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ON UNIQUENESS OF EQUILIBRIUM IN THE KYLE MODEL

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Abstract. A longstanding unresolved question is whether the one-period Kyle Model of an informed trader and a noisily informed market maker has an equilibrium that is different from the closed-form solution derived by Kyle (1985). This note advances what is known about this open problem.

1. Introduction

In his famous paper Kyle (1985) studies a one-period model with one asset, one informed trader, and a market maker. This has become a workhorse model, in both the theoretical and applied literature. (cf. Back and Baruch (2010).) We feel that the one-period model is so simple that it is appropriate to describe it in full detail in this introduction. There is an *informed trader* who observes the liquidation value V of an asset. After observing V, the informed trader submits a market order of size D. The *market maker* observes $D+U$, where U is interpreted as the demand of noise traders, and then sets a price. It is assumed that $U \sim \mathcal{N}(0, \sigma_U^2)$ and $V \sim \mathcal{N}(\mu, \sigma_V^2)$ are normally distributed and independent random variables.

A *strategy* of the informed trader is a measurable function $\phi\colon \mathbb{R} \to \mathbb{R}$ specifying a demand $\phi(v)$ for each observation v of the liquidation value of the asset. For each strategy ϕ the market maker's *competitive price rule* $P_\phi\colon \mathbb{R} \to \mathbb{R}$ is given by the conditional expectation

$$P_\phi(x) = E[V|\phi(V) + U = x]\,,$$

which is an inverse-regression computing the expectation of the "independent" value of the asset from observation of the "dependent" aggregate demand; cf. Krutchkoff (1967) and Li (1991). The competitive price rule is perfectly competitive in the sense that it ensures that the expected profits of the market maker are zero.[1]

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We are greatly indebted to Alexandre Eremenko for discussions. We are grateful for his patience in answering our many questions. We are also indebted to him for an improvement of our main result.

[1]Although it is linguistically natural and convenient to treat the market maker as a single individual, one should really imagine a continuum of market makers competing with each other for order flow.

Given a competitive price rule P_ϕ the informed trader considers the *expected price rule* $\widetilde{P}_\phi \colon \mathbb{R} \to \mathbb{R}$, which is given by $\widetilde{P}_\phi(x) = E[P_\phi(x+U)]$. If the informed trader announces strategy ϕ, observes v, and demands x of the asset, then his expected payoff is $v \cdot x - \widetilde{P}_\phi(x) \cdot x$, the difference between the liquidation value and the expected costs of x. The strategy ϕ is an *equilibrium* if for every observed value v we have

$$v \cdot \phi(v) - \widetilde{P}_\phi(\phi(v)) \cdot \phi(v) \geq v \cdot x - \widetilde{P}_\phi(x) \cdot x$$

for all $x \in \mathbb{R}$. That is, the informed trader has no positive incentive to deviate from demand $\phi(v)$ for each v.

Kyle demonstrates uniqueness of equilibrium within the class of affine equilibria. He notices that if we assume that ϕ is of the affine form $\phi(v) = av + b$, then the inverse-regression P_ϕ is also affine. So the expected payoff maximization conditions boil down to a quadratic equation in the parameters of the affine functions.

Theorem 1.1 (Kyle (1985))**.** *A strategy ϕ is an equilibrium and an affine function if and only if*

$$\phi(v) = (v - \mu)\sqrt{\sigma_U^2/\sigma_V^2}$$

for all values v.

Whether Kyle's closed form affine equilibrium is the only equilibrium of the model is a well known open problem that has emerged as a notoriously intractable theoretical problem.[2] A considerable body of literature simply analyzes affine equilibria while ignoring the possibility that these might not be unique. In addition there are variants of the Kyle model, and similar models such as the one due to Hellwig (1980), for which the corresponding uniqueness issue is unresolved. Thus the issue is, in principle, quite important. Explicit recognition of the problem goes back at least to the theoretical work of Back (1992). A number of researchers have attempted to resolve the question; one attempt appears in Boulatov et al. (2005) and the September 2013 version of that working paper (henceforth BKL). That this question is mathematically difficult is unsurprising in one sense: inverse regressions with non-linear models are rather complicated, and are associated with a number of unresolved theoretical questions.

In this note we extend the uniqueness result of Kyle to a wider class of equilibrium demand functions. A *single-valued function* is an analytic function on a connected open subset of the complex plane $\mathbb{C}$ that

[2] We should emphasize that the problem of uniqueness of equilibrium in the dynamic versions of the Kyle model is different in an essential way from the question of uniqueness in the one-period model. Back (1992) shows that in the continuous time setting the derived equilibrium is unique within a nice class of functions. In the dynamic setting the uniqueness of the Kyle equilibrium was studied by Takayama (2013) for the class of *tame* equilibria, where theoretical results and computational experiments suggest the presence of multiple equilibria!

has an unambiguously defined maximal analytic continuation. Every real valued smooth function that coincides with its Taylor-series expansion centered at zero is single valued. Further, single-valued functions include meromorphic functions and their compositions. In this note we prove the following which states that Kyle's closed-form solution is unique in the class of equilibrium strategies that coincide somewhere with a single valued function.

Theorem 1.2. *A strategy ϕ is an equilibrium and coincides with a single-valued function on some non-empty interval $(\alpha, \beta) \subseteq \mathbb{R}$ if and only if*

$$\phi(v) = (v - \mu)\sqrt{\sigma_U^2/\sigma_V^2}$$

for all values v.

The proof of this uniqueness theorem relies on two novel results concerning the market maker's competitive price rule and the associated expected price function. These are proved in Section 2. Section 3 we prove a stronger version of Theorem 1.2 by studying the first order conditions of the expected utility maximization from of the informed trader. In Section 4 we gather further results the equilibria of in the Kyle model.

2. The competitive and expected price rule

In this section we study P_ϕ and $\widetilde{P}_\phi$ *without imposing any conditions on ϕ beyond measurability.* Remarkably, we are able to obtain results that are somewhat stronger than the corresponding results in BKL. Also, the proofs do not use techniques from complex analysis, but the expected price function is seen to be entire, so methods of complex analysis, and perhaps the theory of inverses of entire functions (cf. the lecture notes of Eremenko (2013)) are potentially applicable to the uniqueness question.

Throughout we will assume that the mean μ of V is zero and that the variances of U and V are one. Since the problem is invariant under affine transformations of the space of possible values and linear rescalings of the noise trader demand, this is without loss of generality. Let $f(x) := f_U(x) = f_V(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ be the associated normal density function. Denote by F_U, F_V the cumulative distribution functions.

Fix a demand function ϕ, which need not be an equilibrium. Let D be the random variable $\phi(V) + U$. Then the joint probability density function for (V, D) is

$$f_{V,D}^{\phi}(a, b) = f_U(b - \phi(a)) f_V(a).$$

To prove this formally observe that the joint distribution function is

$$F(a, b) = \int_{-\infty}^{a} F_U(b - \phi(v)) f_V(v) dv = \int_{-\infty}^{a} \int_{-\infty}^{b-\phi(v)} f_U(x) dx f_V(v) dv$$

$$= \int_{-\infty}^{a}\int_{-\infty}^{b} f_U(x-\phi(v))f_V(v)dxdv.$$

From this we obtain that the conditional density is

$$f^{\phi}_{V|D}(v|d) = \frac{f_U(d-\phi(v))f_V(v)}{\int f_U(d-\phi(x))f_V(x)dx},$$

and that the price function is given by

$$P_\phi(d) = \int v\frac{f_U(d-\phi(v))f_V(v)}{\int f_U(d-\phi(x))f_V(x)dx}dv\,.$$

Equivalently, $P_\phi(x) = D_\phi(x)/C_\phi(x)$ where

$$C_\phi(x) = \int e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv \quad \text{and} \quad D_\phi(x) = \int ve^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv\,.$$

Since $x\phi(v) - \phi^2(v)/2 = x^2/2 - (x-\phi(v))^2/2 \le x^2/2$, these integrals are well defined for any measurable ϕ.

Our first result strengthens Corollary 1 of BKL, which asserts that $|P_\phi(x)| \le a|x| + b$ for some constants $a, b > 0$. Here this result is attained without imposing any assumptions on ϕ beyond measurability.

Theorem 2.1. For any measurable $\phi\colon \mathbb{R} \to \mathbb{R}$ there is a $K > 0$ such that for all $x \in \mathbb{R}$,

$$|P_\phi(x)| \le |x| + K.$$

Proof. We compute that

$$|D_\phi(x)| \le \int |v|e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv = \left(\int_{|v|\le|x|} + \int_{|v|>|x|}\right)|v|e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv$$

$$\le |x|\int_{|v|\le|x|} e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv + \int_{|v|>|x|}|v|e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv$$

$$\le |x|C_\phi(x) + \int_{|v|>|x|}|v|e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv.$$

Notice that

$$e^{x\phi(v)-\frac{\phi^2(v)}{2}} = e^{-\frac{(\phi(v)-x)^2-x^2}{2}} = e^{-\frac{(\phi(v)-x)^2}{2}}e^{\frac{x^2}{2}} \le e^{\frac{x^2}{2}}.$$

Since $\int_{|v|>|x|}|v|e^{-\frac{v^2}{2}}dv = 2\int_{|x|}^{\infty} ve^{-\frac{v^2}{2}}dv = 2e^{-\frac{x^2}{2}}$, we obtain

$$\int_{|v|>|x|}|v|e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}}dv \le e^{\frac{x^2}{2}}\int_{|v|>|x|}|v|e^{-\frac{v^2}{2}}dv \le 2.$$

Therefore $|D_\phi(x)| \le |x|C_\phi(x) + 2$. There are now three cases:

(a) ϕ is almost everywhere non-negative.
(b) ϕ is almost everywhere non-positive.
(c) neither (a) or (b) hold.

First suppose that (c) holds. Setting

$$K = \min\left\{\int_{\phi(v)\geq 0} e^{-\frac{\phi^2(v)+v^2}{2}} dv, \int_{\phi(v)\leq 0} e^{-\frac{\phi^2(v)+v^2}{2}} dv\right\} > 0,$$

we have

$$C_\phi(x) \geq \int_{\mathrm{sgn}[x]\cdot\phi(v)\geq 0} e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}} dv \geq \int_{\mathrm{sgn}[x]\cdot\phi(v)\geq 0} e^{-\frac{\phi^2(v)+v^2}{2}} dv \geq K\,,$$

so $|P_\phi(x)| \leq |x| + 2/K$.

Case (a) follows from case (b) applied to $-\phi$ because $P^{-\phi}(x) = P_\phi(-x)$, so it suffices to consider case (b). Let $K_- = \int e^{-\frac{\phi^2(v)+v^2}{2}} dv$. If $x \leq 0$, then $C_\phi(x) \geq K_-$ and therefore $|P_\phi(x)| \leq |x| + 2/K_-$. If $x > 0$, then

$$\begin{aligned} |D_\phi(x)| &\leq \int_{|v|\leq x} |v| e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}} dv + \int_{|v|>x} |v|\, e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}} dv \\ &\leq |x| C_\phi(x) + \int_{|v|>x} |v|\, e^{-\frac{v^2}{2}} dv = |x| C_\phi(x) + 2e^{-\frac{x^2}{2}}. \end{aligned}$$

Let k be large enough that $\lambda\left(|\phi| < k\right) > 0$. Then

$$C_\phi(x) \geq \int_{|\phi|<k} e^{x\phi(v)-\frac{\phi^2(v)+v^2}{2}} dv \geq e^{-xk-\frac{k^2}{2}} \int_{|\phi|<k} e^{-\frac{v^2}{2}} dv,$$

so

$$|P_\phi(x)| \leq x + \frac{2e^{k^2-(x-k)^2/2}}{\int_{|\phi|<k} e^{-\frac{v^2}{2}} dv} \leq x + \frac{2e^{k^2}}{\int_{|\phi|<k} e^{-\frac{v^2}{2}} dv}.$$

□

A function $f\colon \mathbb{R} \to \mathbb{R}$ is *real entire* if it is smooth and coincides with its Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n\,,$$

for every $x \in \mathbb{R}$. The next result is part 1 of Result 2 of BKL. Again, there are no assumptions aside from the measurability of ϕ, and our argument is simpler and more direct.

Theorem 2.2. For any measurable $\phi\colon \mathbb{R} \to \mathbb{R}$ the expected price function $\widetilde{P}_\phi$ is real entire.

Proof. Notice first that

$$\widetilde{P}_\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_\phi(x+u)\, e^{-\frac{u^2}{2}} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_\phi(u)\, e^{-\frac{(u-x)^2}{2}} du$$

for all $x \in \mathbb{R}$. Since $e^{-\frac{(u-x)^2}{2}} = e^{-\frac{x^2}{2}} e^{ux} e^{-\frac{u^2}{2}}$ we have

$$\begin{aligned}\widetilde{P}_\phi(x) &= e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} P_\phi(u)\, e^{ux} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &= e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{x^n}{n!} \int_{-\infty}^{\infty} P_\phi(u)\, u^n \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.\end{aligned}$$

Let $\alpha_n := \int_{-\infty}^{\infty} P_\phi(u)\, u^n \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du$. It suffices to prove that $\sum_{n=0}^{\infty} \frac{x^n}{n!}\alpha_n$ is entire, which follows if we can show that $\lim_{n\to\infty} \sqrt[n]{\frac{|\alpha_n|}{n!}} = 0$. The last result gives a constant $K > 0$ such that $|P_\phi(x)| \le |X| + K$, so

$$|\alpha_n| = \left| \int_{-\infty}^{\infty} P_\phi(u)\, u^n \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \right| \le \int_{-\infty}^{\infty} (|u| + K)\, |u|^n \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.$$

For $n \ge 2$ we have $u^{n-1} e^{-\frac{u^2}{2}} \Big|_0^{\infty} = 0$, so integration by parts gives

$$\int_0^{\infty} u^{n-1} \cdot u e^{-\frac{u^2}{2}} du = (n-1) \int_0^{\infty} u^{n-2} e^{-\frac{u^2}{2}} du.$$

Since $\int_0^{\infty} e^{-\frac{u^2}{2}} du = \sqrt{\pi/2}$ and $\int_0^{\infty} u e^{-\frac{u^2}{2}} du = 1$, $\int_0^{\infty} u^n e^{-\frac{u^2}{2}} du$ is

$$\sqrt{\pi/2} \cdot 1 \cdot 3 \cdot \ldots \cdot (n-1) \quad \text{or} \quad 1 \cdot 2 \cdot 4 \cdot \ldots \cdot (n-1)$$

according to whether n is even or odd. There are two cases according to whether $|u| < K$, and various minor details, but it is easy to see that, up to small factors, $\sqrt[n]{\frac{|\alpha_n}{n!}}$ is bounded by $(((n-1)/2)!)^{-1/n}$ or $((n/2)!)^{-1/n}$, which converge to zero as desired. □

3. Affine demand from first order conditions

We now review some basic aspects of complex analysis. The *radius of convergence* of a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ with complex coefficients $c_0, c_1, \ldots$ centered at $a \in \mathbb{C}$ is $R = 1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$. The power series converges absolutely on the open ball in $\mathbb{C}$ of radius R centered at a and uniformly on compact subsets of this ball. A *region* D in the complex plane $\mathbb{C}$ is a nonempty (path) connected open subset of $\mathbb{C}$. A function $f : D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$ is a region, is (complex) *analytic* if, for each $a \in D$, there is a power series with positive radius of convergence centered at a that agrees with f in some neighborhood of a. A necessary and sufficient condition for this is that f is complex-differentiable at each point of D.

Turning to our main theorem. Let D be a region, and let $g\colon D \to \mathbb{C}$ be an analytic function. A *curve* is a continuous function $\pi\colon [0,1] \to \mathbb{C}$. An *analytic continuation of g along π* consists of a finite collection $0 = r_0 < r_1 < \cdots < r_{N-1} < r_N = 1$, regions $D_1, \ldots, D_N$, and analytic functions $\{g_n : D_n \to \mathbb{C}\}_{n=1}^N$, such that $\pi(0) \in D$, $\pi([r_{n-1}, r_n)]) \in D_n$ for all $n = 1, \ldots, N$, g_1 coincides with g on $D \cap D_1$, and g_{n-1} coincides with g_n on $D_{n-1} \cap D_n$ for all $n = 2, \ldots, N$.

Definition 3.1. We say that an analytic function $g\colon D \to \mathbb{C}$ is *single-valued* if $g_N(x) = g(x)$ for any $x \in D$, any curve π satisfying $\pi(0) = \pi(1) = x$, and any analytic continuation $\{g_n\}_{n=1}^N$ of g along π.

An *entire function* is an analytic function whose domain is all of $\mathbb{C}$; the power series of such a function centered at any point of $\mathbb{C}$ has an infinite radius of convergence. Notice that an entire function maps real numbers to real numbers if and only if its restriction to the real line is a real entire function. A *meromorphic function* is a ration of two entire functions g/h where h not everywhere zero.

Examples of single valued functions include finite compositions of meromorphic functions and lacunary functions[3]. The logarithm function, restricted to the open unit disk centered at 1, is not single valued, because the value at 1 of an analytic continuation along a path π with $\pi(0) = \pi(1) = 1$ depends on the number of times the path goes around the origin clockwise. Unfortunately, except for the case of entire functions, there do not appear to be tests for single valuedness using the Taylor's series coefficients at a point, or other local information.

Now let Ψ be the class of all analytic functions $g\colon D \to \mathbb{C}$ that are either single-valued or non-injective ($g(z) = g(z')$ for some distinct $z, z' \in D$). We are ready to state and prove the result of the section whose immediate consequence is the Theorem 1.2.

Proposition 3.2. *Let ϕ be a strategy satisfying for each v the first order condition of the payoff maximization condition of the informed trader:*

$$v - f(\phi(v)) = 0\,,$$

where $f(z)$ is the first derivative of $x \mapsto \widetilde{P}_\phi(x) \cdot x$ at $z \in \mathbb{R}$. If there is $g \in \Psi$, $g\colon D \to \mathbb{C}$, and a sequence $\{v_n\} \subseteq D$ of distinct points converging to a point in D, such that $\phi(v_n) = g(v_n)$ for all v_n, then ϕ is affine. If in addition the second order condition $f'(\phi(v)) \geq 0$ holds for all v, then

$$\phi(v) = (v - \mu)\sqrt{\sigma_U^2/\sigma_V^2}$$

for all values v.

Proof. Clearly $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function. Therefore, $f \circ g$ is an analytic function with domain D. Because $f(g(v)) - v = 0$ for all $v \in \{v_n\}$ we see that $f(g(v)) = v$ for all $v \in D$. In particular, g must be injective on D. We show that f must also be injective on all of $\mathbb{C}$.

[3]A lacunary function is a function defined by a Taylor series at a point that cannot be analytically continued outside of the disk given by the radius of convergence.

Now f cannot be a constant function on the non-empty open set $g(D)$ because it must be one-to-one on $g(D)$. Thus, f is not constant and its critical points $F = \{z \colon f'(z) = 0\}$ are discrete. So assume by way of contradiction that that there are two complex numbers $z_1 \neq z_2$ satisfying $f(z_1) = f(z_2)$. We can assume that $z_1, z_2 \notin F$. Pick $z \in g(D) \setminus F$ and a path $\pi \colon [0,1] \to \mathbb{C}$ such that $\pi(\alpha) \notin F$ for all α $\pi(0) = z$, $\pi(1/2) = z_1$ and $\pi(1) = z_3$.

Let $\gamma \colon [0,1] \to \mathbb{C}$ be the path $\alpha \mapsto f(\pi(\alpha))$. Notice that $\gamma(0) = f(z)$ and $\gamma(1/2) = \gamma(1) = f(z_1) = f(z_2)$ Because g is the inverse of f restricted to $g(D)$, the inverse function theorem tells us that g has an analytic continuation along the path γ, which is impossible because it is single valued. This contradiction establishes that f is injective on its whole domain $\mathbb{C}$. But an injective entire function is affine (see Theorem 10.10 of Howie (2003)), which is to say that $(\widetilde{P}_\phi(x) \cdot x)' = ax + b$, for all x. Returning to the first order condition, we see that $\phi(v) = (v - b)/a$.

From this we see that the informed trader's payoff function is a quadratic function and that the second order condition guarantees that ϕ is also an equilibrium. Appealing to Theorem 1.1 completes the proof. □

We conclude with an example in which $\widetilde{P}$ is an non-affine entire function and the maximization problem of the informed trader is given uniquely by a strictly monotone non-affine but real analytic function ϕ, thus satisfying all the elementary properties that we have used in the previous results. The function, however, is not an equilibrium demand function because the zero profit condition market maker is not satisfied. The example indicates that further advances on the uniqueness of equilibrium require that we study properties of equilibria that are not apparent to us from the work in this present note.

Example 3.3. *Suppose that for every $x \in \mathbb{R}$ we have*

$$\widetilde{P}(x) = \sum_n \frac{x^{2n+1}}{(2n+2)(2n+1)n!},$$

which is an entire function. Letting $f(x) = x\widetilde{P}(x)$ we get

$$f(x) = \sum_n \frac{x^{2n+2}}{(2n+2)(2n+1)n!}$$

and

$$f'(x) = \sum_n \frac{x^{2n+1}}{(2n+1)n!} \quad \text{and} \quad f''(x) = e^{x^2}.$$

we note that f' is strictly monotone and $f''(x) \neq 0$ for all x. For each v, the maximization problem

$$\max_x xv - x\widetilde{P}(x)$$

has a unique solution $\phi(v) = (f')^{-1}$. *We note that* ϕ *is real analytic on the whole of* $\mathbb{R}$ *but is not an entire function, because* f' *is not affine.*

4. Further simple consequence

In this section we describe simple properties of equilibrium that are refined by the results of Section 2.

Lemma 4.1. *An equilibrium* ϕ *is increasing.*

Proof. If $v' \geq v$, then $\phi(v')(v' - \widetilde{P}_\phi(v')) \geq \phi(v)(v' - \widetilde{P}_\phi(v))$, and the same inequality with v and v' reversed also holds, so

$$(\phi(v') - \phi(v))v' \geq \phi(v')\widetilde{P}_\phi(v') - \phi(v)\widetilde{P}_\phi(v) \geq (\phi(v') - \phi(v))v$$

and thus $(\phi(v') - \phi(v))(v' - v) \geq 0$. □

Let $\kappa_\phi(x)$ be the derivative of $x \mapsto x\widetilde{P}(x)$ at x.

Corollary 4.2. *An equilibrium* ϕ *is strictly increasing.*

Proof. The last result implies that κ_ϕ is well defined and real entire. For all v and v' the first order condition gives $v = \kappa_\phi(\phi(v))$ and $v' = \kappa_\phi(\phi(v'))$, so $\phi(v') = \phi(v)$ is impossible if $v \neq v'$. □

Froda's theorem implies that the set of discontinuities of ϕ is countable. There are increasing functions whose sets of discontinuities are everywhere dense, but ϕ cannot exhibit such pathologies:

Proposition 4.3. *An equilibrium* ϕ *is locally piecewise continuous, in the sense that any bounded interval contains only finitely many discontinuities.*

Proof. Suppose that v is a point of discontinuity for ϕ, and let $x_- = \lim_{v' \to v^-} \phi(v')$ and $x_+ = \lim_{v' \to v^+} \phi(v')$. Then x_- and x_+ both maximize $x(v - \widetilde{P}_\phi(x))$, so the second order condition $\frac{d\kappa_\phi}{dx}(x) \geq 0$ for maximization is satisfied at both of these points. Between x_- and x_+ there is a local minimum for $x(v - \widetilde{P}_\phi(x))$, and at this point the second order condition $\frac{d\kappa_\phi}{dx}(x) \leq 0$ for minimization holds. The intermediate value theorem implies that $\frac{d\kappa_\phi}{dx}(x) = 0$ for some $x \in [x_-, x_+]$.

If the set of discontinities of ϕ had an accumulation point, there would be an accumulation point of the zero set of the entire function $\frac{d\kappa_\phi}{dx}$. In general, if the zero set of an analytic function has an accumulation point in its domain, then it is identically zero, so this would imply that κ_ϕ was a constant function. In turn it would follow that $x\widetilde{P}_\phi(x)$ was an affine function of x, and since $\widetilde{P}_\phi$ is entire, this is only possible if $\widetilde{P}_\phi$ is a constant function, which is obviously inconsistent with equilibrium. □

If $\frac{d\kappa_\phi}{dx}$ is not identically zero, then it has only finitely many zeros in any bounded interval because its zero set cannot have an accumulation point. If there are no discontinuities of ϕ in the interval (a, b) and $\frac{d\kappa_\phi}{dx}(x) > 0$ for all $x \in (\inf_{a'>a} \phi(a'), \sup_{b'<b} \phi(b'))$, then the analytic implicit function theorem applied to the equation $v = \kappa_\phi(\phi(v))$ implies that ϕ is analytic in (a, b). In sum, on any bounded interval ϕ is analytic except at finitely many points, each of which is either a point of discontinuity or a zero of $\frac{d\kappa_\phi}{dx}$.

It makes intuitive sense that if ϕ is strictly increasing, then P_ϕ should be strictly increasing. We can actually say something stronger.

Proposition 4.4. *If ϕ is strictly increasing and $d' > d$, then the distribution of v conditional on observing d' strictly first order stochastically dominates the distribution of v conditional on observing d.*

Proof. Our goal is to show that for any v,

$$\frac{\int_{-\infty}^{v} f_U(d - \phi(w)) f_V(w) dw}{\int_{-\infty}^{\infty} f_U(d - \phi(x)) f_V(x) dx} > \frac{\int_{-\infty}^{v} f_U(d' - \phi(w)) f_V(w) dw}{\int_{-\infty}^{\infty} f_U(d' - \phi(x)) f_V(x) dx}.$$

Multiplying by the denominators, regarding the two resulting expressions as double integrals, and recognizing that their restrictions to a certain subdomain agree, we find that this is equivalent to

$$\int_{-\infty}^{v} \int_{v}^{\infty} f_U(d - \phi(w)) f_U(d' - \phi(x)) f_V(x) f_V(w)\, dx\, dw >$$
$$\int_{-\infty}^{v} \int_{v}^{\infty} f_U(d' - \phi(w)) f_U(d - \phi(x)) f_V(x) f_V(w)\, dx\, dw.$$

In turn this will follow if we can show that

$$f_U(d - \phi(w)) f_U(d' - \phi(x)) > f_U(d' - \phi(w)) f_U(d - \phi(x))$$

whenever $w < v$ and $x > v$. Applying the formula for the normal density leads quickly to the conclusion that this inequality is equivalent to

$$e^{-(d-\phi(w))^2 - (d'-\phi(x))^2} > e^{-(d-\phi(w))^2 - (d'-\phi(x))^2},$$

which (after a bit of algebra) is equivalent to $(d' - d)(\phi(x) - \phi(w)) > 0$. Since ϕ is strictly increasing, this is true. □

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