A SYSTEMATIC COMPONENT OF THE JUMP-RISK PREMIUM IN AN AJD MODEL

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Abstract

We develop an affine jump diffusion (AJD) model with the jump-risk premium being determined by both idiosyncratic and systematic sources of risk. While we maintain the classical affine setting of the model, we add a finite set of new state variables that affect the paths of the primitive, under both the actual and the risk-neutral measure, by being related to the primitive's jump process. Those new variables are assumed to be common to all the primitives. We present simulations to ensure that the model generates the volatility smile and compute the “discounted conditional characteristic function” transform that permits the pricing of a wide range of derivatives.

Key words: Jump - Systematic Risk - Affine model - Macroeconomic Risk - Derivative pricing

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1 Introduction

The development of financial economics and asset pricing theory in recent decades has led researchers to accept the idea that asset prices dynamics contain important information regarding agents preferences and beliefs. The “weight” that agents put in different states of nature when evaluating assets reflect not only the effect of different endowments or technologies on their utility in equilibrium, but also their perception of the likelihood of different scenarios. Literature in empirical finance has been interested in developing techniques to assess such perception through the observation of asset prices, both derivatives and its primitives. In the case of using derivatives, studies (to be named in the next pages) indicate that if we assume those primitives to be driven by continuous processes, we fail to explain the prices (or implied volatility) pattern observed in real world data. Adding a jump component to the primitive’s process helps to reconcile the theoretical prices and the real-world prices, as its distribution changes (especially in what concerns its volatility).

Adding the jump component partially solves a question, but it also gives rise to another one: what drives the premium investors demand for facing the possibility of jumps? As we said, prices dynamics contain information of agents perception of different scenarios. In the moment we add a jump component in the conditional distribution of a primitive, we instantly add many potential different scenarios in the future and so we have to evaluate how agents “weight” those new scenarios. Looking at the risk neutral probability measure generated by agents preferences and beliefs is a traditional way literature has developed to quantify those “weights”. In fact, a more technical way to pose the same question would be: under the risk neutral probability measure, what determines the joint distribution of jumps and the other sources of uncertainty? The aim of this work is to marginally contribute in the answer of this question by presenting an affine jump-diffusion (AJD) model that highlights the influence of systematic state variables on the jump-risk premium.

In subsection 1.1 we will define and explain what we are now calling a systematic state variable. The basics of the model (the AJD setting) were chosen for its popularity and easiness of implementation. Although we acknowledge the fact that literature has advanced with respect to the affine structure, we believe that its high influence and relevance justify the adaptation of the classical model to incorporate the role of systematic state variables, at least as a small contribution in the attempt to answer the question posed. Besides, the setting of an AJD model is relatively simple to be read. Hence, we will be able to clearly highlight the role played be the variables we introduce, which we believe to be a gain.

We present an affine jump-diffusion (AJD) model that falls into the class studied in the influential work of Duffie, Pan & Singleton (2000). Our idea is to link the systematic state variable process to the primitive jumps’ distribution and to explore a possible relationship between the nominal interest rate and the systematic variable to identify the role that such systematic variable plays.

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1The relation between the nominal interest rate and the systematic variable might be very useful if we model a macroeconomic variable as our state variable. We will discuss an application of the model to macroeconomics in subsection 4.1.
in the price of a derivative. In order to accomplish such a goal, we have to face the heavy analytics involved in solving the differential equations that characterize the derivative’s prices in AJD models. In turn, however, having such prices allows us to identify the premia related to shocks on those systematic variables, which would be our final goal. Of course, systematic risks are systematic. Therefore, the parameters that define its process should be the same for every primitive process (although the way it affects the primitive’s conditional distribution should be allowed to differ) and so panels of primitives should prove themselves useful to estimate such parameters. The estimation, however, will not be a part of this preliminary work.

This paper is organized as follows. The remainder of section 1 discusses the definition and role of systematic state variables (subsection 1.1) and gives a brief literature review on related asset pricing theory and empirics (subsection 1.2). To express our results, we first develop a general model in section 2 that is able to incorporate systematic variables. In section 3, we propose a method to account for the issue of processes driven by correlated shocks. In section 4, we give an application of the model to incorporate macroeconomic variables (or variables that we assume to have linear relation with the interest rate process). In section 5, we present results from simulations that show that the model is able to reproduce the volatility smile. In section 6 we compute the transform that can be used to price different kinds of derivatives. Section 7 concludes.

1.1 Systematic state variables

The presence of jumps was originally introduced in stochastic volatility (SV) models in Bates (1996). In the original setting, the jump process (its intensity and size) was considered independent of the Brownian shocks and its distribution stationary. The work of Pan (2002) maintains the general setting of Bates model, but the jump’s intensity is allowed to be dependent on the current stock’s volatility, on both the data-generating conditional distribution (or “real world”) and the risk-neutral conditional distribution. Consequently, the work captures the premium investors demand for being exposed to the jumps in periods with different levels of market volatility.

While many other variables could be assumed to impact the jump-risk premium, we focus on a narrower class of variables. We define systematic state variables as variables characterized, as the name suggests, by the fact that its path is the same for every stock or asset considered and its distribution is not independent from the prices processes. Unlike volatility, whose component on the jump-risk premium can be considered idiosyncratic (each stock has its own underlying volatility process), a systematic variable component is not stock specific. Shocks on these variables happen in the same way for every primitive we consider in the cross section. What can vary, nevertheless, is the premium related to this risk in each stock.

A systematic state variable can have several interpretations. For instance, consider a manufacturing sector that relies heavily on the supply of a commodity in order to produce. Shocks in the price of this commodity will be equal for every company in the sector. If we believe the commodity’s price can affect the tail risk of the stocks in this sector, we can assume it to be a systematic

\[ \text{A interesting exercise would be to test whether the premium varies across stocks.} \]
state variable in a model that contemplates the dynamics of the sector’s stocks. When investors buy stocks from one company of such sector, they get exposed to the risk that the stock price will jump. Such a risk can be assessed by investors very differently depending on the prices of the commodity. If a shock causes the commodity’s price to soar, investors might demand a greater premium for the possibility of jumps.

Another possible interpretation for the state variables is a macroeconomic process such as inflation or public debt. A macroeconomic variable shock will typically be the same for every asset, which guarantees the definition of a systematic variable is satisfied. Applying our setting for the use of a macroeconomic variable can be very useful. While recent literature has been focused on identifying the influence of macroeconomic variables through a narrow class of assets, it would allow us to diverge from such trend by introducing a model that relates macroeconomic variables to usually studied derivatives, without focusing on any particular one. The introduction of one or more macroeconomic variables would allow us to test the perception of agents regarding the paths of such variable. In this work, we present an application of the model to capture the features of Taylor-type interest rate rules.

As the use of classical arbitrage-free models such as the Black & Scholes (1973) and Heston (1993) suggests, it is indeed possible to use derivatives prices to assess the market's perception of risk related to their primitives. Our model fits the gap of an arbitrage-free model that presents a primitive’s distribution directly influenced by systematic factors instead of only idiosyncratic risks, such as stochastic volatility or stock jumps. Pricing the derivatives from the assumed distribution of the primitives allows for the extraction of the risk premia related to the systematic variable through the same techniques widely used in empirical finance to estimate classical models. Therefore, although we do not touch data, our calculations result into a transform, the discounted conditional characteristic function, that might be used to elaborate inference on the derivative’s prices and, thus, extract the premia related to the systematic risk.

1.2 Literature Review

This theoretical work is focused on a model that can be applied to a general class of assets. We'll be very in touch with literature of arbitrage-free asset pricing, given that the variety of models gives us the tools to develop a model as broad as possible.

Literature in asset pricing theory and empirical finance has been especially interested in finding hypothesis on asset returns distributions that are consistent with observed prices movements. Classical models have been tested not only through time series analysis of stock panels but also from derivatives cross-sections and panels. In fact, recent literature reveals that derivatives data may provide crucial information on the risks related to their primitives. For instance, Broadie, Chernov and Jarrow & Yildirim (2003), Hinnerich (2008) and Chen, Liu & Cheng (2010) use the price of TIPS (Treasury Inflation Protected Bonds) to extract information on inflation. Barro (2006) documents that the possibility of rare economics disasters explains a part of asset pricing problems. Ang, Piazzesi & Wei (2006) extract information from GDP from yield curves. Piazzesi (2005) relates bond yields to Federal Reserve decisions.

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Johannes (2007) and Bollerslev & Todorov (2011) use respectively parametric and nonparametric models to study the impact of jumps on the risk premium of equity. Their work is important to us for two reasons. First, because it corroborates the importance of analysing derivatives to extract information of the risk premia. Second, because our work rely on the assumption that jumps are influenced by systematic factors. If jumps were found not to be important to explain real world data, our use of jumps in the model wouldn’t be well motivated.

It is already well accepted by researchers the idea that most stock returns distributions can be modelled through both stochastic volatility and jump events on their generating process. The existence of those risks, especially of sudden shifts of the stocks prices, and their complex relationship has strengthened the hypothesis that markets are incomplete, as opposed to the assumption from classical Mehra & Prescott (1985) equity premium puzzle. In this context, several asset pricing models based on no-arbitrage arguments to relate the derivative to its underlying asset have been developed. Many of these models, especially the classic ones, fall in the category of affine models studied by Duffie, Pan & Singleton (2000). That’s also going to be the case for our model. Perhaps the most simple of such models, Black & Scholes (1973) assume stock’s volatility to be constant and that there is a single risk factor to be taken into account when pricing options. Heston (1993) assumes a stochastic volatility model and thus present two risk factors: the stock’s diffusion shocks and the volatility’s diffusion shocks. Bates (2000) model incorporates a jump process (and thus a jump premium) as a third factor into its previous counterpart. Finally, Duffie, Pan & Singleton (2000) derive a double jump model, with jumps on both the stock and the volatility’s diffusions. For that last model, one can also look at Eraker, Johannes & Polson (2003) for a more general case. As it has already been highlighted, recent literature covers for example the case of jumping volatility, which we ignore in our model.

This work is located in the gap between Bates (2000) and Duffie et al. (2000) models. We assume the presence of stochastic volatility and jumps in the stock generating process. Pan (2002) adapts Bates model by making the jump-risk premium to be linearly dependent on the market volatility. We adapt this model by incorporating a finite number of processes in the jump-risk premium. By this modification we perform a generalization of the conditional distribution of the jumps occurrences, still leaving it as a “state-dependent” process. It is important to underline, however, that the model treated in this work is far from being the most general possible.

The idea of capturing the effect of systematic risks isn’t new also. For more recent work on systematic state variables, one can look at Bollerslev, Todorov & Li (2013) and Barigozzi, Brownlees, Gallo & Veredas (2014).

2 The general model

In this section, we will characterize the assumptions concerning the dynamics of the primitive (which we shall call “stock”) return process. As classical models analysis suggests, different hypothesis on the stock’s conditional distribution result in differences in the capability of the model to fit
the data. Thus, it is important to be clear about how we model the stock return process.

As already mentioned, our model belongs to the class of AJD models. We incorporate stochastic volatility, jumps, non-constant interest rate and dividend processes. The model presented in Pan (2002) would be a great starting point to understand the stock return dynamics presented in this work. The difference of our work, however, is that we assume the existence of one or more systematic state variables. It is straightforward to realize that there are tons of ways to model the impact such variable in asset pricing, even if we restrict ourselves to the class of AJD models. For this reason, we shall discuss briefly the assumption on the systematic variable before introducing the environment setting with the data-generating processes and risk-neutral processes.

As we want to identify the market’s perception of the likelihood of different scenarios of the systematic variable, it is crucial to incorporate its influence in the state price density. By that, we are implicitly assuming that the way individuals price the asset’s payoff will depend on how they evaluate the systematic variables possible paths. In comparison with the data-generating process of the systematic variable, its risk-neutral process can also present a “premium” component that impacts the pricing of derivatives.

The other fundamental hypothesis is that the systematic variable is linked to the stock’s jump process, in both the data-generating and the risk-neutral distribution. After all, our main motivation is to answer the question of what determines the jump-risk premium. By linking the systematic variable to the jump’s intensity we gain two features: first, we explicitly identify the risk premia for future econometric work; second, it allows our model to be used in the study of what processes influence a stock’s tail risks, which is a still open question in empirical finance. In fact, including the systematic variable as a determinant of the jump’s intensity constitute a nice robustness test to the hypothesis that the intensity is time invariant or to the hypothesis that it depends on other time variant processes, such as the volatility.

Given the discussion above, it is important to highlight the set of properties we will require the systematic variable to possess when entering the primitive’s conditional distribution in our model. There will be two basic properties:

1. **Pricing kernel property.** We want the systematic variable to affect the risk-neutral conditional distribution of the stock process, or its kernel price. If that isn’t true, then agents do not price its risk. In that case, even if we assume that the “data-generating” process is affected by the macro variable, derivative’s price won’t be, and computing its price will be a meaningless exercise.

2. **Jumps property.** As our main goal is to study the jump-risk premium and the impact it receives from systematic variables, it is clear that the systematic variable must be related to the distribution of the jump process. We will satisfy this property by assuming that the systematic variable impacts the intensity of the jump process.

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5As it will be noted, in our models markets are incomplete, and thus the equivalent martingale measure is not unique.
Properties 1 and 2 will be standing in the setting we present in next subsection.

Another property that we are going to use, although not one worthwhile to appear in the list above is that the variable at hand moves in cycles. Although this property is not fundamental, we believe that such setting would be able to capture the dynamics of many autonomous variables.

2.1 Stock return dynamics: data-generating process

In this subsection we mathematically characterize the assumptions on the stock return dynamics. We start by fixing a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a filtration \(\mathcal{F}_t\). Probability measure \(\mathbb{P}\) is considered the “actual” probability measure. We will discuss the conditional distribution of the stock return under a risk neutral mesure \(\mathbb{P}^\ast\) in subsection 2.2.

Let \(S_t\) be the ex-dividends stock price, \(V_t\) its instantaneous volatility, \(q_t\) the rate at which dividends are paid for this stock, \(r_t\) the interest rate process and \(m_{it}\) the \(i\)-th systematic variable, all of them evaluated at period \(t\).

The dynamics of the state variables under \(\mathbb{P}\) are given below

\[
\begin{align*}
    d\ln S_t &= \left[ r_t - q_t + \eta_s V_t + (\mu - \bar{\mu}) \left( \lambda_0 + \lambda_1 V_t + \sum_{i=1}^{N} \gamma_i m_{it} \right) - \frac{V_t}{2} \right] dt + \sqrt{V_t} dW_t^1 \\
    &+ \ln(1 + Y)dN_t - \mu \left( \lambda_0 + \lambda_1 V_t + \sum_{i=1}^{N} \gamma_i m_{it} \right) dt \\
    dV_t &= \kappa_v (\bar{\nu} - V_t) dt + \sigma_v \sqrt{V_t} dW_t^1 + \sigma_v \sqrt{1 - \rho^2} \sqrt{V_t} dW_t^2 \\
    dq_t &= \kappa_q (\bar{\eta} - q_t) dt + \sigma_q \sqrt{q_t} dW_t^3 \\
    dr_t &= \kappa_r (\bar{\mu} - r_t) dt + \sigma_r \sqrt{r_t} dW_t^4 \\
    dm_{it} &= \kappa_{im} (\bar{m}_i - m_{it}) dt + \sigma_{im} \sqrt{m_{it}} dW_t^{4+i} \quad i = 1, \ldots, N
\end{align*}
\]

where \(W_t = [W_t^1, \ldots, W_t^{N+4}]^T\) is an adapted standard multivariate brownian motion under \(\mathbb{P}\) and \(N_t\) is a Poisson process with instantaneous state-dependent intensity. \(\lambda_t \equiv \lambda_0 + \lambda_1 V_t + \sum_{i=1}^{N} \gamma_i m_{it}\). We assume \(\lambda_0, \lambda_1, \gamma_i > 0\) for \(i = 1, \ldots, N\). In order to keep things as simple as possible, we also assume \(\bar{m}_i > 0\) for every \(i\) and that the initial conditions are positive for each systematic variable.

As one can see, our model is an extended version of Bates (1996) model. It captures two major features of the stock return dynamics: stochastic volatility and price shifts. We now discuss the main differences from traditional models.

We start by analysing equation (1). It is basically the Itô’s lemma application of the log function on the original stock price process below:

\[
    dS_t = [r_t - q_t + \eta_s V_t + (\mu - \bar{\mu}) \lambda_t] S_t dt + \sqrt{V_t} S_t dW_t^1 + S_t - Y dN_t - \mu \lambda_t S_t dt
\]

\(^6\) We use the informal notation \(\lambda_t\) for the jump’s intensity even though it depends on the path of the state variables besides the period \(t\).

\(^7\) Notice that the case in which the systematic variable is assumed to take only negative levels is only a particular case of the above setting - we can multiply it by \(-1\) and model it as a positive process.
Under $\mathbb{P}$, the ex-dividend stock process must have the mean return of the interest rate minus the rate on which dividends are paid plus the premia for brownian shocks and for jumps. The $\mu \lambda_t$ term accounts for the expected change in the jump process $S_t Y dN_t$, which we now explain. The jumps occurrences are modelled through a Poisson process $N_t$. Its instantaneous intensity is linearly depend on both the volatility $V_t$ and the systematic variables $m_{it}$, $i = 1, \ldots, N$. The dependency on the volatility isn’t new in literature (see Pan (2002)). We introduce the dependency on the systematic variable $m_{it}$ so that property 2 from the last subsection is satisfied.

Conditional on the occurrence of a jump, the stock price shifts from $S_{t^-}$ to $S_{t^-}(1 + Y_k)$, where $k = N_t$ (i.e., $Y_k S_{t^-}$ is the jump size of the k-th jump). The jump sizes are independent of the Poisson process, as it is usually true in AJD models. The jump size factor $1 + Y$ is assumed to be log-normal under the risk-neutral measure, with constant mean $\mu_j$ and variance $\sigma_j^2$. Therefore, the jump size $Y$ has mean $\mu = e^{\mu_j + \sigma_j^2/2} - 1$. It should now be clear that the $\mu \lambda_t S_t$ compensates for the trend the jump process imposes on the stock price path.

Under the risk-neutral measure $\tilde{\mathbb{P}}$ we assume to be valid in subsection 2.2, random variable $1 + Y$ is still assumed to be log-normal, but with different mean $\tilde{\mu}_j$ and variance $\tilde{\sigma}_j^2$. Parameter $\tilde{\mu}$ accounts for the expected jump size of variable $Y$, conditional on the occurrence of a jump, under $\tilde{\mathbb{P}}$ (so that we have $\tilde{\mu} = e^{\tilde{\mu}_j + \tilde{\sigma}_j^2/2} - 1$). As for the intensity $\lambda_t$, we assume it to be the same under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$, following Pan (2002). Hence, the stock risk premia associated with the jump is given by the term $(\mu - \tilde{\mu}) \lambda_t S_t$. It must be clear now that, with this setting we allow the risk premia associated with the jump component of the stock prices to vary with more or less volatile markets as well as with the current state of the systematic variables. We are, therefore, satisfying property 1.

Finally, the term $\eta_s V_t$ represents the premium for the risk of brownian shocks. Overall, the state variable $\ln S_t$ captures two premiums. The first one, $\eta_s V_t$, corresponds to the risk of brownian shocks and the second one, $(\mu - \tilde{\mu}) \lambda_t S_t$, corresponds to the risk of jumps. Under the actual probability measure $\mathbb{P}$, the expected rate of growth of the stocks is given by the sum of the interest rate, the two premia for risk minus the dividend payment rate. Following empirical findings, we always expect to find positive premia for both types of shocks. That means we expect $\eta_s > 0$ and $\mu > \tilde{\mu}$.

We now turn our attention to the volatility process $V_t$, the dividend rate processes $q_t$ and the interest rate process, defined by equations (2), (3) and (4), respectively. All of them are autonomous mean reverting processes, as modelled in Cox, Ingersoll & Ross (1985). Their long run averages are given by parameters $\tilde{V}$, $\tilde{q}$ and $\tilde{r}$, while the mean-reversion rates are $\kappa_u$, $\kappa_q$ and $\kappa_r$, all assumed to be positive. Provided that their long term means and their initial conditions are all greater than zero, $V_t$, $q_t$ and $r_t$ are all going to be non negative with probability one, which seems to be a desirable property. Parameters $\sigma_u$, $\sigma_q$ and $\sigma_r$ give the expected standard deviation of those processes. Notice, that the volatility diffusion is allowed to be correlated with the stock diffusion. Parameter $\rho$ represents the correlation between the two diffusions. The dividends and the interest rate diffusions, on the other hand, are both assumed to be independent from the stock price diffusion, although allowing for a correlation seems to be an interesting case to be studied that doesn’t add much difficulty in finding the transforms. Before moving on, it is worth noting that, in order to maintain the
internal consistency of the model, the interest rate process parameters should be equal for every stock in the cross-section, unlike the volatility and the dividends-rate parameters, which should vary from stock to stock. In fact, strictly speaking, the interest rate is a systematic variable if we use the definition provided in subsection 1.1.

We now investigate the systematic variables $m_{it}$. First thing to be noticed is that we assume only a finite number of systematic variables to be relevant in the dynamics of the stock. Second, as already pointed out, we want to provide our systematic variables a “cycle-behavior” character, although other linear settings would be an easy alternative to our model. As we modelled in equation (5), $m_{it}$ will be mean reverting provided that $\kappa_{im} > 0$. We can say that $m_{it}$ is also an autonomous mean reverting process. Indeed, it will be characterized by the fact that it is expected to go up whenever it is below the long run average $\bar{m}_i$ and to go down when it is above $\bar{m}_i$. Its diffusion part is not allowed to correlate with any other diffusion, although it might reasonable to think that some stock (and other primitives) prices vary along systematic shocks, which makes the correlation case an interesting one (and not analytically costly).

2.2 The risk-neutral dynamics

In addition to the actual probability measure, we assume the existence of an equivalent martingale measure $\tilde{P}$. Since the market at hand is incomplete due to the log-normal jump size, there are infinite equivalent martingale measures, and so we can advantage of that fact to calibrate the model to fit the data.

Under the risk-neutral probability measure we choose, $\tilde{P}$, the dynamics of the state variables $\ln S_t$, $V_t$ and $m_{it}$, for $i = 1, \ldots, N$ are given bellow:

\[
\begin{align*}
    d\ln S_t &= \left[r_t - q_t - V_t^2 / 2\right] dt + \sqrt{V_t} d\tilde{W}_1^1 + \ln(1 + Y) dN_t - \tilde{\mu}_t \lambda_t dt \\
    dV_t &= [\kappa_v (\bar{v} - V_t) + \eta_v V_t] dt + \sigma_v \sqrt{\bar{V}_t} d\tilde{W}_1^1 + \sigma_v \sqrt{1 - \rho^2} \sqrt{V_t} d\tilde{W}_1^2 \\
    dm_{it} &= [\kappa_{im} (\bar{m}_i - m_{it}) + \eta_{im} m_{it}] dt + \sigma_{im} \sqrt{m_{it}} d\tilde{W}_4^i \quad i = 1, \ldots, N
\end{align*}
\] (6) (7) (8)

Under $\tilde{P}$, $\tilde{W}_1$, $\tilde{W}_2$ and $\tilde{W}_4$, $i = 1, \ldots, N$ are all independent standard Brownian motions. Also, $W^1$ and $W^4$ still are considered Brownian motions, so that the dynamics of the processes $q_t$ and $r_t$ are still given by equations (3) and (4) respectively. $N_t$ is still a Poisson process with intensity $\lambda_0 + \lambda_1 V_t + \sum_{i=1}^N \gamma_i m_{it}$ under $\tilde{P}$, which guarantees that property 1 given in the beginning of section 2 is satisfied, for the systematic variables are a part of the pricing kernels. As argued in subsection 2.1, the moments of the jump size change under $\tilde{P}$. The expected jump size on $\ln S$ conditional on the occurrence of a jump is given by $\tilde{\mu}_j$ and the variance is $\tilde{\sigma}_j^2$. Correspondingly, the mean of $Y$, conditional on the jump occurrence, is $\tilde{\mu}$.

Under the risk-neutral measure, the ex-dividend stock price is expected to grow at rate $r_t - q_t$, as it should be by the definition of a risk-neutral probability measure. The term $-\tilde{\mu}_t \lambda_t dt$ compensates for the jumps coming from the term $\ln(1 + Y) dN_t$, so that the sum of them do not add trend to
the overall process, only volatility.

The volatility process under $\tilde{P}$ is represented by equation (7). Once more, we model it as a mean-reverting process, but now we add a premium $\eta_t V_t$ that compensates for the stochastic component of the volatility. Since volatility per se cannot be traded as a primitive asset, the premium for the fact that the stock’s volatility is volatile appears only in the dynamics of option prices. In other words, while stock prices won’t reflect the term $\eta_t V_t$, derivatives’ prices will. Previous literature has already acknowledged the existence of this premium (one can see evidence of this term in Lamoreux & Lastrapes (1993)). Once again, the volatility diffusion is allowed to be correlated with the stock diffusion. We assume parameter $\rho$ to be same considered in the data-generating process.

The analysis of the systematic variables under $\tilde{P}$ is pretty much the same we just did for $V_t$. Equation (8) provides the dynamics of $m_{it}$. Again, we consider the existence of a premium $\eta_{im} m_{it}$ investors demands for incurring in risks embedded in the volatility that the diffusion parts of $m_{it}$ add to the stock price process. These diffusions are again assumed to be independent of the diffusion part of $\ln S_t$. The existence of these premia, however, creates a serious problem of identification of the components of the overall jump-risk premium. It is likely that additional hypothesis are to be made in eventual work to estimate the parameters of the model.

Finally, as we have already said, the dynamics of the interest process and the dividends process are not changed under the risk-neutral measure.

In the next section we discuss a procedure that allows the implementation of this model in the case in which the systematic variables are not mutually independent or independent from the volatility.

3 A two stage orthogonalization procedure

In this section we turn to the matter of econometric estimation. There are two problems related to estimation in the model presented in section 2. The first problem, quite common in empirical finance literature, is how to extract the volatility series. Unlike the underlying asset price, the dividends, the interest rate and the systematic variables, all of which we assume to be observable, the instantaneous volatility path is, in general, unknown. If we don’t have a volatility series, we won’t be able to perform estimation of the model’s parameters.

The second problem is more particular of the model presented in this paper. It relates to the quite strong assumption that the systematic variables Brownian shocks are mutually independent and, also, independent from the volatility shocks. We must acknowledge that such hypothesis might not apply in most cases, especially if we are dealing with macroeconomic aggregates, which will typically present some degree of correlation. If we cannot convert the systematic series we are dealing with into independent Brownian-motion driven processes, the model developed in the section 2 will not be applicable.

We now propose a methodology that can be applied in the attempt to partially solve both of these drawbacks. To make things simpler, we shall consider, in the course of this section, the case of
a single systematic variable\textsuperscript{8}. The method consists on a two stage estimation procedure. To sum up, in the first stage we ignore the role of the systematic variable and identify a volatility path. In the second stage, we make use of the (now available) volatility path to project the systematic variable series and, by that, we create “shock” series independent, by construction, of the instantaneous volatility series. We can then estimate the parameters of the model and, consequently, the evolution of the risk-premia.

The procedure above separates the treatment of the estimation problem in two. Subsections 3.1 and 3.2 explain the procedure in detail.

3.1 Extracting the volatility path

The instantaneous volatility, which we denoted $V_t$ in equation (2), is not observable in most cases. This is a problem because we need a volatility path to perform the systematic variable projection (explained in subsection 3.2) and because it is required to estimate the model’s parameters.

In order to extract a volatility path from the data, we can, in a first moment, ignore the existence of the systematic variable and estimate a alternative, simpler version of the baseline model. Notice that the no-systematic-variable case of our model is applied in Pan (2002). However, the volatility series can be extracted through even simpler and more intuitive models. Consider for instance a modified version of Bates (1996) model:

$$
d\ln S_t = \left[ r_t - q_t + \eta_s V_t + (\mu - \tilde{\mu})\lambda - \frac{V_t}{2} \right] dt + \sqrt{V_t} dW_t^1 + \ln(1 + Y)dN_t - \mu\lambda dt
$$

$$
dV_t = \kappa_v (\bar{V} - V_t) dt + \sigma_v \rho \sqrt{V_t} dW_t^1 + \sigma_v \sqrt{1 - \rho^2} \sqrt{V_t} dW_t^2
$$

with $q_t$ and $r_t$ still given by equations (3) and (4), respectively.

The underlying asset price series is not sufficient to identify the volatility path. The observation of option prices is required. Using the model above, a volatility path can be extracted using the fact that, given the asset price path, the option price is a function of the current volatility state. Different techniques have emerged in empirical finance literature to perfom the extraction using the underlying asset and the option price series. Most recently, Andersen, Fusari & Todorov (2013) presented a estimation procedure for the volatility based on the observation of option panels.

3.2 Systematic variable projection

We already argued that a very strong and not always reliable assumption made in the baseline model is that the Brownian shocks driving the systematic variables are mutually independent and independent of the Brownian shocks driving the underlying asset price (although this particular hypothesis could be relaxed), the instantaneous volatility, the interest rate and the dividends processes. Since we would like to apply our framework to macroeconomic variables and other variables

\textsuperscript{8}The case of more than one systematic variable represents a rather subtle modification of the procedure we explain in here.
typically correlated to each other and correlated to the state processes, it is mandatory to develop a procedure to convert those variables to independent Brownian motion driven ones.

The procedure we propose consists of converting the systematic variables “as they are” into their “shocks” counterpart. In the case of a single systematic variable, the idea would be to convert the original systematic variable time series by projecting it into the volatility, the interest rate and the dividends time series. Such procedure is extensively used in empirical macroeconomics.

Formally, the procedure consists on the least squares estimation of the following reduced form model:

\[ m_t = \nu_0 + \nu_v \hat{V}_t + \nu_q q_t + \nu_r r_t + \varepsilon_t \]  

(9)

where \( m_t \) is our single systematic variable\(^9\) and \( \hat{V}_t \) is the volatility series, estimated according to the procedure described in subsection 3.1.

Consider the series of least squares estimated errors \( \{\hat{\varepsilon}_t\} \) calculated from model (9). By construction, the errors are orthogonal to the state variables series. The orthogonalization procedure consists of replacing the correlated systematic variable by this error series. The error’s interpretation is somewhat different from its correlated counterpart. It represents shocks on the systematic variable conditional on the occurrence of the instantaneous volatility, the interest rate and the dividend payment, rather than the systematic variable per se. In a few words, the procedure consists of replacing the systematic variable by the shocks it suffers. Those shocks represent the moves of the systematic variables unspanned by the other state variables.

It is to be expected that the parameters related to the systematic variable change significantly, given that their interpretations is somewhat different. Nevertheless, the estimation of the risk-premia will still be meaningful, since, as already argued, we are removing from the systematic variable series only the movements spanned by the other state variables. So, the shocks suffered by the systematic variable (given all the other Brownian shocks) correspond to the only “new information” added by the use it. More than that, the assumption of the baseline model will be satisfied, for the systematic variable will now be independent from the other state variables.

With the volatility series and the systematic variables orthogonalized, the econometrician is ready to estimate the model’s parameters. Again, literature offers many different techniques to perform the estimation which we will not discuss. Our only goal in this work is to present the dynamics of the model with systematic variables and to focus on the particular problems that its setting generates in the estimation front.

We now provide an example of application of the general model just developed.

\(^9\)In the case of multiple systematic variables, we can replace model (9) by multiple projections for each systematic variable:

\[ m_{it} = \nu_0 + \nu_v \hat{V}_t + \nu_q q_t + \nu_r r_t + \sum_{j<i} \nu_{ij} \hat{\varepsilon}_{jt} + \varepsilon_{it} \]

where \( \hat{\varepsilon}_{jt} \) now representing the shocks impinging on the \( j \)th systematic variable.
4 An application to macroeconomic processes

Before we proceed to the calculations that lead to the pricing of derivatives, we want to use our general setting to evaluate the case of macroeconomic variables. When macroeconomic variables are considered stochastic and affect the fundamentals of a group of companies, they constitute by nature a systematic source of risk from the point of view of investors. We can apply the model we just developed to assess the premium investors charge to face the risk related to macroeconomic variables shocks. Not only that, we can also deepen the theory and make small changes in the original model so that it covers eventual relationships we assume to exist between the macroeconomic variables and the interest rate (as it is predicted by macroeconomic theory).

As it is indeed predicted by theory, it is reasonable to assume that the perception of the macroeconomic variable by agents is connected to the their perception of the interest rate process. For example, if agents believe that inflation will rise in the future, they are more likely to attribute greater probability to higher interest rate. Likewise, analysts might predict smaller interest rates if they believe unemployment will rise. In order to capture this possibility, we will require the systematic variable to fulfil one additional property, besides the ones presented in the beginning of section 2.

3 Interest-rate relation property. Rather than modelling the interest rate as a completely independent process, we want to open the possibility of agents to relate the paths of the interest rate process to the paths of one of the systematic variables. As discussed, this property might be important to capture features from macroeconomic variables. Moreover, this relation should not differ from stock to stock.

This property will be very important if we use the setting to model macroeconomic variables. As the interest rates interfere directly on the pricing of a stock (through both the discounting process and the risk-neutral drift), it is an important channel to identify the macroeconomic risk premium. As we would be treating a macro variable that is proposed to account for the systematic risk, a reasonable restriction would be to require this relationship macro variable/interest rate to be constant in the stocks cross-section.

We now start to adapt the original model. We begin by assuming that the expectations on the interest rate process $r_t$ and the macro variable $M_t$ (which we consider to be $m_{1t}$ without loss of generality) have a relationship characterized by the measurable function $f : \mathbb{R} \mapsto \mathbb{R}$:

$$\mathbb{E}[r_t|\mathcal{F}_s] = \mathbb{E}[f(M_t)|\mathcal{F}_s]$$

where $\mathbb{E}[,]$ is the expectation taken with respect to $\mathbb{P}$.

In order to maintain the linear setting of the model, we assume that $f$ is an ordinary linear function:

$$f(x) = f_0 + f_1 x$$
Regarding equation (10), although the deterministic relationship \( r_t = f(m_t) \) case seems an interesting one, we decided to look to the stochastic case, with the relationship standing only when expectations are taken. It’s important to be said that the deterministic case would also be much more costly analytically, since the Ricatti equations that would have to be solved in order to find the price of a derivative would have nonconstant (and nonlinear) coefficients. We shall discuss more about the pricing process later in section 6. In what concerns equation (11), breaking the linearity would be very valuable in the sense that many macroeconomic variables can be modelled with a nonlinear relationship with the interest rate. Breaking the linearity, however, would make calculations much more breathtaking as we would depart from the standard affine setting. For this reason and the fact the many macro variables can be modelled through equation (11), we’ll refrain from breaking linearity.

Another point to be made is that, in order to assure that property 3 stands, the parameters \( f_0 \) and \( f_1 \) in equation (11) are to be assumed constant across different assets.

The dynamics of the state variables are now presented. Under \( \tilde{\mathbb{P}} \), state variables \( \ln S_t, V_t \) and \( q_t \) are still driven by (6), (7) and (3), respectively. The interest rate and the systematic variables are modelled as follows:

\[
\begin{align*}
    dr_t &= \zeta \kappa_M (\bar{r} - r_t) \, dt + \sigma_r \sqrt{r_t} \, dW^4_t & \text{for } \bar{r} = f(\bar{M}) \tag{12} \\
    dM_t &= dm_{1t} = [\kappa_M (\bar{M} - M_t)] \, dt + \sigma_M \sqrt{\bar{M}} \, d\tilde{W}^5_t \tag{13} \\
    dm_{it} &= [\kappa_{im} (\bar{m}_i - m_{it}) + \eta_{im} m_{it}] \, dt + \sigma_{im} \sqrt{m_{it}} \, d\tilde{W}^4_{i+t} & i = 2, \ldots, N \tag{14}
\end{align*}
\]

We use the same notation of section 2, so that \( \tilde{W}^1, \tilde{W}^2 \) and \( \tilde{W}^i, i = 1, \ldots, N \) are all independent standard Brownian motions, \( N_t \) is a Poisson process and so on.

We assume \( M_t \) to be a macroeconomic variable. The other systematic variables can have any interpretation. All of them are still autonomous mean-reverting processes, and their stochastic differential equations are the same they were in (8). The interpretation of the parameters also remain the same. Notice, however, that we assume that \( \eta_{im} = 0 \).

The big difference is in the interest rate process described by equation (12). It is connected to the macro variable \( M_t \) through its drift component. Nonetheless, the interest-rate is still a mean-reverting process. We assume its long-run average \( \bar{r} \) is given by \( f(\bar{M}) \) which is the same as assuming that, in the long-run, the relation between the macro variable and the interest rate given by function \( f \) will stand. Assuming \( \bar{r} = f(\bar{M}) \) to be true gives parameter \( \zeta \) a useful interpretation when it comes to verifying if equation (10) stands. It represents the deviation of the interest adjustment rate to the macro variable adjustment rate. If we assume the interest rate/macro variable relationship (10) to be true, then \( \zeta \) can only take on single value. It follows proposition 1:

**Proposition 1.** Assume that the dynamics of \( S_t, V_t, q_t, r_t \) and \( m_{it}, i = 1, \ldots, N \), are given by equations (6), (7), (3), (12), (13) and (14), and that the relationship between the interest rate and the macro variable given by (10) also stands. Then \( \zeta = 1 \).

**Proof.** Consider the process \( x_t = f_0 + f_1 M_t - r_t \). If equation (10) stands, \( x_t \) must be a martingale,
which means that its drift term must equal zero. Then, for $t > 0$ we have:

$$dx_t = f_1 dM_t - dr_t$$

$$\implies \mathbb{E}[dx_t] = \mathbb{E}[f_1 dM_t] - \mathbb{E}[dr_t]$$

$$= \left[ \kappa_M \left( f_1 \bar{M} - \mathbb{E}[f_1 M_t] \right) - \zeta \kappa_M \left( \bar{r} - \mathbb{E}[r_t] \right) \right] dt$$

(applying Fubini’s theorem)

$$= \left[ \kappa_M \left( f_0 + f_1 \bar{M} - \mathbb{E}[f_0 + f_1 M_t] \right) - \zeta \kappa_M \left( \bar{r} - \mathbb{E}[r_t] \right) \right] dt$$

$$= \left[ \kappa_M \left( \bar{r} - \mathbb{E}[r_t] \right) - \zeta \kappa_M \left( \bar{r} - \mathbb{E}[r_t] \right) \right] dt$$

The term in brackets is zero for every value of $t$ if and only if $\zeta = 1$. \hfill \Box

All Proposition 1 states is that if the interest rate and the macro variable are expected to have the linear relation (10), then they must have the same mean-reversion rate. Intuitively, testing $\zeta = 1$ is testing whether the interest rate and the macro variable will typically have cycles of equal length. It is crucial to keep in mind that the random variables at hand are characterized under the risk-neutral measure, which means that the processes presented in equations (12) through (14) (along with (6) and (7)) characterize the perception of agents when evaluating the states of nature rather than the actual relationship between the variables. Thus, what the model proposes to study isn’t the determinants of macroeconomic policy or the true link between a macroeconomic variable and the interest rate evolution. In fact, such relation might not even exist. What the model wants to identify is whether agents take into account any relation of that sort when pricing, and, as it was our original goal, whether the macroeconomic variable $M_t$ impacts the risk premium relative to the jump component of the stock’s prices.

Lastly, as it was already assumed to be true for $f_0$ and $f_1$, we must also assume that the other parameters associated with the $r_t$ and $m_t$ processes, namely $\kappa_m$, $\bar{m}$, $\sigma_m$ and $\sigma_r$, do not vary from asset to asset. Although it doesn’t constitute an effective restriction on our work, because we will not perform estimation, such restriction must be taken into account in any future statistical exercise performing an application of the model.

Next, we deepen our current application of macroeconomic variables by giving exact meaning to the macro variables.

4.1 Taylor rule application

Literature in macroeconomics has always been concerned with the role played by interest rates in the evolution of the product and unemployment, for example. Also, researchers have pointed out several potential variables that determine the interest rate or, more importantly for us, are taken into account by central banks when choosing the monetary policy and the nominal rate.

Although there are indeed many possibilities of macro variables to be modelled as linked to the interest rate, as an example we choose the Taylor rule for its simplicity and intuition. We consider
the following rule:

$$r_t = \delta_0 + \delta_\pi (\pi_t - \bar{\pi}) + \delta_y (y_t - \bar{y})$$  \hspace{1cm} (15)$$

for $\delta_\pi, \delta_y > 0$. We call $\pi_t$ the current inflation rate, $\bar{\pi}$ the inflation target fixed by the central bank, $y_t$ the logarithm of the real GDP and $\bar{y}$ the logarithm of the potential real GDP.

It would be interesting to test the inflation rate and the GDP independently as the macro variable $m_t$ by setting $\bar{\pi}$ and $\bar{y}$ respectively as the long-run average $\bar{m}$ in equation (13). Given that typically $\delta_\pi, \delta_y > 0$, we would have a mean reverting process as predicted in the last subsection. However, extending the model presented in this work to a model that incorporates two macro processes with stochastic differential equation of the mean-reverting form would be a simple exercise. Instead of (12), we would then model the interest rate as:

$$dr_t = \zeta_1 \kappa_1 m (\bar{\pi} - m_{1t}) dt + \zeta_2 \kappa_2 m (\bar{y} - m_{2t}) dt + \sigma_r \sqrt{r_t} dW_t^4$$

where $m_{1t}$ and $m_{2t}$ would be respectively inflation and log-GDP. The condition analogous to (10) would then imply $\zeta_1 = \delta_\pi$ and $\zeta_2 = \delta_y$ with a proof analogous to the proof of Proposition 1, which would have to be tested with observed market prices. In the case above, we would have to change the price of the derivatives in relation to what is done in section 6. The alteration, however, isn't too complicated.

Alternatively, if we assume the inflation and the log-GDP to have the same mean-reversion rate (say, $\kappa_\pi$), we would still be able to model the interest rate as a single factor Cox-Ingersoll-Ross process:

$$dr_t = \zeta \kappa_\pi (\bar{r} - r_t) dt + \sigma_r \sqrt{r_t} dW_t^4$$

In that case, we would set $\bar{r} = \delta_0 - \delta_\pi \bar{\pi} - \delta_y \bar{y}$ and the test would be the same as before: $\zeta = 1$.

If we change the model in one of the ways considered above, we can fit our Taylor rule (15) to the setting of the model. Then, we would be able to identify whether agents associate the paths of inflation rate and log-GDP in the future with a path of interest rate that satisfies the Taylor rule and, if they do, whether this affects the jump-risk premium.

5 Simulations

We now apply the techniques so far developed in a numerical example. Our goal is to generate the volatility smile. We go back to the case of a macroeconomic variable, which, once again, we call $M_t$. We assume that, under the risk-neutral measure $\tilde{P}$, the dynamics of the state variables $(\ln S, V, q, r, M)$ are given by equations (6), (7), (3), (12) and (13), respectively. We calibrate the model using the parameters given in table 1. We want to perform our analysis assuming $\zeta = 1$, that is, we assume agents believe in the macro variable/interest rate relationship $^{10}$.

$^{10}$Notice that we set $\eta_M = 0$ in the calibration.
Using the transform calculated in section 6, we compute the price of a put that pays

$$\max (k - S_T, 0)$$

at period $T$ and calibrate $T = 1$, assuming we start from period 0.

We calculate the surface of the put prices for different starting values of $\frac{k}{S}$ and $M$. The state variables initial values were fixed at the levels:

- $S_0 = 10$
- $V_0 = 0.25$
- $q = 0.05$
- $r_0 = 0.1$
- $\gamma_1 = 4$
- $\eta_M = 0$
- $\bar{M} = 0.06$

The prices we present are given in terms of the implied Black-and-Scholes volatility, so that we can observe the volatility smile directly. The surface is exhibited in figure 1.

It is clear that the implied volatility is much greater when the put is out-of-the-money, which happens for small values of the $y$-axis. For those small values, the constant volatility that generates the price is well above the volatility that generates the price for larger values of the strike (when the option is “more in-the-money”). Figure 1 shows that our model is capable of generating the phenomenon know in literature as the volatility smile: options fixed at different strike levels will have their market prices justified by different constant volatilities, if evaluated by the Black-and-Scholes formula.

Different starting values of the systematic variable also change the implied volatilities, but in a smoother way, almost imperceptible in figure 1. We present in figure 2 the curve of implicit volatilities for different levels of $M_0$. We fixed $\frac{k}{S} = 1$. The greater the current value of the systematic macro variable, the greater the implied volatility. We can interpret this result through the fact that

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_0$</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>1</td>
</tr>
<tr>
<td>$\tilde{\mu}_j$</td>
<td>-0.04</td>
</tr>
<tr>
<td>$\tilde{\sigma}_j$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\sigma_v, \sigma_q, \sigma_r, \sigma_M$</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_v$</td>
<td>1</td>
</tr>
<tr>
<td>$\kappa_v$</td>
<td>2</td>
</tr>
<tr>
<td>$\kappa_q, \kappa_r, \kappa_M$</td>
<td>1</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>0.25</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\bar{r}$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.5</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>4</td>
</tr>
<tr>
<td>$\eta_M$</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{M}$</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Table 1: Calibration
higher levels of the macro variable imply higher risk demanded by investors to compensate for being exposed to the jumps, given that they occur with bigger intensity. The risk of jumps, however, is not captured by the constant volatility of the Black and Scholes model. As a result, extraction of implied volatility results in greater volatility for high levels of jump-risk, that is, high levels of the current systematic variable.

We now proceed to the calculation of the transform that characterizes different derivatives prices.

6 Calculating the discounted characteristic function

In this section, we derive the transform that permits the pricing of derivatives and the computation of other relevant information, such as the density of the state variables. We won’t be concerned with any specific function, as our immediate goal is only to introduce a parsimonious model capable of extrating the systematic component of the jump-risk premium.

In order to expose the transform, we follow the steps and notation of Duffie, Pan & Singleton (2000). In their work, they provide a guideline to calculate the price of contigent claims when the state variables have an affine setting, as they have in our model.

For now on, we denote $X_t$ the vector containing the state variables described in subsection 2.2 in the order they have been stated, $X_0$ a given initial condition for $X_t$ that, for simplicity, assures the mean-reverting processes to start in positive territory, and $\chi$ the set of exogenous parameters that determine the dynamics of the model$^{11}$. Notice that we are dealing with the dynamics under the risk-neutral measure $\tilde{P}$. We also denote $D \equiv \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^N$ the space on which the state variables are defined. We want to price an asset that pays the value $e^{a \cdot X_T}$ at period $T$.

$^{11}$For more detail, see Duffie, Pan & Singleton (2000).
contigent on $b \cdot X_T \leq y$. We compute such price with the function $G_{a,b}(\cdot; X_0, T, \chi) : \mathbb{R} \mapsto \mathbb{R}_+$, which, by risk-neutral pricing is given by:

$$G_{a,b}(y; X_0, T, \chi) = \mathbb{E} \left[ \exp \left( - \int_0^T r_t dt \right) e^{a \cdot X_T} \mathbf{1}_{b \cdot X_t \leq y} \right]$$

Expectations are taken with respect to the same risk-neutral probability that we assume to generate the dynamics described in subsection 2.2.

As it is well documented in Duffie, Pan & Singleton (2000), one can use function $G_{a,b}$ to price a wide variety of derivatives. An expression for $G_{a,b}$ can be obtained through the inversion of its Fourier transform. The Fourier transform, in turn, can be calculated from the discounted conditional characteristic function $\psi : \mathbb{C}^5 \times D \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{C}$ defined by

$$\psi(u, X_t, t, T) = \mathbb{E} \left[ \exp \left( - \int_t^T r_t dt \right) e^{u \cdot X_T} \bigg| \mathcal{F}_t \right]$$

Under certain conditions that guarantee that the integrals converge, function $\psi$ can be written in the following way:

$$\psi(u, s, v, q, r, m_1, \ldots, m_N, t, T) = \exp \left( \alpha(u, t, T) + \beta_s(u, t, T) s + \beta_v(u, t, T) v + \beta_q(u, t, T) q \\
+ \beta_r(u, t, T) r + \sum_{i=1}^N \beta_{im}(u, t, T) m_i \right)$$

for functions $\alpha, \beta_s, \beta_v, \beta_q, \beta_r, \beta_{im} : \mathbb{C}^5 \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{C}$. Notice that in the rest of the text, we'll
deal with square root and logarithms of complex numbers, which are not uniquely defined. Our calculations are done according to MatLab (and traditional) standards. For every complex $c \in \mathbb{C}$, we define $\ln(c) = \ln|c| + i \arg(c)$ and $\sqrt{c} = |c|^{1/2} \exp \left( \frac{i \arg(c)}{2} \right)$ where $\arg(c)$ is defined by $c = |c| \exp (i \arg(c))$ for $-\pi < \arg(c) < \pi$.

For fixed $u = (u_s, u_v, u_q, u_r, u_m)$, $t$ and $T$, the ordinary differential equations (ODE) that these functions must satisfy are:

\[
\begin{align*}
\dot{\beta}_s &= 0 \\
\dot{\beta}_v &= \left[-\frac{\sigma_v^2}{2}\right] \beta_v^2 + \left[- (\eta_v - \kappa_v) - \sigma_v \rho \beta_s\right] \beta_v + \left[\left(\tilde{\mu} \lambda_1 + \frac{1}{2}\right) \beta_s - \frac{\beta_v^2}{2} - \lambda_1 \phi(\beta)\right] \\
\dot{\beta}_q &= \left[-\frac{\sigma_q^2}{2}\right] \beta_q^2 + [\kappa_q] \beta_q + [\beta_s] \\
\dot{\beta}_r &= \left[-\frac{\sigma_r^2}{2}\right] \beta_r^2 + [\zeta \kappa_m] \beta_r + [1 - \beta_s] \\
\dot{\beta}_{im} &= \left[-\frac{\sigma_{im}^2}{2}\right] \beta_{im}^2 + [- (\eta_{im} - \kappa_{im})] \beta_{im} + [\tilde{\mu} \gamma_i \beta_s - \gamma_i \phi(\beta)] \quad \text{for } i = 1, \ldots, N
\end{align*}
\]

where arguments have ommited, $\dot{\beta}$ denotes the derivative of $\beta$ with respect to current time $t$ argument, $\phi(a) \equiv e^{\tilde{\mu} a_1 + \frac{a_2^2}{2}} - 1$ and $\beta$ denotes the vector $(\beta_s, \beta_v, \beta_q, \beta_r, \beta_1m, \ldots, \beta_{Nm})$. The boundary condition for each of the ODE is $\beta_i(u, T, T) = u_i$ with $i = s, v, q, r, 1m, \ldots, Nm$.

Function $\alpha$ in turn has to satisfy

\[
\dot{\alpha} = -\lambda_0 \phi(\beta) + \tilde{\mu} \lambda_0 \beta_s - \kappa_v \bar{\beta}_v - \kappa_q \bar{\beta}_q - \kappa_r \bar{\beta}_r - \sum_{i=1}^{N} \kappa_{im} \bar{m}_i \beta_{im}
\]

with boundary condition $\alpha(T) = 0$. Once we find $\beta_i$ for $i = s, v, q, r, m$, we'll be able to easily calculate it.

The differential equations relative to the elements of $\beta$ are of Riccati type, which in general allows for relatively easy solutions. A deeper look into the equations shows that their solution might be analitically costly since its coefficients are not constant. However, the coefficient depend only on $\beta_s$, which has a constant solution.

Now we state the solutions. First ODE implies that

\[
\beta_s(u, t, T) = u_s \quad (17)
\]

For the ODE's relative to $\beta_v, \beta_q, \beta_r, \text{ and } \beta_{im}$ for $i = 1, \ldots, N$, general solution can be stated as

\[
\begin{align*}
\beta_k(u, t, T) &= \frac{2d_k(u)u_k - \bar{\beta}_k(u) \left[2d_k(u) + \sigma_k^2 (u_k - \bar{\beta}_k(u))\right]}{2d_k(u) - \left[2d_k(u) + \sigma_k^2 (u_k - \bar{\beta}_k(u))\right]} (1 - e^{-d_k(u)(T-t)}) \\
\beta_k(u, t, T) &= \frac{2d_k(u)u_k - \bar{\beta}_k(u) \left[2d_k(u) + \sigma_k^2 (u_k - \bar{\beta}_k(u))\right]}{2d_k(u) - \left[2d_k(u) + \sigma_k^2 (u_k - \bar{\beta}_k(u))\right]} (1 - e^{-d_k(u)(T-t)}) \\
k = v, q, r, 1m, \ldots, Nm
\end{align*}
\]
where we define the following auxiliary functions:

\[ a_v(u) \equiv [(\eta_0 - \kappa_v) + \sigma_v u_s] \quad a_{im}(u) \equiv \{ a_v(u)^2 + \sigma_v^2 [(2\bar{\mu} \lambda_1 + 1) u_s - u_s^2 - 2 \lambda_1 \varphi(u)] \}^{\frac{1}{2}} \]

\[ \bar{\beta}_v(u) \equiv \frac{d_v(u) - a_v(u)}{\sigma_v^2} \]

\[ d_q(u) \equiv (\kappa_q^2 + 2\sigma_q^2 u_s)^{\frac{1}{2}} \quad \bar{\beta}_q(u) \equiv \frac{d_q(u) + \kappa_q}{\sigma_q^2} \]

\[ d_r(u) \equiv (\kappa_r^2 + 2\sigma_r^2 (1 - u_s))^2 \quad \bar{\beta}_r(u) \equiv \frac{d_r(u) + \kappa_r}{\sigma_r^2} \]

\[ a_{im} \equiv \eta_{im} - \kappa_{im} \quad d_m(u) \equiv \{ a_{im}^2 + 2\sigma_{im}^2 \bar{\mu}_v (\bar{\mu}_v - \varphi(u)) \}^{\frac{1}{2}} \]

\[ \bar{\beta}_{im}(u) \equiv \frac{d_{im}(u) - a_{im}}{\sigma_{im}^2} \]

The last three definitions stand for \( i = 1, \ldots, N \).

In the definitions above, the \( \bar{\beta}_i \) terms are particular solutions of the Ricatti differential equations and can be calculated by solving an ordinary second order polynom. The \( d_i, a_v \) and \( a_{im} \) terms are auxiliary and serve to maintain notation as simple as possible.

We now proceed to the solution of \( \alpha \), which can be found by simple integration of the \( \beta_i \) functions for all the \( i \):

\[ \alpha(u, t, T) = \lambda_0 \varphi(u)(T-t) - \bar{\mu}_0 \int_t^T \beta_s ds + \kappa_v \bar{v} \int_t^T \beta_v ds + \kappa_q \bar{q} \int_t^T \beta_q ds + \kappa_r \bar{r} \int_t^T \beta_r ds + \sum_{i=1}^N \kappa_{im} \bar{m}_i \int_t^T \beta_m ds \]

Integration of \( \beta_s \) is simple. For the other elements of \( \beta \), integration leads to a function with the same face since the general solution (18) applies for each one of those \( \beta_i \). We define functions \( g_i(u, t, T) \) to compute the integrals above:

\[ g_k(u, t, T) = -\left( \frac{2d_k(u)}{\sigma_k^2} - \bar{\beta}_k \right) (T-t) \]

\[ -\frac{2}{\sigma_k^2} \ln \left[ \frac{\sigma_k^2}{2d_k(u)} (\bar{\beta}_k(u) - u_k) \left( 1 - e^{-d_k(u)(T-t)} \right) + e^{-d_k(u)(T-t)} \right] \]

for \( k = v, q, r, \) \( 1m, \ldots, Nm \).

So finally \( \alpha \) is given by:

\[ \alpha(u, t, T) = \lambda_0 \varphi(u) - \bar{\mu} u_s \right) (T-t) + \kappa_v \bar{v} g_v(u, t, T) + \kappa_q \bar{q} g_q(u, t, T) + \kappa_r \bar{r} g_r(u, t, T) + \sum_{i=1}^N \kappa_{im} \bar{m}_i g_{im}(u, t, T) \]

With equations (16), (17), (18), (19) and (20), we have the closed form solution for \( \psi \), which was exactly what we needed in order to assess the Fourier transform of \( G_{a,b} \). From this point,
finding the solution for $G_{a,b}$ is only a matter of calculating an integral. We shall not go into details, as the step-by-step has already been documented in Duffie, Pan & Singleton (2000) with a useful example of the process in Pan (2002).

What is important for us right now is to highlight once more that through the price of a derivative, we can use observed derivatives prices to extract information on agents perception of a macroeconomic factor.

7 Conclusion

Our goal in this work was to address the question of what factors determine stock jumps. We introduced a model capable of integrating systematic risks into the stock's risk premia for jumps source of uncertainty. By assuming that the systematic variables influenced the pricing kernel and the jumps intensity, we were able to incorporate them in a classic AJD setting.

While at first sight the systematic variables do not appear to have a very different meaning from idiosyncratic variables, such as volatility, proper application of the model can result into interesting analysis. We proposed an application of the general model to the particular case of macroeconomic variables, or at least variables that we assume agents perceive to be linearly related to the interest rate. In fact, the model can be used to estimate whether agents believe in such relationship.

In the course of this work, we assumed the systematic variables to be driven by uncorrelated Brownian shocks. In many applications, this seems to be a rather strong assumption. In the view of this problem, we also proposed a method to avoid it. Following related literature, the main idea was to work with the shocks in each systematic variable given the realization of the state variables (instantaneous volatility, dividends and interest rate). In order to do that, first we propose a method to extract a volatility series from the data (and this is why we call the whole method a two stage procedure). Although the method leads to a change in the interpretation of some of the coefficients, as it should, by using it we are able to fit a wider class of applications to the framework developed in the baseline model.

The setting of the model was selected so that its functionality was kept as simple and intuitive as possible. In turn, we do recognize that the estimation of the parameters and the premia for inclusion of systematic variables will require many efforts, which, in our defense, seems to be an issue in most part of modern work in empirical finance. Pricing, on the other hand, is a less breathtaking effort. We provide the transform that, as it is shown in Duffie, Pan & Singleton (2000), allows us to get to the price of different derivatives. Using this transform, we showed, through a numeric example, that the phenomenon of the volatility smile is generated by the model. This discussion allows us to conclude that, despite requiring additional empirical validation of the hypothesis that systematic variables in deed influence the paths of the underlying asset, the model developed here has been worked to the point that it is ready to be applied in the computation of derivatives prices.
References


