Buy-or-sell auctions

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"Buy-or-Sell" Auctions

(Preliminary and Incomplete)

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Abstract

This paper develops a game theoretic model of a "Buy-or-Sell" auction. Participants have to submit both a bid and an offer price for up to one of the many units of the good being auctioned. The bid-ask

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spread is set in advance by the auctioneer. Such an auction was used by the Central Bank of Brazil to intervene in the foreign exchange market during the exchange rate crawling-peg regime (1995-1999). I investigate whether such mechanism is more effective than standard intervention auctions to prevent speculative attacks in the context of managed exchange rate regimes.
1 Introduction

During the exchange rate crawling-peg regime (1995-1999) the Central Bank of Brazil (CBB) used a new auction mechanism (it was named “Spread Auction”) to intervene in the foreign exchange (FX) market by buying and selling foreign currency reserves. Participants had to submit both a bid and an ask price to exchange between foreign currency and local currency. The “bid-ask spread” was set in advance by the CBB. The informal justification for such type of auction was that it would increase the firing-power against would-be speculators. This would, in turn, reduce the probability of a speculative attack against the peg. This paper sets up a game theoretic model of such “Buy-or-Sell” (BOS) auctions to assess whether such conclusions stand a formal test.

There is a large literature that studies the occurrence of speculative attacks in the context of currency pegs (either crawling or fixed pegs, with or without an exchange rate band). In general, intervention is not modelled at the microeconomic level. (mention literature on FX market microstructure).

The issues involved in speculative attacks on a currency peg are typically dynamic. They include the sustainability of the official exchange rate level, which is in turn determined by the interaction between the country’s fiscal
and monetary policies. Expectations play a crucial role. I do not pursue these issues here. I do not intend to evaluate whether a different intervention mechanism is capable of altering the "fate" of an exchange rate regime in the long, nor in the medium run. Instead, I look at the auction of foreign currency reserves in the event of a speculative attack as a one-shot game. If the BOS auction succeeds, ceteris paribus, in reducing the ex-ante probability that a devaluation occurs in a one-shot game, when compared to the standard intervention auction, then I'll say that it has stand the formal test.

I proceed in steps, starting with a simple model of a standard FX reserves auction, such as the ones usually carried out by a Central Bank (CB) facing a speculative attack on its currency. I use a framework of common values, with conditionally independent and identically distributed private signals. I first solve for the Nash Equilibria in the case where the fundamental value of the exchange rate is observable, and for a symmetric Bayesian Nash Equilibrium in the case where the fundamental value is unobservable. Finally I set up and solve the model of the BOS auction in the case of unobserved fundamental value.
2 Basic Setup

There are \( N + 1 \) participants in a market for foreign currency, in which quotes are in units of local currency per unit of foreign currency. The zero\(^{th} \) player is the CB, and the other \( N \) players are "banks" that trade in the market. Banks are assumed to be risk-neutral profit maximizers. The CB intervenes in the market by buying or selling foreign currency in order to affect its price. For that purpose it uses its "international reserves", which consist of \( k \geq N \) units of foreign currency, and its monopoly power to issue unlimited amounts of local currency.

The CB's goal is to maintain the exchange rate within a "target zone" (or band) \( [p, \overline{p}] \). This is a publicly announced goal and all \( N \) banks fully believe that the CB will intervene in the market using its reserves and its monopoly power as issuer of local currency to try to keep the FX market price inside the band. Formally, in the context of mechanism design, one could model the CB's problem as that of maximizing the ex-ante probability of maintenance of the band. Restricting attention to BOS auctions, the problem would be that of choosing the bid-ask spread to maximize such probability.

I want to model intervention auctions when the exchange rate is trading at \( \overline{p} \) (or \( p \)). In "standard" intervention auctions, the CB ends up either selling
foreign currency at $p$ or buying at $\bar{p}$. Since it can issue unlimited amounts of local currency to buy foreign currency, defending the floor of the target zone is always feasible (although it may not be desirable). Therefore, I focus on the case in which the CB has to defend the ceiling (i.e., try to prevent a devaluation).

I assume that there exists a "fundamental" or "equilibrium" value for the exchange rate, denoted by $V_t$, which in general may differ from the market price. Such fundamental value is taken to be a real valued random variable $V_t$, which will be assumed to be observable or not depending on the model being analyzed. I assume that $V_t$ can take any value on the real line with cdf given by $F_v(v)$ and density $f_v(v) > 0$ for all $v$.

I assume that in a freely floating exchange rate regime the market would trade at all times at a price $p_t$ equal to the "fundamental value" (if it were observable) or to $E[V_t|\Omega_t]$ (if unobservable). I do not model the price formation mechanism when the exchange rate is within the band. This is replaced with all information available at time $t$ in aggregate terms (that is, it includes the union of the banks' information sets). This is an analogy with a fully revealing rational expectations equilibrium, but still an assumption given that this is not a general equilibrium model.

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1 Further assumptions on $F_v(v)$ will be specified below.

2 $\Omega_t$ represents all information available at time $t$ in aggregate terms (that is, it includes the union of the banks' information sets). This is an analogy with a fully revealing rational expectations equilibrium, but still an assumption given that this is not a general equilibrium model.

3 One possibility here would be to incorporate results from the literature on exchange
by an assumption about the behavior of the exchange rate after unsuccessful speculative attacks: if the band is maintained, after the auction the exchange rate trades at $\bar{p} - c$, where $0 < c < \bar{p} - p^4$.

I model the CB as a "big" player, in the sense that each bank is allowed to buy at most one unit of foreign currency (or to sell at most one unit when short selling is allowed). So, in principle the CB could always meet all demand for FX in the one-shot auction. But the CB is also a "small" player in the following sense: it can only prevent a devaluation and continue to intervene if, after the auction, it has at least $k_{\text{min}}$ units of FX left, with $0 < k_{\text{min}} < k$. That is, $k^* = k - k_{\text{min}}$ is the maximum amount of reserves it rate target zones, initiated by Krugman (1991).

4This is a shortcut in the one-shot game to model the fact that after unsuccessful speculative attacks, agents that bought foreign currency in anticipation of a devaluation typically lose money. One alternative which doesn't change qualitatively the results of the paper is to assume that in this case the market price after the auction is distributed in $[\bar{p}, \bar{p}]$ according to some exogenous distribution. Or that $c$ depends positively on the amount of reserves that the CB has left after the unsuccessful attack. A more interesting alternative is to have a new draw of $v_t$ and then restart the same game with the new level of reserves, allowing banks to buy one unit per auction (and to have more than 1 unit in stock). This would lead us into dynamic models, which are beyond the scope of this paper.
can sell at the auction and still prevent a devaluation. This makes it possible for a speculative attack to occur, but only if there is "enough coordination" by the banks (a number of banks strictly greater than \( k^* \) decides to bet against the currency)\(^5\).

I now turn to the analysis of auctions which serve the purpose of this type of intervention, modelling them as one-shot games.

### 3 Standard Auction

In this section I analyze standard intervention auctions, handling both the observable- and the unobservable-fundamental-value cases. I restrict the banks' problem to a binary choice: either bid \( \bar{p} \) or not bid at all. This deserves some comments. Given the objective of the CB, it does not make

\(^5\)A more natural set of assumptions would be \( k_{\text{min}} = 0, \ k < N \). The problem in this case is that if more than \( k^* \) (= \( k \)) banks decide to buy one full unit at the auction there may be quantity rationing (depending on the bid prices). A reasonable assumption in this case is that the stock of reserves would be split evenly (up to one unit per bank) across all banks that bid the same price. In this case the model becomes substantially more complex. So, in order to avoid rationing, we assume that the CB meets all demand at the auction (even if that means selling more than \( k^* \)) but then abandons the peg if reserves fall below \( k_{\text{min}} \).
sense for it to sell at any price lower than \( p \), since I assume that a devaluation only occurs if it does not have "enough" reserves to sell at \( p \). So, bidding less than \( p \) is equivalent to not bidding at all. On the other hand, not allowing bids above \( p \) clearly could be restrictive. One reason banks could be willing to bid above \( p \) is to avoid quantity rationing. But in the model without the possibility of rationing there is no reason to bid more than \( p \), since any bank is guaranteed to get 1 unit if it bids \( p \). So there is no loss of generality from restricting the strategy space.

### 3.1 Observable Fundamental Value

In this first simple case, I assume that the fundamental value of the exchange rate is observable just prior to the auction. So, I am interested in analyzing an intervention auction after it becomes public that a realization \( v_t > p \) has occurred.

The timing of the game is as follows:

1) The CB announces an auction to sell reserves; it is assumed that it will only sell at \( p \). Its stock of reserves, \( k \), as well \( k_{\text{min}} \) are assumed to be common knowledge.

2) Banks submit bids for up to 1 unit of foreign currency (at price \( p \)); in
fact, due to the risk neutrality assumption, each bank will either bid for 1 full unit or not bid at all\textsuperscript{6}.

3) The CB satisfies all bids at price $\bar{p}$. I denote “cumulative aggregate demand” at any price $p$, in general, by $N_+(p)$, and note that it equals the number of banks that bid $p$ or more. In the current setting the only object of interest is $N_+(\bar{p})$, which equals the number of banks that bid in the auction.

4) Payoffs accrue to the banks\textsuperscript{7}. I assume that:

a) if $N_+(\bar{p}) > k^*$, then the exchange rate floats and the market price after the auction equals the fundamental value. All banks that bid $\bar{p}$ earn $(v_t - \bar{p}) > 0$. Banks that did not bid $\bar{p}$ earn zero.

b) if $N_+(\bar{p}) \leq k^*$, then the band is maintained and the exchange rate trades at $\bar{p} - c$, where $0 < c < \bar{p} - p$. Banks that bid $\bar{p}$ therefore earn $-c < 0$. Banks that did not bid $\bar{p}$ earn zero.

I focus on pure strategy Nash Equilibria. This leads us to:

\textit{Result 1:} In this game there are only 2 pure strategy Nash Equilibria (NE): one in which all banks bid for 1 unit of FX at price $\bar{p}$, and another in

\textsuperscript{6}We assume they bid for 1 unit when they are indifferent.

\textsuperscript{7}Specifying payoffs to the CB is immaterial in this simple model, since its role is completely passive. It cares about the ex-ante probability of a devaluation, but there is nothing it can do to affect it.
which no bank bids in the auction.\footnote{In addition, there is a mixed strategy NE in which banks bid with probability \( \pi \) (don't bid with probability \( 1 - \pi \)).}

Proof: First consider the case in which every bank bids for 1 unit of FX at price \( \bar{p} \). If a bank deviates and does not bid, it gets zero instead of \( (v_t - \bar{p}) > 0 \). In the other equilibrium, if a bank deviates and bids for one unit at price \( \bar{p} \) it earns \( -c < 0 \). Suppose now that there is another pure strategy NE with \( N' \) banks bidding for 1 unit each in the auction, with \( 0 < N' < N \). If \( N' > k^* \), then banks which are not bidding and therefore earning zero could do better by deviating and earning \( (v_t - \bar{p}) > 0 \). On the other hand, if \( N' < k^* \), then banks which are bidding and earning \( -c \) could deviate and earn zero instead. Finally, if \( N' = k^* \), either type of deviation is profitable.\\

This result is standard in the literature on speculative attacks, although in general little or no attention is given to the actual FX market micro-structure. It corresponds to the self-fulfilling equilibria in models of speculative attacks under perfect foresight (for example, Obstfeld, 1996), in the spirit of Diamond and Dybvig (1983) bank runs. Next I turn to the Bayesian extension of this game.
3.2 Unobservable Fundamental Value

Now I assume that the fundamental value of the exchange rate is unobservable. Formally, I model this situation as a game of incomplete information with the following timing:

1) An unobserved realization $v_t$ occurs, which causes the exchange rate to trade against the ceiling of the band\(^9\). The CB announces an auction to sell reserves; again, it is assumed that it will only sell at $\bar{p}$. Its stock of reserves, $k$, as well as $k_{\min}$ are assumed to be common knowledge.

2) Each bank $i = 1, \ldots, N$ receives a private signal $V_i$, which summarizes the information it has available to make its probabilistic assessment of the fundamental value just prior to the auction. I assume that, conditional on $v_t$, the $V_i$'s are independently and identically distributed over $(-\infty, \infty)$, with cdf given by $F(v_i|v_t)$ and density $f(v_i|v_t)$. Moreover, I assume that $f(\cdot|\cdot)$\(^9\)Once again the natural case to think of is $v_t > \bar{p}$. Arguably, it could also be the case in reality that the exchange rate trades against the edges of the band even when the fundamental value is strictly within the band's limits. But since we are not modelling the formation of the market price explicitly, it is more natural to think of $v_t > \bar{p}$. Also, since we don't have such explicit model, we abstract from the fact that banks could try to infer the fundamental value from the actual market price by modelling the information structure as in 2).
satisfies the Monotone Likelihood Ratio Property (MLRP).

3) Banks submit bids for up to 1 unit of foreign currency (at price $\bar{p}$); again, due to the risk neutrality assumption, each bank will either bid for 1 full unit or not bid at all\(^{10}\).

4) The CB satisfies all bids at price $\bar{p}$. Again, $N_+(p)$ denotes "cumulative aggregate demand" at price $p$.

5) Payoffs accrue to the banks according to\(^{11}\):

a) if $N_+(\bar{p}) > k^*$, then the exchange rate floats and the price after the auction equals $E[V_i|\Omega_i]$, where $\Omega_i = \cup_{i=1}^{N} \{v_i\}$. All banks that bid $\bar{p}$ earn $(E[V_i|\Omega_i] - \bar{p}) > 0$. Banks that did not bid $\bar{p}$ earn zero.

b) if $N_+(\bar{p}) \leq k^*$, then the band is maintained and the exchange rate trades at $\bar{p} - c$. Banks that bid $\bar{p}$ therefore earn $-c < 0$. Banks that did not bid $\bar{p}$ earn zero.

I focus on a symmetric Bayesian Nash Equilibrium (BNE) in which banks follow a “cut-off” strategy: bank $i$ bids for 1 unit of FX if and only if the realization of its signal satisfies $v_i \geq v_{sa}^*$. I start by deriving some auxiliary results which are needed for the characterization of the equilibrium\(^{12}\). These

\(^{10}\)Once more, we are assuming that they bid for 1 unit when they are indifferent.

\(^{11}\)Specifying payoffs to the CB is still immaterial in this extension.

\(^{12}\)For notational simplicity, whenever we need to take the point of view of a particular
are:

Auxiliary Result 1: The random variables \( V_t, V_1, V_2, \ldots, V_N \) are affiliated. They are also pairwise-affiliated.

Proof: Given the assumption that \( f(\cdot, \cdot) \) satisfies the MLRP, the proof follows directly from Milgrom and Weber (1982), Theorems 1 and 4.\(^{13}\)

Auxiliary Result 2: Using Bayes’ Rule, the Law of Total Expectations and the fact that the \( V_t \)'s are conditionally independent given \( v_t \), one can derive the following:

(i) The distribution of the fundamental value given \( v_1 \):

\[
G_v(v|v_1) = \Pr[V_t \leq v|v_1] = \frac{\Pr[V_t \leq v, v_1]}{\Pr[v_1]} = \frac{\int_{\mathbb{R}} \Pr[V_t \leq v, v_1|v_t] dF_v(v_t)}{\int_{\mathbb{R}} \Pr[v_1|v_t] dF_v(v_t)} = \frac{\int_{\mathbb{R}} \Pr[V_t \leq v|v_1] f(v_1|v_t) dF_v(v_t)}{\int_{\mathbb{R}} f(v_1|v_t) dF_v(v_t)} = \frac{\int_{-\infty}^{v} f(v_1|v_t) dF_v(v_t)}{\int_{\mathbb{R}} f(v_1|v_t) dF_v(v_t)}
\]

(ii) The distribution of other banks’ signals given bank 1’s signal \( v_1 \):

\(^{13}\)We assume that the MLRP holds with strict inequality, so that strict affiliation obtains.
\[
G(v_2, \ldots, v_N|v_1) = \frac{\Pr[V_2 \leq v_2, \ldots, V_N \leq v_N|v_1]}{\Pr[v_1]}
\]
\[
= \int_{\mathbb{R}} \Pr[V_2 \leq v_2, \ldots, V_N \leq v_N, v_1|v_1] \, dF_v(v_1)
\]
\[
= \int_{\mathbb{R}} \frac{F(v_2|v_1) \cdots F(v_N|v_1) f(v_1|v_1) f_v(v_1) \, dv_1}{\Pr[v_1]}
\]
\[
= \int_{\mathbb{R}} F(v_2|v_1) \cdots F(v_N|v_1) \, dG_v(v_1|v_1)
\]

iii) Let \(N^*\) denote the number of banks other than bank 1 that bid \(\bar{p}\) in the auction (ie, that receive a signal \(\geq v_{a0}^*\)):

a) The distribution of \(N^*\) given \(v_1\):

\[
\Pr[N^* = q|v_1] = \binom{N-1}{q} (1 - F(v_{a0}^*|v_1))^q (F(v_{a0}^*|v_1))^{N-1-q}
\]

b) The distribution of \(N^*\) given \(v_1\):
c) The distribution of the fundamental value given $v_1$ and $N^*$:

\[
\Pr[N^* = q|v_1] = \frac{(N - 1) \int_{v_{a1}}^{\infty} \cdots \int_{v_{an}}^{\infty} \int_{-\infty}^{v_a} \int_{-\infty}^{v_n} \frac{\partial^{N-1} G(v_2, \ldots, v_N|v_1)}{\partial v_2 \cdots \partial v_N} dv_2 \cdots dv_N}{Pr[v_1]}
\]

\[
= \frac{(N - 1) \int_{v_{a1}}^{\infty} \cdots \int_{v_{an}}^{\infty} \int_{-\infty}^{v_a} \int_{-\infty}^{v_n} \frac{\int_{\mathbb{R}} \prod_{i=1}^{N} F(v_i|v_1) dF_{v_i}(v_i)}{\int_{\mathbb{R}} f(v_1|v_1) dF_{v_1}(v_1)} dv_2 \cdots dv_N}{Pr[v_1]}
\]

\[
= \frac{\int_{\mathbb{R}} (N-1) (1 - F(v_{a1}|v_1))^q (F(v_{a1}|v_1))^{N-1-q} f(v_1|v_1) f_0(v_1) dv_1}{Pr[v_1]}
\]

Auxiliary Result 3: Assuming all required derivatives exist,
i) \( \frac{\partial \Pr[N^* > q | v_i]}{\partial v_i} > 0, \quad q = 0, \ldots, N - 1. \)

ii) \( \frac{\partial E[V_i | v_i, N^* > q]}{\partial v_i} > 0, \quad q = 0, \ldots, N - 1. \)

**Proof:** The results follow from Theorem 5 in Milgrom and Weber (1982).

**Result 2:** With this results I can characterize the above mentioned symmetric BNE, in which banks follow a “cut-off” strategy: bank \( i \) bids for 1 unit of FX if and only if \( v_i \geq v^{\text{sa}}_i \).

**Proof:** The trigger value \( v^{\text{sa}}_i \) is such that bank \( i \) is indifferent between bidding for 1 unit of FX and not bidding. Denote the profit (or loss) from bidding \( \bar{p} \) by \( \pi(\bar{p}) \). The expected payoff for bank 1 from bidding for 1 unit at price \( \bar{p} \), given its signal \( v_1 \) and the fact that all other banks are bidding according to the cut-off value \( v^{\text{sa}}_1 \), can be written as

\[
E[\pi(\bar{p}) | v_1, v^{\text{sa}}_1] = \Pr[N^* \geq k^* | v_1, v^{\text{sa}}_1] \cdot E[V_1 - \bar{p} | v_1, v^{\text{sa}}_1, N^* \geq k^*] - c \cdot \Pr[N^* < k^* | v_1, v^{\text{sa}}_1].
\]

Using the auxiliary results, \( E[\pi(\bar{p}) | v_1, v^{\text{sa}}_1] \) can be expanded as

\[
\sum_{q=k^*}^{N-1} \int_{\mathbb{R}} \left( \begin{array}{c} N - 1 \\ q \end{array} \right) (1 - F(v^{\text{sa}}_1 | v_1))^{q} (F(v^{\text{sa}}_1 | v_1))^{N-1-q} dG_v (v_1 | v_1) (E[v_1 | v_1, v^{\text{sa}}_1, q] - \bar{p})
\]

\[
- c \cdot \sum_{q=0}^{k^*-1} \int_{\mathbb{R}} \left( \begin{array}{c} N - 1 \\ q \end{array} \right) (1 - F(v^{\text{sa}}_1 | v_1))^{q} (F(v^{\text{sa}}_1 | v_1))^{N-1-q} dG_v (v_1 | v_1)
\]
So, \(v_{sa}^*\) satisfies

\[
\sum_{q=k}^{N-1} \int_{\mathbb{R}} \left( \frac{1 - F(v_{sa}^* | v_t)}{q} \right)^q \left( F(v_{sa}^* | v_t) \right)^{N-1-q} dG_v(v_t | v_{sa}^*) \left( E[v_t | v_{sa}^*, v_{sa}^*, q] - \bar{p} + c \right) = c
\]

Due to Auxiliary Result 3, the expected payoff from bidding \(\bar{p}\) is (strictly) increasing in a bank’s signal. So there is no profitable deviation from the situation in which all banks are deciding whether to bid or not based on the cut-off value \(v_{sa}^*\). Bidding after a signal \(v_i < v_{sa}^*\) yields strictly negative expected profits, while not bidding after a signal \(v_i > v_{sa}^*\) means foregoing strictly positive expected profits.

A qualitatively similar equilibrium would also obtain if the signals were positively correlated conditional on \(v_t\), in which case a signal \(v_i < v_{sa}^*\) would imply an even lower probability that the other banks have received a signal above \(v_{sa}^*\).

With this equilibrium, given distribution functions \(F_v(v_t)\) and \(F(v_i | v_t)\), I can compute the (unconditional) probability of a devaluation occurring and also the conditional probability of a devaluation after any given realization of the fundamental value.
3.2.1 Existence and Uniqueness

A formal analysis of existence and uniqueness of equilibria in general is beyond the scope of this paper. Based on some preliminary analysis, I believe that existence in general can be guaranteed by the results in Athey (2001), since the framework seems to satisfy the sufficient single-crossing and monotonicity conditions.

Nevertheless, in this subsection I sketch an analysis of existence and uniqueness of the particular equilibrium which I consider. Neither existence or uniqueness are guaranteed for arbitrary parameter values and probability distributions. This has been verified through numerical analysis using gaussian distributions for both $F_v(\cdot)$ and $F(\cdot|\cdot)$.

Mathematically, existence and uniqueness of the kind of equilibrium which I consider correspond to existence and uniqueness of roots of the (highly non-linear) equation $E[\pi(\overline{F})|v_1, v^*_a]|_{v_1=v^*_a} = 0$, with $v^*_a$ viewed as a single unknown. A formal analysis of these issues involve understanding the properties of the primitives of the model which determine the nature of this equation.

Intuitively, the reasons for non-existence or existence of multiple equilibria can be understood by inspection of the equation which defines $v^*_a$: 19
The problem arises due to the term \( \Pr [N^* \geq k^* | v_1, v_{sa}^*] \). Auxiliary Result 3 shows that \( \Pr [N^* \geq k^* | v_1, v_{sa}^*] \) and this would seem to suffice for existence (and uniqueness!), given that \( E [V_t - \bar{p}|v_1, v_{sa}^*, N^* \geq k^*] |_{v_1=v_{sa}^*} \) is clearly increasing in \( v_{sa}^* \). Unfortunately, this is not the case. Auxiliary Result 3 is a statement about how this probability changes when \( v_1 \) changes, holding fixed the cut-off value for all other banks at \( v_{sa}^* \). Existence and uniqueness, on the other hand, have to do with the behavior of this probability when the same cut-off value varies for all banks simultaneously. This generates two opposite effects: a higher \( v_{sa}^* \) i) increases the probability of the event \( [N^* \geq k^*] \) due to its effect on the conditional distribution of the other banks’ signals given \( v_1 = v_{sa}^* \), but ii) it decreases such probability, since it becomes less likely that enough signals exceed the higher \( v_{sa}^* \). These effects might dominate the behavior of \( E [V_t - \bar{p}|v_1, v_{sa}^*, N^* \geq k^*] |_{v_1=v_{sa}^*} \).

The second effect is driven essentially by the “prior distribution” of \( V_t \), \( F_v(\cdot) \). Given \( F(\cdot|\cdot) \), it determines the prior distribution of \( (V_2, \ldots, V_N) \) and the posterior distribution \( G(v_2, \ldots, v_N|v_1) \). To gain intuition, imagine that
$F_v(\cdot)$ displays very little dispersion (relative to $F(\cdot|\cdot)$). In this case, a higher signal $v_1$ has little effect on the posterior distribution of $V_t$ given $v_1, G_v(v|v_1)$, and therefore also has little effect on the posterior distribution $G(v_2, \ldots, v_N|v_1)$. This tends to make the probability of the event $[N^* \geq k^*]$ decrease with $v^*_a$. If, on the other hand, $F_v(\cdot)$ displays a lot of dispersion relative to $F(\cdot|\cdot)$, the effect of a higher signal $v_1$ on the conditional distribution $G(v_2, \ldots, v_N|v_1)$ should mitigate the probability-reducing effect of a higher cut-off value. In the limit, if $F_v(\cdot)$ is a "flat prior", this second effect should vanish and the function $E[\pi(P)|v_1, v^*_a]|_{v_1=v^*_a}$ should become increasing in $v^*_a$.\footnote{This intuition has been confirmed by simulations using gaussian distributions for both $F_v(\cdot)$ and $F(\cdot|\cdot)$.}

On a different direction, increasing $N$ relative to $k^*$ should also contribute to the existence of equilibria, since for any $v^*_a$ the event $[N^* \geq k^*]$ becomes more likely. In fact, by a law of large numbers argument

$$
\lim_{N \to \infty} \Pr[N^* \geq k^*|v^*_a] = 1.
$$

(Derive formally conditions under which $E[\pi(P)|v_1, v^*_a]|_{v_1=v^*_a}$ is increasing in $v^*_a$.)
4 Buy or Sell Auction

In this section I analyze the BOS auction. I focus on the unobservable-fundamental-value case, since with observable fundamentals the reasoning is essentially the same as in the standard auction. The critical difference with respect to the standard auction is that in the BOS auction, when placing a bid at price \( p \) the bank is simultaneously placing an offer of equal size at price \( p + \varepsilon \), where \( \varepsilon \) is a bid-ask spread set in advance by the CB and satisfies \( 0 < \varepsilon < c \). I therefore say that banks place “bid-offers” and always use \( p \) (or \( p_i \)) to refer to the bid price. The settlement of the auction is as follows:

1) The CB is never a net buyer in the auction\(^{15}\).

2) The CB always fulfills all bids at prices \( \bar{p} \) or higher (up to 1 unit per bank), even if that means selling more than \( k^* \). It abandons the peg if reserves fall below \( k_{\text{min}} \). The difference now is that it can act as an “intermediary”, buying from some banks to sell to others.

3) The “intermediation rules” specify a sequential procedure. As long as there are “unfilled” bids at prices that exceed the price of some “unfilled” offer, the CB buys from the bank with the best (i.e. lowest) unfilled offer price and sells to the bank with the best (i.e highest) unfilled bid price.

\(^{15}\)This is a natural assumption since we are interested in studying the risk of devaluation.
It continues in this fashion until it exhausts the trade possibilities amongst banks\textsuperscript{16}. If there are any unfulfilled bids at prices greater than or equal to $\bar{p}$ remaining, the CB sells to these banks from its reserves. A devaluation occurs if and only if the CB needs to sell strictly more than $k^*$ units from its reserves.

So, using $N_-(p)$ to denote “cumulative aggregate supply” at $p$ (ie, the number of banks that offer at price $p$ or less\textsuperscript{17}), there are 2 cases:

a) if $N_+(\bar{p}) > k^* + N_-(\bar{p})$ there is a devaluation. In this case the CB buys from all $N_-(\bar{p})$ banks which agreed to sell at or below $\bar{p}$ and sells to the $N_+(\bar{p})$ banks that bid $\bar{p}$ or higher.

b) if $N_+(\bar{p}) \leq k^* + N_-(\bar{p})$ there is no devaluation.

Given that many events will be stated in terms of $N_+(p)$ and $N_-(p)$, it is useful to derive the following:

\textit{Auxiliary Result 4:} $N_+(p) + N_-(p + \epsilon) = N + N(p)$, where $N(p)$ denotes

\textsuperscript{16}In case there are multiple banks with the same “best” offer (or bid), we assume that the CB breaks ties by randomizing amongst them, attaching the same probability to all.
\textsuperscript{17}Once again we are using the fact that, although banks can submit bid-offers for up to 1 unit of foreign currency, due to the risk neutrality assumption they will always place bid-offers for 1 full unit (or not participate in the auction at all). Indifference is resolved in favor of participation.
the number of banks that bid exactly $p$ (not $p$ or higher).

Proof: It may seem that $N_+(p) + N_-(p) = N$, but this is not true. $N_+(p)$ sums all banks that bid $p$ or more while $N_-(p)$ encompasses all banks that offer at price $p$ or less. So, banks with bids $\in (p - \epsilon, p)$ and associated offers at prices $\in (p, p + \epsilon)$ are not taken into account (in general $N_+(p) + N_-(p) \leq N$). Therefore the decomposition that is valid for all $p$ is $N_+(p) + N_-(p + \epsilon) = N + N(p)$, where $N(p)$ must be added to the right-hand side since the left-hand side accounts twice for all banks that bid exactly $p$ (and therefore offer at exactly $p + \epsilon$)\(^{18}\).

4.1 Unobservable Fundamental Value

Apart from the differences in the rules of the auctions, the game is identical to the one in subsection 3.2. I start by analyzing the bid-ask space and ruling out some alternatives that will not be used in the equilibrium that I consider. I use Figure 1 to facilitate reference to the possible cases, which are indicated by the numbers attached to the bid-offers. For a bid-offer of type $q$ I denote the bid price by $p_q$.

\(^{18}\)Nonetheless, we show in the next subsection that in the equilibrium that we consider $N_+(\bar{p}) + N_-(\bar{p}) = N$. 

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**Auxiliary Result 5:** Not placing bid-offers of types 9, 7 and 2, 1 (for any realization of the signal) is consistent with equilibrium\(^{19}\).

*Proof:* I assume that banks 2...N are not placing bid-offers of types 9, 7 and 2, 1 (for any realization of the signal) and consider whether bank 1 could find it profitable to place any such bid-offer.

**Type 9:** Given that any bank is guaranteed 1 unit of FX if it bids \(\bar{p}\), such bid-offer would only be more profitable than a type 3 bid-offer if bank 1 could end up selling at \(p_0 + \varepsilon\) (in a no devaluation outcome). This is impossible if no other bank bids more than \(\bar{p}\).

**Type 7:** A bid at \(p_7\) can only be fulfilled in a no devaluation outcome and the possibility of selling at \(p_7 + \varepsilon\) (in a no devaluation outcome) is ruled out by the same argument as above.

**Type 2:** Placing such a bid-offer would only make sense if there was some non-zero probability of buying at \(p_2\). This is impossible if no other bank places this type of bid-offer.

**Type 1:** Necessary conditions for such a bid-offer to make sense are: i) a non-zero probability of buying at \(p_1\); or ii) zero probability of a devaluation.

\(^{19}\)Nonetheless, in equilibrium, type-2 bid-offers will be placed in the limit as \(v_1 \to -\infty\), since we assume that \(F_\nu(\cdot)\) and \(F(\cdot)\) are such that

\[
\forall (v_2, ..., v_N) \lim_{v_1 \to -\infty} G(v_2, ..., v_N|v_1) = 1
\]
(in which case selling at $\bar{p} - c$ is harmless). Buying at $p_1$ is a zero probability event, since no other bank places this type of bid-offer; and there is always a non-zero probability of a devaluation in the equilibrium that I consider (given any finite signal). ■

This result allows us to derive the following:

**Auxiliary Result 6**: $N_+(\bar{p}) + N_-(\bar{p}) = N$.

**Proof**: Given that bid-offers of type 7 are not used in the equilibrium under consideration, the result follows directly from Auxiliary Result 4. ■

I focus on a symmetric BNE characterized by an increasing (bid-price) function $\beta(v)$ and two cut-off values $(v^{*}_{bos}, v^{**})$, with $v^{*}_{bos} \geq v^{**}$. In such equilibrium, bank $i$ chooses its bid price according to the increasing function $\beta(v_i)$ (and therefore places its offer at $\beta(v_i) + c$). The trigger values are such that $v \in [v^{**}, v^{*}_{bos}) \implies \beta(v) = \bar{p} - \epsilon$, and $v \in [v^{*}_{bos}, \infty) \implies \beta(v) = \bar{p}$. The general shape of the "bidding function" is illustrated in Figure 2.

When a bank places a bid-offer at $(p, p + c)$ there are five possible outcomes: i) it does not trade at all; ii) it buys (b) and there is a devaluation ($d$); iii) it buys and there is no devaluation ($nd$); iv) it sells (s) and there is a devaluation; v) it sells and there is no devaluation. Since the payoff from not trading is zero, one can decompose the expected payoff from placing a
bid-offer at \((p, p + \varepsilon)\), given \(v_1\), as in Table 1. It displays each of the terms of \(E[\pi(p) | v_1]\) in the top and the possible ranges for \(p\) in the second column. Most events for which probabilities need to be calculated in this table will be rewritten in terms of order statistics. So, it is useful to state the following:

**Auxiliary Result 7:** Let \(\mathbf{Y} = (Y_1, Y_2, \ldots, Y_{N-1})\) denote the order statistics associated with the conditional distribution of \((V_2, \ldots, V_N)\) given \(V_1\). The joint density of \(\mathbf{Y}\) given \(V_1\) is

\[
    f_{\mathbf{Y}}(y_1, y_2, \ldots, y_{N-1} | v_1) = \begin{cases} 
    (N-1)! g(y_1, y_2, \ldots, y_{N-1} | v_1) & \text{if } y_1 \geq y_2 \geq \ldots \geq y_{N-1} \\
    0 & \text{otherwise}
    \end{cases}
\]

where \(g(y_1, y_2, \ldots, y_{N-1} | v_1) = \frac{\partial^{N-1} g(y_2, \ldots, y_N | v_1)}{\partial y_2 \cdots \partial y_N}\).


Calculating the relevant probabilities in the general case is a very hard task. Therefore I restrict attention to the particular case \(k^* = N - 1\), which means that a devaluation occurs if and only if all banks bid \(\bar{p}\) in the auction.

Referring to each term in Table 1 by its row and column, it is useful to derive the following\(^20\):

**Auxiliary Result 8:** Let \(N_+(p)\) and \(N^*(p)\) denote, respectively, the number of banks other than bank 1 that bid \(p\) and offer at \(p\) in the auction. Recall the notation for the events in Table 1: \(b = \text{buy}, s = \text{sell}, d = \text{devaluation} \)

\(^{20}\)For the terms C3) and C7) I also set \(N=3\).
and \( nd = \text{no devaluation} \).

- **D1)** \( \Pr[b,d|v_1] = \Pr[d|v_1] \), since bidding \( \bar{p} \) implies buying with certainty. So, \( \Pr[d|v_1] = \Pr[N^*_+(\bar{p}) = N - 1|v_1] \). Given the equilibrium strategies, this translates into

\[
\Pr[d|v_1] = \int_{v_{bos}^*}^{\infty} \cdots \int_{v_{bos}^*}^{\infty} g(v_1, v_2, \ldots, v_{N-1}|v_1) dv_2 \cdots dv_N
\]

\[
= \int_{v_{bos}^*}^{\infty} \cdots \int_{v_{bos}^*}^{\infty} \prod_{j=1}^{N} f(v_j|v_1) dF_v(v_k) \int_{\mathbb{R}} f(v_1|v_1) dF_v(v_k) dv_2 \cdots dv_N
\]

\[
= \int_{\mathbb{R}} (1 - F(v_{bos}^*|v_1))^{N-1} dG_v(v_1|v_1)
\]

\[
= \int_{\mathbb{R}} \Pr[N^*_+(\bar{p}) = N - 1|v_1] dG_v(v_1|v_1)
\]

- **D3)** Analogously, \( \Pr[b,nd|v_1] = \Pr[nd|v_1] \). So, \( \Pr[b,nd|v_1] = 1 - \Pr[d|v_1] \).

- **A5), B5) and C5) In all 3 cases the probability is zero, since a devaluation is impossible if bank 1 does not bid \( \bar{p} \).**

- **B3)** \( \Pr[b,nd|v_1] = \Pr[b|v_1] \), since "no devaluation" is certain, given that bank 1 does not bid \( \bar{p} \). In order to calculate such probability, it is useful to partition the event "buy" as

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"buy" = \( (p \geq \beta(Y_1) \cap p \geq \beta(Y_{N-1}) + \varepsilon) \cup (p < \beta(Y_1) \cap p \geq \beta(Y_2) \cap p \geq \beta(Y_{N-2}) + \varepsilon) \cup \ldots \cup \left( p < \beta\left(Y_{\text{int}\left[\frac{q}{N}\right]} - 1\right) \cap p \geq \beta\left(Y_{\text{int}\left[\frac{q}{N}\right]}\right) \right) \) + \varepsilon) 

= \( (p \geq \beta(Y_1) \cap p \geq \beta(Y_{N-1}) + \varepsilon) \cup \bigcup_{j=2}^{\text{int}\left[\frac{q}{N}\right]} \left( p < \beta(Y_{j-1}) \cap p \geq \beta(Y_j) \cap p \geq \beta(Y_{N-j}) + \varepsilon\right) \)

So,

\[ \Pr[b|v_1] = \Pr[(p \geq \beta(Y_1) \cap p \geq \beta(Y_{N-1}) + \varepsilon) | v_1] + \]

\[ + \sum_{j=2}^{\text{int}\left[\frac{q}{N}\right]} \Pr[(p < \beta(Y_{j-1}) \cap p \geq \beta(Y_j) \cap p \geq \beta(Y_{N-j}) + \varepsilon) | v_1] \]

This expression can be computed using Auxiliary Result 7.

- A7), B7) Following the same reasoning as in the previous derivation,

\[ \Pr[s, nd|v_1] = \Pr[s|v_1] \]

Partitioning the event "sell" analogously, this translates into

\[ \Pr[s|v_1] = \Pr[(p + \varepsilon \leq \beta(Y_{N-1}) + \varepsilon) \cap (p + \varepsilon \leq \beta(Y_1))] | v_1] + \]

\[ + \sum_{j=2}^{\text{int}\left[\frac{q}{N}\right]} \Pr[(p + \varepsilon > \beta(Y_{N-j+1}) + \varepsilon) \cap (p + \varepsilon \leq \beta(Y_{N-j}) + \varepsilon) \cap (p + \varepsilon \leq \beta(Y_1))] | v_1] \]

- C3) In this case ties must be taken into account. I assume tie-breaking by randomizing with equal probability among all banks that bid \( \bar{p} - \varepsilon \).

Again, \( \Pr[b, nd|v_1] = \Pr[b|v_1] \). Using the Law of Total Probability\(^{21}\),

\[ \Pr[b|v_1] = \sum_{q=0}^{N-2} \Pr[b|v_1, "bid with q"] \cdot \Pr["bid with q" | v_1], \]

\(^{21}\)Note that \( \Pr[b|v_1, "bid with N - 1"] = 0. \)
where the event "bid with q" means that q other banks bid \( \bar{p} - \varepsilon \). I restrict attention to the case \( N=3 \). Given the equilibrium strategies, the relevant terms are:

i)

\[
\Pr [b|v_1, "bid with 0"] = \Pr [(p \geq \beta (Y_1) \cap p \geq \beta (Y_2) + \varepsilon) | v_1, "bid with 0"]
\]

\[
= \frac{\Pr [p \geq \beta (Y_1) \cap p \geq \beta (Y_2) + \varepsilon \cap (Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 \notin [v^{**}, \nu_{bos}]) | v_1]}{\Pr [(Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 \notin [v^{**}, \nu_{bos}]) | v_1]}
\]

\[
= \frac{\Pr [(Y_1 < v^{**}) \cap p \geq \beta (Y_2) + \varepsilon \cap (Y_2 \notin [v^{**}, \nu_{bos}]) | v_1]}{\Pr [(Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 \notin [v^{**}, \nu_{bos}]) | v_1]}
\]

ii)

\[
\Pr [b|v_1, "bid with 1"] = \frac{1}{2} \Pr [(p \geq \beta (Y_1) \cap p \geq \beta (Y_2) + \varepsilon) | v_1, "bid with 1"]
\]

\[
= \frac{1}{2} \Pr [(p \geq \beta (Y_1) \cap p \geq \beta (Y_2) + \varepsilon \cap (Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 \notin [v^{**}, \nu_{bos}]) \cup (Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 \notin [v^{**}, \nu_{bos}]) | v_1]}
\]

\[
= \frac{1}{2} \Pr [(Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 \notin [v^{**}, \nu_{bos}]) \cup (Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 \notin [v^{**}, \nu_{bos}]) | v_1]
\]

\[
= \frac{1}{2} \Pr [(Y_1 \notin [v^{**}, \nu_{bos}]) \cap (Y_2 < v^{**}) \cup (Y_1 \geq \nu_{bos}) \cap (Y_2 \in [v^{**}, \nu_{bos}]) | v_1]
\]
C7) Following the steps from the previous derivation,

$$\Pr[s, nd|v_1] = \Pr[s|v_1] = \sum_{q=0}^{N-2} \Pr[s|v_1, \text{"offer with } q\text{"}] \cdot \Pr[\text{"offer with } q\text{"}|v_1],$$

where the event "offer with q" means that q other banks offer at $\bar{p}$. In this case the relevant terms are:

i) Write derivation

ii) Write derivation.

(Write Auxiliary Result 9: distribution to compute $E[V_t - \bar{p}|v_1, d], v_1 \geq v_{bos} \Rightarrow \text{bid } \bar{p})$

(Write Auxiliary Result 10: for the case $p = \bar{p} - \epsilon$ (row C) in Table 1),

$$\frac{\partial \Pr[s|v_1, v^*]|_{v_1 = v^*}}{\partial v^*} < 0, \text{ which implies } \frac{\partial E[\pi(\bar{p} - \epsilon)|v_1, v^*]|_{v_1 = v^*}}{\partial v^*} > 0).$$

(Write FOC of which characterizes $E[\pi(p)|v_1], \text{ for } v_1 < v^{**}$)

Result 3: With this results I can characterize the above mentioned symmetric BNE, summarized by an increasing (bid-price) function $\beta(v)$ and two cut-off values $(v^*_{bos}, v^{**})$, with $v^*_{bos} \geq v^{**}$. In such equilibrium, bank $i$ chooses its bid price according to the increasing function $\beta(v_i)$ (and therefore places its offer at $\beta(v_i) + \epsilon$). The trigger values are such that $v \in [v^{**}, v^*_{bos}) \Rightarrow \beta(v) = \bar{p} - \epsilon$, and $v \in [v^*_{bos}, \infty) \Rightarrow \beta(v) = \bar{p}$.

Proof: Characterize $v_{bos}, v^{**}$ etc.
5 Comparing the Mechanisms

There are two channels through which the BOS auction could be more effective than standard auctions in preventing speculative attacks. The first is the direct effect of the number of banks that bid \( \bar{p} \) required for a devaluation to take place. Given \( v^*_{sa} = v^*_{bos} \), in the general case in which \( k^* < N - 1 \) a devaluation occurs under the standard auction when the number of banks which receive a signal above \( v^*_{sa} \) exceeds \( k^* \). Under the BOS auction, a devaluation occurs when the number of banks which receive a signal above \( v^*_{bos} \) exceeds the sum of \( k^* \) and the number of banks which receive a signal below \( v^*_{bos} \). That is, under the BOS auction a devaluation occurs if and only if

\[
N_+ (\bar{p}) > N_-(\bar{p}) + k^* \iff N_+ (\bar{p}) > N - N_+ (\bar{p}) + k^* \iff N_+ (\bar{p}) > \frac{N + k^*}{2} > k^*.
\]

The second channel is through differences in equilibrium values of \( v^*_{sa} \) and \( v^*_{bos} \). In the special case analysed in this paper, in which \( k^* = N - 1 \), the first channel does not exist: under both auction mechanisms a devaluation occurs if and only if every bank bids \( \bar{p} \). So, eventual differences in the implications of the two auction mechanisms for the probability of a devaluation can only arise due to the second channel.

I state below the main result of the paper, which shows that the BOS auction does imply a lower probability of devaluation when compared to the
standard auction (under a set of assumptions).

Result 4: Assume \( k^* = N - 1 \), and that the conditions for \( E [\pi (\bar{p}) | v_1, v^*] | v_1 = v^* \) and \( E [\pi (\bar{p} - \varepsilon) | v_1, v^*_{bos}] | v_1 = v^*_{bos} \) to be increasing in \( v^* \) hold. Then \( v^*_{sa} \leq v^*_{bos} \), which means that the ex-ante probability of devaluation under the BOS auction is smaller than under the standard auction.

Proof: \( E [\pi (\bar{p}) | v^*_{bos}] = E [\pi (\bar{p} - \varepsilon) | v^*_{bos}] \), while \( E [\pi (\bar{p}) | v^*_{bos}] = 0 \). By Auxiliary Result 10, \( \frac{\partial E [\pi (\bar{p} - \varepsilon) | v_1, v^*] | v_1 = v^*}{\partial v^*} > 0 \) and by an individual rationality (or participation) constraint augment, \( E [\pi (\bar{p} - \varepsilon) | v^*] \geq 0 \). So, \( E [\pi (\bar{p}) | v^*_{bos}] = E [\pi (\bar{p} - \varepsilon) | v^*_{bos}] > E [\pi (\bar{p} - \varepsilon) | v^*] \geq 0 \). This implies \( v^*_{bos} \geq v^*_{sa} \), since \( E [\pi (\bar{p}) | v_1, v^*] | v_1 = v^* \) is an increasing function of \( v^* \).

(This result should hold in more general cases \( k^* < N - 1 \) and/or \( N > 3 \), as long as the relevant functions remain monotonic (increasing in \( v^* \)).

6 Concluding Remarks

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\(^{22}\)To be more precise, I also impose \( N=3 \).
References


Figure 1:
Figure 2:
| $E[\pi(p)|v_1]$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|
| $Pr[b,d|v_1]$   |   |   |   |   |   |   |   |   |
| $E[\pi(v_1,b,d)]$ |   |   |   |   |   |   |   |   |
| $Pr[b,nd|v_1]$  |   |   |   |   |   |   |   |   |
| $E[\pi(v_1,b,nd)]$ |   |   |   |   |   |   |   |   |
| $Pr[s,d|v_1]$   |   |   |   |   |   |   |   |   |
| $E[\pi(v_1,s,d)]$ |   |   |   |   |   |   |   |   |
| $Pr[s,nd|v_1]$  |   |   |   |   |   |   |   |   |
| $E[\pi(v_1,s,nd)]$ | p + $\varepsilon - \bar{p} + c$ |   |   |   |   |   |   |   |
| $p \in [\bar{p} - c - \varepsilon, \bar{p} - c]$ | 0 |   |   |   |   |   |   |   |
| $p \in [\bar{p} - c, \bar{p} - \varepsilon)$ | 0 |   |   |   |   |   |   |   |
| $p = \bar{p} - \varepsilon$ | 0 |   |   |   |   |   |   |   |
| $p = \bar{p}$ | D1) | D3) |   |   |   |   |   |   |

buy
sell
devaluation

$= $ no devaluation

$\pi(p)$