An SDF Approach to Hedge Funds’ Tail Risk: Evidence from Brazilian Funds
Laura Simonsen Leal

An SDF Approach to Hedge Funds’ Tail Risk: Evidence from Brazilian Funds

Dissertação submetida a Escola de Pós-Graduação em Economia como requisito parcial para a obtenção do grau de Mestre em Economia.

Orientador: Caio Almeida

Rio de Janeiro
21 de Março de 2016
Leal, Laura Simonsen
An SDF approach to hedge funds’ tail risk: evidence from Brazilian funds / Laura Simonsen Leal. – 2016.
29 f.

Dissertação (mestrado) - Fundação Getulio Vargas, Escola de Pós-Graduação em Economia.
Orientador: Caio Almeida.
Inclui bibliografia.


CDD – 332.64524
LAURA SIMONSEN LEAL

TAIL RISK AND HEDGE FUNDS

Dissertação apresentada ao Curso de Mestrado em Economia da Escola de Pós-Graduação em Economia para obtenção do grau de Mestre em Economia.

Data da defesa: 21/03/2016.

ASSINATURA DOS MEMBROS DA BANCA EXAMINADORA

Cão-Ibsen Rodrigues de Almeida
Orientador (a)

Daniela Kubudi Glasman

Axel André Simonsen
Abstract

The main purpose of this paper is to propose a methodology to obtain a hedge fund tail risk measure. Our measure builds on the methodologies proposed by Almeida and Garcia (2015) and Almeida, Ardison, Garcia, and Vicente (2016), which rely in solving dual minimization problems of Cressie Read discrepancy functions in spaces of probability measures. Due to the recently documented robustness of the Hellinger estimator (Kitamura et al., 2013), we adopt within the Cressie Read family, this specific discrepancy as loss function. From this choice, we derive a minimum Hellinger risk-neutral measure that correctly prices an observed panel of hedge fund returns. The estimated risk-neutral measure is used to construct our tail risk measure by pricing synthetic out-of-the-money put options on hedge fund returns of ten specific categories. We provide a detailed description of our methodology, extract the aggregate Tail risk hedge fund factor for Brazilian funds, and as a by product, a set of individual Tail risk factors for each specific hedge fund category.

KEYWORDS Asset Pricing; Stochastic Discount Factor; Risk-Neutral Probability; Tail Risk; Hedge Funds.
Contents

1 Introduction .......................................................... 7

2 Literature Review .................................................. 9
  2.1 The SDF approach .............................................. 9
  2.2 Risk-Neutral Distribution ...................................... 11
  2.3 Tail Risk .......................................................... 13
  2.4 Hedge Fund Performance ...................................... 15

3 Methodology ......................................................... 16
  3.1 Dual Problem ................................................... 16
    3.1.1 Choice of Gamma ........................................ 18
  3.2 Estimation of the Risk Neutral Distribution ................. 18
  3.3 Tail Risk Measure ............................................. 19
  3.4 Empirical Aspects of the Methodology ...................... 20

4 Empirical Performance ............................................ 21
  4.1 Data Description ............................................... 21
  4.2 Results ........................................................ 23
    4.2.1 Checkpoints .............................................. 23
    4.2.2 Tail Risk Time Series .................................... 26
    4.2.3 Tail Risk for Individual Subcategories of Hedge Funds (Within the Aggregate Framework) ............... 29

5 Conclusion ................................................................ 33

Bibliography ............................................................. 35

Appendix A ................................................................ 38

Appendix B ................................................................ 39
1 Introduction

The motivation for this paper comes from the human fear of losing money. Investors all over the world dread the possibility that their portfolios will face extreme downturns and strive to protect against them. Therefore, it is only natural that one would search for ways of identifying and hedging against potential tail risks. What we endeavor to present in this paper is a first step for an identification of such potential risks. We suggest a new methodology to measure hedge fund tail risks.

Given observed returns from a set of basis assets, Hansen and Jagannathan (HJ, 1991) introduced a methodology to estimate minimum variance stochastic discount factor bounds. The methodology offers, as a by-product, implied Stochastic Discount Factors (SDFs) that are linear functions of these observed returns. Since then, their methodology has been applied in many empirical analysis including but not limited to diagnostic of asset pricing models, portfolio management, predictability issues, and performance analysis of mutual and hedge funds (see Ferson (2003)).

Almeida and Garcia (2015) generalized the quadratic loss function in HJ to a whole family of convex functions related to the Cressie Read family of discrepancies. In their work, also as a by-product, they obtain a corresponding family of implied admissible SDFs that are hyperbolic functions of observed returns.

This family of implied SDFs, which captures as a particular case the linear HJ SDF, has been adopted in Almeida and Garcia (2013) to calculate ranges of performance for international hedge funds. Since each given admissible SDF presents a different structure for risk compensation in different states of nature, Almeida and Garcia (2013) show that performance is SDF dependent. That is, depending on each specific hedge fund class (i.e., the structure of its returns), it is possible that a whole range of different values for its performance is obtained. More specifically, for some particular hedge fund classes, performance may consistently vary across different members of the Cressie Read family. This variability is related to how much weight each implied SDF puts in higher order moments (like skewness and kurtosis) of observed returns of the risky factors being priced as basis assets.

Since each strictly positive admissible SDF can be associated with a risk-neutral measure (see Duffie (2001)), implied Cressie Read SDFs provide a way of risk-neutralizing
measures, objects, and/or moments. This was precisely the insight pursued by Almeida, Ardison, Garcia, and Vicente (2016) (AAGV), who adapted the methodology proposed by Almeida and Garcia (2015) to estimate a tail risk factor extracted from portfolios of U.S. stocks. AAGV show that this tail risk factor is priced in the cross-section of individual stocks in the U.S., and is able to forecast a number of important U.S. macroeconomic variables.

Kelly and Jiang (2014) measure tail risk from a cross-section of stock returns adopting a dynamic power law structure for the shape of the tails. It is interesting to note that their tail risk factor has a strong predictive power for aggregate market returns. Building on this work, Fernandes and Santos (2015) use the same power-law structure to extract a tail risk factor from a cross-section of U.S. hedge fund returns. Part of the rationale behind the idea of Santos and Fernandes is based on the fact that hedge funds contain, in general, rich portfolios, which include assets that go far beyond stocks and linear instruments. For this reason, a tail risk factor directly extracted from hedge fund returns should be able to better explain future hedge fund returns than a tail risk factor extracted from pure equity returns. Indeed, their hedge fund tail risk factor has clearly stronger predictive ability to forecast future returns for different hedge fund categories, than the original tail risk factor extracted by Kelly and Jiang.

All above-mentioned work was taken into account when designing the research proposal for this paper. We intended to somehow explore the richness of the Cressie Read family of implied SDFs to extract a hedge fund tail risk factor. Naturally, such factor could have been extracted from a set of risky factors observed on primary markets (equities, currencies, interest rates, options, etc...) as done in Almeida and Garcia (2013) (or in Kelly and Jiang (2014), who only adopt stocks). However, the approach proposed by Fernandes and Santos (2015) in adopting hedge fund returns to obtain a tail risk factor, inspired a different path for the use of Cressie Read SDFs within the hedge fund industry.

The contribution of this paper, although specific is made clear here. We work in the intersection of the research proposed in Almeida and Garcia (2015), Almeida and Garcia (2013) and AAGV (2015) to propose a way to measure Tail Risk for the Brazilian hedge fund industry. This factor is directly extracted from hedge fund returns as
done in Fernandes and Santos (2015).

Our approach is the meeting point of a few different branches of specialized literature: the SDF methodology in asset pricing (Cochrane, 2001), risk-neutralization procedures, tail risk literature, and hedge fund performance. In this juncture, we offer a way to measure hedge fund tail risk using the SDF approach to price a set of returns of funds that use non-linear financial instruments.

In order to better organize our work, we first provide a literature review on the topics whose intersection inspired our research project. Following that, we present a methodological section describing how to obtain our Tail Risk measure, and an empirical section where we effectively estimate such a measure making use of Brazilian hedge funds data.

Going through a more specific description, Section 2.1 describes the SDF approach to asset pricing problems. Risk-neutral probabilities, their relation to SDFs, and the way to obtain them are provided in Section 2.2. Section 2.3 presents some adequate ways of measuring tail risk adopted in the literature. Section 2.4 presents a brief description of the hedge funds performance literature. In Section 3, our methodology is presented in details. In section 4, we estimate both an aggregate tail risk measure as well as a tail risk measure for each individual hedge fund category, using hedge funds return data from ANBIMA. In section 5, we conclude, offering a short description of this paper’s main contribution.

2 Literature Review

2.1 The SDF approach

The Stochastic Discount Factor approach provides a very general framework for pricing assets. This approach, whose non-parametric form used in this paper was introduced by Hansen and Jagannathan (1991), relies on the idea that the price of an asset must equal its expected payoff discounted by a certain stochastic factor. Its structure is so general that encompasses both one-factor models like the CAPM and

---

1Risk-neutral probabilities are very important for asset pricing because they allow us to discount returns using the risk-free rate, sometimes making estimation a lot easier.

2We thank very much both the technical and the data teams at ANBIMA for providing access to their private dataset on hedge fund returns.
multiple factor models such as Fama-French. What differentiates one model from
the other is the specific functional form for the SDF that each one assumes.

We can summarize the asset pricing equations as:

\[ p_t = E[m_{t+1}x_{t+1}] \]
\[ m_{t+1} = f(data, parameters) \] (1)

where \( p_t \) is the asset price, \( x_{t+1} \) is the asset payoff and \( m_{t+1} \) is the stochastic dis-
count factor.

Or, equivalently, we can write the above equation in the form of expected returns:

\[ 1 = E[m_{t+1}R_{t+1}] \]
\[ m_{t+1} = f(data, parameters) \] (2)

where \( R_{t+1} = \frac{x_{t+1}}{p_t} \) is the gross return on time \( t + 1 \).

Non-parametric approach based on a generic stochastic discount factor instead
of an SDF implied by a particular model has the advantage of requiring very weak
conditions, such as the Law of One Price of No-Arbitrage conditions.

**Definition.** Law of One Price: If two portfolios have the same payoffs (in every state of
nature), then they must have the same price.

**Definition.** No-Arbitrage: A payoff space \( X \) and a pricing function \( p(x), x \in X \), leave
no arbitrage opportunities if every payoff \( x \) that is always non-negative, \( x \geq 0 \) (almost
surely), and positive, \( x \geq 0 \), with some positive probability, has a positive price, \( p(x) \geq 0 \).

There is no need for more restrictive hypothesis such as market completeness, in-
formation homogeneity, aggregation, independence, probability distributions, mar-
ket equilibrium, or a specific kind of preference structure. This is what makes the
SDF approach so general.

Although market completeness is not a necessary hypothesis for the existence of
the stochastic discount factor, it is crucial to determine its uniqueness. If markets are
not complete, we will have an infinity of SDFs. It can be shown that at least one of
them is (almost surely) positive in the absence of arbitrage (see Hansen and Jagannathan (1997)).

Moreover, as Hansen and Richard (1987) have shown, if markets are complete and the Law of one Price is effective, then the No-Arbitrage condition is satisfied if, and only if, the SDF is (almost surely) positive.

Hansen and Jagannathan (1997) have also suggested that an appropriate way to compare the performance of asset pricing models is to evaluate functions of their implied pricing errors on corresponding Euler equations. They use a least-squares projection of a proxy model on a family of admissible SDFs to compare models that are possibly misspecified.

Critics of the methodology proposed by Hansen and Jagannathan (1997) observe that the quadratic form of their metric only considers the mean and variance of a distribution, not taking into account higher moments, such as skewness and kurtosis, that have been proved important in asset pricing.

Almeida and Garcia (2012) build on the idea of Hansen and Jagannathan (1997) and propose that we use a general convex function to measure the distance between the proxy model and the family of admissible SDFs. They minimize the convex discrepancy function and use the minimum discrepancy distance to compare models, the best one being the one with smallest distance. By doing this, they account for all other moments of the return distribution.

2.2 Risk-Neutral Distribution

Now, as in Ross (2015), consider a one period world with asset payoffs $x(\theta)$ at time $T$, contingent on the realization of a state of nature $\theta \in \Omega$. As stated above, no-arbitrage implies the existence of positive state space prices, i.e., Arrow-Debreu contingent claims prices $p(\theta)$ (or the measure $P(\theta)$) paying $1$ in state $\theta$ and $0$ in other states.

The current value of the asset is:

$$ p = \int x(\theta) dP(\theta) \quad (3) $$

Letting $r$ denote the risk-free rate, we can rewrite the above equation as:
\[ p = \int x(\theta) dP(\theta) = \left( \int dP(\theta) \right) \int x(\theta) \frac{dP(\theta)}{dP(\theta)} \equiv e^{-rT} \int x(\theta) d\pi^*(\theta) \equiv e^{-rT} \mathbb{E}^*[x(\theta)] = \mathbb{E}[x(\theta)m(\theta)] \] (4)

where an asterisk denotes the expectation in the risk-neutral or martingale measure and where the pricing kernel or stochastic discount factor, \( m(\theta) \), is the Radon-Nikodym derivative of \( P(\theta) \) with respect to the natural measure.

Ross (2015) shows that it is possible to separate risk aversion from the natural distribution and estimate each of them from market prices. He defines the states from the filtration of the stock value, so that the kernel is the projection of the kernel across the broader state space onto the more limited space defined by the filtration of the asset price.

In our paper, we will also interpret the stochastic discount factor as a transformation to risk-neutral probabilities. However, we will follow Cochrane (2001) and use a discrete setting instead, which conforms better with our data sample.

The first step is to specialize our state space to be finite-dimensional, i.e., suppose that there are \( S \) states of nature that can occur tomorrow. As before, an Arrow-Debreu contingent claim is one that pays one dollar in only one state \( s \) tomorrow. The price today of this contingent claim is \( pc(s) \). Letting \( x(s) \) denote an asset’s payoff in state of nature \( s \), the asset’s price must then be \( p(x) = \sum_s pc(s)x(s) \).

Next, if \( \pi(s) \) is the probability that state \( s \) will occur, we can write

\[ p(x) = \sum_s \pi(s) \left( \frac{pc(s)}{\pi(s)} \right) x(s) \] (5)

And, defining \( m \) as the ratio of contingent claim price to probability,

\[ m(s) \equiv \frac{pc(s)}{\pi(s)} \] (6)

we can write:

\[ p(x) = \sum_s \pi(s)m(s)x(s) = E[mx] \] (7)

Now, to find the risk-neutral probabilities, we first define
\[ R_f \equiv \frac{1}{\sum_s pc(s)} = \frac{1}{\mathbb{E}[m]} \quad (8) \]

and

\[ \pi^* \equiv R_f m(s) \pi(s) = R_f pc(s) = \frac{pc(s)}{\sum_s pc(s)} \quad (9) \]

As we can see from equation (9), \( 0 \leq \pi^* \leq 1 \) and \( \sum \pi^* = 1 \), so \( \pi^* \) is indeed a set of probabilities.

The asset pricing formula can then be written as:

\[ p(x) = \sum_s pc(s)x(s) = \frac{1}{R_f} \sum_s \pi^*(s)x(s) = \frac{\mathbb{E}[x]}{R_f} \quad (10) \]

In short, we initially had a setting on which agents were risk-averse and made decisions based on state probabilities. Now, we have risk-neutral agents that consider risk-neutral probabilities. It would be natural to expect - and this is actually the case - that risk-neutral probabilities give greater weight to states of nature on which the average marginal utility is higher than average. Those are precisely our negative tail risk events.

From equations (6), (8) and (9), it is easy to see that the transformation from actual to risk-neutral probabilities is given by:

\[ \pi^*(s) = \frac{m(s)}{\mathbb{E}[m]} \pi(s) \quad (11) \]

This result will be very useful when we calculate our risk-neutral probabilities from estimated stochastic discount factors.

### 2.3 Tail Risk

Tail risk is the probability of extremely large losses in portfolio returns. Because investors are tail risk averse, there is a positive relation between tail risk and future returns. Naturally, the return required by investors to hold assets increases when tail risk increases.

One of the first researchers to describe this relation between tail risks and asset prices was Rietz (1988), who extends the Mehra and Prescott (1985) model to include
a rare disaster state. With this model, he hoped to solve the equity premium puzzle. Critics, however, claimed that his estimates were not compatible with historical U.S. consumption growth. Barro (2006) later extended his model to international data and found that this new calibration finds reasonable estimates for the equity premium.

In recent literature, time series analysis has given way to cross-section approaches of tail risk estimation. Bollerslev and Todorov (2011) use high-frequency data for S&P 500 futures and closing bid and ask quotes for S&P 500 options to estimate a model-free index of investor's fear. Their empirical findings suggest that historically large equity and variance risk premia are probably compensation for tail events. They, however, have a different definition of tail risk from the one stated above. For them, tail risk is seen as daily jumps in asset prices, which are usually not as large.

Siriwardane (2013) constructs a disaster risk measure based on option prices. It consists of subtracting an OTM call from and OTM symmetrical put, normalized by the firm's current stock price. His measure proxies for the ex-ante disaster risk of a firm's stock. Firms that present higher disaster risk should require higher equity returns in equilibrium, if this risk is priced in. Indeed, they find that a zero-cost equity portfolio that is exposed to high disaster risk stocks earns excess annualized returns of 12.13%, even after controlling for standard factors such as Fama-French, momentum, liquidity and volatility risk.

Kelly and Jiang (2014) measure tail risk using the cross-section of returns. They use the fact that some assets are probably going to go through tail events in a given month to create a global measure of common fluctuations in tail risk. Their tail risk measure builds from a dynamic power law structure that considers two parameters, one representing the systematic tail risk and the other representing asset-specific tail risk. They show that tail risk has strong predictive power for aggregate market returns.

Almeida, Ardison, Garcia, and Vicente (2016) propose a novel way to estimate tail risk incorporating risk-neutral information using a panel of stock returns. This approach is very interesting because it precludes the use of options data, which are not available, or do not have enough liquidity, in many markets. In their paper, they use 25 Fama-French size and book to market portfolios to estimate tail risk and forecast
aggregate U.S. macroeconomic activity indexes and yield spreads.

In short, here are three ways to measure tail risk:

(i) Using option prices, as in Bollerslev and Todorov (2011) and Siriwardane (2013)

(ii) Using the cross-section of returns, as in Kelly and Jiang (2014)

(iii) Using the SDF approach and a synthetic put to estimate tail risk, as in Almeida, Ardison, Garcia, and Vicente (2016)

2.4 Hedge Fund Performance

In normal times, hedge funds provide liquidity and increase the efficiency of the financial markets. In times of crisis, on the other hand, they might contribute to increased market volatility and downfalls. It is important to keep in mind that hedge funds can be a great source of financial risk to a country’s economy. They face liquidity risks, volatility risks, investor protection and same strategy risks, which happen when hedge funds look at the same macroeconomic scenario, build the same strategies and get squeezed by the lack of counterparts to the operation. The extent to which hedge funds are exposed to tail risk can have a great impact on the overall economy and, therefore, it is very important to have a measure that helps us understand this exposure. There have been some very interesting contributions in the literature in this regard.

Using their power law methodology, Kelly and Jiang (2012) find that tail risks are the main drivers of hedge fund returns in both cross-section and time series. This indicates that a large fraction of hedge fund returns can be interpreted as compensation for providing insurance against disasters. That is, whether a fund is protected or not against tail risks directly influences hedge fund returns.

Building on the approach presented in Kelly and Jiang (2014) that uses Extreme Value Theory to approximate lower tail, Fernandes and Santos (2015) proposes a new factor to control for extreme downside events risk. Using information embedded in a panel of hedge funds performance, they create a Hedge Fund Tail Risk measure (HFTR) and show that this new measure has greater forecasting explanatory power than the one presented by Kelly and Jiang (2014). Their approach can account for over-crowding and over-leverage effects, as well as for equity market risks.
Adrian, Brunnermeier, and Nguyen (2011) link hedge funds to the risk of systemic crisis and study interdependencies between different styles in times of distress. Using quantile regressions, they find that tail sensitivities between strategies are higher in difficult times, and identify seven factors that explain a large part of tail risks. They stress that offloading tail risks that derive from these seven factors might come at the cost of lower returns for individual funds.

Capocci and Hübner (2004) investigate hedge funds performance and persistence using a combination of various asset pricing models, including an extension of Carhart (1997), Fama and French (1998), Agarwal and Naik (2002) and an additional factor that takes investment in emerging market bonds into account. They find that only one fourth of hedge funds have significant positive excess returns and that over-performance is usually constant over time. However, they also find that the great majority of these funds suffered from the Asian crisis, which tells us that they are exposed to global tail risks.

The approach followed by Bailey, Li, and Zhang (2004) analyses hedge funds performances by using stochastic discount factors. They use Hansen and Jagannathan (1997) distance measure within the SDF framework in a way that allows them to take into account the non-arbitrage requirement. Since hedge funds are able to trade derivatives and have dynamic strategies, they are often able to take advantage of negative SDFs. By imposing non-arbitrage and requiring the SDFs to be positive, they are able to take into account hedge fund trading strategies and non-linearity in returns and thus surpass linear asset pricing models.

In this paper, we build on the stochastic discount factor approach used by Bailey, Li, and Zhang (2004), but instead of using Hansen and Jagannathan (1997) distance measure, we use the generalization of their model for general convex functions as presented in Almeida and Garcia (2015). In the next section, we present their methodology, which relies on the solution of a dual problem and proceed to building our tail risk measure for hedge funds.
3 Methodology

3.1 Dual Problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $R$ denote a $K$-dimensional random vector on this space representing the returns of $K$ primitive basis assets. In the static setting we are considering, an admissible SDF is a random variable $m$ for which $E[mR]$ is finite and the Euler equation $E[mR] = 1^K$ is satisfied, where $1^K$ is a vector of ones with $K$ dimensions.

As stated in section 1.2, the set of admissible SDFs will vary according to the market structure. Complete markets will give us an unique SDF and incomplete markets will give us an infinity of SDFs. We can restrict the latter to be positive if we impose the absence of arbitrage. This is exactly what we will do. Brazilian markets are incomplete, i.e., the number of states of nature ($T$) is larger than the number of basis assets, and in the absence of arbitrage we will get an infinity of positive stochastic discount factors. For each one of them there will be a corresponding risk neutral probability, following the result in (4).

In their seminal paper, Hansen and Jagannathan (1991) minimize a quadratic function in the space of non-negative admissible stochastic discount factor with fixed mean $a$ to find a minimum variance bound for the SDFs. They do so in the context of (2). Assuming that the process $(m_t, R_t)$ is sufficiently regular (e.g., stationary and ergodic) such that a time series version of the law of large number applies, sample moments will converge to population counterparts as the sample size $T$ becomes large.

Now, given a discrepancy function $\phi$, Almeida and Garcia (2015) find that the (in sample) generalized minimum discrepancy problem can be stated as:
\[ \hat{m} = \arg\min_{m_1, \ldots, m_T} \frac{1}{T} \sum_{i=1}^{T} \phi(m_i) \]

subject to
\[ \frac{1}{T} \sum_{i=1}^{T} m_i \left( R_i - \frac{1}{a} 1_K \right) = 0_K \]
\[ \frac{1}{T} \sum_{i=1}^{T} m_i = a \]
\[ m_i > 0, \forall i \]

where the first and second restrictions are the sample equivalent to \( \mathbb{E}[m(R - \frac{1}{a} 1_K)] = 0_K \) and \( \mathbb{E}[m] = a \), respectively, and the last restriction imposes positivity of the stochastic discount factor, which makes it compatible with the absence of arbitrage.

Furthermore, Almeida and Garcia (2015) make use of the results in Borwein and Lewis (1991) to prove that, in general, the optimization problem stated in (12) can be solved in a simpler finite dimensional dual space. The problem then becomes:
\[ \hat{\lambda} = \arg\sup_{\alpha \in \mathbb{R}, \lambda \in \Lambda} a * \alpha - \frac{1}{T} \sum_{i=1}^{T} \phi^* + \left( \alpha + \lambda' \left( R_i - \frac{1}{a} 1_K \right) \right) \]

where \( \Lambda \subseteq \mathbb{R}^K \), \( \lambda \) is a vector of \( K \) Lagrange multipliers that comes from the Euler equations for the primitive basis assets, \( \alpha \) is the Lagrange multiplier that comes from the original restriction \( \mathbb{E}[m] = a \) and \( \phi^* + \) is the convex conjugate of \( \phi \) restricted to the positive real line.

\[ \phi^* + = \sup_{w > 0} zw - \phi(w) \]

Assuming that the discrepancy \( \phi \) above belongs to the Cressie and Read (1984) discrepancy family, i.e.,
\[ \phi^\gamma(m) = \frac{m^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma + 1)} \]

with \( \gamma \in \mathbb{R} \). The use of this family of discrepancy functions allows for the generalization of several particular restrictions on the space of SDFs, such as those derived by Hansen and Jagannathan (1991), Snow (1991), Stutzer (1995), Bansal and Lehman (1997) and Cerny (2003).
Considering $\gamma < 0$, we arrive at closed form formulas for $\lambda$ and for the empirical estimates of minimum discrepancy SDFs ($\hat{m}_{MD}$):

$$\hat{\lambda} = \operatorname{arg sup}_{\lambda \in \Lambda_{CR}} \frac{1}{T} \sum_{i=1}^{T} \left( a^{\gamma+1} \gamma + 1 \right) - \frac{1}{\gamma+1} \left( a^{\gamma} + \gamma \lambda' \left( R_i - \frac{1}{a} 1_K \right) \right)^{\frac{\gamma+1}{\gamma+1}} \right) \tag{16}$$

where $\Lambda_{CR} = \lambda \in \mathbb{R}^K \ | \ \forall i = 1, \ldots, T; (a^{\gamma} + \gamma \lambda' (R_i - \frac{1}{a} 1_K)) > 0$ and

$$\hat{m}_{MD}^i = a \frac{\left( a^{\gamma} + \gamma \hat{X}_i' \left( R_i - \frac{1}{a} 1_K \right) \right)^{\frac{1}{\gamma}}}{\frac{1}{T} \sum_{i=1}^{T} \left( a^{\gamma} + \gamma \hat{X}_i' \left( R_i - \frac{1}{a} 1_K \right) \right)^{\frac{1}{\gamma}}} \tag{17}$$

### 3.1.1 Choice of Gamma

In this paper, we will work with $\gamma = -\frac{1}{2}$, following the robustness results of Kitamura, Otsu, and Evdokimov (2013). Their main theoretical result shows that the minimum Hellinger distance estimator (MHDE) possesses optimal min-max robust properties and that it remains semi-parametrically efficient when the model assumptions hold. Therefore, we choose the Hellinger estimator because it gives us higher chances of obtaining a tail risk factor that will be robust to the choice of the SDF used to calculate synthetic put prices.

### 3.2 Estimation of the Risk Neutral Distribution

To estimate the risk neutral distribution, we need both the state probabilities and the stochastic discount factor. From section 2.1, we have our estimated SDF for $\gamma = -\frac{1}{2}$. For the state probabilities, we will follow an approach that is less than perfect, but is traditionally used in finance, which assumes that each state of nature can occur with probability $\frac{1}{T}$.

This assumption allows us to compute the risk neutral probability quite easily by using equation (11)

$$\pi_{RN} = \frac{m_{MD}^i (1 + r_f)}{T} \tag{18}$$

where $\pi(s) = \frac{1}{T}$, $E[m] = \frac{1}{T}$, $\frac{1}{1 + r_f}$ and $m(s)$ becomes the stochastic discount factor estimated in (17), with $\gamma = -\frac{1}{2}$. 

19
These risk-neutral probabilities will allow us to estimate our tail risk measure.

### 3.3 Tail Risk Measure

To estimate a tail risk measure, we rely on the fact that prices usually reflect future expectations, i.e., what investors think about what might happen in future states. In particular, prices for out-of-the-money puts reveal information about the negative tail of the distribution of returns.

Therefore, it would be easier to use options data as a way to measure investors’ perceptions. Unfortunately, in Brazil, we do not have enough options data available to estimate tail risks. In particular, out-of-the-money options are not very liquid when they exist. Thus, a way to get around this is to follow AAGV (2015) in constructing a tail risk measure that will be the average price of out-of-the-money synthetic puts for a pre-determined set of basis assets returns. This measure is obtained using the previously estimated risk-neutral probabilities.

First, we estimate the price of a synthetic put for each of the subcategories of hedge funds:

\[
P = \sum_{i=1}^{T} \pi_{i}^{RN} \left[ \max(K - R_{i}, 0) \right]
\]  \hspace{1cm} (19)

where \( \pi_{i}^{RN} \) are the risk-neutral probabilities of each possible state, \( K \) is the (risk-neutral) strike price, which is the 10\textsuperscript{th} percentile of the risk-neutral distribution, and \( \max(K - R_{i}, 0) \) is the payoff of the synthetic put in each possible state.\(^3\)

The Tail Risk measure is the average of the synthetic puts that we have just obtained.

### 3.4 Empirical Aspects of the Methodology

Once the methodology has been set, we approach the data with care. We have return data from over six thousand hedge funds provided by ANBIMA\(^4\). This is a huge

\(^3\)AAGV (2015) provide a long robustness section, showing, among other things, that the 10\textsuperscript{th} percentile has the most adequate sensitivity to derive a tail risk measure whenever the estimated SDFs present a small number of states, which is the case that we have in this paper with 30 states of nature.

\(^4\)Brazilian Association of Financial and Capital Market Institutions or, in Portuguese, “Associação Brasileira das Entidades dos Mercados Financeiros e de Capitais”
number on its own and it seems even larger when compared to the fifteen hundred and sixty five days in our sample.

Therefore, we felt the need to aggregate the returns in some way. We chose to aggregate them into the ten subcategories of hedge funds defined by ANBIMA. This helped us avoid the problem of over-specification generated by having more types of funds than data occurrences for each individual fund.

In this paper, tail risk is not measured as standard-deviations or as the probability of a huge historical loss. Instead, as defined above in Section 2, it depends on the average price of synthetic puts (on which we have defined the strike as a return quantile) that are generated on an thirty-day window. This approach implies that every month some subcategory of hedge funds will experience tail events and we will be capturing information related to the aggregate level of tail risk within a thirty-day moving window.

With this in mind, we transformed the return panel in a 30x10x1565 matrix representing 1565 moving windows of thirty days for the ten types of hedge funds. On each of these windows, we calculated the ten Lagrange multipliers and, subsequently, the thirty entries for the Hellinger implied SDF using the methodology described before. The implied SDF will allow us to obtain the discrete risk-neutral probability given by a thirty-dimensional vector containing probabilities for each of the thirty scenarios within each moving window.

Within each moving window, to obtain the strike of the synthetic put priced for each of the ten individual hedge fund categories (or types), we extract the first decile of each class returns under the Hellinger-implied risk-neutral probability measure. The price of the synthetic put is obtained for each of the ten categories of hedge funds as being the expected value of the payoffs of our synthetic puts, under the Hellinger implied risk-neutral measure. Finally, our tail risk measure is obtained as the average of the prices obtained for the ten synthetic puts.

On a preliminary study using this new methodology, we had followed previous literature and used a fixed value for the interest rate throughout the analyzed period. This would have worked perfectly in current European or American markets due to their very low interest rate values, however, it is a strong generalization when using Brazilian interest rates data.
Interest rates in Brazil have changed considerably between 2009 and 2015, ranging from 7.12 in 01/16/2013 to 14.15 at the end of our sample. In our study, we have noticed that our tail risk measure is very sensitive to interest rate changes. Therefore, we use the whole time series for interest rates when calculating the tail risk measure. For each moving window (as defined above) we use the daily value for the interest rate that relates to the first day of the window.

Using this methodology, we are sure to find a tail risk measure that takes into account the contemporary interest rate value, instead of depending on a single unconditional expected value that could potentially bias our results.

4 Empirical Performance

4.1 Data Description

Our data for hedge funds was provided by ANBIMA\textsuperscript{5}. According to their classification, Brazilian hedge funds are called multi-market funds. These funds have investment policies that involve several risk factors. They may invest in fixed income, exchange markets, stocks, etc. In this paper, we will use data for multi-market funds to measure tail risk.

Multimarket funds are subdivided in ten categories: Balanced, Protected Capital, Long and Short - Neutral, Long and Short - Directional, Macro, Trading, Multi-strategy, Funds of Funds, Interest and Currency and Specific Strategy\textsuperscript{6}. The characteristics of these types of funds can be found in Appendix A and the daily return time series for each individual subcategory of hedge funds can be found in Appendix B.

To give the reader a better idea of the data that we are analyzing, we present a table with summary statistics.

This table reports summary statistics for the ten hedge fund style daily returns. For each type of fund we have 1594 observations. The Sharpe ratio is the ratio of mean excess returns to the standard deviation of returns, where we have used the daily equivalent of the 14.15% risk-free annual rate to obtain the mean excess re-

\textsuperscript{5}Brazilian Association of Financial and Capital Market Institutions or, in portuguese, “Associação Brasileira das Entidades dos Mercados Financeiros e de Capitais”

\textsuperscript{6}Their respective codes in ANBIMA’s database are: 207, 208, 327, 328, 329, 330, 331, 332, 333 and 334.
<table>
<thead>
<tr>
<th>Type of Fund</th>
<th>Sharpe</th>
<th>Mean</th>
<th>Median</th>
<th>Std Dev</th>
<th>Skew</th>
<th>Kurt</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balanced</td>
<td>0.2006</td>
<td>0.0374</td>
<td>0.0393</td>
<td>0.1839</td>
<td>-0.2711</td>
<td>5.1903</td>
<td>-0.9104</td>
<td>0.8230</td>
</tr>
<tr>
<td>Protected Capital</td>
<td>0.1104</td>
<td>0.0238</td>
<td>0.0312</td>
<td>0.2113</td>
<td>-0.2145</td>
<td>4.3465</td>
<td>-0.9738</td>
<td>0.7938</td>
</tr>
<tr>
<td>Long/Short - Neutral</td>
<td>0.4587</td>
<td>0.0434</td>
<td>0.0399</td>
<td>0.0934</td>
<td>0.1197</td>
<td>7.5071</td>
<td>-0.5772</td>
<td>0.6386</td>
</tr>
<tr>
<td>Long/Short - Directional</td>
<td>0.4173</td>
<td>0.0412</td>
<td>0.0374</td>
<td>0.0974</td>
<td>-0.0577</td>
<td>4.4684</td>
<td>-0.3628</td>
<td>0.5068</td>
</tr>
<tr>
<td>Macro</td>
<td>0.2494</td>
<td>0.0487</td>
<td>0.0470</td>
<td>0.1931</td>
<td>0.0644</td>
<td>6.9432</td>
<td>-0.9442</td>
<td>0.9930</td>
</tr>
<tr>
<td>Trading</td>
<td>0.1498</td>
<td>0.0324</td>
<td>0.0420</td>
<td>0.2125</td>
<td>-0.4425</td>
<td>7.7800</td>
<td>-0.9637</td>
<td>0.9401</td>
</tr>
<tr>
<td>Multistrategy</td>
<td>0.2619</td>
<td>0.0430</td>
<td>0.0410</td>
<td>0.1621</td>
<td>0.0488</td>
<td>5.8150</td>
<td>-0.7131</td>
<td>0.8787</td>
</tr>
<tr>
<td>Funds of Funds</td>
<td>0.4475</td>
<td>0.0390</td>
<td>0.0402</td>
<td>0.0860</td>
<td>-0.4288</td>
<td>8.3538</td>
<td>-0.6609</td>
<td>0.5278</td>
</tr>
<tr>
<td>Interest and Currency</td>
<td>0.6699</td>
<td>0.0386</td>
<td>0.0434</td>
<td>0.0568</td>
<td>-1.2477</td>
<td>15.6700</td>
<td>-0.4505</td>
<td>0.3825</td>
</tr>
<tr>
<td>Specific Strategy</td>
<td>0.3270</td>
<td>0.0446</td>
<td>0.0451</td>
<td>0.1347</td>
<td>-0.5301</td>
<td>8.5035</td>
<td>-0.9520</td>
<td>0.7338</td>
</tr>
</tbody>
</table>

returns, considering the year as having 252 working days.

As we can see from the summary statistics, Interest and Currency is the subcategory with the highest Sharpe ratio, even though the mean returns are not as high as in other subcategories. This is due to the fact that it has the lowest standard deviation. Readers familiar with the hedge fund literature might recall that the Sharpe ratio is not the best way to evaluate hedge funds. This is due to the fact that they are able to leverage their portfolios and, by means of buying and short-selling assets, they are able to mask the value of their Sharpe ratio. Therefore, even though the Interest and Currency subcategory might look very interesting as an investment, it is important to check its exposure to tail risk before including it in a portfolio, as an investor. We will see later that, indeed, Interest and Currency not only presents the highest Sharpe Ratio but also is the subcategory that presents smallest correlation (i.e., 30%) with our aggregate measure of Tail Risk.

From the summary statistics, we can also see that most of the funds have negative or very low skewness. We also find that all of them have positive, sometimes high, kurtosis, which indicates that the return distribution has fat tails when compared to the normal distribution. We test the data for normality using Royston's Multivariate Normality Test, as done by Adrian, Brunnermeier, and Nguyen (2011). The results are presented below.

---

We have used the Matlab code provided by Trujillo-Ortiz, Hernandez-Walls, Barba-Rojo, and Cupul-Maganad (2007) to execute Royston's test for normality.
Therefore, we have checked that, indeed, the daily return data for hedge funds does not follow a normal distribution and has positive kurtosis. With this in mind, we are now sure that we have to take into account higher moments of the distribution when considering hedge fund returns. Our methodology, described in section 2, comes in very handy at this point, as we are able to evaluate an aspect of the return distribution of the utmost importance: tail risk.

4.2 Results

4.2.1 Checkpoints

Using the methodology described in this paper, we obtain for each moving window described in section 2: ten Lagrangian weights, one for each subcategory of hedge fund, representing the linear combination that produces the portfolio obtained in the dual optimization problem; thirty states for the Hellinger stochastic discount factor, one for each date in the moving window; and correspondingly, thirty states for the Hellinger risk-neutral probability.

Before presenting our tail risk series, we will present checkpoints that indicate if our empirical results are theoretically consistent. For that, we will pick the last moving window as an example for the reader, but keeping in mind that the results
are fairly stable throughout the sample, and most importantly, that we checked for consistency considering all different windows.

First, we check that the stochastic discount factors are all positive and, therefore, that the no-arbitrage condition is being satisfied. Their positivity comes from the fact that they come from solutions to (strictly increasing) utility maximization problems. Below we present, as an example, the SDF implied using the last window of data.

**Table 2:** An example of Hellinger’s implied SDF - last window of data

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDF</td>
<td>1.2474</td>
<td>1.2913</td>
<td>1.3680</td>
<td>0.5537</td>
<td>0.2455</td>
<td>0.2880</td>
<td>0.4888</td>
<td>0.3753</td>
<td>0.9245</td>
<td>2.3638</td>
</tr>
<tr>
<td>Period</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>SDF</td>
<td>0.7091</td>
<td>0.2114</td>
<td>0.1779</td>
<td>3.3520</td>
<td>1.6089</td>
<td>0.9874</td>
<td>1.7820</td>
<td>0.2963</td>
<td>1.2530</td>
<td>1.8691</td>
</tr>
<tr>
<td>Period</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
</tr>
<tr>
<td>SDF</td>
<td>1.4165</td>
<td>1.8778</td>
<td>0.8121</td>
<td>0.2948</td>
<td>0.5518</td>
<td>0.5148</td>
<td>0.6987</td>
<td>0.7522</td>
<td>1.4779</td>
<td>0.1943</td>
</tr>
</tbody>
</table>

We can also observe (see graphic below) that the stochastic discount factor follows a hyperbolic structure. This structure is directly related to the proposed Cressie-Read family of discrepancy functions, since as shown by Almeida and Garcia (2016), it restricts the analysis to a specific subset of admissible SDFs that captures non-linear patterns of returns via hyperbolic functions of returns.

For the risk neutral probabilities, we have that \( \pi_{RN} \) and in all of the moving windows the risk-neutral probabilities sum to one. We present the result for the last moving window. Some risk-neutral probabilities are higher, depending on the risk scenario, and others are lower, but we can guarantee in our model, by construction, that they sum to one in a given moving window.

**Table 3:** An example of Hellinger’s implied RN probabilities - last window of data

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>RNP</td>
<td>0.0416</td>
<td>0.0431</td>
<td>0.0456</td>
<td>0.0185</td>
<td>0.0082</td>
<td>0.0096</td>
<td>0.0163</td>
<td>0.0125</td>
<td>0.0308</td>
<td>0.0788</td>
</tr>
<tr>
<td>Period</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>RNP</td>
<td>0.0236</td>
<td>0.0071</td>
<td>0.0059</td>
<td>0.1118</td>
<td>0.0537</td>
<td>0.0329</td>
<td>0.0594</td>
<td>0.0099</td>
<td>0.0418</td>
<td>0.0623</td>
</tr>
<tr>
<td>Period</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
</tr>
<tr>
<td>RNP</td>
<td>0.0472</td>
<td>0.0626</td>
<td>0.0271</td>
<td>0.0098</td>
<td>0.0184</td>
<td>0.0172</td>
<td>0.0233</td>
<td>0.0251</td>
<td>0.0493</td>
<td>0.0065</td>
</tr>
</tbody>
</table>

Moreover, the same hyperbolic shape that we found for the SDFs can be observed
in the structure of the risk-neutral probabilities (see graphic below), since the last object is simply a normalization of the former. It is important to notice, at this point, that this hyperbolic structure is not valid for the returns on individual funds or types of funds. It only works for the endogenous portfolio, which can be obtained by multiplying the matrix of returns by the weights obtained from the Lagrangian and the result by $\gamma = -\frac{1}{2}$. It is very clear from the graphic that physical and risk-neutral probabilities differ a lot, especially for tail returns. In addition, it is also interesting to note that lower returns receive more weight when we use these risk-neutral probabilities, while higher returns are attached to lower probabilities.

### 4.2.2 Tail Risk Time Series

As interestingly put by Fernandes and Santos (2015), hedge fund managers can often pursue “picking up nickels in front of a steamroller” strategies. That is to say that they seek ways to produce high alphas by investing in tail risk. These strategies tend to produce small positive gains most of the time, but may also sporadically result in huge losses.

Throughout this paper, we have defended that it is of the utmost importance to
take into account tail risk when analyzing hedge funds and we have presented a very straightforward methodology to do so. Below, we plot our tail risk time series for the aggregate hedge fund index, which includes all the ten subcategories of hedge funds.

The tail risk time series is very volatile and it might be difficult to distinguish whether tail risk is trending higher or lower just by looking at it. Therefore, we have used the Hodrick-Prescott filter to smooth the time series in order to reduce sensitivity and emphasize long term fluctuations instead of short term patterns. We have used the HP parameter as 14400, which is the regular value used for monthly analysis. The HP filter corresponds to the line shown in red in the figure. Our tail risk measure is the blue line.

Furthermore, we would like to know how our hedge fund tail risk measure compares with total market volatility, and a tail risk measure for the Brazilian market index Ibovespa, which will be introduced in the next section.

First, we have to define the way by which we are measuring volatility. In this paper, we have used an Exponentially Weighted Moving Average (EWMA) model to obtain volatility for the market index IBOV. The initial value used in the EWMA model was 40.52%, corresponding to an approximate standard deviation for the first ob-
servations. The lambda used in the model was approximately 0.9143, which corresponds to the value that minimizes the daily mean squared errors between the squared returns for the IBOV Index and the EWMA estimate within our sample. Below, we plot the volatility time series for the IBOV Index.\(^8\)

**Figure 4: An Estimate for the Volatility of the Ibovespa Index**

To see how our tail risk measure relates to total market volatility, we obtain the

\(^8\)The original return time series for the IBOV Index can be found on Appendix B.
very down-to-earth measure of correlation. The correlation between the two time series is 0.1368. As expected, this number is positive, indicating that the tail risk measure tends to move in the same direction as market volatility. However, it is rather low, telling us that the two measures contain different information. This is not difficult to reconcile since tail risk, coming from the prices of out-of-the money puts must contain some additional information about jumps in return, not directly captured by the EWMA measure of volatility.

If, on the other hand, we compare the HP filtered tail risk time series with the market volatility, we find a correlation of 0.7768, which is rather high. This tells us that the tendency for our tail risk time series moves along with the IBOV Index volatility. Even though it is not an obvious result, it is rather intuitive to have higher tail risk when volatility is higher. High volatility means that we get more spread out returns. In turn, this can lead to a higher chance of getting very low returns and, consequently, to increased tail risks.

In the next section, we offer measures of Tail risk implied for each individual hedge fund category and briefly analyze their correlation structure and also their correlations with Hellinger's implied Ibovespa tail risk.

4.2.3 Tail Risk for Individual Subcategories of Hedge Funds (Within the Aggregate Framework)

The next three pictures present tail risk for the ten different hedge fund categories. Note that by just looking at those pictures we conclude that their behavior varies a lot across subcategories. Some categories like “Interest and Currency”, “Fund of Funds”, “Macro”, and “Trading” have clearly a smallest number of spikes in the series than the others. Despite the fact that they do have a smaller number of spikes, those spikes usually achieve values two to three times higher than the average value of the corresponding tail risk time series, indicating a large variability of the tail risk measure. The tail risk series of the remaining categories have a large number of jumps and some of these jumps are very extreme. For instance, the Multi-strategy category presents jumps four to five times higher than the average tail risk of this

---

9 Of course, since we are talking about tail events, that a linear measure like correlation should not be the most appropriate one to compare Tail risk to volatility. On the other hand, it gives an idea of how much they vary together on average, on a daily basis.
category.

**Figure 5:** Tail Risk for Individual Hedge Fund Categories - Part I

**Figure 6:** Tail Risk for Individual Hedge Fund Categories - Part II

**Figure 7:** Tail Risk for Individual Hedge Fund Categories - Part III
The table below presents the correlation structure of these individual series of tail risk as well as the corresponding correlations between these series and the aggregate measure of tail risk derived above. Since each of the individual measures is calculated by projecting, on each window of data, the implied SDF on the specific sub-category being analyzed, the correlation matrix will indicate how aligned (or not) are the individual returns under the Hellinger risk neutral probability measure obtained with the implied Lagrange multipliers that solve the HARA utility problem described in the methodological section. We can see that, in general, most different subcategories have implied tail risk that is positively correlated across subcategories. An interesting point to mention is that categories like “Protected Capital”, “LS-Neutral”, and “Specific Strategy” that try to somehow work with hedging schemes indeed present the overall smallest correlations with respect to the tail risk of other subcategories. On the other hand, clearly “Balanced”, “Macro”, “Multi-strategy”, and “Fund of Funds” appear to be among the categories most exposed to tail risk, both at individual level as well as at aggregate level.\textsuperscript{10}

\textbf{Table 4: Tail Risk Correlation Table for Individual Hedge Fund Categories.}

<table>
<thead>
<tr>
<th></th>
<th>207</th>
<th>208</th>
<th>327</th>
<th>328</th>
<th>329</th>
<th>330</th>
<th>331</th>
<th>332</th>
<th>333</th>
<th>334</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>207</td>
<td>1.0000</td>
<td>0.3383</td>
<td>0.2264</td>
<td>0.1696</td>
<td>0.3176</td>
<td>0.1147</td>
<td>0.4479</td>
<td>0.5557</td>
<td>0.2994</td>
<td>0.1362</td>
<td>0.6978</td>
</tr>
<tr>
<td>208</td>
<td>0.3383</td>
<td>1.0000</td>
<td>0.0973</td>
<td>0.0426</td>
<td>0.1309</td>
<td>-0.0144</td>
<td>0.1981</td>
<td>0.2737</td>
<td>0.1356</td>
<td>0.1056</td>
<td>0.4637</td>
</tr>
<tr>
<td>327</td>
<td>0.2264</td>
<td>0.0973</td>
<td>1.0000</td>
<td>0.1798</td>
<td>0.1468</td>
<td>0.2325</td>
<td>0.1178</td>
<td>0.2517</td>
<td>0.1180</td>
<td>-0.0060</td>
<td>0.3842</td>
</tr>
<tr>
<td>328</td>
<td>0.1696</td>
<td>0.0426</td>
<td>0.1798</td>
<td>1.0000</td>
<td>0.1480</td>
<td>0.0951</td>
<td>0.0839</td>
<td>0.1034</td>
<td>0.1150</td>
<td>0.0997</td>
<td>0.2960</td>
</tr>
<tr>
<td>329</td>
<td>0.3176</td>
<td>0.1309</td>
<td>0.1468</td>
<td>0.1480</td>
<td>1.0000</td>
<td>0.0944</td>
<td>0.4721</td>
<td>0.2897</td>
<td>0.1285</td>
<td>0.1718</td>
<td>0.6579</td>
</tr>
<tr>
<td>330</td>
<td>0.1147</td>
<td>-0.0144</td>
<td>0.2325</td>
<td>0.0951</td>
<td>0.0944</td>
<td>1.0000</td>
<td>0.0077</td>
<td>0.1624</td>
<td>-0.0184</td>
<td>0.0698</td>
<td>0.4588</td>
</tr>
<tr>
<td>331</td>
<td>0.4479</td>
<td>0.1981</td>
<td>0.1178</td>
<td>0.0839</td>
<td>0.4721</td>
<td>0.0077</td>
<td>1.0000</td>
<td>0.4302</td>
<td>0.2259</td>
<td>0.1799</td>
<td>0.6618</td>
</tr>
<tr>
<td>332</td>
<td>0.5557</td>
<td>0.2737</td>
<td>0.2517</td>
<td>0.1034</td>
<td>0.2897</td>
<td>0.1624</td>
<td>0.4302</td>
<td>1.0000</td>
<td>0.2390</td>
<td>0.1553</td>
<td>0.6191</td>
</tr>
<tr>
<td>333</td>
<td>0.2994</td>
<td>0.1356</td>
<td>0.1180</td>
<td>0.1150</td>
<td>0.1285</td>
<td>-0.0184</td>
<td>0.2259</td>
<td>0.2390</td>
<td>1.0000</td>
<td>-0.0158</td>
<td>0.2948</td>
</tr>
<tr>
<td>334</td>
<td>0.1362</td>
<td>0.1056</td>
<td>-0.0060</td>
<td>0.0997</td>
<td>0.1718</td>
<td>0.0698</td>
<td>0.1799</td>
<td>0.1553</td>
<td>-0.0158</td>
<td>1.0000</td>
<td>0.3439</td>
</tr>
<tr>
<td>Total</td>
<td>0.6978</td>
<td>0.4637</td>
<td>0.3842</td>
<td>0.2960</td>
<td>0.6579</td>
<td>0.4588</td>
<td>0.6618</td>
<td>0.6191</td>
<td>0.2948</td>
<td>0.3439</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

\textsuperscript{10}Of course, our analysis is just of qualitative order since exposure to tail risk, with tail risk being clearly a nonlinear measure based on the left extreme of the returns’ distribution, can not be properly measured by a linear relationship like correlation.
Indeed, since our aggregate measure of tail risk is obtained by an equally-weighted average of the ten subcategories, by looking at the last line of the previous table we are able to understand which subcategories contribute more to the aggregate tail risk. We quickly observe that the categories Balanced, Macro, Multi-Strategy and Fund of Funds, are the ones with the largest exposure to the aggregate tail risk index, with correlations close to 70% (except for the Fund of Funds, which presents a correlation of 61.91%). On the other hand, categories like LS-Directional and Interest and Currency have much smaller correlations, of the order of 30%. The interpretation here is easy: In terms of contribution to the aggregate measure (or index) of hedge fund tail risk, LS-Directional and Interest and Currency are the most idiosyncratic categories presenting less dependent extreme behavior than the remaining categories.

We also compare our measures of hedge fund tail risk, both aggregate as well as for each individual category, to the corresponding Hellinger's implied measure of tail risk for the Ibovespa.

**Table 6: Tail Risk Correlations Between Individual Hedge Fund Categories and Ibovespa**

<table>
<thead>
<tr>
<th>Balanced</th>
<th>Protected Capital</th>
<th>LS-Neutral</th>
<th>LS-Directional</th>
<th>Macro</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.553</td>
<td>0.312</td>
<td>-0.044</td>
<td>0.044</td>
<td>0.105</td>
</tr>
<tr>
<td>Trading</td>
<td>Multi-Strategy</td>
<td>Funds of Funds</td>
<td>Interest and Currency</td>
<td>Specific Strategy</td>
</tr>
<tr>
<td>0.132</td>
<td>0.078</td>
<td>0.178</td>
<td>0.033</td>
<td>0.159</td>
</tr>
</tbody>
</table>

By looking at the table above, we can see that the “Balanced”, and “Protected Capital” strategies are the ones that present highest correlation with the Ibovespa tail
risk, all others presenting correlations smaller than 20%, and in most cases very close to zero. The aggregate measure of hedge fund tail risk has a 9.03 % correlation with the Ibovespa tail risk. Since those measures of tail risk capture in part the behavior of jumps expected by investors within each of these different series of returns, we can see that expectations are very different across hedge fund returns and the Ibovespa aggregate market return index. In part, this should be expected since hedge funds have in general dynamic portfolios generating trading strategies whose dynamics is much richer than that of an aggregate equity index. Therefore, a nonlinear transformation of the returns implied by such strategies may potentially be very different from the corresponding transformation obtained for the aggregate equity index.

All in all, the most important contribution of the present paper was to introduce a tail risk measure for hedge funds in Brazil. We believe that it is important for both managers and financial regulators to have an objective measure of tail risk. The former needs to know whether the high returns are present due to compensation for the probability of extreme events, which the manager eagerly wishes to avoid. And the latter is interested in constraining global risk to avoid huge market downfalls that might affect the aggregate economy. We find that our measure is able to fill in the
existing gap of measuring tail risk for these hedge funds in Brazil, especially considering the lack (and/or small number) of options data available for this market.

5 Conclusion

The results in this paper support the perception that hedge funds are structurally exposed to tail risk. We measure tail risk for Brazilian hedge funds building on the methodology proposed by Almeida and Garcia (2015). It relies on solving the dual problem of minimizing a discrepancy function. In our case, this discrepancy function was of the Cressie-Read type, with gamma = $-\frac{1}{2}$.

Solving this minimization problem, we find closed-form formulas for both Lagrangian weights$^{11}$ and stochastic discount factors for a sequence of moving windows. The SDFs are later used to compute risk neutral probabilities for each of the scenarios being considered. Then, based on the previously obtained RN probabilities, we build synthetic puts for each of the hedge fund subcategories to construct our tail risk measure.

Since Brazil's interest rate (one of our inputs) has changed a lot during the observed period, we have incorporated the risk-free interest rate time series into our analysis. This precludes the use of comparisons between different interest rate scenarios and solves the problem of having an interest rate sensitive tail risk.

The methodology that we have used to build our tail risk measure has one very important characteristic: It does not depend on options data. We use nothing but the daily returns on hedge funds. Therefore, it can be extended to calculate tail risk in different markets on which options data are not easily available. All in all, we have presented a very useful tool for both managers and regulators to control the exposure of portfolios and institutions to extreme events.

$^{11}$The Lagrange multipliers represent the weights given to each of the individual hedge fund categories when solving the dual maximization problem of the HARA utility function.
References


Appendix A

According to ANBIMA, hedge funds can be rated as:

1. Balanced: funds that seek long-term return by investing in a diversified portfolio (fixed income, stocks, exchange rate, etc.) with the possibility of rebalancing in the short-term.

2. Protected Capital: funds that seek return while protecting the principal invested, partially or totally. This type of fund does not admit leverage.

3. Specific Strategy: funds that adopt an investment strategy that implies specific risks, such as commodity risk or future index risk. Leverage is permitted.

4. Interest and Currency: funds that seek long-term return by investing on fixed income while admitting strategies that imply interest risk, price index risk and currency risk. Strategies that include exposure to variable income (stocks, etc.) are excluded. Leverage is permitted.

5. Long and Short - Directional: funds that operate assets and derivatives on the variable income market, using both long and short positions.

6. Long and Short - Neutral: funds that operate assets and derivatives, using both long and short positions, with the objective of keeping neutral exposure to the stock market.

7. Multi-strategy: funds that can adopt more than one investment strategy, without committing to a particular strategy. Leverage is permitted.

8. Macro: funds define strategies based on medium and long-term macroeconomic scenarios. These funds operate very diverse classes of assets and act in a directional manner. Leverage is permitted.

9. Funds of funds: have the objective of investing in more than one fund, managed by distinct people. The main competence of this type of fund is selecting other funds. Leverage is permitted.
10. Trading: funds that concentrate their investment strategies in different markets or asset classes, exploring opportunities originated by short-term movement on asset prices.

Appendix B

Below, we present the time series for daily returns for all the ten subcategories of hedge funds. We have separated them into two distinguished groups: those with lower range of return and those with higher range of return.

Amongst the funds with a narrow range of daily returns, we can find: Long and Short - Neutral, Long and Short - Directional, Funds of Funds and Interest and Currency.
Likewise, amongst the funds with a wider range of daily returns, we can find: Balanced, Protected Capital, Macro, Trading, Multi-strategy and Specific Strategy.
Last, but not least, we present the return time series for the IBOV Index.