). Also, let \( q = q(\tilde{M}) \). There is no local feasible deviation \( M \) from \( \tilde{M} \) such that \( \Pi(M|v) > \Pi(\tilde{M}|v) \) if and only if:

\[
(u'(x_H) - v) + (v \sum_i \lambda_i \mu_i - \mu_L u'(x_L)) \frac{u''(x_H)}{E_H u''(X_H)} \leq 0
\]

This proposition provides a necessary and sufficient condition for there to be a profitable and feasible deviation from the optimal pooling menu that is local. It’s expression is convoluted, but it is solely written in terms of first and second derivatives of the utility function, and of the - endogenous - best pooling coverage, \( \tilde{w} \).

It is not the aim of that proposition to be a characterization. Rather, it is useful for understanding the following heuristics. First, it should be noted that the above expression, for any optimal pooling contract, can be rewritten as:

\[
\lambda_H \mu_H \left[ u'(\tilde{w} + \theta_H(\tilde{w}, q)) - v \right] + \left[ \theta_L (w - d, q) - \theta_L(\tilde{w}, q) \right] \frac{\partial q}{\partial \tilde{w}_H} \leq 0
\]

\(^{31}\)Or, equivalently, as we proved in Proposition (3), if IC and IR are satisfied for \( q(M) \)
The first term measures the distance from efficient insurance for the high-risk individual at the best pooling contract. This term is always positive in a standard economy. Therefore, if price impacts disappear, such inequality cannot be satisfied and pooling cannot be optimal. That recovers the qualitative result of the basic framework, represented by standard economies.

The second term is the key. By the Lemma above, it must be always negative, so that the inequality might be satisfied for some economies. We will argue that it measures the amount of the general equilibrium externality, or the extent to which such externality poses an additional layer of costs for separation. In order to see heuristically the source of such term, assume that the economy is at a pooling menu \( \mathcal{M} \) with coverage \( \hat{w} > w - d \), such that the low-risk agents’ individual rationality constraints are binding.\(^{32}\)

\[
V_L(p, \hat{w}|q) - V_L(0, w - d|q) = 0
\]

Assume, then, that the monopolist considers a new menu providing more coverage to the high-risk agents and keeping the low-risks at the same contract. Then, if this new menu is sufficiently close to the original one, the individual rationality constraint for safer individuals is changed to:

\[
\frac{\partial V_L(p, \hat{w}|q)}{\partial q} - \frac{\partial V_L(0, w - d|q)}{\partial q} = [\theta_L(w - d, q) - \theta_L(\hat{w}, q)] \frac{\partial q}{\partial \hat{w}} < 0
\]

Where we used the envelope theorem to compute the differentials of the indirect utility functions and the inequality comes from the Lemma and \( \hat{w} > w - d \). The rationale above shows that price variations lead to a violation of an individual rationality constraint. In words, when the firm increases insurance for riskier individuals, these save less through the risk-free assets, resulting in a reduction in the price of saving. Such change in prices affects the decision of the low-risk individuals between accepting their contract or not.

As trading in the market becomes cheaper, price variations benefit disproportionately the outcome obtained through the outside option vis-à-vis the utility of accepting the contract.\(^ {33}\) As a consequence, price variations impose an externality

\(^{32}\)It is proved in the appendix that this is indeed the case in the best pooling menu. See Lemma (8).

\(^{33}\)This is so because the direct effect of a price change is to get savings cheaper. For any individual,
on the attractiveness of the menu, making it less appealing for the safer agents. Naturally, then, if the principal insists in trying to separate agents, it must deal with this additional burden. This is the **market effect** in our model. It is preventing separation by affecting the willingness of individuals to stay accepting a contract after price changes.

Proposition (7), then, helps us to identify the source of pooling as arising from individual rationality constraints, rather than incentive compatibility. Exactly because it comes from individual rationality, the market effect should not be expected to vanish for any of the individuals, in contrast with the result obtained for the standard case. Our next and last proposition affirms that it is indeed the case that efficient insurance cannot be achieved in a simple economy. Further, it shows that monotonicity in allocations, a key result of standard insurance problems, is maintained.\(^{34}\)

**Proposition 8.** Let \(\mathcal{M} = \{(p_i, \tilde{w}_i)\}_{i \in \{L,H\}}\) be a menu of equilibrium, with associated price \(q\). Assume also that \(w > \tilde{w}_H > w - d\). Then, \(u'(\tilde{w}_L + \theta_L(\tilde{w}, q)) > u'(\tilde{w}_H + \theta_H(\tilde{w}, q)) > v\). In particular, the allocation is less than first-best efficient for the riskiest agent.

---

\(^{34}\)A stronger result is true. Not only allocations are increasing with risk, but also the wealth guaranteed by insurance. That is, \(\tilde{w}_H > \tilde{w}_L\)

\(^{38}\)her asset holding are higher when she is not covered - that is, in the outside option - than when she is. Consequently, she gains more through the devaluation of savings when she does not accept any contract.
9 Conclusion

We presented a model that couples a competitive market for savings in a financial economy framework with the problem of a monopolistic insurer. Our purpose was to understand how these two markets interrelate and, particularly, the influence of savings as imperfect competitors to insurance in the problem of the firm. First of all, it is shown that equilibrium always exists in that model, that is, that noncompetitive and competitive sides of the market aggregate well and are capable of functioning in an orderly way.

Following, we focus on understanding the problem of the firm as a variation of the canonical problem of an insurance company. It is clear that, even though these models are related, the one presented here brings about serious technical difficulties, just as non-convexities that are absent in the basic contract theory case.

Then, some qualitative description is provided. We emphasize the most important feature distinguishing our model from the canonical one: the existence of pooling equilibria. Particularly, we focus on identifying and describing the driving force in favour of pooling, which is a direct externality generated by the presence of a financial market together with the market power of the firm. In contrast to more usual models of insurance, extending coverage for one type affects not only the attractiveness of the contract being accepted for the other one through incentive constraints, but also through participation. That is so because individuals are linked among themselves through financial markets.

Our pooling result arises because, as the firm tries to separate types, prices in the financial market change. Because the natural direction of separation is increasing coverage for the individuals who are more willing to pay for it - the riskiest - this effect is in the direction of reducing prices. Now, individuals save more when they have insurance than when they do not. As a consequence, savings are higher under the outside option than under the inside outcome. Once the direct effect of price changes is exactly the amount of savings held, the outside option is disproportionately benefited by prices dropping. Therefore, it becomes costlier for the firm to guarantee attractiveness of the menu.

Such result is *per se* interesting, because it could inform decisions from large
players like the government and big insurance companies which deal directly with financial markets as imperfect competition. However, it does not depend on the monopoly role of the insurer as deeply as it seems to. Rather, assume competition à la Rothschild and Stiglitz (1976). In that setting, if one of the competing firms considers offering a menu that attracts an agent from any type, it also anticipates that the whole fraction of individuals of that type will be attracted, implying price changes. Therefore, any rational firm would take into account its own price impacts when making any decision. That sheds light on the somewhat sensitive definition of competition as price-taking behaviour.

A second feature of the model presented here, and that is also distinct from traditional models, is that even the high-risk agent has his allocation distorted at an equilibrium. This is so because core distortions do not arise from incentives, but, rather, from individual rationality. What we called market effect is exactly the amount the company must give up from in order to provide an additional unit of insurance. That is true whether incentive issues are relevant or not. It ends up being true that whatever is the situation of incentives, both agents are less than fully insured by the firm internalizing its effects on the price of savings.

Finally, we can understand this dissertation as a first step towards a convenient approach for modelling insurance and financial markets simultaneously, without the need of giving up either of competitive markets or of contract theory. We hope that further research in this area could perfect the modelling and get rid of some of the shortcomings we have chosen.
References


10 Appendix

Proof of Proposition 1. Take any menu $\mathcal{M} = [c_\beta]_{\beta \in B}$, $B$ being an arbitrary index set. Let $e = \{[c_x, \theta_x, q]\} \in WI(\mathcal{M})$. Then, for each type $i \in I$ there is a contract $c_i \in \mathcal{M}$ and portfolio $\theta_i$ such that, $c_x = c_i$, $\theta_x = \theta_i$ for all $x$ whose type is $i$, by the definition of $WI(\mathcal{M})$.

Now, call $\tilde{\mathcal{M}} = \{c_i\}$. Notice that, because $\tilde{\mathcal{M}} \subseteq \mathcal{M}$, and since $e$ was a WISE in which each $i$-agent accepts $c_i$, optimization in the menu $\tilde{\mathcal{M}}$ given $q$ is guaranteed. That is so because optimization in the menu compares the utility of the contract offered, which is the same as it was before, with that provided by any other contract in the menu. But now the comparison must be made against fewer contracts, all of which were present in the former menu. In other words, any inequality that should hold in the former case still holds, but the new menu, as it is smaller, asks for less inequalities to be satisfied.

Besides, also given $q$, taking as given that each $i$-agent accepts $c_i$ - as it were so before, each agent of type-$i$ is optimising on the market by choosing the exact same portfolios $\theta_i = \theta_x$. Since these portfolios clear the market, $q$ is a Walrasian Equilibrium price. It follows that $e \in WI(\tilde{\mathcal{M}})$.

Proof of Proposition 2. The argument consists in showing that the monopolist’s problem - which determines equilibrium existence - is one of maximizing a continuous function over a compact set. In the process, we make extensive use of quasilinearity properties. We break the demonstration in a number of lemmas. First, it is shown that $E(\mathcal{M})$ is either the null set or a singleton. Next, it is pointed out that, given incentive compatibility, asset prices $q$ are a function of coverages only - they are independent of premia. Subsequently it becomes clear that premia can then be chosen in a compact set. That establishes the proof.

In order to fix notation, let $\mathcal{C} = \mathbb{R}^I_+ \times [0, 1]^{I \times S \Sigma}$ be the set from which menus contracts are extracted. Also, we will still use $z$ as a notation for firm’s portfolios given a menu $\mathcal{M}$. Finally, for an $i - type$ individual accepting contract $(p, \eta)$, we call his wealth in state $(s, \sigma)\tilde{w}_i(s, \sigma|\eta)$, and the whole vector of wealth is $\tilde{w}_i(\eta)$. When $\eta = \eta_i$, then $\tilde{w}_i(s, \sigma|\eta) = \tilde{w}_i(s, \sigma)$ and the whole vector of wealth is $\tilde{w}_i$. 

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Lemma 1. For any $\mathcal{M} \in \mathcal{C}$, $\#E(\mathcal{M}) \in \{0, 1\}$.

Proof. First, assume each $i$-type is accepting $c_i \in \mathcal{M}$. Consider an economy with endowments given by $(c_i)_{i \in \mathcal{I}}$. Having fixed such endowments, by Geanakoplos and Polemarchakis (1985), existence of a Walrasian Equilibrium is guaranteed by continuity and concavity assumptions, besides positivity of endowments in the second period and positivity of probabilities of falling in any aggregate state, which ensures monotonicity.

We claim such Walrasian Equilibrium is unique. Let $c_i = (p_i, \tilde{w}_i)$. Indeed, because in any Walrasian Equilibrium agents should be maximizing, the following necessary and sufficient first order conditions should hold:

$$
\pi_s \sum_{\sigma \in \Sigma} \mu_i(s, \sigma) u'(\tilde{w}_i(s, \sigma) + \theta_s) = q_s
$$

(3)

For all $i \in \mathcal{I}$, $s \in \mathcal{S}$. Now, seeking a contradiction, assume that $q \neq \tilde{q}$ are Walrasian Equilibria prices, and associate with them the portfolios list that satisfies first order conditions and market clearing, $\{\theta^i\}_{i \in \mathcal{I}}$ and $\{\tilde{\theta}^i\}_{i \in \mathcal{I}}$, respectively. Without loss of generality, assume $q_s > \tilde{q}_s$ for some $s \in \mathcal{S}$. Notice that, in the first order conditions, the only adjusting variables are portfolios.

If $q_s > \tilde{q}_s$, then it means that the left hand side of all the first order conditions in $s$ should be higher for $q$ than for $\tilde{q}$. By concavity of utility functions, the only way for that to be true is that for all types $i$, $\theta^i_s < \tilde{\theta}^i_s$. But then, it is impossible for market clearing to be satisfied for both, and we get a contradiction. Therefore, $q = \tilde{q}$ and Walrasian Equilibrium is unique, given $(c_i)_{i \in \mathcal{I}}$.

Now, let $\{(\theta_i)_{i \in \mathcal{I}}, q\}$ be this unique Walrasian Equilibrium. Then, by the definition of an ISE, if we define $e = \{(c_i, \theta_i)_{i \in \mathcal{I}}, q\}$, $E(\mathcal{M}) \subseteq \{e\}$, and the lemma follows. □

Corollary 1. $q$ can be written as a function of $\mathcal{M}$ by choosing $q(\mathcal{M})$ to be the unique Walrasian Equilibrium price for $\mathcal{M}$.

The corollary above shows that $q$ can be written as a function of $\mathcal{M}$. The next step is to show that $q$ is actually not a function of the whole menu of contracts but, rather, of coverages only.

Inada and concavity guarantee the equality.\footnote{Inada and concavity guarantee the equality.}
Lemma 2. Let \( \mathcal{M} \) and \( \mathcal{N} \) be two menus of contracts such that, for all \( i \in I, (p_i, \tilde{w}_i) \in \mathcal{M} \) and \( (f_i, \tilde{u}_i) \in \mathcal{N} \) imply \( \tilde{w}_i = \tilde{u}_i \). Then \( q(\mathcal{M}) = q(\mathcal{N}) \).

Proof. Inspecting first order condition in Equation (??), it clearly does not depend upon premia. Besides, market clearing is also not dependent on premia. It readily follows that \( q \) should not depend on premia too.

There is one more property of the competitive price function that is relevant to us: continuity. That is relevant because prices being continuous and depending only on the compact set of coverages, as we have shown above, imply that the monopolist has not infinite power over the financial market. Bounds on the monopolist's power are exactly what is demanded for the proof to be completed, as will become clear soon.

Lemma 3. \( q \) is a continuous function.

Proof. By quasilinearity and (3), we can use the implicit function theorem to see that \( \theta^i_s \) is a continuously differentiable function of \( \{w_i(s, \sigma)\}_{\sigma \in \Sigma} \) and \( q \). Explicitly:

\[
\frac{d\theta^i_s}{dq_k} = \begin{cases} 
\frac{1}{\pi_k \sum_{\sigma \in \Sigma} \mu_i(s, \sigma)u''(w_i(s, \sigma) + \eta_s)} , & s = k \\
0 , & s \neq k
\end{cases}
\]

Now, we know that, at equilibrium, given a menu of contracts \( \mathcal{M} \) there is \( q \) such that, for each \( s \in S \)

\[
\sum_{i \in I} \lambda_i \theta^i_s(\mathcal{M}, q) = 0
\]

Analysing the Jacobian of the left hand side with respect to \( q \), using the derivatives of \( \theta^i_s \) above, it is clear that it is a diagonal matrix and, therefore, has full rank in the whole domain - as each entry is negative due to strict concavity of \( u \). Applying the implicit function theorem again and knowing that \( q \) is a function, we get that \( q \) is a continuously differentiable function, and, therefore, continuous.

The next lemma establishes that the monopolist is actually constrained to choose contracts in a compact set.

Lemma 4. The menu of contracts can be chosen from a compact set \( \mathcal{K} \subseteq \mathcal{C} \).
Proof. Fix a menu $\mathcal{M}$. Recall that $W$ are the endowments and that $w = \max W$. Let $\bar{w}$ be the coverage policy giving $w$ for an agent in each state of period two. First, notice that, by quasilinearity, $V_i(p, \bar{w}|q(\mathcal{M})) = -p + V_i(0, \bar{w}|q(\mathcal{M}))$. Moreover, by monotonicity of the original utility functions, $V_i(p, \bar{w}|q) \leq V_i(p, \bar{w}|q)$; that is, at the same prices for assets and the same premium, $i$ is always better off being completely covered.

A necessary condition for a price $p$ to be a feasible choice for the insurance company is that individual rationality constraints are satisfied for at least one type, say $i$:

$$p \leq V_i(0, \bar{w}|q(\mathcal{M})) - V_i(0, W|q(\mathcal{M})) \leq V_i(0, \bar{w}|q(\mathcal{M})) - V_i(0, W|q(\mathcal{M}))$$

We can use Berge’s Maximum Theorem to prove that $V_i$ is a continuous function for every $i \in \mathcal{I}$. Then, as $q$ is a continuous function only of coverages, that are chosen in a compact set and $\mathcal{I}$ is a finite set, we find a uniform upper bound for $p$ by noticing that

$$V_i(0, 1|q(\mathcal{M})) - V_i(0, 0|q(\mathcal{M})) \leq \max_{M \in K, i \in \mathcal{I}} \{V_i(0, 1|q(A)) - V_i(0, 0|q(A))\}.$$

Therefore, premia can be embedded in a compact set. As coverages are already picked from a compact set, our proof is complete.

Now we are equipped with the necessary information to complete the proof of the existence theorem. We conclude with this last lemma.

**Lemma 5.** We can rewrite the company’s problem as:

$$\max_{\mathcal{M} \in K} \left\{ \Pi(\mathcal{M}) : V_i(p_i, \bar{w}_i|q(\mathcal{M})) \geq V_i(p, u|q(\mathcal{M})) \forall i \in \mathcal{I}; (p, u) \in \mathcal{M} \cup (0, W) \right\} \quad (4)$$

**Proof.** The original problem is stated under the condition that $E(\mathcal{M}) \neq \emptyset$. Assume that holds, then $\exists e = \{(c_i, \theta_i), q\} \in E(\mathcal{M})$. Because of the definition of an ISE, $\{(\theta_i), q\}$ is a Walrasian Equilibrium for an economy in which endowments are $c_i$. Therefore,
\( q = q(\mathcal{M}) \). So, all that it takes for \( e \) to be an equilibrium is, by the definition:

\[
V_i(p_i, \tilde{w}_i|q(\mathcal{M})) \geq V_i(p, u|q(\mathcal{M})) \quad \forall i \in \mathcal{I}; (p, u) \in \mathcal{M} \cup (0, W)
\]

Conversely, if a menu \( \mathcal{M} = (c_i) \) satisfies the constraint in (4), then \( \{ (\theta_i), q(\mathcal{M}) \} \) are a Walrasian Equilibrium, by definition, and because that inequalities hold, \( \{(c_i, \theta_i), q \} \in E(\mathcal{M}) \), implying the latter is not the empty set.

Now, using equation (4), as argued before, \( V_i \) is a continuous function by Berge's maximum theorem. In turn, we proved that \( q \) is a continuous function and that the set of menus can be chosen from a compact set. As a consequence, the image of such compact set by \( V_i(p, \eta|q(A)) \) is compact, because such function is continuous. Besides, the incentive and rationality inequalities are intersections between such compact sets and closed sets being, therefore, compact.

Finally, the profit function is trivially continuous.\(^{36}\) The problem is one of maximizing a continuous function over a compact set and, by Weierstrass’ theorem, has a solution.

\[\square\]

**Proof of Proposition 3.** It is proved within the last proof. It suffices to combine Lemmas (3) and (5).

\[\square\]

**Proof of Proposition 4.** The full proof is omitted. Here is a sketch. It closely parallels the characterization of an optimal contract in the canonical model.

First, it can easily be shown that individual rationality for the riskiest type is redundant. In sequence, we assume that the incentive compatibility constraint for the safest type is not binding. From that, it follows that the two nonredundant constraints bind.

By taking first order conditions, it is seem that, the riskiest agents are efficiently insured. That efficient insurance is achievable is guaranteed by our assumption that \( u'(w - d) > v > u'(w) \).

\[\square\]

**Proof of Proposition 5.** That separation entails incentive separation is trivial.

\(^{36}\)It is linear in a finite-dimensional space.
Now, assume a standard economy $\Gamma$ is incentive separable. Then, take a pooling menu $M = \{p, \tilde{w}\}$ that is individually rational and such that $w - d < \tilde{w} < w$. Assume we constructed a menu $\tilde{M} = \{p_i, \tilde{w}_i\}_{i \in \{L, H\}}$ as in the definition of incentive separability. We know that $\Pi(M) \leq \Pi(\tilde{M})$ and that $\tilde{M}$ is incentive compatible. Besides, for each $i \in \{L, H\}$

$$B_i(p_i, \tilde{w}_i) \geq B_i(p, \tilde{w}) \geq B_i(0, W)$$

Where the first inequality follows from the construction of $\tilde{M}$ and the second one by the fact that the original pooling contract was individually rational. As a consequence, $\tilde{M}$ is also individually rational. Because $M$ was an arbitrary individually rational pooling contract, we conclude that $\Gamma$ allows for separation.

Proof of Lemma. For the first part of the Lemma, we apply the implicit function theorem. By the market clearing condition, we know that any portfolios $(\theta_i)_{i \in \{L, H\}}$ compatible with an ISE satisfy: $\sum_{i \in \{L, H\}} \lambda_i \theta_i = 0$.

Allowing for variation in $w_i$, for some $i$, we know that portfolios must change in order to maintain market clearing. Therefore, differentials must satisfy:

$$-\frac{\lambda_i}{\lambda_j} d\theta_i = d\theta_j \quad (5)$$

We also know that first order conditions equalize between agents, so that the total differential for individual optimality is:

$$\mu_i u''(\tilde{w}_i + \theta_i) d\tilde{w}_i + \mathbb{E}_i[u''(\tilde{W}_i + \theta_i)] d\theta_i = \mathbb{E}_j[u''(\tilde{W}_j + \theta_j)] d\theta_j = -\mathbb{E}_j[u''(\tilde{W}_j + \theta_j)] \frac{\lambda_i}{\lambda_j} d\theta_i \quad (6)$$

where, in the last equality, we used equation (5). Reorganizing that, we find:

$$d\theta_i = -\lambda_j \frac{\mu_i u''(\tilde{w}_i + \theta_i)}{\lambda_j \mathbb{E}_i[u''(\tilde{W}_i + \theta_i)] + \lambda_i \mathbb{E}_j[u''(\tilde{W}_j + \theta_j)]} d\tilde{w}_i \quad (7)$$

Now, recalling that, in any ISE, first order conditions must always equalize average marginal utilities to prices, for the $j$-type individual:
\[-\mathbb{E}_j[u''(\bar{W}_j + \theta_j)] \frac{\lambda_i}{\lambda_j} d\theta_i = d_q\]

Substituting \(d\theta_i\) from (7) above:

\[
\frac{dq}{d\bar{w}_i} = \mu_i \mathbb{E}_j[u''(\tilde{W}_j + \theta_j)] \lambda_i \left\{ \frac{u''(\tilde{w}_i + \theta_i)}{\lambda_j \mathbb{E}_i[u''(\tilde{W}_i + \theta_i)] + \lambda_i \mathbb{E}_j[u''(\tilde{W}_j + \theta_j)]} \right\}
\]

And negativity follows directly from the concavity of \(u\).

For the second part, just notice that, at the same prices, for the same type, let \(\tilde{W}, \bar{W}\) be the wealth variables related to coverages \(\tilde{w} > \bar{w}\), respectively. Then, for the same \(\theta\), it follows by concavity that:

\[
\mathbb{E}_i u'(\tilde{W} + \theta) < \mathbb{E}_i u'(\bar{W} + \theta)
\]

But at equilibrium portfolios, these two quantities must be equal, what implies \(\theta_i(\tilde{w}, q) < \theta_i(\bar{w}, q)\), as we wished to demonstrate.

\(\square\)

Proof of Proposition 6. First of all, notice that a given economy is incentive separable if and only if there exist separating menus that dominate each most profitable feasible pooling menu. To see that, let \(M\) be a feasible menu that gives maximum profits for the firm. Then, \(M\) is incentive compatible, individually rational and gives more profits for the firm than any other pooling menu that is not profits maximizing.

As a consequence, it suffices to show that, for any profit maximizing feasible pooling menu, there is a deviation satisfying the terms of the definition of incentive separation. Because \(M\) is a profit maximizing menu, it must be the case that the deviation is separating.

We shall divide this proof in a number of lemmas. The strategy is to prove first that any simple economy satisfies a stronger version of the ordering property. Then, we establish that any profit maximizing pooling menu satisfies a technical condition. This condition, in addition to the ordering property, is shown to imply incentive separability, as a last step.

Lemma 6. If \(\Gamma\) is a simple economy, then it satisfies the ordering property everywhere.
Proof. This is a matter of computing derivatives. Consider any point \((p, \tilde{w})\). By using the envelope theorem we get, for any \(q\):

\[
\frac{\partial V_i}{\partial \tilde{w}}(p, \tilde{w} \mid q) = \mu_i u'(\tilde{w}^{i} + \theta^{i}(\tilde{w}^{i}, q)) .
\]

Let us show that such derivatives are ordered now.

Claim 1. \(\mu_H u'(x + \theta_H(x, q)) > \mu_L u'(x + \theta_L(x, q))\) for any \(x \in [w - d, w]\) given a price \(q\). Moreover, \(\theta_H(x, q) > \theta_L(x, q)\).

Proof. Let \(X\) be the random variable paying \(x\) in the accident state and \(w\) otherwise. Optimality and the price \(q\) demand the first order conditions to be equalized between the agents:

\[
\mathbb{E}_H[u'(X + \theta_H(x, q))] = \mathbb{E}_L[u'(X + \theta_L(x, q))] \quad (8)
\]

Let us assume by contradiction that the opposite of what is stated holds: \(\mu_H u'(x + \theta_H(x, q)) \leq \mu_L u'(x + \theta_L(x, q))\). But then, as \(\mu_H > \mu_L\) and because \(u\) is concave, that must be true that \(\theta_H(x, q) > \theta_L(x, q)\). For ((8)) to be possible given the contradiction assumption:

\[
(1 - \mu_H)u'(w + \theta_H(x, q)) > (1 - \mu_L)u'(w + \theta_L(x, q))
\]

Again, because \(\mu_H > \mu_L\) and by concavity, if \(\theta_H(x, q) > \theta_L(x, q)\), such inequality cannot be satisfied, yielding a contradiction.

For the second part, just notice that \(u'(w + y) < u'(x + y)\), for \(x \in [w - d, w]\) and any \(y \in \mathbb{R}\). Then, because \(\mu_H > \mu_L\), \(\mathbb{E}_H[u'(X + y)] > \mathbb{E}_L[u'(X + y)]\). Therefore, for first order conditions to be equal in equilibrium, \(\theta_H(x, q) > \theta_L(x, q)\) is a necessary condition.

Now, the above claim got us that, for any point,

\[
\frac{\partial V_H}{\partial \tilde{w}}(p, \tilde{w} \mid q) > \frac{\partial V_L}{\partial \tilde{w}}(p, \tilde{w} \mid q).
\]

To conclude the lemma, just notice that, as it holds for every \(q\), it implies the ordering property.
Lemma 7. For any $q$ and $w - d \leq \tilde{w} \leq w$, let $\Delta V^i(p, \tilde{w}) \equiv V^i((p, \tilde{w}) | q) - V^i((0, w - d) | q)$. Then, $\Delta V_H(p, \tilde{w}) > \Delta V_L(p, \tilde{w})$.

Proof. By the envelope theorem, we can write

$$\Delta V^i(p, \tilde{w}) = -p + \int_{w - d}^{\tilde{w}} \mu_i u'(x + \theta^i(x, q)) dx.$$ 

But then, by Claim 1, we have that, $\mu_H u'(x + \theta_H(x, q)) > \mu_L u'(x + \theta_L(x, q))$. Using this information,

$$\Delta V_H(p, \tilde{w}) = -p + \int_{w - d}^{\tilde{w}} \mu_H u'(x + \theta_H(x, q)) dx > -p + \int_{w - d}^{\tilde{w}} \mu_L u'(x + \theta_L(x, q)) dx = \Delta V_L(p, \tilde{w}).$$

Now, let us adopt the notation $D_q = \partial q / \partial \tilde{w}_L + \partial q / \partial \tilde{w}_H$. Besides, say that a contract is interior if $\tilde{w}$ satisfies $w - d < \tilde{w} < w$.

Lemma 8. If a best pooling menu $N' = \{(p, \tilde{w})\}$ consists of an interior contract, it satisfies, for $q = q(N')$

$$\mu_L u'(\tilde{w} + \theta_L(\tilde{w}, q)) + [\theta_L(w - d, q) - \theta_L(\tilde{w}, q)] D_q = v(\mu_L \lambda_L + \mu_H \lambda_H).$$

and $p = \Delta V_L(0, \tilde{w})$.\(^{37}\)

Proof. Solving for the best pooling menu consists of facing the problem of the firm when the latter is searching for optima in the space of pooling contracts. In that space, incentive constraints hold trivially. Additionally, by Lemma (7), in such space the individual rationality constraint for the riskiest type is redundant. Therefore, ((PF')) can be rewritten as

$$\max_{M = \{p \geq 0, \tilde{w} \in [w - d, w]\}} \left\{ p - v \sum_i \lambda^i \mu_i w : -p + V_L(0, \tilde{w}|q(M)) \geq V_L(0, W|q(M)) \right\}.$$ 

\(^{37}\) That is, $\Delta V_L(p, \tilde{w}) = 0$. 

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Because we know that, for fixed coverage, $q$ is a constant, it is always in the interest of the principal to set equality in the individual rationality constraint of the $L$-types, what establishes the second claim in the lemma. That also allows one to substitute $p$ in the objective function of the insurance company.

The first claim then follows from taking the necessary first order conditions of the problem above, assuming interiority.\footnote{Without interiority, the same condition holds with the left hand side being less or equal to the right hand side}

Now let us show that the lemmas above imply the proposition. As argued before, it suffices to show that for any feasible pooling menu that is profit maximizing, there is another menu that dominates it. Additionally, by the definition of incentive separation, we just have to show it for those menus that are interior.

Let $\mathcal{M} = \{p, \tilde{w}\}$ be a profit maximizing pooling menu. We shall fix notation to avoid cumbersome expressions. Define $u'_i = u'(\tilde{w} + \theta^i(\tilde{w}, q))$, $E_i[u'_i] = E_i[u'(\tilde{W} + \theta^i(\tilde{w}, q))]$, $\beta_i = \theta^i(w - d, q) - \theta^i(\tilde{w}, q)$ and $\Delta V^i = V^i((p, \tilde{w}) \mid q) - V^i((0, w - d) \mid q)$. Finally, let us denote $D_i q = \partial q / \partial \tilde{w}^i$. Notice that any local deviation of this kind must satisfy the inequalities that follow. First, it must increase profits for the firm:

$$\lambda^L(dp^L - \mu_Lvd\tilde{w}^L) + \lambda^H(dp^H - \mu_Hvd\tilde{w}^H) > 0.$$  

Now, notice that incentive compatibility constraints are binding at the original menu, $\mathcal{M}$. Therefore, for them to hold in the deviation menu, for each $i$,

$$-dp^i + \frac{\partial V^i(0, \tilde{w} \mid q)}{\partial \tilde{w}}d\tilde{w}^i \geq -dp^i + \frac{\partial V^i(0, \tilde{w} \mid q)}{\partial \tilde{w}}d\tilde{w}^j.$$  

Because derivatives of $V^i$ relative to prices appear in both sides of the inequality and, hence, cancel out. By the envelope theorem, that could be written as

$$-dp^i + \mu_i u'_i d\tilde{w}^i \geq -dp^i + \mu_i u'_i d\tilde{w}^j.$$  

Besides, let us consider deviations $\mathcal{\tilde{M}}$ such that:

$$V_i(p^i, \tilde{w}^i \mid q(\mathcal{\tilde{M}})) \geq V_i(p, \tilde{w} \mid q(\mathcal{\tilde{M}}))$$
So, in differential form,

\[-dp^i + \mu_i u'_i d\hat{w}^i \geq 0\]

It is possible to arrange this system of five linear inequalities in matrix form, as

\[Ax \geq b\]

where

\[A = \begin{pmatrix}
\lambda_H & \lambda_L & -\lambda_H \mu_H v & -\lambda_L \mu_L v \\
-1 & 1 & \mu_H' u'_H & -\mu_H' u'_H \\
1 & -1 & -\mu_L u'_L & \mu_L u'_L \\
-1 & 0 & \mu_H' u'_H & 0 \\
0 & -1 & 0 & \mu_L u'_L 
\end{pmatrix},
\]

\[x = \begin{pmatrix}
dp_H \\
dp_L \\
dw_H \\
dw_L
\end{pmatrix}
\]

and

\[b = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

We use Lemma (9) for concluding the proof.

Now, we find a necessary condition for the existence of a vector \(\gamma\) as in Lemma (9). Write \((a^i)_{i \leq 4}\) as the columns of the matrix \(A\).

Assume that such \(\gamma\) exists. Then, \(\gamma(a^1 + a^2) = 0\) implies \(\gamma_1 = \gamma_4 + \gamma_5\) and \(\gamma a_1 = 0\) implies \(\gamma_2 = \lambda^H \gamma_1 + \gamma_3 - \gamma_4\).

With these information, \(\gamma(a^3 + a^4) = 0\) imply, by use of Claim 1, \(\mu^L u'_L \leq v(\lambda^H \mu_H + \lambda^L \mu_L)\). We show next that this necessary condition is not satisfied for an optimal pooling menu, so that a solution for the system of inequalities we have is guaranteed by Lemma (9) above. Now, by Lemma (8) and Lemma (??):

\[\mu^L u'_L - v(\lambda^H \mu_H + \lambda^L \mu_L) = -[\theta_L(w - d, q) - \theta_L(\hat{w}, q)] Dq > 0\]

As a consequence, we proved through Lemma (9) that there is a menu that domi-
nates our initial profit maximizing pooling one, that is incentive compatible and that, under the prices it induces on the market, makes both agents better off compared to the original pooling. That settles the proof.

\[ \Box \]

**Proof of Proposition 7.** We wish to use a version of Farkas’ Lemma - as an alternative for the KKT conditions - to deduce the result. A few previous facts will be helpful, that help us characterize the best pooling contract.

Now, let us establish the notation \( D_q = \sum_{i \in \{L,H\}} \frac{\partial q}{\partial \tilde{w}_i} \). Besides, an interior contract is a contract such that its coverage, \( \tilde{w} \) satisfies \( w - d < \tilde{w} < w \).

For the remaining of the proof, let us take any best pooling contract that is interior, say \((p, \tilde{w})\). We shall again fix notation to avoid cumbersome expressions. Define \( u'_i = u'(\tilde{w} + \theta_i(\tilde{w}, q)) \), \( E_i[u'_i] = E_i[u(\tilde{w} + \theta_i(\tilde{w}, q))] \), \( \beta_i = \theta_i(w - d, q) - \theta_i(\tilde{w}, q) \) and \( \Delta V_i = V_i((p, \tilde{w})|q) - V_i((0, w - d)|q) \). Finally, let us define \( D_i q = \frac{\partial q}{\partial \tilde{w}_i} \).

If a deviation is close enough to the original contract, it must satisfy in differential form all the constraints. If it is profitable for the firm, then it must also satisfy

\[ \sum_{i \in \{L,H\}} \lambda_i \left\{ dp_i - \mu_i vd\tilde{w}_i \right\} > 0 \]

Now, notice that incentive compatibility constraints are binding at the original menu, \( M \). Therefore, for them to hold in the deviation menu, for each \( i \):

\[ -dp_i + \frac{\partial V_i(0, \tilde{w}|q)}{\partial \tilde{w}} d\tilde{w}_i \geq -dp_j + \frac{\partial V_i(0, \tilde{w}|q)}{\partial \tilde{w}} d\tilde{w}_j \]

Because derivatives of \( V_i \) relative to prices appear in both sides of the inequality and, hence, cancel out. By the envelope theorem, that could be written as:

\[ -dp_i + \mu_i u'_i d\tilde{w}_i \geq -dp_j + \mu_i u'_i d\tilde{w}_j \]

For individual rationality, we must have for each \( i \):\[^{39}\]

\[ -dp_i + \mu_i u'_i d\tilde{w}_i + \beta_i (D_i q d\tilde{w}_i + D_j q d\tilde{w}_j) \geq -\Delta V_i \]

\[^{39}\text{We could rightfully ignore individual rationality for the H-types. We opt not to do so just to maintain everything clear.}\]
It is possible to arrange this system of 5 linear inequalities in matrix form as \( Ax \geq b \) where

\[
A = \begin{bmatrix}
\lambda_H & \lambda_L & -\lambda_H \mu_H q & -\lambda_L \mu_L q \\
-1 & 1 & \mu_H u_H' & -\mu_H u_H' \\
1 & -1 & -\mu_L u_L' & \mu_L u_L' \\
-1 & 0 & \beta_H D_H q(\bar{w}) + \mu_H u_H' & \beta_H D_L q(\bar{w}) \\
0 & -1 & \beta_L D_H q(\bar{w}) & \beta_L D_L q(\bar{w}) + \mu_L u_L'
\end{bmatrix}
\]

\[
\begin{bmatrix}
dp_H \\
dp_L \\
dw_H \\
dw_L
\end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix}
0 \\
0 \\
-\Delta V_H \\
0
\end{bmatrix}.
\]

The following lemma, which will be stated without proof is what is left for establishing our proposition. This is a variation of Farkas’ Lemma. 40

**Lemma 9.** Let \( A \) be a matrix with dimensions \( m \times n \) and \((a_i)_{i \leq m}\) its lines. Also, consider \( b \in \mathbb{R}^n \). Suppose that, for the matrix \( \bar{A} = (a_i)_{2 \leq i \leq m} \) there is \( x \in \mathbb{R}^{m-1} \) such that \( \bar{A}x \geq \bar{b} \) - equality being a possibility -, where \( \bar{b} = (b_i)_{2 \leq i \leq m} \). Then, one and only one of the two possibilities is true:

(A) There exists a vector \( y \in \mathbb{R}^m \) such that \( Ay \geq b \) with \( a_1x > b_1 \);

or

(B) There exists \( \gamma = (\gamma_i)_{i \leq m} \), no entry being negative, such that at least \( \gamma_1 > 0 \) and with the property that

\[
\sum_i \gamma_i a_i = 0
\]

and

\[
\sum_i \gamma_i b_i \geq 0
\]

We shall find a necessary and sufficient condition for the existence of \( \gamma \) as in (B) in Lemma (9). First of all, notice that the condition on the statement of Lemma (9) is

\[40\text{See Rockafellar (1970)[pg. 198, Theorem 22.2]}\]
satisfied. Notice that $x = 0 \in \mathbb{R}^4$ is such that $\bar{A}x \geq \bar{b}$. Now, we can safely apply the Lemma.

Assume there is a $\gamma$ satisfying (B), that is, the pooling contract allows no local deviation. We shall show that this is so whenever $(p, \tilde{w})$ is in the conditions of the theorem. Along the way, the reciprocal will be proved. That is, that for any pooling contract that satisfies the conditions of the theorem, a $\gamma$ satisfying (B) in Lemma (9) can be constructed.

By the second part of property (B) we know that:

$$\gamma b = \gamma_4 (-\Delta V_H) \geq 0$$

By Proposition (??), $\Delta V_H > \Delta V_L = 0$, where the last equality is a consequence of Lemma (8), since we are assuming $(p, \tilde{w})$ to be a best pooling contract that is interior. Therefore, because $\gamma_4 \geq 0$, it must be the case that $\gamma_4 = 0$.

Write $(a_i)_{i \leq 4}$ as the columns of the matrix $A$. By property (A) in the Lemma, $\gamma a^i = 0$ for all $i \in \{1, \ldots, 4\}$. By taking $\gamma(a^1 + a^2) = 0$ we get $\gamma_1 = \gamma_5$. Substituting that in $\gamma a^1$, we also get $\gamma_2 = \gamma_1 \lambda_H + \gamma_3$.

Taking $\gamma(a^3 + a^4)$ and using $\gamma_1 = \gamma_5$ we get to:

$$\gamma_1 \left\{ \mu_L u'_L + \beta_L Dq - \sum_{i \in \{L, H\}} \mu_i \lambda_i \right\} = 0$$

(9)

Because we are assuming $(p, \tilde{w})$ to be an best, interior pooling contract, by Lemma (8), the term in braces is zero. Therefore, this equation is satisfied automatically. So, the last equation to be checked is either $\gamma a^3 = 0$ or $\gamma a^4 = 0$. Choosing the first one gives us:

$$\gamma_1 \{ \lambda_H \mu_H(u'_H - q) + \beta_L D_H q \} = \gamma_3 \{ \mu_L u'_L - \mu_H u'_H \}$$

As we proved in Claim 1, $\mu_L u'_L - \mu_H u'_H < 0$, so that for $\gamma_1$ and $\gamma_3$ to be greater or equal to zero, that must be true that:

$$\lambda_H \mu_H(u'_H - q) + [\beta_L - \xi (\tilde{w} - (w - d))] D_H q \leq 0$$

In order to conclude the proof of sufficiency, it suffices to show that such inequal-
ity is identical to the one in the statement of the theorem. That is done by using the necessary condition a best pooling contract must satisfy by Lemma (8) to substitute for $\beta_L$ in the above inequality.

Notice, however, that necessity is also proved. If $(p, \bar{w})$ is a best pooling contract that allows no local deviations that are profitable for the firm, then a $\gamma$ satisfying the conditions in (B) must exist. Therefore, the inequality above must hold.

Proof of Proposition 8. First, we shall prove monotonicity. Then, we follow with the demonstration of inefficient insurance.

Lemma 10. For any menu $M = \{p_i, \bar{w}_i\}_{i \in \{L,H\}}$ that is incentive compatible, $\bar{w}_H \geq \bar{w}_L$

Proof. Let a menu like in the statement be incentive compatible with an associated price $q$. We can then reorder incentive constraints to get:

$$V_L(0, \bar{w}_H|q) - V_L(0, \bar{w}_L|q) \leq p_H - p_L \leq V_H(0, \bar{w}_H|q) - V_H(0, \bar{w}_L|q)$$

Using the envelope theorem, we can rewrite the first and third terms in integral form - for fixed $q$ - as:

$$\int_{\bar{w}_L}^{\bar{w}_H} \mu_L u'(x + \theta_L(x,q))dx \leq \int_{\bar{w}_L}^{\bar{w}_H} \mu_H u'(x + \theta_H(x,q))dx$$

(10)

By Claim 1, we know that, for any $x$: $\mu_L u'(x + \theta_L(x,q)) < \mu_H u'(x + \theta_H(x,q))$

It follows that for the inequality (10) to hold, $\bar{w}_H \geq \bar{w}_L$.  

Now, we show that, even though savings are related inversely to insurance and riskier agents are more insured, any optimal allocation is larger for low-risks.

Lemma 11. Let $w - d \leq x < y \leq w$. Then $x + \theta_i(x,q) < y + \theta_i(y,q)$ for any $i \in \{L,H\}$ and $q > 0$.

Proof. It suffices to use the first order conditions to differentiate $\theta_i(x,q)$ given prices. That solves as:

$$\frac{d\theta_i(x,q)}{dx} = -\frac{\mu_i u''(x + \theta_i(x,q))}{E_i[u''(X + \theta_i(x,q))]} < 1$$

It follows that $x + \theta_i(x,q)$ is an increasing function for any $i \in \{L,H\}$ and $q > 0$.
That is a direct consequence of the last two lemmas that $u'(\tilde{w}_H + \theta_H(\tilde{w}_H, q)) < u'(\tilde{w}_L + \theta_L(\tilde{w}_L, q))$ for any $\tilde{w}_H, \tilde{w}_L$ and $q$ in an ISE.

For inefficient insurance let us look at KKT conditions in the firm’s problem. First, the next lemma shows us that we can get rid of the individual rationality constraint for $H$.

**Lemma 12.** Assume incentive compatibility for $H$ and individual rationality for $L$ hold. Then, individual rationality for $H$ is redundant.

**Proof.** Let $\mathcal{M} = \{p_i, \tilde{w}_i\}$ is a menu such that the conditions above hold. Assume $q = q(\mathcal{M})$. By incentive constraints for $H$:

$$V_H(p_H, \tilde{w}_H|q) - V_H(0, W|q) \geq V_H(p_L, \tilde{w}_L|q) - V_H(0, W|q)$$

Now, by Lemma (7)

$$V_H(p_L, \tilde{w}_L|q) - V_H(0, W|q) > V_L(p_L, \tilde{w}_L|q) - V_L(0, W|q)$$

And, last, because individual rationality holds for the low type:

$$V_L(p_L, \tilde{w}_L|q) - V_L(0, W|q) \geq 0$$

Combining the three inequalities, we get that individual rationality holds - strictly - for $H$.

For the KKT conditions, we can restrict attention to the following two inequalities that naturally arise from the set of first-order conditions, in which $\gamma_i$ is the multiplier associated with the IC of an $i$-type. We use notation settled in the proof of Proposition 7:

$$\lambda_H = \gamma_H - \gamma_L$$

and\footnote{The next inequality comes from the first order condition for $\tilde{w}_H$. It comes as an equality by adding}
First, notice that, by what we have already proved, \( \theta_i(\tilde{w}_L) - \theta_i(\tilde{w}_H) > 0 \). That is sufficient. First, assume \( \gamma_L = 0 \), then (11) implies that \( \gamma_H = \lambda_H \) and, in (12):

\[
-v \lambda_H \mu_H + \mu_H u'_H \gamma_H + \gamma_H (\theta_H(\tilde{w}_L) - \theta_H(\tilde{w}_H)) D_H q \\
-\gamma_L \mu_L u'(\tilde{w}_H + \theta_L(\tilde{w}_H, q)) + \gamma_L (\theta_L(\tilde{w}_H) - \theta_L(\tilde{w}_L)) D_H q > 0
\]

By \( \theta_i(\tilde{w}_L) - \theta_i(\tilde{w}_H) > 0 \), \( u'_H > v \). Analogously, if we assume \( \gamma_H = 0 \), we can use (11) again and by the same procedure we get that:

\[
\mu_L u'(\tilde{w}_H + \theta_L(\tilde{w}_H, q)) - \mu_H v > 0
\]

But using Claim 1, that implies again \( u'_H - v > 0 \).

Finally, if both \( \gamma_L \) and \( \gamma_H \) are strictly positive, both IC hold and, from the proof of Lemma (10) it is easy to see that the optimal menu must be pooling. In this case, though, an argument similar to the proof of Proposition 6 shows that \( u'_H > v \). The proof is complete, then.

\[\square\]