Asymmetric Auctions and Risk Aversion Within Independent Private Values.

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Contests with Many Heterogeneous Agents*

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Abstract

We study tournaments with many ex-ante asymmetric (heterogeneous) contestants as an independent-private-values all-pay auction. The asymmetry is either with respect to the distribution of valuations for the prize or the risk preferences. By characterizing equilibria in monotone strategies we show that tournaments with many heterogeneous contestants are qualitatively distinct. First, with two (or many ex-ante identical) participants, a contestant always exerts some effort with positive probability. In contrast, with many asymmetric participants, one might not exert any effort at all, even if there is a positive probability that he has the highest valuation among all. Second, in tournaments with two (or many ex-ante homogenous) contestants, equilibrium effort densities are decreasing. This prediction is at odds with experimental evidence that shows the empirical density might be increasing at high effort levels. With many heterogeneous contestants, however, the increasing bid density is consistent with an equilibrium behavior.

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Key words: all-pay auctions, tournaments, asymmetric bidders

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1 Introduction

It is hard to imagine an area of human activity that does not involve contests. Students striving to be the best in their class, employees awaiting promotion, sportsmen fighting for a gold medal, R&D firms racing to capture monopoly profits, researchers competing for grants — all can be viewed as players in games with a single winner. Typically, the rest of the participants are losers, who have to absorb the cost of the invested effort. To formulate predictions of individual behavior in such environments, we analyze equilibria of all-pay auctions with more than two heterogeneous agents.

Indeed, rarely do the contestants look alike: their background, previous experience, gender, age and other observable characteristics vary, and so, each might be perceived as being different from his opponents. In particular, the contestants might be viewed as having distinct risk attitudes or different valuations for the prize. However, a precise value attached to the prize by a rival is usually hard to induce from the observables, so that some residual uncertainty still remains. In-line with most of the literature on the subject, we assume that contestants view the values of others as random variables, every two contestants agree on the distributions of values for the rest of the players, and this is common knowledge.

Ex-ante heterogeneity, being a good positive assumption, has strong implications on equilibrium behavior. We show that if there are more than two contestants, some potential participants might decide to exert zero effort, no matter what their valuation for the prize is, even though there is a positive chance that they value (desire) the prize the most of all those who take part in the contest, that is, the model might generate a complete drop-out behavior. However, this is inconsistent with an equilibrium\(^1\) in a contest with ex-ante symmetric contestants,\(^2\) or if there are only two participants.\(^3\) We also show that in those cases effort density should be decreasing in equilibrium, this conclusion being no longer true in contests with more than two heterogeneous participants: different distributions of abilities or different risk attitudes might cause some to choose highest effort more frequently than the intermediate levels of effort, generating the so-called workaholic behavior. These findings are in accord with the observation that:

\(^1\)In all statements ‘equilibrium’ stands for ‘equilibrium in monotone strategies’.
\(^2\)As in Gavious et al. (2002) or Fibich et al. (2006).
\(^3\)As in Amann and Leininger (1996).
"many organizations are characterized by a bifurcation of effort among workers. While one subset appears to not be able to stop themselves from working (the fast track) the other seem alienated and exerts no effort at all." Müller and Schotter (2007, p. 2).

Recent experimental literature paid close attention to individual behavior in contests, providing some support for bimodality (bifurcation) of effort distributions, thus, replicating 'workaholics' and 'drop outs' in actual organizations. In experiments with six subjects, whose valuations (or marginal cost of effort) are uniformly distributed, Müller and Schotter (2007) and Nousair and Silver (2006) report that a significant number of subjects display bimodal empirical distributions of effort, with a high mode located near the lowest effort level and, a smaller mode near the highest effort level.4

Most importantly, the experimental literature stresses that individual behavior should not be subsumed under the aggregate behavior.5 It must be time to provide more general theoretical foundations for the experimental work by accounting for bidder heterogeneity.

So, we present the model next, characterize an equilibrium in section 2.1, and formulate our main results in section 3. Proofs omitted in the text are collected in the appendix, additional related literature is cited in the text.

2 The Model

There are \( N \geq 2 \) individuals competing for a prize. The prize is allocated to the contestant who demonstrates the top performance or achieves the best result. We assume that one's performance fully reflects own effort. Simply put, effort is observable. The contestants have different values associated with receiving the prize, or the desire to win. The payoff to the winner,
who exerts costly effort $b \geq 0$, is $u_k(v_k + w - b)$, while the losers get $u_j(w - b)$, $j \neq k$, where $w > 0$ is the initial wealth of a contestant, substantial enough, so that she is never resource-constrained, $w > \max_i \bar{v}_i$, with $\bar{v}_i$ being $i$'s highest possible valuation of the prize. We assume the contestants are weakly risk averse with $u_i : \mathbb{R} \to \mathbb{R}_+$, twice differentiable, strictly increasing and concave.

When deciding on one’s effort, contestant $i$ knows his desire to win, $v_i$. In the eyes of all, values of the rivals are distributed independently, but not necessarily identically, $V_i \sim F_i$ on $[\underline{v}, \overline{v}_i]$, where $\underline{v} \geq 0$, with corresponding density $f_i$, which is continuous and is bounded away from zero for all $v \in [\underline{v}, \overline{v}_i]$. To choose an optimal level of effort, or, simply, a bid, $b$, contestant $i$ maximizes the payoff resulting from placing that bid,

$$\Pi_i(b|v_i) = W_i(b; b_{-i}) u_i(v_i + w - b) + (1 - W_i(b; b_{-i})) u_i(w - b)$$

where $W_i(\cdot)$ is $i$'s probability of winning given the effort levels of the rivals.

A strategy for individual $i$ is a Lebesgue-measurable function that maps valuations into effort levels, $b_i : [\underline{v}, \overline{v}_i] \to \mathbb{R}_+$. We restrict attention to equilibria in which contestants with higher valuations for the prize expend (weakly) higher effort, or, simply, bid higher. Existence of a Bayes-Nash equilibrium in non-decreasing strategies follows from Athey (2001, theorem 7, p. 881). Moreover, for bids above zero, equilibrium strategies are strictly increasing, as follows from lemma 1 in the appendix. This observation enables us to formulate (generalized) inverse bid functions for each contestant: $\phi_i(b) : b \mapsto v_i$

$$\phi_i(b) \equiv \max \{v, \sup \{v : b_i(v) \leq b\}\} , \ i = 1, \ldots, N$$

This function agrees with the inverse bid $b_i^{-1}$ whenever the latter is well-defined; it is constant at any point of discontinuity (jump) of the bid function, and it returns $\underline{v}$ for any bid $b$ strictly below the lowest equilibrium bid; it is continuous, and it is differentiable almost everywhere since it is a bounded, non-decreasing function. Finally, let $G_i(b) \equiv \text{Prob}[b_i(V_i) \leq b] = F_i(\phi_i(b))$ be the probability that contestant $i$ bids at or below $b$. Then the probability of winning by contestant $i$ who bids $b$ can be expressed as the product of cumulative distributions of equilibrium bids, $W_i(b) \equiv \prod_{j \neq i} G_j(b)$.

\footnote{Lemma 2 in the appendix, section A.2.}
2.1 Equilibrium

Fix the bidding behavior of all the contestants, but $i$. For contestant $i$ with valuation $v_i$, an interior best response, $b > 0$, must satisfy the first order condition:

$$MB_i(b) = MC_i(b)$$

$$MB_i(b) = [u_i(v_i + w - b) - u_i(w - b)] W_i'(b)$$

$$MC_i(b) = u_i'(w - b)(1 - W_i(b)) + u_i'(v_i + w - b)W_i'(b)$$

where the marginal probability of winning is $W_i'(b) = \sum_{j \neq i} \prod_{k \neq i, j} G_k(b)g_j(b)$. If the marginal benefit, $MB_i$, is below marginal cost, $MC_i$ for any choice of $b \in [0, v_i]$, then if at least one of the rivals is bidding above zero for any $v > v_i$, contestant $i$ with valuation $v_i$ should drop out, $b_i (v_i) = 0$.

Lemma 4 in appendix A.4 provides a convenient formulation of the necessary first-order conditions in terms of the rate of growth of bid distribution $G_i$ of an active contestant.

The necessary conditions also implicitly describe, although not exhaustively, the set of active participants for each effort $b > 0$, that is, the set of contestants who choose this effort for some realizations of their valuations,

$$J(b) = \{i \in \{1, ..., N\} \mid \exists v \in [\underline{v}, \overline{v}] : b_i (v) = b\}$$

In contrast to a model with ex-ante identical or two arbitrary participants, who have to bid in the same interval, this set does not necessarily include all contestants for the bids chosen in an equilibrium. Indeed, as we demonstrate in the next section, it can happen that some participants exert no effort (bid zero) irrespective of their valuation for the prize in an equilibrium; it is also possible that some will never bid as high as their rivals. We also provide a numerical example (fig. 5) of discontinuous bidding, illustrating bifurcation of effort mentioned in the introduction, with only zero and high bids placed with positive probability by one of the contestants.

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7 Since contestants are weakly risk-averse, $MB_i(b) - MC_i(b)$ is strictly increasing in $v_i$ for $b > 0$. In other words, $\Pi_i(b|v_i)$ satisfies the strict single-crossing property.

8 If this last condition is violated, i.e., all the rivals’ strategies have an ‘atom’ at zero, then bidding zero is never optimal. But if $MB_i(b) < MC_i(b)$ for any $b > 0$, then no positive bid is optimal and therefore, the best response function is not well-defined.
3 Qualitative Predictions

3.1 Lack of Competitiveness and Drop Outs

A contestant decides whether to enter a race. He looks at his rivals and realizes that he will face a fierce competition. Should he bother to fight for the prize? And if he does enter, how hard should he try? Were the valuations commonly observed,\(^9\) — apart from knife-edge cases — the two individuals with the highest values for the prize would enter the competition, while the rest would drop out. While this is ‘good news’ in such a contest as an additional participation is a waste of effort, the drop-out in our model is less attractive. The (ex-post) most-willing-to-contribute contestant might stay out, in which case the prize is mis-allocated.

**Proposition 1.** Assume preferences of contestant \(i \in \{1, \ldots, N\}\) are represented by the constant absolute risk aversion (CARA) utility function, 
\[
\begin{align*}
&u_i(x) = \exp(-\rho x_{1,i}), \\
&\rho_{1,i} \in (0, \rho_i) \\
&b_j([v_i, \overline{v}_j]) = [0, b_j] \text{ for all } j \neq 1; \\
&\sum_{j \neq 1} \frac{\rho_j}{\exp(\rho_j \overline{v}_j) - 1} < (N - 2) \frac{\rho_1}{\exp(\rho_1 \overline{v}_1) - 1}; \\
\end{align*}
\]

Then, \(b_1(\overline{v}) < \overline{b}\). If, in addition,

3. for some \(j \neq 1\) \(b_j(v) > 0, \text{ if } v > \overline{v},\)\(^{10}\)

4. for all \(j \neq 1\) \(F_j(v) \leq H_j(v) = \frac{1 - \exp(\rho_j v_{1,j})}{1 - \exp(\rho_j \overline{v}_j)} \text{ for } v \in [0, \overline{v}_j];\)

5. \(\sum_{j \neq 1} \frac{\rho_j}{1 - \exp(-\rho_j \overline{v}_j)} \leq (N - 2) \frac{\rho_1}{\exp(\rho_1 \overline{v}_1) - 1}\)

Then, \(b_1(v) = 0 \text{ for all } v \in [0, \overline{v}]\).

**Remark 1.** Proposition 1 can be applied to rule out participation of several contestants in a recursive manner.

\(^9\)Hillman and Riley (see 1989); Baye et al. (see 1993).

\(^{10}\)If this assumption is dropped the best response of the first player might not be well-defined. See footnote 8.
The first contestant is not going to exert the highest equilibrium effort, if the rivals think his highest possible valuation, \( \bar{v}_1 \), is low or if he is sufficiently less risk averse than the others (assumption 2). He could have bid at the top and get the prize with certainty, by assumption 1, but he does not, since assumption 2 implies that his marginal cost of bidding at the top is lower than the marginal benefit.

Indeed, the difference between one’s marginal benefit and marginal cost of bidding at \( b \) (see equation (2)) is decreasing in his valuation. Thus, if the upper valuation of 1 is sufficiently low, he should refrain from ever paying the highest price offered by his rivals. Another reason for the lack of competitiveness, stems from the differences in risk attitudes. The more risk-averse are 1’s rivals, the more aggressive is their bidding at the top since they want to insure themselves against loosing. If one is substantially less risk-averse, it is more profitable for him to shade his bid and bear the risk of loosing rather than place the top bid.\(^{11}\) see figure 1. However, further increasing the gap

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\(\text{Fig. 1: Lack of competitiveness. Six contestants, five of which are identical with } \rho_j = 1 \text{ (their bid density is dashed) and one is risk neutral, whose bid density is a solid line. All have uniform valuations drawn from } [0, 1].\)

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\(^{11}\)The result is robust: by corollary 3 and lemma 4 in the appendix, if a contestant’s rivals are sufficiently risk averse, she should not bid the same top bid as they do.
are satisfied, which, in case of risk neutrality are reduced to a single requirement, \( N - 2 > \sum_{j \neq 1} \frac{\bar{v}_j}{\bar{p}_j} \). We use it to construct the next example, in which valuations are uniformly distributed in \([0, \bar{v}_1]\), see fig. 2.

Fig. 2: Complete drop-out. Left: bid density of the first player facing the same two rivals \((\bar{v}_2 = 3, \bar{v}_3 = 6)\) in three different contests \((\bar{v}_1 = 6: \text{upper line}, \bar{v}_1 = 3: \text{lower line}, \bar{v}_1 = 2: \text{density is zero})\). Right: the equilibrium payoff of players as \(\bar{v}_1\) varies from 0 to 7.

Another way to disentangle the role of risk aversion and the range of possible values for the prize is to use a different set of sufficient conditions for the complete drop-out.

**Claim 1.** *Condition 5 can be replaced by \( \sum_{j \neq 1} \rho_j \leq (N - 2) \rho_1 \).*

Having established the 'partial drop-out', we can assure, using assumption 4, that reducing the bid will not cut the costs by more than the drop in benefits, as valuations of 1's competitors are not very likely to be low. This is easy to see in case all contestants are risk-neutral, as 1's marginal benefit of bidding is proportional to his marginal winning probability, while the marginal cost of bidding is constant. Assumption 4 then guarantees that \( W'_1(b) \leq W'_1(\hat{b}) \).

Common-knowledge of differences in contestants' highest possible valuations is crucial for the complete drop-out result. In particular, if any two contestants have identical risk preferences, and if the distributions of their valuations have common support, they should bid in the same interval, see lemma 5 in appendix B.1. This is true even if the distribution of valuations
of one is first-order stochastically dominated by the other, though this should
affect the 'aggressiveness' of bidding, see proposition 6.

Interestingly, the drop-out results provide a lower bound on the upper
equilibrium bid.

**Corollary 1.** Assume \( N \) contestants with CARA utilities; that assumptions
1 and 4 in proposition 1 hold for all contestants while assumption 3 holds for
at least one contestant. Then,

\[
\bar{b} \geq \frac{1 - N}{\sum_{i=1}^{N} \rho_i} \ln \left( 1 - \frac{\sum_{i=1}^{N} \rho_i}{\sum_{i=1}^{N} \frac{1}{1 - \rho_i}} \right)
\]

In particular, if contestants are risk-neutral then \( \bar{b} \geq \frac{N-1}{\sum_{i=1}^{N} 1/\rho_i} \); while for
homogeneous, but not necessarily risk-neutral, contestants, \( \bar{b} \geq \frac{N-1}{N} \).

### 3.2 Workaholics

Now let us assume there are no drop-outs: contestants' values are drawn
from the same interval and they choose the same top bid. How high should
be one's valuation to bid near the top? If there are only two contestants or
if all are ex-ante identical, bidding is 'cautious' in a sense that higher bids
are chosen less frequently (proposition 3), whereas asymmetry can lead to an
'aggressive' behavior prescribed by an increasing equilibrium bid density, as
we show next.

**Proposition 2.** Assume \( N > 2 \) contestants with CARA utility functions and
the same range of possible valuations \([\underline{v}, \overline{v}]\) bid in the same interval. Then
contestant 1 is workaholic, \( g_1'(\bar{b}) > 0 \), if either

1. contestants have distinct distributions of values and contestant 1's like-
   likelihood of having the highest valuation, \( f_1(\overline{v}) \) is below a threshold, which
   is weakly increasing in likelihood of the other contestants having the
   highest valuation;

2. or distributions of valuations are the same, but contestant 1's coefficient
   of absolute risk aversion is sufficiently high, and his rivals are risk-
   neutral.
Here, as in the previous section, contestants' perceptions about each other play a crucial role. In case 1 of this proposition, it is the 'weak' contestant — the rivals of whom dismiss almost completely the possibility of her having the top value for the prize — it is she who might exhibit workaholic behavior.

To see why, let individual preferences be identical. As all the players with the top valuation bid the same (top bid) and have the same chance to win (regardless of their identity), in this case, marginal probability of winning at $\tilde{b}$ must be the same, which implies their bid densities are identical as well, $g_i(\tilde{b}) = \tilde{g}$. Moreover, with CARA preferences $\tilde{g}$ is independent of the level of the top bid. The definition of the bid density, $\phi_i(\tilde{b}) = \frac{\tilde{g}}{g_i(\tilde{b})}$, implies then that the 'weak' contestant should be aggressive when her valuation is near the top: her inverse bid function is steeper near the top bid compared to the rivals.\textsuperscript{12} It follows that her bid density must increase near the top bid, which easy to see by inspecting characterization of equilibrium bid density in lemma 4 for the risk-neutral case: the density increases in own inverse bid and decreases in that of the rivals (while the other factors grow at comparable rates near the top), see, e.g., fig. 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig3.png}
\caption{Three risk-neutral contestants. $F_1(x) = 2x - x^2$, $F_2 = F_3 = U$ on $[0, 1]$. The density of 1 is increasing at the top.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig4.png}
\caption{Five risk-neutral contestants (their bid density is dashed) and one CARA with $\rho_1 = 1$, $F_1 = U$ on $[0, 1]$.}
\end{figure}

The underlying intuition for the second part of proposition 2 is similar: bidding is risky, as losers are not compensated; so a risk-averse contestant chooses to bid at the top more frequently, as it 'almost assures' the prize

\textsuperscript{12}Equivalently, her bid function is flatter near $\tilde{v}$, that is, $b_j(v) > b_j(\tilde{v})$ for $j \neq 1$ in a neighborhood of $\tilde{v}$. 
(proposition 5). Since distributions of valuations are the same, she must be more aggressive near the top.

Bidder heterogeneity can give rise to bifurcation of effort in an equilibrium, as defined in the experimental literature mentioned in the introduction: some contestants choose zero and high bids more frequently than moderate ones. We provide a numerical example, figure 5, demonstrating bimodal distribution of effort with discontinuous equilibrium strategies, which could not arise in an equilibrium if there are only two players, or if all the players are ex-ante identical. Notice the bifurcation of effort of the risk-averse contestants: they bid zero with positive probability, never place low bids, and place high bids with an increasing frequency. A caveat: although valuations fol-

![Fig. 5: The bifurcation. Four risk-neutral (dashed line) and two CARA competitors with $\rho = 0.3$ (solid line). Valuations are uniform on $[1/250, 4]$.](image)

low the same distributions as the monetary valuations in the experiments by Noussair and Silver (2006), the absolute risk-aversion parameter is excessive compared with the estimates in the literature, see Cohen and Einav (2007, table 5).

It is crucial to have more than two ex-ante different contestants to support either workaholic behavior or bifurcation of effort in an equilibrium.

**Proposition 3.** The equilibrium bid probability density function of any contestant is non-increasing if contestants have CARA utility functions and at least one of the following conditions is satisfied:
1. there are only two contestants;

2. both the distribution of valuations and the utility are the same for all contestants;

This is a clear test of behavior in an all-pay auction: CARA contestants should choose lower efforts with higher frequency if there are only two of them (no matter what is the distribution of their valuations and what is the risk aversion parameter) or if they are ex-ante identical.

Restriction to CARA preferences can be relaxed, and in the appendix\(^\text{13}\) we describe the set of utility functions, for which equilibrium bid density is decreasing. In particular, the condition holds true for constant relative risk aversion as long as parameters are in a reasonable range.

To gain an intuition for why the density is decreasing in contests with symmetric or only two agents, assume they are risk-neutral.\(^\text{14}\) The second order condition for an optimal bid is \(W''_i(b) v_i < 0\). The density decreasing as the marginal winning probability is \((N-1) g(b) G(b)^{N-2}\) for contests with homogeneous agents and \(W'_i(b) = g_i(b)\) for contests with two competitors.

For general risk-averse preferences, lemma 6 in the appendix provides sufficient conditions for the marginal winning probability to be diminishing. The conditions are: linear bidding costs (as we assumed throughout this paper) and that contestants are not 'too' risk-averse. If any of these assumptions were violated, that is, if bidding costs were convex enough or if contestants were 'too' risk-averse then it may be possible to satisfy the second-order condition with increasing marginal winning probability.

### 3.3 Further Characterization of Individual Behavior

In this section we provide additional restrictions on behavior that might arise in equilibrium in which contestants use continuous strategies, so that any bid below the top one can be chosen.

Let \(n\) be the number of active contestants, that is, those who might place a positive bid.

**Proposition 4.** Assume that for all \(i\), \(b_i(v) > 0\) for all \(v > v > 0\). Then, either

\(^{13}\)See lemma 6.

\(^{14}\)We would like to thank the anonymous referee who suggested the explanation.
1. All but one, \( n - 1 \), active contestants choose zero bid with positive probability or

2. None of the active contestants does, so that \( G_i(0) = 0 \), and all are infinitely more likely to choose the lowest bid than any other bid, that is, \( \lim_{b \to 0} g_i(b) = +\infty \) for all \( i \).

Next result demonstrates that more risk-averse contestants bid at the top more often and under CARA preferences the differences in bidding behavior are more pronounced when risk aversion parameters of the players are further apart. Note that first result is `in the same spirit as' proposition 2 in Fibich et al. (2006), who compare behavior of more and less risk averse players in two different symmetric equilibria, though in our case the contestants with distinct risk attitudes face each other in the same auction.

**Proposition 5.** Assume that \( b_k((v, \bar{v}_k)) = (0, \bar{b}) \) for \( k = i, j \) with \( j \neq i \).

1. If contestant \( i \) is strictly more risk averse than contestant \( j \), then in a neighborhood of zero and in a neighborhood of the top bid, \( \bar{b} \), bid distributions are ordered, \( G_i(b) < G_j(b) \).

2. With CARA contestants individual rate of growth of bid distribution \( g_i(b)/G_i(b) \) grows with \( \rho_i \) and falls with \( \rho_j \) in a neighborhood of \( \bar{b} \); the reverse is true in a neighborhood of zero, provided \( \bar{v} > 0 \).

Finally we show that first-order stochastic dominance of values distributions implies the dominance of equilibrium bid distributions.

**Proposition 6.** Assume all contestants have identical risk-preferences, two of them choose continuous bidding functions and their valuations are driven from the distributions with the same support, \( [v, \bar{v}] \), satisfying \( F_j(v) < F_i(v) \) for all \( v \in (v, \bar{v}) \). Then, \( G_j(b) < G_i(b) \) for all \( b \in (v, \bar{v}) \).

### 4 Conclusions

We have studied individual equilibrium behavior in winner-take-all tournaments with many heterogeneous participants. Our methodology can also be applied to the War-of-Attrition contests, or any auction format where the highest bid wins, by modifying lemma 4 to reflect the corresponding incentives.
We derive testable equilibrium restrictions for the case of homogeneous contestants and show that heterogeneity of contestants has important equilibrium implications.

Further characterization of equilibrium in all-pay auctions with asymmetric bidders, including tighter conditions for continuity of bidding strategies, remains to be done. Main difficulty lies in identifying regions of bids where players are active, which is implicit in characterization given in lemma 4 that we used to derive the results.

A Auxiliary Results

A.1 Positive Bids Are Strictly Increasing

For any Borel set \( A \subset \mathbb{R} \), define \( \mu_G(A) = \Pr[b_i(V_i) \in A] \) and \( \mu_W(A) = \Pr[\max_{j \neq i} b_j(V_j) \in A] = \prod_{j \neq i} G_j(A) \). That is, \( \mu_G \) (respectively \( \mu_W \)) is the measure associated to the cumulative probability distribution function, \( G_i \) (respectively \( W_i \)).

Lemma 1. The measure \( \mu_G \) has no atoms at \( b > \hat{b} \).

Proof. If a positive mass of types of contestant \( i \) bids \( b \) then \( \lim_{\varepsilon \to 0} W_j(\varepsilon) < W_j(b) < W_j(b + \delta) \) for any \( \delta > 0 \) as in case of a tie any contestant who exerts the highest effort has equal chance of getting the prize. So, the left and the right derivatives 'explode', that is, \( W_j^-(b) = W_j^+(b) = +\infty \). Therefore, the type of contestant \( j \) who bids \( b \) should raise his bid marginally above \( b \): the marginal cost, \( MC(b) = u_j'(w - b)(1 - W_j(b)) + u_j'(v_j + w - b)W_j(b) \) is discontinuous, but bounded, while the marginal benefit is unbounded, \( MB(b) = [u_j(v_j + w - b) - u_j(w - b)] W_j^+(b) \).

\[ \square \]

Corollary 2. (Inverse) bid functions \( b_i(\phi_i) \) are strictly increasing for \( v > \phi_i(0) \) (\( b > 0 \)).

A.2 Generalized Inverse Bids are Continuous

Lemma 2. For any contestant \( i, \phi_i \) is continuous.

Proof. In the view of lemma 1 it is sufficient to establish \( \phi \) is right continuous at \( \hat{b} \). If not, then there is a \( \delta > 0 \) such that \( \phi_i(\hat{b}) < \phi_i(\hat{b} + \delta) < \phi_i(\hat{b} + \varepsilon) \) for any \( \varepsilon > 0 \). In other words, type \( \phi_i(\hat{b}) + \delta \) bids strictly above \( \hat{b} \) and strictly below \( \hat{b} + \varepsilon \) for any \( \varepsilon > 0 \), a contradiction. \[ \square \]
A.3 The Lowest Bid is Zero

Lemma 3. Let \((b_1, \ldots, b_N)\) be an equilibrium then \(b_i(y) = 0\) for all \(i\).

Proof. Assume that for \(b_i(y) = \beta > 0\). Since the equilibrium is monotone, the probability contestant \(j \neq i\) wins by bidding at or below \(\beta\) is zero. As a result, either \(b_i(v) = 0\) or \(b_i(v) \geq \beta\). In sum, \(G_j(\beta) = G_j(0)\) for \(j \neq i\) (\(G_i(\beta) = \Pr[b_i(V) < \beta]\) since \(G_i\) is non-atomic). Now, if \(G_j(\beta) = G_j(0) > 0\), for all \(j \neq i\), then \(W_i(\beta - \varepsilon) = W_i(\beta)\) for some \(\varepsilon > 0\). Therefore, bidding \(\beta - \varepsilon\) yields a higher payoff than bidding \(\beta\). On the other hand, when \(G_j(\beta) = G_j(0) = 0\), for some \(j\), then \(W_i(\beta) = 0\) and so, bidding zero yields a higher payoff than bidding \(\beta\).

A.4 The System of First Order Conditions

Recall the bidding strategies of the contestants are denoted by \(b_j : [y, \bar{y}_j] \rightarrow \mathbb{R}_+\). For every bid \(b \geq 0\) denote the set of active contestants at \(b\), that is, contestants that choose this bid for some of their type realizations:

\[
J(b) = \{ j \in \{1, \ldots, N\} | \exists y_j \in [y, \bar{y}_j] : b_j(v_j) = b \}.
\]

By definition, if contestant \(j\)'s highest equilibrium bid, \(\bar{b}_j\), is strictly below \(b\), then \(G_j(b) = 1\).

Lemma 4. For almost all bids, \(b > 0\), the system of first order conditions (2) can be represented as

\[
\frac{g_i(b)}{G_i(b)} = \left\{ \begin{array}{ll}
\frac{1}{K(b)-1} \left( \sum_{j \in J(b) \setminus \{i\}} S_j(b) - (K(b) - 2)S_i(b) \right), & i \in J(b) \\
0, & \text{otherwise}
\end{array} \right.
\]

(3)

where \(K(b) = \#J(b)\) is the number of contestants with a type who bids \(b\), and the rate of growth of winning probability of active contestant \(i\) is

\[
S_i(b) = \frac{u_i(\phi_i(b) + w-b) + \left( \frac{1}{\bar{y}_i(v_i)} - 1 \right) u_i(w-b)}{u_i(\phi_i(b) + w-b) - u_i(w-b)}.
\]

More precisely, (3) holds for all \(b > 0\) where the inverse bid functions are differentiable and the set of active contestants, \(J(b)\), is constant in some neighborhood of \(b\).
Proof. Consider contestant \(i \in \{1, \ldots, N\}\). Fix the strategies of other contestants. Using the definition of winning probability \(W_i(b) = \prod_{j \neq i} G_j(b)\),

\[
\sum_{j \neq i} \frac{g_j(b)}{G_j(b)} = \frac{W_i'(b)}{W_i(b)}
\]  \hspace{1cm} (4)

The system (4) is linear in the rate of growth of \(G_j(b)\), implying

\[
\left( \begin{array}{c} \frac{g_j(b)}{G_j(b)} \\ \frac{W_j'(b)}{W_j(b)} \end{array} \right) = M^{-1} \left( \begin{array}{cccc} - (N-2) & 1 & \ldots & 1 \\ 1 & - (N-2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & 1 & -(N-2) \end{array} \right)
\]  \hspace{1cm} (5)

Note that to be consistent with an equilibrium the solution to the system has to be non-negative, thus constraining the range for the rate of the growth of winning probabilities, which also are to satisfy the first order conditions (2):

\[
\frac{W_i'(b)}{W_i(b)} \leq S_i(b) = \frac{u_i'(\phi_i(b) + w - b) + \left( \frac{1}{W_i(b)} - 1 \right) u_i'(w - b)}{u_i(\phi_i(b) + w - b) - u_i(w - b)} > 0
\]  \hspace{1cm} (6)

Having fixed the strategies of all the players but \(i\) we have also fixed the set \(J_i(b) \setminus \{i\}\) for any \(b\). If at some given \(b > 0\), the first-order condition (6) is satisfied with equality, then let \(b_i(v_i) = b\). Otherwise, let \(b_i(v_i) = 0\), and therefore, \(g_i(b) = 0\). Therefore, \(S_i(b)\) is the rate of growth of the winning probability \(W_i(b)\) only for an active contestant, \(i \in J_i(b)\).

As we restrict attention to non-decreasing strategies, the highest bid of contestant \(i\) is the optimal bid for the highest type of that contestant, \(b_i(\overline{v}_i) = \overline{b}_i\). In addition, it does not pay-off for any of the contestants to bid above the highest bid of all the rivals, no matter how high his valuation is, which implies that for any strictly positive bid below or equal the highest equilibrium bid, \(\overline{b}_i, K(b) \geq 2\). It follows that condition (5) implies (3). \(\square\)
A.5 Continuity of Bids and Uniqueness

Let \( v = (v_1, \ldots, v_N) \in \mathbb{R}^N \) and write,

\[
S_i(b, v) = \frac{u'_i(v_i + w - b) + u'_i(w - b)}{u_i(v_i + w - b) - u_i(w - b)} \left( \frac{1}{\prod_{j \neq i} F_j(v_j)} - 1 \right).
\]

**Proposition 7.** Assume that: for all \( i, f_i \) is continuous and uniformly bounded above zero in its support, \([v, \bar{v}]\); then, the system of differential equations

\[
\dot{b}_i = \frac{f_i(\phi_i(b))}{f_i(\phi_i(b))} \left( \sum_{j \neq i} S_j(b, \phi(b)) - (N - 2)S_i(b, \phi(b)) \right), \quad i = 1, \ldots, N,
\]

has a unique solution that satisfies the terminal condition, \( \phi(\bar{b}) = (\bar{v}, \ldots, \bar{v}) \).

**Proof.** There is a neighborhood of \((\bar{b}, \phi(\bar{b}))\) such that the system satisfies the Lipschitz condition because, for all \( i, f_i \) is continuous and bounded away from zero and; \( S_i \) is continuous in \((b, \phi)\) and bounded in a small neighborhood of \((\bar{b}, \phi(\bar{b}))\). Consequently, the solution \( \phi(b) \) is locally (restricted to this neighborhood) unique.

Furthermore, as long as \( \phi_i(b) > \bar{v} \) for all \( i \), there is a neighborhood of \((\bar{b}, \phi(b))\) where the Lipschitz condition is satisfied. Therefore, \( \phi(b) \) can be further extended by continuity, in a unique way, from \( \bar{b} \) to \( \beta \) where, \( \beta \) is defined as the largest \( b < \bar{b} \) such that there is at least one contestant, say \( k \), such that \( \phi_k(\beta) = \bar{v} \).

Albeit the uniqueness of the solution of (7) for the terminal condition \( \phi(\bar{b}) = (\bar{v}, \ldots, \bar{v}) \) has been established, the uniqueness of \( \bar{b} \) has not been proven yet. We do not know whether \( \bar{b} \) is unique if arbitrary risk-preferences are allowed. For CARA preferences, however, the system of differential equations (7) is autonomous, that is, its solution is invariant with respect the value of \( \bar{b} \). Thus in this case we can solve the system (7) setting an arbitrary value for the terminal bid, say \( \bar{b} = \bar{v} \). The solution determines \( \beta \), as defined above, \( \phi_k(\beta) = \bar{v} \). Lemma 3 above establishes that the lowest equilibrium bid is zero, \( \bar{b} = 0 \) and that allows us to pin-down the value of the top bid as the difference, \( \beta = \bar{v} - \beta \).

It is important to notice that there is no guarantee that the above unique solution corresponds to an equilibrium. For example, it is conceivable that...
\( \phi'(b) < 0 \) for some \( b \) and \( i \). One needs to prove that all contestants are active in order to show that the above solution corresponds to an equilibrium, that is, \( J(b) = \{1, \ldots, N\} \) for any \( b \in (0, \bar{b}) \). Indeed, as long as contestants have the same risk preferences and \( \bar{v}_i = \bar{v} \) for any contestant \( i \), lemma 5 implies that \( b_i(\bar{v}) = \bar{b} \). In addition, provided the strategies are continuous and since strategies are strictly increasing there is \( v \) such that \( b_i(v) = \bar{b} \). As a consequence, any equilibrium in continuous strategies must coincide with the unique solution of (7).

\section*{B Main Results}

\subsection*{B.1 Participation}

\textit{Proof of proposition 1.} Assume to the contrary, that contestant 1 does bid the highest bid, \( \bar{b} \), that is, his first order conditions hold as equality at that point. Since the growth rate of the winning probability of contestant \( i \) who bids \( \bar{b} \) is \( S_i(\bar{b}) = \frac{\rho_i \exp(-\rho_i v_i)}{1 - \exp(-\rho_i v_i)} \), then by lemma 4, 1's bid density should have the same sign as \( \sum_{j \neq 1} S_j(\bar{b}) - (N-2)S_1(\bar{b}) \) which, by assumption 2 of the proposition, is negative, a contradiction.

Similarly, if assumptions 4-5 of the proposition hold, the first contestant should never bid any other positive bid, if his valuation is at the top, \( v_1 = \bar{v}_1 \), as for any \( b > 0 \) at which the rivals are active \( \sum_{j \neq 1} S_j(b) - (N-2)S_1(b) < 0 \). Indeed, this difference is proportional to

\[
\sum_{j \neq 1} \left[ \frac{G_j(b) \rho_j}{1 - e^{-\rho_j v_j(b)}} - \rho_j \prod_{k=1}^{N} G_k(b) \right] - (N-2) \left[ \frac{G_1(b) \rho_1}{1 - e^{-\rho_1 v_1(b)}} - \rho_1 \prod_{k=1}^{N} G_k(b) \right]
\]

(8)

Given \( G_1(b) = 1 \) (\( b \) is chosen by the highest valuation type of contestant 1); weak risk aversion, \( \rho_i \geq 0 \); assumption 4: \( \sum_{j \neq 1} \frac{G_j(v)}{1 - e^{-\rho_j v_j}} \leq \sum_{j \neq 1} \frac{1}{1 - e^{-\rho_j v_j}} \) for any \( v_j \); and, finally, \( \prod_{k=1}^{N} G_k(b) = W_1(b) < 1 \), this expression is majorised by \( \sum_{j \neq 1} \frac{1}{1 - e^{-\rho_j v_j}} - (N-2) \frac{1}{1 - e^{-\rho_1 v_1}} \), which is negative by assumption 5. In the view of footnote 8, assumption 3 assures \( b_1(\bar{v}_1) = 0 \). Finally, since the equilibrium is monotone, \( b_1(v) = 0 \) for all \( v \).

\textit{Proof of claim 1.} Expression 8 is negative provided the alternative sufficient condition is satisfied along with 4. \hfill \Box

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Proof of corollary 1. To calculate the highest equilibrium bid, \( \overline{b} \), let us introduce a ‘dummy’ contestant zero, whose risk aversion parameter and highest valuation satisfy \( \sum_{i=1}^{N} \rho_i = (N-1) \rho_0 \) and

\[
\sum_{i=1}^{N} \frac{\rho_i}{1 - \exp(-\rho_i \overline{v}_i)} = \frac{(N-1) \rho_0}{1 - \exp(-\rho_0 \overline{v}_0)}.
\] (9)

Then, the inequalities in assumption 2 of proposition 1 and in claim 1 are satisfied as equalities and the rest of the assumptions of proposition 1 hold for \( N + 1 \) potential participants. Furthermore, the right hand side of (9) is decreasing in \( \overline{v}_0 \), therefore if the ‘dummy’ contestant top valuation were lower than \( \overline{v}_0 \), he would drop out by claim 1, so \( \overline{b} \) is above \( \overline{v}_0 \). In case the contestants are risk-neutral, (9) is \( \sum_{i=1}^{N} 1/\overline{v}_i = (N-1)/\overline{v}_0 \).

\[\Box\]

Lemma 5. Assume contestants \( i \) and \( j \) have the identical risk attitudes and their valuations have common support, that is, \( \overline{v}_i = \overline{v}_j = \overline{v} \). Then \( b_i(\overline{v}) = \overline{b} \), if and only if \( b_j(\overline{v}) = \overline{b} \).

Proof. Assume to the contrary, there exists a \( \beta < \overline{b} \) such that \( b_j(\overline{v}) = \beta \) and \( b_i(\overline{v}) = \overline{b} \), or equivalently, \( G_i(\beta) < G_j(\beta) = 1 \). By revealed preference arguments,

\[
W_j(\beta) u(\overline{v} + w - \beta) + (1 - W_j(\beta)) u(w - \beta) \geq u(\overline{v} + w - \overline{b}) \geq W_i(\beta) u(\overline{v} + w - \beta) + (1 - W_i(\beta)) u(w - \beta).
\] (10)

From \( W_i(\beta) G_i(\beta) = W_j(\beta) G_j(\beta) \) and hypothesis \( G_i(\beta) < G_j(\beta) = 1 \), we obtain \( W_i(\beta) > W_j(\beta) \), which contradicts (10).

\[\Box\]

B.2 No Bifurcation of Effort

B.2.1 CARA case

Proof of proposition 3. Under CARA \( S_i \), as defined in (6), becomes

\[
S_i(b) = \frac{\rho_i}{(1 - e^{-\rho_i \phi_i(b)}) W_i(b)} - \rho_i
\] (11)

so, by lemma 4, if \( N = 2 \)

\[
g_j(b) = \frac{\rho_i}{1 - \exp(-\rho_i \phi_i(b))} - \rho_i G_j(b), \; i \neq j
\] (12)
Since both \( \phi(\cdot) \) and \( G_j(\cdot) \) are increasing in \( b \), it follows that \( g_j \) is decreasing in \( b \), thus part 1.

When \( \rho_i = \rho \) for all \( i \in \{1, \ldots, N\} \), then in a symmetric equilibrium, inverse bids are the same, \( \phi_i = \phi \), and so is the effort density, \( g_i = g \) for all \( i \in \{1, \ldots, N\} \). Then lemma 4 implies

\[
g(b) = \frac{\rho}{[1 - \exp(-\rho \phi(b))](N - 1)G^{N-2}(b)} - \frac{\rho G(b)}{N - 1}
\]

Again, the result follows from monotonicity of \( \phi(\cdot) \) and \( G(\cdot) \).

\( \square \)

\section*{B.2.2 Other preferences specifications}

\textbf{Notation.} For any \( f : \mathbb{R} \rightarrow \mathbb{R} \), \( \bar{f} = \sup_{v \leq x \leq \bar{v}} f(x) \), \( \underline{f} = \inf_{v \leq x \leq \bar{v}} f(x) \).

\begin{align*}
\rho_i(x) &= \frac{-u''(x)}{u'(x)}
\end{align*}

\textbf{Lemma 6.} Assume the distributions of valuations share a common support and contestant \( i \) is risk-averse. If \( i \) is risk-neutral or \( \bar{\bar{\tau}}_i < \ln(2u_i''/u_i'') \), then his marginal probability of winning is diminishing, \( W_i''(b) < 0 \).

\textbf{Remark 2.} If the contestant has CRRA preferences given by \( u(x) = \frac{x^{1-\sigma}}{1-\sigma} \), the sufficient condition for \( W_i'' \) to be decreasing is

\[
\sigma \frac{\bar{v}}{w - \bar{v}} + (1 + \sigma) \ln \left( \frac{w + \bar{v}}{w - \bar{v}} \right) < \ln(2).
\]

\textbf{Proof.} For any \( b > 0 \) at which contestant \( i \) is active and lemma 4 holds, the first order conditions (2), have to be satisfied as equality. Differentiating these conditions and noting that at that point the inverse bid function is strictly increasing, we get

\[
A + B > W_i''(b)[u_i(\phi_i(b) + w - b) - u_i(w - b)],
\]

with

\[
A = 2W_i''(b)[u_i'(\phi_i(b) + w - b) - u_i'(w - b)]
\]

\[
B = -W_i'(b)u_i''(\phi_i(b) + w - b) - (1 - W_i(b))u_i''(w - b)
\]

In case of risk-neutral contestants, \( B = 0 \) and \( A < 0 \), thus the result. If all are strictly risk averse, \( A < 0 \) and \( B > 0 \). In order to prove that \( W_i'' < 0 \), it is sufficient to establish that \( 0 > A + B \).

By the first order conditions, \( W_i'(b) \geq \frac{u_i'(\phi_i(b) + w - b)}{u_i(\phi_i(b) + w - b) - u_i(w - b)} > 0 \). And so,

\[
A \leq \frac{2u_i'(\phi_i(b) + w - b)}{u_i(\phi_i(b) + w - b) - u_i(w - b)} \int_{w - b}^{\phi_i(b) + w - b} u_i''(x)dx \leq
\]

\[
\frac{2u_i'(\phi_i(b) + w - b)}{u_i(\phi_i(b) + w - b) - u_i(w - b)} \phi_i(b) u''_i.
\]
As $-u_i'' \geq B$, the inequality $A + B < 0$ follows from

$$2 u_i' (\phi_i (b) + w - b) \overline{u_i} = \frac{u_i (\phi_i (b) + w - b) - u_i (w - b) \phi_i (b)}{u_i (\phi_i (b) + w - b) - u_i (w - b)} \phi_i (b) > \frac{u_i''}{\overline{u_i}}$$

Using Pratt (1964)'s representation, $u_i (x) = \int_0^\infty \exp \left( - \int_0^x r_i (z) \, dz \right) \, dy$,

$$\frac{u_i (\phi_i (b) + w - b)}{u_i (\phi_i (b) + w - b) - u_i (w - b)} \frac{1}{\int_{-\phi_i (b)+w-b}^{\phi_i (b)+w-b} \exp \left( \int_{y}^{\phi_i (b)+w-b} r_i (z) \, dz \right) \, dy} \geq \frac{\phi_i (b) \exp (\phi_i (b) \overline{\nu_i})}{1} \geq \frac{\phi_i (b) \exp (\overline{\nu_i})}{1}$$

Thus, the sufficient condition can be expressed as

$$\frac{2}{\exp (\overline{\nu_i})} > \frac{u_i''}{\overline{u_i}} \iff \overline{\nu_i} < \ln \left( \frac{2u_i''}{u_i''} \right)$$

\[\square\]

### B.2.3 Bifurcation of Effort

**Proof of proposition 2**. The first order conditions in this case (lemma 4) imply

$$\frac{g_i (b)}{G_i (b)} = \frac{1}{N - 1} \left( \sum_{j \neq i} S_j (b) - (N - 2) S_i (b) \right) \quad (14)$$

By definition of $S_i$ (11)

$$S_i' (\overline{b}) = -\frac{\rho_i}{1 - \exp (-\rho_i \phi_i (\overline{b}))} \left( W_i' (\overline{b}) + \rho_i \phi_i' (\overline{b}) \exp (-\rho_i \phi_i (\overline{b})) \right).$$

By definition of the winning probability of contestant $i$, $W_i' (\overline{b}) = \sum_{j \neq i} g_j (\overline{b}) = S_i (\overline{b})$ and using identities $\exp (-\rho_i \overline{\nu}) = \frac{S_i (\overline{b})}{S_i (\overline{b}) + \rho_i}$ and $g_i (b) = \phi_i' (b) f_i (\phi (b))$, $S_i' (\overline{b}) = -(1 + \frac{g_i (\overline{b})}{f_i (\overline{b})}) (S_i (\overline{b}) + \rho_i) S_i (\overline{b}) \quad (15)$
Differentiating (14),
\[ \frac{g_i'(\bar{b})G_i(\bar{b}) - g_i(\bar{b})^2}{G_i(\bar{b})^2} = \frac{\sum_{j \neq i} S_j'(\bar{b}) - (N - 2)S_i'(\bar{b})}{N - 1} \]
and substituting (15),
\[ g_i'(\bar{b}) = g_i^2(\bar{b}) + \frac{1}{N - 1} \left[ (N - 2) \left( 1 + \frac{g_i(\bar{b})}{f_i(\bar{b})} \right) (S_i(\bar{b}) + \rho_i)S_i(\bar{b}) \right. \]
\[ - \sum_{j \neq i} \left( 1 + \frac{g_j(\bar{b})}{f_j(\bar{b})} \right) (\rho_j + S_j(\bar{b}))S_j(\bar{b}) \] (16)

The first term is always positive, so to show \( g_i'(\bar{b}) > 0 \) using expression (16) it is sufficient to have the difference between the second and the third term to be positive, which can be assured by choosing sufficiently low value for \( f_i(\bar{b}) \). This condition is equivalent to
\[ \frac{g_i(\bar{b})}{f_i(\bar{b})} > \frac{\sum_{j \neq 1} S_j'(\bar{b})}{(N - 2)(S_1(\bar{b}) + \rho_1)S_1(\bar{b})} - 1 \] (17)

Note that the right hand side of this inequality is bounded: for any \( i \), \( S_i(\bar{b}) \) is bounded, and so is \( g_i(\bar{b}) \). \( S_j(\bar{b}) \) are bounded as long as \( f_j(\bar{b}) > 0 \).

To prove the last part of the proposition, assume valuations of all the contestants are the same, risk aversion parameter for contestants 1 is \( \rho_1 \) and the rest of the contestants \( j \neq 1 \) are risk-neutral, \( \rho_j = 0 \), so \( S_j(\bar{b}) = 1/\bar{b} \). In this case \( g_j(\bar{b}) = \frac{S_j(\bar{b})}{N-1} \) and \( g_1(\bar{b}) = 1/\bar{b} - \frac{N-2}{N-1}S_1(\bar{b}) \). Then applying (16) to calculate the derivative of the bid density for contestant 1:
\[ g_1'(\bar{b}) = \left( \frac{1}{\bar{b}} - \frac{N-2}{N-1}S_1(\bar{b}) \right)^2 - \left( 1 + \frac{S_1(\bar{b})}{(N-1)f(\bar{b})} \right) \frac{1}{\bar{b}^2} \]
\[ + \frac{N-2}{N-1} \left( 1 + \frac{1}{f(\bar{b})} - \frac{N-2}{(N-1)f(\bar{b})}S_1(\bar{b}) \right) (\rho_1 + S_1(\bar{b}))S_1(\bar{b}) \]
\[ = S_1(\bar{b})A(\bar{b}) + S_1(\bar{b})\rho B(\bar{b}), \]

where \( B(\bar{b}) = \frac{N-2}{N-1} \left( 1 + \frac{1}{f(\bar{b})} - \frac{N-2}{(N-1)f(\bar{b})}S_1(\bar{b}) \right) \) and \( A(\bar{b}) = \frac{N-2}{N-1} \left( \frac{N-2}{N-1}S_1(\bar{b}) - \frac{\rho}{\bar{b}^2} \right) - \frac{1}{(N-1)f(\bar{b})\bar{b}^2} + S_1(\bar{b})B(\bar{b}) \). As \( \rho \to \infty \), \( \rho S_1(\bar{b}) \to 0 \); \( B(\bar{b}) \) converges to a positive constant. Therefore, \( g_1'(\bar{b}) > 0 \) for sufficiently high \( \rho \), although it converges to zero in the limit. \( \Box \)
C  Additional Results

Lemma 7. Let the utility of contestant i be represented as in Pratt (1964),

\[ u_i(x) = \int_0^x \exp \left( - \int_0^y r_i(z) dz \right) dy. \]

Then,

\[ \frac{\pi_i}{e^{\theta_i} - 1} \leq W_i'(\theta) = \frac{1}{\int_{\theta_i}^{\theta_i + \theta - \theta} \exp \left( - \int_y^{\theta_i + \theta - \theta} r_i(z) dz \right) dy} \leq \frac{\pi_i}{e^{\theta_i} - 1}. \]

\[ W_i'(0) = \frac{1}{\int_{\theta_i}^{\theta_i + \theta - \theta} \exp \left( - \int_y^{\theta_i + \theta - \theta} r_i(z) dz \right) dy}. \]

Corollary 3. As contestant i becomes infinitely risk averse the growth rate of his winning probability at the top, \( S_i(\theta) = W_i'(\theta) \) converges to zero, provided \( \theta = \theta_i(\pi_i) \)

Proof.

\[ W_i'(\theta) = S_i(\theta) = \frac{1}{\int_{\theta_i}^{\theta_i + \theta - \theta} \exp \left( - \int_y^{\theta_i + \theta - \theta} r_i(z) dz \right) dy}. \quad (18) \]

The conclusion then follows from,

\[ \frac{1}{\int_{\theta_i}^{\theta_i + \theta - \theta} c(\theta_i + \theta - \theta - y) \pi_i \ dy} \leq S_i(\theta) \leq \frac{1}{\int_{\theta_i}^{\theta_i + \theta - \theta} c(\theta_i + \theta - \theta - y) \pi_i \ dy}, \]

coupled with the following qualification. Because \( \theta \) is endogenous, the domain of integration in the above expression depends on \( r_i(\cdot) \). The result could fail if the measure of the domain converged to zero as \( r_i \to +\infty \). Nevertheless, since the measure of the domain is bounded away from zero, the integrand diverging to infinity guarantees the integral diverges to infinity.

Analogously, for the marginal probability of winning at zero,

\[ W_i'(0) = \frac{1}{\int_{\theta_i}^{\theta_i + \theta - \theta} \exp \left( - \int_y^{\theta_i + \theta - \theta} r_i(z) dz \right) dy}, \quad (19) \]

provided, \( \phi_i(0) = \pi_i \). The bounds for \( W_i'(0) \) are obtained in a similar manner.

\[ \square \]

Remark 3. \( W_i'(0) \) is increasing in \( r_i(\cdot) \) while, \( W_i'(\theta) \), holding \( \theta \) constant, is decreasing in \( r_i(\cdot) \).
Proof of proposition 4. First, the lowest effort should be zero by the lemma 3. There should be at least one contestant, $k$, who does not choose $b = 0$ with strictly positive probability. If there are only two active contestants, the proposition reduces to lemma 5 of Amann and Leininger (1996, p. 6). So, assume there are at least 3 active contestants and, for simplicity, that all contestants are active. Consider a pair of active contestants $i, j$ distinct from $k$. For these contestants, their respective winning probability approaches zero as $b \to 0$ in the presence of $k$. By definition, $W_i(b)G_i(b) = W_j(b)G_j(b)$, so

$$\lim_{b \to 0} G_j(b) = \lim_{b \to 0} \frac{W_i(b)}{W_j(b)} = \lim_{b \to 0} \frac{W_i'(b)}{W_j'(b)}$$

(20)

where the last equality follows from the L'Hôpital's Rule.

Notice that even when $G_i(0) > 0$, the first order condition holds with equality for type $\phi_i(0)$, provided his bidding function is continuous. Thus, using the first order conditions (2)

$$\lim_{b \to 0} \frac{W_i'(b)}{W_j'(b)} = \frac{u_i'(w)}{u_j'(w)} \frac{u_j(\phi_j(0) + w) - u_j(w)}{u_i(\phi_i(0) + w) - u_i(w)} < \infty$$

since $\phi_i$ and $\phi_j$ are right continuous, $\phi_i \geq \phi_j \geq \phi$ and, $\phi > 0$ by assumption. As a result,

$$\lim_{b \to 0} \frac{G_j(b)}{G_i(b)} = \frac{u_i'(w)}{u_j'(w)} \frac{u_j(\phi_j(0) + w) - u_j(w)}{u_i(\phi_i(0) + w) - u_i(w)} < \infty$$

(21)

It follows that only two scenarios are possible. First, both bid distributions might have an atom at zero, $G_i(0) > 0$, $G_j(0) > 0$ or secondly, it might happen that $i$ and $j$ start bidding at zero, so that $G_i(0) = G_j(0) = 0$. For a given choice of $k$ and $i$, since the choice of $j$ (distinct from $i$ and $k$) can be made arbitrarily, either $n - 1$ contestants choose zero bid with positive probability or none does.

In the case where, $G_i(0) = 0$ for all $i$, the first-order conditions imply the marginal winning probabilities are be strictly greater than zero, $W_i(0) = \frac{u_i(0)}{u_i(0) - u_i(b)} > 0$. The last observation coupled with the fact that, for all $b > 0$, $W_i'(b) = \sum_{j \neq i} \prod_{k \neq i,j} G_k(b) g_j(b)$, implies that, at least for some $j \neq i$, the density $g_j$ must ‘explode’ at zero. Otherwise, if all densities were finite at zero, we would have $W_i'(0) = 0$, which is not possible. Furthermore, if at least one density ‘explodes’, all densities must ‘explode’ as well. Using L'Hôpital's
Rule again, for any $i$, $\lim_{b \to 0} \frac{g_i(b)}{g_i(\bar{b})} = \lim_{b \to 0} \frac{g_i(b)}{g_i(\bar{b})} < +\infty$, we conclude that $\lim_{b \to 0} g_i(b) = +\infty$, if and only if, $\lim_{b \to 0} g_i(b) = +\infty$. □

Proof of proposition 5. 1. All the players are bidding the top bid, so, evaluating equilibrium bid density at that point, by lemma 4 given $W_i(\bar{b}) = 1$ for all $i$, to get the result, $g_i(\bar{b}) > g_j(\bar{b})$, it is sufficient to show that $S_i(\bar{b}) < S_j(\bar{b})$, but the latter inequality follows from lemma 7, as $i$ is more risk averse than $j$.

By continuity of bidding functions and given $\phi_i(0) = \bar{u}$, we can use (20), so $\lim_{b \to 0} \frac{g_i(b)}{g_i(\bar{b})} = \frac{W_i'(0)}{W_j'(0)}$. Then the conclusion follows from lemma 7, implying $W_i'(0) > W_j'(0)$.

2. With CARA participants bidding at the top, $S_i(\bar{b}) = \frac{\phi_i}{\phi_i - \bar{u}}$, which decreases in $\rho_i$. So, the first part of this claim follows from equilibrium density representation in lemma 4 and the continuity of bid functions. As for the vicinity of zero, it is sufficient, again, to evaluate $S_i(0) = \frac{\phi_i}{1 - \rho_i}$, which is increasing in $\rho_i$, given $\bar{u} > 0$. □

Proof of proposition 6. Notice that $F_j$ first-order stochastically dominates $F_i$ implies the weak inequality $f_i(\bar{v}) \leq f_j(\bar{v})$. To simplify this proof, we assume the strict inequality, $f_i(\bar{v}) < f_j(\bar{v})$.

1. Suppose that either: $G_i$ and $G_j$ cross at some point in the interior of support of equilibrium effort levels, $b^* \in (0, \bar{b})$, or that they are tangent at $\bar{b}$. In this case, it follows that $G_i(b^*) = G_j(b^*)$ together with $F_j$ strictly first-order dominates $F_i$ imply that $\phi_i(b^*) < \phi_j(b^*)$. Moreover, from $G_i(b^*) = G_j(b^*)$, $\phi_i(b^*) < \phi_j(b^*)$, we obtain that $S_i(b^*) > S_j(b^*)$.

Using the characterization of the effort densities, we get $g_i(b^*) < g_j(b^*)$. In sum, $G_i$ and $G_j$ can not be tangent at any $b \in (0, \bar{b})$ and moreover, if $G_i$ and $G_j$ cross then $G_j$ must intersect $G_i$ from below.

2. At the boundaries of the support of the equilibrium effort levels, the distributions of effort may be tangent. In particular since $S_i(\bar{b}) = S_j(\bar{b})$, they are tangent at $\bar{b}$, that is, $G_i(\bar{b}) = G_j(\bar{b}) = 1$ and $g_i(\bar{b}) = g_j(\bar{b})$, where $\bar{b} = b_i(\bar{v}) = b_j(\bar{v})$ as established by lemma 5. These equalities and the expression for the derivative of the effort density, (16), yield $g_i(\bar{b}) \geq g_j(\bar{b})$, if and only if, $f_i(\bar{v}) \leq f_j(\bar{v})$. But, $F_j$ first-order dominates $F_i$ implies $f_i(\bar{v}) \leq f_j(\bar{v})$. Moreover, by assumption $f_i(\bar{v}) < f_j(\bar{v})$ and
therefore \( g'_i(\tilde{b}) > g'_j(\tilde{b}) \). Therefore, at the top, \( G_i \) must intersect \( G_j \) from below.

The conclusions of 1 and 2 above imply that \( G_i \) and \( G_j \) can never intersect in the interior of the support and, \( G_i \) is always above \( G_j \). \( \square \)

References


