Revisiting Modern Portfolio Theory

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Abstract

This paper revisits Modern Portfolio Theory and derives eleven properties of Efficient Allocations and Portfolios in the presence of leverage. With different degrees of leverage, an Efficient Portfolio is a linear combination of two portfolios that lie in different efficient frontiers - which allows for an attractive reinterpretation of the Separation Theorem. In particular a change in the investor risk-return preferences will leave the allocation between the Minimum Risk and Risk Portfolios completely unaltered - but will change the magnitudes of the tactical risk allocations within the Risk Portfolio. The paper also discusses the role of diversification in an Efficient Portfolio, emphasizing its more tactical, rather than strategic character.

Key Words: Modern Portfolio Theory, Separation Theorem, Leverage, Efficient Frontier, Strategic Allocation, Tactical Allocation.

JEL codes: G10, G11, G12, G14
1 Introduction

The main conclusions of Modern Portfolio Theory have been extensively discussed in the literature and are widely known and used in the practice of Asset Management. This paper revisits the topic and, in the spirit of Robert Merton (1972), re-derives the main conclusions of the theory by the usage of matrix algebra rather than graphical analysis. In addition, the paper also allows for different degrees of portfolio leverage and, in this sense, different efficient frontiers. By doing this, it brings new intuition to the traditional analysis and reinterprets the Minimum Variance and Risk Portfolios as, in practitioners’ jargon, Strategic and Tactical Portfolios – helping bring the theory and practice of Asset Management closer together.

The paper has two major conclusions. The first one is a reinterpretation of the Separation Theorem. Rather than defining an efficient portfolio \( P^* \) as a combination of a risk free rate and a Risk Portfolio at a given efficient frontier; this paper reinterprets the Efficient Portfolio as a combination of two efficient portfolios which, because of different degrees of leverage, lie in different efficient frontiers. The first of these portfolios is the Minimum Variance Portfolio, \( P^*_{\text{MIN}} \), which lies in a frontier where the sum of the portfolio weights of all the securities in the portfolio is exactly equal to the leverage of the portfolio. That is all the value of the portfolio \( P^* \) is actually invested in this Minimum-variance Portfolio. The second of these portfolios is the Risk Portfolio, \( P^*_{\text{RISK}} \), where all tactical allocations are made. The Risk Portfolio lies in a frontier where the sum of the portfolio weights is exactly equal to zero. That is, every long tactical position in the \( P^*_{\text{RISK}} \) portfolio is compensated by short tactical positions - in a manner that all positions end up summing up to zero. It is exactly in this sense that, in the day-to-day jargon of Asset Management, the Minimum Variance Portfolio would be the Strategic Portfolio - where the more stable allocations are made - whereas the Risk Portfolio – comprising of the over/underweight tactical allocations that sum up to zero – would be the Tactical Portfolio.

Still related to this first conclusion, the paper also shows that an increase (decrease) in the leverage of the efficient portfolio \( P^* \) will be completely allocated in the Minimum Variance or Strategic Portfolio, \( P^*_{\text{MIN}} \), leaving the tactical allocations in the Risk Portfolio \( P^*_{\text{RISK}} \) completely unaltered. On the other hand, a change in the return/risk preferences will increase the magnitude of the tactical allocations within the Risk Portfolio – leaving the
allocation in the Minimum Variance or Strategic Portfolio completely unchanged. This is in contrast with the traditional interpretation of the Separation Theorem, in which changes in investor’s return/risk preferences do alter the allocation between the Risk Free rate and the Risk Portfolio – as traditionally defined.

The second major conclusion of the paper has to do with Diversification. While the Strategic or Minimum Variance Allocation for each security in portfolio $P^*$ is solely a function of the variances and covariances of these securities, the Tactical Allocations within the Risk Portfolio are a linear function of the expected returns of all securities in the portfolio. In this sense, in order to take an over/underweight position on a single security, the investor needs to have a view - and actually take tactical positions - on all securities in the portfolio. That is, while minimum variance optimization usually implies a concentrated Strategic Portfolio, the Risk Portfolio will usually have a tactical allocation in most of the securities of the portfolio. The Efficient Portfolio $P^*$ thus, will be well diversified even in those circumstances where the Minimum Variance Portfolio is not.

The paper is organized as follows. Section II describes the investor’s problem in a mean-variance framework and then derives the first order conditions for an optimal allocation. Section III states and proves eight properties of an Efficient Allocation while Section IV deals with three properties of the Mean-variance Efficient Portfolio. Section V reinterprets the Separation Theorem in light of the properties of Efficient Allocations and Portfolios. Section VI discusses the new role for Diversification that arises within the Risk Portfolio– rather than the Minimum Variance. Finally, Section VII concludes.

2 The Investor`s Problem

Consider a portfolio $P$ consisting of $N$ securities with the expected return on the $i^{th}$ security denoted by $r_i$; the covariance between the returns on the $i^{th}$ and $j^{th}$ securities denoted by $\sigma_{ij}$, and the variance of the $i^{th}$ security denoted by $\sigma_{ii} = \sigma_i^2$.

Let $a_i$ be the percentage value of portfolio $P$ that is invested in security $i$. Under these definitions, the return and variance of portfolio $P$ are given by the following expressions:

$$1) R_p = \sum_{i=1}^{N} a_i r_i$$
2) \( \sigma_p^2 = \sum_{i=1}^{N} a_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} a_i \sum_{j=i+1}^{N} a_j \sigma_{ij} \)

The investor’s problem is to find a vector of asset allocations \( a_i; \ i = 1, \ldots, N \), that maximizes his/her utility subject to the constraint that the sum of these allocations is equal to the degree of leverage of the portfolio \( k \). That is:

3) \( \sum_{i=1}^{N} a_i = k \)

Where \( k \) is the sum of the weights of all assets within the portfolio. Notice that for \( k = 1 \), there is no leverage in the portfolio, while for \( k > 1 \) or \( k < 1 \) the portfolio has, respectively, a positive or negative degree of leverage.

The utility of the investor is defined over the expected return and the variance of the portfolio, with the derivatives of the utility function having the usual signs: \( U_1 \acute{>} 0; U_2 \acute{<} 0; U_{11} \acute{<} 0; U_{22} \acute{>} 0 \). That is, the utility function of the investor is given by:

4) \( U(E[R_p], \sigma_p^2) \)

Where \( E[R_p] \) is the expected return of the portfolio and:

\[
U_1 = \frac{\partial U}{\partial E[R_p]}; U_2 = \frac{\partial U}{\partial \sigma_p^2}
\]

\[
U_1 = \frac{\partial U^2}{\partial^2 E[R_p]}; U_2 = \frac{\partial U^2}{\partial^2 \sigma_p^2}
\]

The Lagrangian for this constrained utility maximization problem is given by:

\[
\Gamma(a; \lambda) = U(E[R_p], \sigma_p^2) + \lambda \left[ k - \sum_{i=1}^{N} a_i \right]
\]

With \( \lambda \) being the Lagrange multiplier

The standard first order conditions for this problem are a system of \( N+1 \) linear equations given by:

5) \( U_1 \frac{\partial E[R_p]}{\partial a_i} + U_2 \frac{\partial \sigma_p^2}{\partial a_i} - \lambda = 0 \hspace{1cm} i = 1, \ldots, N \)
3) \( k - \sum_{i=1}^{N} a_i = 0 \)

The system of equations in 5) states that, at an optimal allocation for security \( i \), the increase (decrease) in utility derived from a higher (lower) expected return for this security, has to be balanced by the decrease (increase) in utility derived from the higher (lower) volatility – the exact balance being equal to the Lagrange multiplier \( \lambda \). In addition, at an optimal, this exact balance has to be the same for all \( N \) securities in the portfolio.

Also notice that when \( \lambda = 0 \) and the maximization problem is unconstrained, the balance has to be an exact one in the sense that the utility gain (loss) derived from higher (lower) expected return on asset \( i \) has to be exactly equal to the utility gain (loss) from lower (higher) variance. On the other hand, when \( \lambda > 0 \) (\( \lambda < 0 \)) and the problem is constrained, the utility gain (loss) derived from higher (lower) expected return has to be greater than the utility gain (loss) from lower (higher) variance by the exact magnitude of the Lagrange multiplier \( \lambda \).

We can rewrite the first order derivation in 5) in the following manner:

\[
5') \quad \frac{\partial \sigma_p^2}{\partial a_i} - \frac{\lambda}{U_2'} = - \frac{U_1'}{U_2'} \frac{\partial E[R_p]}{\partial a_i} \quad i = 1, \ldots, N.
\]

Where we define, \((- U_1'/U_2')\) - the marginal rate of substitution between expected return and variance - as the risk-return preference parameter.

In addition, from equations 1) and 2), we have that:

6) \( \frac{\partial E[R_p]}{\partial a_i} = E[R_i] \quad i = 1, \ldots, N. \)

7) \( \frac{\partial \sigma_p^2}{\partial a_i} = 2 \left( a_i \sigma_i^2 + \sum_{j \neq i}^{N} a_j \sigma_{ij} \right) \)

Thus, we can rewrite the first order system for a maximum in 5') and 3) in matrix form in the following manner:
3.1 Property 1:

Every Optimal Allocation $a^*_i$, $i = 1, \ldots, N$ for any individual security in the portfolio, is a linear combination of two allocations: a Strategic or Minimum Variance Allocation, $a_{i,MIN}$, and a Tactical or Risk Allocation, $a_{i,RISK}$; with the linear coefficient of the Strategic Allocation, $a_{i,MIN}$, being the degree of leverage of the portfolio, $k$, and the linear coefficient of the Risk Allocation, $a_{i,RISK}$, being the return/risk parameter $U^1/U^2$.

Proof:
From Cramer’s rule we know that the optimal allocations \( a^*_i \) are given by:

\[
9) a^*_i = \frac{\text{Det}(Q_i)}{\text{Det}(Q)}, \quad i = 1, \ldots, N
\]

Where \( Q_i \) is the same as matrix \( Q \) with its \( i^{th} \) column substituted by the elements of the right-hand-side vector \( R \); and \( \text{Det}(\cdot) \) is just the determinant of a given matrix.

Now, by the properties of determinants, we can add a \( N + 1 \) columns of zeros to the \( i^{th} \) column and write the optimal solution \( a^*_i \) as:

\[
9') a^*_i = \frac{\text{Det}(Q_{i}^{0,k})}{\text{Det}(Q)} + \frac{\text{Det}(Q_{i}^{(-U_1',U_2')E[r_i],0})}{\text{Det}(Q)}, \quad i = 1, \ldots, N
\]

Where the matrix \( Q_{i}^{0,k} \) is the same as matrix \( Q \), with its \( i^{th} \) column substituted by a column of \( N \) zeroes in the first \( N \) rows and the degree of leverage of the portfolio, \( k \), in the \( N + 1^{th} \) row. Similarly, the matrix \( Q_{i}^{(-U_1',U_2')E[r_i],0} \) is the same as matrix \( Q \), with its \( i^{th} \) column substituted by a column of \( N \) terms, \( (-U_1'/U_2')E[r_i] \), \( i = 1, \ldots, N \), in the first \( N \) rows, and a zero in the \( N + 1^{th} \) row.

One can once again use the properties of determinants to factor out the \( k \) term in the \( N + 1^{th} \) row, and \( i^{th} \) column of \( \text{Det}(Q_{i}^{0,k}) \) and factor out the risk/return preference parameter \( (-U_1'/U_2') \), in the first \( N + 1 \) rows of the \( i^{th} \) column of \( \text{Det}(Q_{i}^{(-U_1',U_2')E[r_i],0}) \). That is:

\[
9'') a^*_i = k \frac{\text{Det}(Q_{i}^{0,1})}{\text{Det}(Q)} + \frac{U_1'}{U_2'} \frac{\text{Det}(Q_{i}^{(-U_1',U_2')E[r_i],0})}{\text{Det}(Q)}, \quad i = 1, \ldots, N
\]

Where the matrix \( Q_{i}^{0,1} \) is the same as matrix \( Q \) with its \( i^{th} \) column substituted by a column of \( N \) zeroes in the first \( N \) rows and the number one in the \( N + 1^{th} \) row. In addition, the matrix \( Q_{i}^{E[r_i],0} \) is the same as matrix \( Q \), with its \( i^{th} \) column substituted by a column of \( N \) terms, \( E[r_i] \), \( i = 1, \ldots, N \), in the first \( N \) rows, and a zero in the \( N + 1^{th} \) row.

Thus, if one defines:
The optimal allocation for security $i$ will be given by the following expression:

$$11) \ a_i^* = k a_i^{MIN} + \frac{U_1'}{U_2'} a_i^{RISK}$$

### 3.2 Property 2:

*The Strategic or Minimum Variance Allocation, $a_i^{MIN}$, $i = 1,\ldots, N$, for any individual security in the portfolio, is the solution to the problem of minimizing the variance of the portfolio, subject to the constraint that $k = 1$.*

**Proof:**

Consider the problem of minimizing the variance of the portfolio, $\sigma_p^2$, subject to the constraint that the leverage of the portfolio is equal to one. The Lagrangian for this problem is given by:

$$\Gamma(a_i; \lambda) = \sigma_p^2 + \lambda \left[ 1 - \sum_{i=1}^{N} a_i \right]$$

Where $\lambda$ is the Lagrange multiplier for this problem.

The first order conditions for a minimum is a system of $N + 1$ equations given by:

$$12) \quad \frac{\partial \sigma_p^2}{\partial a_i} - \lambda = 0$$

$$13) \quad 1 - \sum_{i=1}^{N} a_i = 0$$

We can write the $N + 1$ system of first order conditions in matrix notation as:

$$13) \quad Q \cdot A^* = Z$$
Where the coefficient matrix $Q$ is defined exactly as before. The parameter vector, $A'$ on the other hand, is exactly equal to vector $A$ in equation 8) except for the $N + 1^{th}$ row, where we now have the term $\lambda'$ rather than $-\lambda/2U'.2$. The right-hand-side vector $Z$ is a ($N + 1, 1$) vector, with zeroes as elements in its first $N$ rows, and one as the element of its $N + 1^{th}$ row.

By Cramer’s rule, we know that the solution to 13) is given by:

$$a_{i, MIN}^{\sigma_p^2} = \frac{Det(Q_i^2)}{Det(Q)}$$

Where $Q_i^2$ is the same as matrix $Q$ with its $i^{th}$ column substituted by vector $Z$.

However, this is exactly the definition of matrix $Q_{i,0.1}$ in equation 10). That is, $a_{i, MIN}^{\sigma_p^2} = a_{i,MIN}$ as we wanted to prove.

### 3.3 Property 3:

*The Risk Allocation* $a_{i, RISK}$, $i=1........N$ for any individual security in the portfolio is the solution to the problem of minimizing the variance of the portfolio subject to the constraint that the leverage of the portfolio, $k$, equals zero and the Lagrange multiplier for the constraint $R = E[R_p]$ equals one.

**Proof:**

Consider the problem of minimizing the variance of the portfolio, $\sigma_p^2$, subject to the constraint that the leverage of the portfolio, $k$, equals to zero and that the expected return on the portfolio $E[R_p]$ equals to $\bar{R}$.

The Lagrangian for their problem is given by:

$$\Gamma(a_i; \lambda; \gamma) = \sigma_p^2 + \lambda \left[ 0 - \sum_{i=1}^{N} a_i \right] + \gamma \left[ \bar{R} - E[R_p] \right]$$

The first order condition for a minimum is a system of $N+2$ equations given by:

$$14) \frac{\partial \sigma_p^2}{\partial a_i} + \lambda = \gamma E[R_i] \quad \text{for } i=1........N$$
\[ 3^{'''} \) \[ 0 = \sum_{i=1}^{N} a_i \]

15) \[ \bar{R} - E[R_p] = 0 \]

Using matrix notation, we can write the system for the first \( N+1 \) equations in 14) and \( 3^{'''} \) as

\[ 16) Q \cdot A'' = R' \]

Where the coefficient matrix \( Q \) is as defined before. The parameter vector \( A'' \) is exactly equal to the \( A \) vector described in equation 8) except for its \( N+1 \) th term, \( \lambda'' \) (rather than \( -\frac{\lambda}{2U'_2} \)). The right hand side vector \( R' \) is exactly the same as \( R \) vector except that we substitute the terms in the \( N \) first rows by \( \gamma E[r_i] \) (rather than \( (-U'_2/U'_2)E[r_i] \)) and in the \( N+1 \) th row we substitute the leverage of the portfolio \( k \), by the new leverage \( k = 0 \).

By Cramer’s rule we know that the solution to the 16) is given by:

\[ a_{i,MIN\gamma^*, k=0} = \frac{\text{Det}(Q_{x[i],0})}{\text{Det}(Q)} \quad i=1, \ldots, N \]

Where \( Q_{x[i],0} \) is the same as matrix \( Q \) with its \( i \) th column substituted by a column of \( N \) adjusted expected returns \( \gamma E[r_i] \) and a zero term in the \( N+1 \) row.

We can now use the property of determinants to factor out the \( \gamma \) from the \( i \) th column to get

\[ a_{i,MIN\gamma^*, k=0} = \gamma \frac{\text{Det}(Q_{x[i],0})}{\text{Det}(Q)} \quad i=1, \ldots, N \]

And, as we want to show, setting \( \gamma = 1 \) gives the exact expression for \( a_{RISK} \), in equation 10).
3.4 Property 4:

The Strategic, or Minimum Variance Allocation, $a_{i,\text{MIN}}$, $i = 1, \ldots, N$, of any individual security in the portfolio is independent of the expected returns of all securities in the portfolio, depending only on the variance and covariance of these securities.

Proof:

From equation 10), the Strategic or Minimum Variance Allocation for the $N$ securities in the portfolio is given by:

$$ a_{i,\text{MIN}} = \frac{\text{Det}(Q_{i,1}^{0,1})}{\text{Det}Q} \quad \text{for } i = 1, \ldots, N $$

Where matrix $Q$, as described before, is a bordered matrix, in which the $(N; N)$ covariance matrix is borded by a column and a row of ones and zeros. So, matrix $Q$ does not possess any expected returns as its components but only variances and covariance terms.

Similarity, the matrix $Q_{i,1}^{0,1}$ is the same as matrix $Q$ with its $i$th column substituted by a column of zeroes and ones. It also does not possess any expected return terms as its components; but only variance and covariance terms.

That is, as we wanted to show, all Minimum Variance Allocations $a_{i,\text{MIN}}$, $i = 1, \ldots, N$ are independent of expected returns and depend only on the variances-covariances of the securities in the portfolio.

3.5 Property 5:

The Tactical, or Risk Allocation of any individual security in the portfolio, is a linear function of the expected returns of all assets in the portfolio - with the linear coefficient depending only on the variances and covariances of these securities. In addition, for asset $i$, the linear coefficient on the $i$th expected return is positive so that an increase (decrease) in the expected return of asset $i$, $\mathbb{E}[r_i]$ will increase (decrease) the magnitude of the Tactical Allocation $a_{i,\text{RISK}}$. 
Proof:

From equation 10) the Tactical or Risk Allocation for security $i$ is given by the following expression:

$$a_{i,RISK} = \frac{\text{Det}(Q_i^{E[r_i],0})}{\text{Det}(Q)} \quad i=1 \ldots \ldots N$$

Where $Q_i^{E[r_i],0}$ is the same as matrix $Q$ with its $i$th column substituted by the right-hand-side vector $R$; $R$ being a column vector composed of $N$ expected returns $E[r_i]; i = 1, \ldots , N$ and with a zero at its $N+1$th row term.

By the property of determinants, one can write the determinant of matrix $Q_i^{E[r_i],0}$ in the following manner:

$$\text{Det}(Q_i^{E[r_i],0}) = E[r_i]\text{Det}(Q_{i,i}) + E[r_{i+1}]\text{Det}(Q_{i,i+1}) + \ldots + E[r_N]\text{Det}(Q_{N,i}) + 0.\text{Det}(Q_{N+1,i})$$

Where the $Q_{j,i}$ terms are the minors $(i,j)$ of matrix $Q$. These minors, however, do not depend on the expected returns of the securities in the portfolio $E[r_i]$, as the $i$th column is eliminated when calculating the determinant of matrix $Q_i^{E[r_i],0}$, and depend only on the variance-covariance terms. In addition, matrix $Q$ also does not depend on the expected returns, depending only on the variance-covariance terms.

That is, as we wanted to show, all Tactical or Risk allocations, $a_{i,RISK}, \ i = 1, \ldots, N$, are linear functions of all the expected returns on the individual securities of the portfolio, with the linear coefficients depending only on the variance-covariance terms.

Finally, let’s show that the linear coefficient of the $E[r_i]$ term, $\text{Det}(Q_{i,i})/\text{Det}(Q)$, is positive – where $Q_{i,i}$ is the minor $(i,i)$ of matrix $Q$. For this, notice that $Q_{i,i}$ is exactly matrix $Q$ without the $i$th security. That is, it is exactly as if portfolio $P$ had one less security - in this case security $i$. In this sense, for the maximization problem to be well defined, for both portfolio $P$ or for portfolio “P-without-the-ith-security”, we would need $\text{Det}(Q)$ to have the same sign as $\text{Det}(Q_{i,i})$. Thus, as we wanted to show, the linear coefficient of the $E[r_i]$ term, for the Tactical or Risk Allocation of security $i$, is positive.
3.6 Property 6:

The Tactical or Risk Allocation, of any individual security in the portfolio can be written as a linear function of its expected excess returns with respect to a Benchmark Security also in the portfolio - with the linear coefficients depending only on the variances and covariances of the securities in the portfolio. In addition, for asset i, the linear coefficient on the \(i\)th expected excess return will be positive, so that an increase (decrease) in the expected excess return of asset i with respect to the Benchmark Security, will increase (decrease) the Tactical Allocation on the \(i\)th asset.

Proof:

Define a Benchmark Security \(B\) in portfolio \(P\).

By the property of determinants, we can subtract row \(i = B\) in matrix \(Q^{[\mathbf{r}]0}\) from all other rows \(i = 1, \ldots, N, \) for \(i \neq B\), while leaving row \(N + 1\) unaltered. Notice that, by doing this, all elements of the \(i\)th column of matrix \(Q^{[\mathbf{r}]0}\) will become a function of the expected excess returns with respect to the benchmark asset \(B\), \(E[r_i - r_B]\) - except for the \(i = B\) row, which will remain \(E[r_B]\).

In addition, by subtracting row \(i = B\) from rows \(i = 1, \ldots, n, i \neq B\); then, the \(N + 1\)th column will become a column of zeros, except for the term in the \(B\)th row, which will actually be a one. So, if we now calculate the determinant from the \(B\)th column we can derive the desired result.

To show that the coefficient on the \(i\)th return, \(E[r_i - r_B]\) is positive, we proceed as in Property 5 when we showed that the sign on \(\text{Det}(Q_{i,i})\) is the same as the sign on \(\text{Det}(Q)\). This is done by simply remembering that, by the properties of the determinants, if one subtracts row \(B\) from every row \(i = 1, \ldots, n, i \neq B\) of matrix \(Q_{i,i}\), it does not alter \(\text{Det}(Q_{i,i})\). That is, its sign remains unchanged and still equals to the sign of \(\text{Det}(Q)\). In other words, the coefficient \(\text{Det}(Q_{i,i})/\text{Det}(Q)\) still has a positive sign, as we wanted to show.
3.7 Property 7:

If all securities in the portfolio have the same expected return, then the Tactical or Risk Allocation for all securities in the portfolio will be equal to zero and the Optimal Allocation will be equal to the Minimum Variance Allocation.

Proof:

This is immediate from Property 6, which states that the Tactical or Risk Allocation of all securities in the portfolio can be written as a linear function of the excess returns with respect to a benchmark asset. That is:

\[ a_{i,RISK} = \phi_{i,1} E[r_1 - r_B] + \phi_{i,2} E[r_2 - r_B] + \ldots + \phi_{i,n} E[r_n - r_B] \], \quad i = 1, \ldots, N

Thus, if \( E[r_i] = E[r_j] = E[r_B] \) for all \( i, j = 1, \ldots, N \), then \( a_{i,RISK} = 0 \) for all \( i \), as we wanted to show.

3.8 Property 8:

If there is, in the portfolio, a security with zero volatility then the Strategic or Minimum Risk Allocation will be equal to one for this security and zero for all other securities in the portfolio.

Proof:

Assume a Risk Free security \( F \) with zero volatility. This implies that, in matrix \( Q \), the row and column \( F \), will be a row and column with \( N \) zero elements, \( \sigma_{F,j} = \sigma_{i,F} = 0 \); \( i, j = 1, \ldots, N \), and a one as its \( N^{th} \) element; so that \( Det(Q_F^{0,1}) = Det(Q) \). Thus, from equation 10), \( a_{F,MIN} = 1 \).

In addition, for all other securities in the portfolio, \( i \neq F \), columns \( i \) and \( F \) of matrix \( Q_F^{0,1} \) will be exactly the same. Thus \( Det(Q_i^{0,1}) = 0 \) and, by equation 10), \( a_{i,MIN} = 0 \), for all \( i \neq F \); as we wanted to prove.
4 Properties of an Optimal Portfolio

Given the properties of an Efficient Allocation, one can now derive the properties of an Optimal Portfolio.

4.1 Property 9:

A Mean-variance Efficient Portfolio is a linear combination of two portfolios: a Strategic or Minimum Variance Portfolio, in which all the value of the portfolio (leveraged or not) is allocated; and a Tactical or Risk Portfolio, in which the magnitude of the Tactical Allocations – that sum to zero – depend on the magnitude of the risk/return preferences of the investor.

Proof:

By Property 1, the optimal allocation is given by:

\[ a_i^* = k a_{i,MIN} + \frac{U_1}{U_2} a_{i,RISK}, \quad i = 1, \ldots, N \]

One can define a Strategic or Minimum Variance Portfolio \( P_{MIN} \), with \( \sum_{i=1}^{N} a_{i,MIN} = 1 \), and with expected return and variance defined by the following equations:

\[ 1') R_{MIN} = \sum_{i=1}^{N} a_{i,MIN} r_i \]

\[ 2') \sigma_{MIN}^2 = \sum_{i=1}^{N} a_{i,MIN}^2 \sigma_i^2 + 2 \sum_{i=1}^{N} a_{i,MIN} \sum_{j=i}^{N} a_{j,MIN} \sigma_{ij} \]

Similarly one can define a Tactical or Risk Portfolio \( P_{RISK} \), with \( \sum_{i=1}^{N} a_{i,RISK} = 0 \), and with expected return and variance defined by the following equations:

\[ 1'') R_{RISK} = \sum_{i=1}^{N} a_{i,RISK} r_i \]

\[ 2'') \sigma_{RISK}^2 = \sum_{i=1}^{N} a_{i,RISK}^2 \sigma_i^2 + 2 \sum_{i=1}^{N} a_{i,RISK} \sum_{j=i}^{N} a_{j,RISK} \sigma_{ij} \]
Thus, from 11), the Return and Variance of the Efficient Portfolio $P^*$, can be written as functions of portfolios $P_{MIN}$ and $P_{RISK}$ in the following manner:

$$17) R^*_p = kR_{MIN} + \left( -\frac{U'_1}{U'_2} \right) R_{RISK}$$

This is what we wanted to prove.

In addition, one can calculate the variance of the Efficient Portfolio as a function of the variances and covariances of the Minimum Variance and Risk Portfolios as follows:

$$18) \sigma^2_p = k^2 \sigma^2_{MIN} + \left( -\frac{U'_1}{U'_2} \right)^2 \sigma^2_{RISK} + 2k \left( -\frac{U'_1}{U'_2} \right) \sigma_{MIN;RISK}$$

Where $\sigma_{MIN;RISK}$ is the covariance between the returns of the Minimum Variance and Risk Portfolios.

4.2 Property 10:

The Efficient Portfolio, as well as the Minimum Variance Portfolio and the Risk Portfolio, are all efficient frontier portfolios.

Proof:

The problem of finding a given frontier portfolio is a quadratic optimization exercise in which one minimizes the portfolio variance subject to the constraints that the portfolio weights sum to a constant $k$ and the expect return of the portfolio equals a constant $\bar{R}$.

The Lagrangian for this problem is given by:

$$\Gamma(a_i; \lambda) = \sigma^2_p + \lambda \left[ k - \sum_{i=1}^{N} a_i \right] + \gamma \left[ -\bar{R} - E[R_p] \right]$$

Where $\lambda$ and $\gamma$ are the Lagrange multipliers for this problem.

The first order conditions comprise a system of $N + 2$ equations given by:

$$14) \frac{\partial \sigma^2_p}{\partial a_i} + \lambda \frac{\partial}{\partial a_i} \left[ k - \sum_{i=1}^{N} a_i \right] + \gamma \frac{\partial}{\partial a_i} \left[ -\bar{R} - E[R_p] \right] = 0 \quad \text{for } i=1, \ldots, N$$
To derive the efficient frontier, one varies the constraint on the expected return of the portfolio $\tilde{R}$, calculates the effects on the optimal portfolio weights $a^*_i, i=1\ldots N$ - given the constraint on the sum of these portfolio weights - and then maps these on the optimal variance and expected return of the portfolio; finally illustrating the frontier in the mean-standard deviation space.

Three things to notice. First, the Efficient Allocations $a^*_i, i=1\ldots N$, of Property 1, are exactly the solution to this problem when $\gamma = \left(-U'_1/U'_2\right)$. Secondly, the Strategic or Minimum Risk Allocations, $a_{i,MIN}, i=1\ldots N$, of Property 2, are also the solution to this exact problem, in this case when $\gamma = 0$. Finally, the Tactical or Risk Allocations, $a_{i,RISK}, i=1\ldots N$ of Property 3, are also the solution to this exact problem, but when $\gamma = 1$ and $k = 0$. That is, all three portfolios are frontier portfolios – as we wanted to prove – but, by having a different degree of leverage, $k$, the Tactical or Risk Portfolio lies in a frontier that is different than that of the Efficient and Minimum Variance Portfolios.

4.3 Property 11:

An increase (decrease) in the degree of leverage of the Efficient Portfolio will all be allocated in the Strategic or Minimum Variance Portfolio, and none in the Risk or Tactical Portfolio. In addition, it will leave the magnitudes of the Tactical Allocations in the Risk Portfolio unaltered. On the other hand, an increase (decrease) in the return-risk presence of the investor will not alter the allocations either in the Minimum Variance or in the Risk Portfolios. It will, however, increase (decrease) the magnitudes of the Tactical Allocations within the Risk Portfolio.

Proof:

This is evident from the solution for the Efficient Allocation in equation 11)
\[ 11) \quad \alpha_i^* = k\alpha^*_{\text{MIN}} + \frac{U_1}{U_2} \alpha^*_{\text{RISK}}, \quad i = 1, \ldots, N \]

Thus, an increase (decrease) in \( k \) will increase (decrease) the allocation in the Minimum Variance Portfolio while leaving the allocation in the Risk Portfolio, as well as the magnitude of the tactical positions within this portfolio, completely unaltered.

On the other hand, a change in the return risk preferences \( -\left(\frac{U'_1}{U'_2}\right) \) will not alter the asset allocation between the Minimum Variance and Risk Portfolios – as \( \sum_{i=1}^{N} \alpha_{i,\text{RISK}} = 0 \) - but will alter the magnitude of the Tactical Allocations within the Risk Portfolio.

5 The Separation Theorem Reconsidered

Property 9, states that a Mean-variance Efficient Portfolio is a linear combination of two other portfolios; the Strategic or Minimum Variance Portfolio and the Tactical or Risk Portfolio. In the Finance literature this property is traditionally called the Separation Theorem – a name conveying the notion that the optimal allocation decision for the \( N \) assets within the portfolio can, in fact, be separated into a simpler allocation decision between two portfolios.

There are a few things to notice about the Separation Theorem as stated in Property 9. The first one is that the problem defined in Section 1) - that of maximizing an Utility function subject to the constraint that the sum of the shares of all securities in the portfolio equals \( k \) - can actually be divided into three separated problems: (i) the problem of minimizing the variance of the portfolio subject to the constraint that all Wealth – leveraged or not - is allocated in the portfolio. (ii) the problem of minimizing the variance of the portfolio subject to an Expected Return constraint as well as the constraint that the portfolio is self-financed – so that all wealth allocated to the portfolio is equal to zero \( (k = 0) \), (iii) and finally the problem of equating the shadow price of the Expected Return constraint in 15), \( \gamma \), to the return-risk preference parameter \( -\left(\frac{U'_1}{U'_2}\right) \).

In other words, by Property 9 - and using Asset Management terminology - the solution to the Utility Maximization Problem in Section 1) can be divided in three parts: (i) the definition of a Strategic Portfolio, in which all the Wealth of the portfolio – leveraged of
not – is allocated. This would be the Minimum Variance Portfolio, (ii) the definition of a Tactical Portfolio, in which the sum of the over/underweight positions will all be equal zero. This would be the Risk Portfolio (iii) the calibration of the magnitudes of the Tactical allocations within the Risk Portfolio, according to the investor’s return-risk parameter $(-U_1/U_2)$.

Notice that the above decomposition – or separation – of the Utility Maximization problem in Section 1 is in contrast to the traditional separation typically stated in Finance textbooks. There, the Minimum Variance Portfolio is actually the Risk Free Rate and the Risk Portfolio is comprised of all the other securities in the portfolio. In addition, the adjustment to the risk-return parameter $(-U_1/U_2)$ is actually made by transferring Wealth between the Risk Free Rate and the Risk Portfolio – rather than keeping all Wealth allocated in the Minimum Variance Portfolio and merely adjusting the tactical allocations within the Risk Portfolio - as in Proposition 11.

The reason for this difference has to do with how “separation” is carried out in each approach. In the traditional Finance Textbooks, once a Risk Free Rate exists, the Variance-covariance matrix of the portfolio cannot be inverted. And the solution to this is simply to take the Risk Free Rate out of the portfolio and then carry on with the optimization problem for the remaining $N-1$ securities; which would result in the Risk Portfolio – a portfolio comprised of all securities that are not the Risk Free Rate. The Risk Portfolio would then be combined with the Risk Free Rate, in accordance to the investor’s risk-return preferences, to form the Efficient Portfolio.

In this paper, however, “separation” is done in a different manner. Even in the presence of a Risk Free Rate, matrix $Q$ in equation 8) is a bordered Variance-covariance matrix and can still be inverted. So the Risk Free Rate remains in the portfolio. Separation is then done by first minimizing the variance of a portfolio in which all the investor’s Wealth is allocated; and then, secondly, by minimizing the variance of a portfolio where the all Tactical Allocations sum to zero – but subject to an Expected Return constraint. Thirdly, according the investor’s Risk Return preferences, the magnitudes of the Tactical Allocations within the Risk Portfolio are then calibrated. Notice that if there is a Risk Free Rate, then as stated in Proposition 8, all the investor’s Wealth would be allocated in it, with the remaining securities being part only of the over/underweight decisions within the Tactical or Risk Portfolio.
6 The Role of Diversification

According to Property 5, the Risk allocation of a given security in the portfolio will be a linear function of the Expected Return of all $N$ securities in the portfolio – not only of its own Expected Return. This is in contrast to the Minimum Variance allocation, which, according to Property 4, will depend only on the variance-covariances among the $N$ securities of the portfolio. Thus, while the Minimum Variance Portfolio can be concentrated – as, for example, in the presence of a Risk Free security - the Risk Portfolio will continue to be diversified; as will then be the Efficient Portfolio.

Let’s take a closer look at this Diversification argument. Firstly, suppose that the investor expects all securities in the portfolio to have the same exact expected return. Then, by Property 7, there will be zero tactical allocations in the Risk Portfolio and the optimal allocation, for all securities, will be the Minimum Variance Allocation. Secondly, suppose that the investor believes that there is only one single security in the portfolio, say security $i$, with a positive (negative) excess return. In this case, by Property 5, the Risk Allocation in this security will be an overweight (underweight). However, still by Property 5, the Risk allocation for all other $N-1$ securities in the portfolio will also be different from zero - as they all are a linear functions of the Expected Return on security $i$. In other words, it suffices for the investor to have a view on the excess return of one single security in the portfolio, for the Risk Portfolio to be well diversified. Portfolio Diversification is thus, the efficient solution for taking risk even for the case in which the investor has a view on the Expected Return of one single security.

The above argument clarifies an important practical issue regarding Diversification. While Finance Theory argues in favor of a well-diversified portfolio, as a mean of reducing idiosyncratic risk; the truth is that a variance minimizing exercise usually results in fairly concentrated portfolios. For instance, a rule of thumb in the practice of Asset Management is that a portfolio comprising of 10 to 15\(^1\) securities would already be sufficiently diversified; with the volatility reduction achieved by further increasing the number of securities in the portfolio being only marginal.

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\(^1\) See Evans and Archer (1968). More recent work by Statman (1987) argues in favor of 30 to 40 securities for a well-diversified portfolio, whereas the work of Campbell et al (2001) defends that the number of assets in a well-diversified portfolio have increased in the USA.
Property 5, however, defines a new role for Diversification in an Efficient Portfolio. Rather than simply reducing portfolio volatility; Diversification is also an important instrument to efficiently increase the risk of a portfolio. In this sense, an over/underweight tactical position in a given security, would be financed by an under/overweight tactical position in many other securities.

7 Conclusion

By using matrix algebra rather than figures, and by allowing for the presence of leverage, this paper derives eleven properties of an Efficient Portfolio. In particular, a Mean-variance Efficient Allocation is a combination of two Efficient Portfolios – each on a different frontier: a Minimum Variance or Strategic Portfolio, where all the leverage of the portfolio is allocated, and a Risk or Tactical Portfolio, where all the tactical over/underweight decisions - that sum to zero - are allocated.

An increase in the leverage of the Efficient Portfolio will be all allocated in the Minimum Variance Portfolio, leaving unchanged the over/underweight allocations in the Risk Portfolio. In contrast, a change in the return-risk preference of the investor will only alter the magnitude of the over/underweight allocations in the Risk Portfolio – which will continue to sum to zero – without altering the allocation in the Minimum Variance Portfolio. This conclusion yields, in a sense, a different perspective to the Separation Theorem, which is traditionally derived for portfolios with the same leverage and, thus, in the same efficient frontier. In this traditional approach, a change in the return-risk preferences would only change the allocation between the Risk Free Rate and the Risk Portfolio, without altering the allocation of securities within the Risk Portfolio. Under the new interpretation, however, the allocation between the portfolios remains unaltered - but the magnitudes of the tactical allocations within the Risk Portfolio are all changed.

This paper also illustrates a different role for diversification – in addition to the traditional role of minimizing the portfolio variance. Within the Risk Portfolio, diversification is simply a manner to efficiently implement a tactical allocation movement and, in fact, if the investor possesses a distinct view on a single excess return, the Risk Portfolio will be diversified.
The paper also derives a few other properties of a Mean-variance Efficient Portfolio:

(i) The Minimum Variance Allocation for any given security in the portfolio, is only a function of the variances and covariances and does not depend on the securities expected returns; (ii) The Risk Allocation for any given security is a linear function of the expected returns of all securities in the portfolio, and will depend positively on the expected return of that security itself. (iii) The Risk Allocation for any given security can be written in terms of excess expected returns of all securities in the portfolio, with respect to a benchmark security also in the portfolio; and will depend positively on the expected excess return of that security itself (iv) if the expected return of all securities in the portfolio are the same, the Risk Allocations will be zero and the optimal allocation will be the Minimum Variance Allocation for all securities in the portfolio (v) if there is a Risk Free security in the portfolio, the Minimum Variance Allocation for this security will be exactly equal to one and will be zero for all other securities in the portfolio. That is, for the Risk Free Rate, the Efficient Allocation will be exactly equal to the Minimum Variance Allocation; while for all other securities in the portfolio it will be exactly equal to the Risk Allocation. (vi) The Minimum Variance as well as the Risk Portfolio are also Efficient Portfolios; but because of different degrees of leveraging, they lie on different efficient frontiers.
References


