Nash Equilibrium under Knightian Uncertainty: A Generalization of the Existence Theorem

Paulo César Coimbra-Lisboa
EPGE - FGV

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Abstract

The most well successful definition of Nash equilibrium for two-person normal form games in the presence of Knightian uncertainty is due to Dow-Werlang [1994]. With the formalization of Gilboa and Schmeidler they proved the existence of equilibrium for the case of an uniform squeeze (let \( \Sigma \) be the power set of a finite state space - \( \Omega \), \( v \) be a convex capacity, \( q \) be a(n) (additive) probability measure and \( c \) a number between 0 and 1, and then, for any \( A \in \Sigma \) - except the whole set - \( v(A) = (1 - c)q(A) \) and in the case of \( A \) being the whole set both \( v \) and \( q \) are 1). Taking a different definition of support from that of Dow-Werlang's Nash Equilibrium under Uncertainty paper, Marinacci [2000] extended the proof of the existence for any given uncertainty aversion function.

The purpose of this paper is to extend the proof of the Dow-Werlang's existence theorem of Nash equilibrium under uncertainty, using the same definition of support in their paper (the most useful, general notion and has some advantages over the others definitions). Let \( \Sigma \), \( v \) and \( q \) as above defined and \( \psi: \Sigma \rightarrow [0,1] \) be an uncertainty aversion function such that, for any \( A \in \Sigma \):
\[
\psi(A) = c(v,A) - \text{the right hand side of this equality is the uncertainty aversion measure of } v \text{ at event } A \text{ (see Dow-Werlang [1992])}.
\]
I will present a restriction over the set of convex capacities, more specifically, I will work with the class of convex capacities that are squeeze of (additive) probability measure (that I will refer as the set \( \Theta(\mathcal{P},\mathcal{P}) \)). Then any capacity \( v \in \Theta(\mathcal{P},\mathcal{P}) \) can be represented as:
\[
v(A) = (1 - \psi(A))q(A), \text{ except for the case when } A \text{ is the whole set (implying } v \text{ and } q \text{ are 1). This enables us to give a parametric approach of the existence result that generalize Dow-Werlang's existence theorem and will be very useful for comparative static exercises over the uncertainty aversion function.}

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1 Preliminary version.
2 PhD student in Economics - EPGE - FGV.
E-mail: coimbra@fgymail.br url: http://www.fgv.br/epge/users/coimbra. 2 Graduate School of Economics at Getulio Vargas Foundation.
Praia de Botafogo, nº 190, sala 1120. CEP: 22253-900. Rio de Janeiro, Brazil.
1. INTRODUCTION

Dow-Welang [1994] extended the notion of Nash equilibrium for two-player finite normal games when players are uncertainty on the behavior of his opponents. They showed the existence of equilibrium for any given degree of uncertainty (however constant over all possible events, except the null and the whole event). Using a different definition of support, Marinacci [2000] proved the existence of Nash equilibrium for any given uncertainty aversion function.

In this paper I will extend Dow-Welang [1994]'s Nash equilibrium under uncertainty using the same definition of support that they used and a parametrical approach, based on the uncertainty aversion function, which enables us to do comparative exercises in a easy way. I will work with convex capacities that are "squeeze" of (additive) probability measures.

1.1. Ellsberg Paradox

In Ellsberg [1961]'s paper there exists a mind experiment that consist of making a choice over lotteries to show that decisions under uncertainty (in the Knight sense) is not consistent with Savage's paradigm. Suppose that there exist an urn that contains 90 balls on 3 different colors: red, black and yellow. Besides this is also known that there exists 30 red balls and that all that is known about the other 60 balls is that they are black or yellow. Ellsberg purpose two mind experiments.

Imagine, first, the two following lotteries: "win $100 if a red ball ball is drawn from the urn and win nothing otherwise" and "win $100 if a black ball is drawn from the urn and win nothing otherwise". Of course it is expected that most of participants of this experiment will prefer to bet on "red ball" lottery, because it is a known fact that 1/3 of the balls are with certainty reds. However, if in each of these lotteries the option "yellow ball" is included in the following way: "win $ 100 if a red ball or a yellow ball is drawn from the urn and win nothing otherwise" and "win $ 100 if a black ball or a yellow ball is drawn from the urn and win nothing otherwise" then it is expected that most of the participants will now prefer to bet on "black ball – yellow ball" lottery. This occurs because it is known with certainty that there exist exactly 60 balls that are blue or yellow. So decisions under uncertainty are different from that ones when the probability distributions are objectively known.

The paper is organized as follows. The next section introduces the required definitions and basic statements on Knightian uncertainty's decision theory. In Section 3 I introduce formally the class of convex capacities that are squeeze of (additive) probability measures, a key concept to the extension that I am purposing in this paper. Section 4 gives the definition of Nash equilibrium under uncertainty and present the theorem on the existence of Nash equilibrium that extend the Dow-Welang[1994]'s existence theorem [1994]. Section 5 present some related results and concludes.

2. SET-UP AND PRELIMINARIES ON CAPACITY INTEGRATION

The decision setting that I will use in the paper are developed in a Savage-style (see Savage [1954]). Let $\Omega$ be a non-empty finite set that contains all the states of the world, $\Sigma$ be an algebra of subsets of $\Omega$ called events and a set $X$ of consequences. I denote by $\mathcal{F}$ the class of all simple acts: finite-valued functions $\omega: \Omega \rightarrow R$ which are measurable with respect to $\Sigma$. For $x \in X$
define \( x \in \mathcal{F} \) to be the constant act such that \( x(\omega) = x \) for all \( \omega \in \Omega \). So, with abuse of notation, the set \( \chi \) of consequences is identified with the subclass of the constant acts in \( \mathcal{F} \).

A set-function \( v : \Sigma \rightarrow \mathbb{R} \) with \( v(\emptyset) = 0 \) is called a capacity (also called a non-additive probability) on \( (\Omega, \Sigma) \) if it is normalized and monotone, that is: i) normalized: \( v(\Omega) = 1 \); ii) monotone: For all \( A, B \in \Sigma \) such that \( A \subseteq B \), \( v(A) \leq v(B) \). I denote by \( V(\Omega, \Sigma) \) the class of all capacities on \( (\Omega, \Sigma) \). A capacity is convex if, besides (i) and (ii) it satisfies the following property: (iii) For all \( A, B \in \Sigma \): \( v(A \cup B) + v(A \cap B) \geq v(A) + v(B) \). I denote by \( \Lambda \) the class of all convex capacities on \( (\Omega, \Sigma) \). A capacity is (finitely) additive (also called a(n) (additive) probability measure) if, besides (i) and (ii) it satisfies the following property: (iii') For all \( A, B \in \Sigma \) such that \( A \cap B = \emptyset \): \( v(A \cup B) = v(A) + v(B) \). We denote by \( \Delta \) the class of all (additive) probability measures on \( (\Omega, \Sigma) \).

2.1. Uncertainty Aversion Measure

If \( \forall \lambda \in \Lambda \), then there exists at least a pair \( A, B \in \Sigma \) such that: \( v(A \cup B) + v(A \cap B) > v(A) + v(B) \). In particular, if \( B = (\Omega \setminus A) \) then \( v(A) + v(\Omega \setminus A) \) may be less than 1, implying that not all probability mass is allocated to an event and its complement. Dow-Werlang [1992] proposed an uncertainty aversion measure of a capacity \( v \) at event \( A \):

**Definition 2.1 (Dow-Werlang [1992])**

Let \( v \in V(\Omega, \Sigma) \) and \( A \in \Sigma \). The uncertainty aversion measure of \( v \) at event \( A \), is defined by:

\[
c(v, A) = 1 - v(A) - v(\Omega \setminus A).
\]

It is easy to prove that if a capacity is convex then for all \( A \in \Sigma \): \( c(v, A) \in [0, 1] \). Convex capacities are also known as non-additive probabilities reflecting uncertainty aversion. Throughout this paper I will restrict attention to convex capacities.

The following proposition presents the properties that are satisfied by the uncertainty aversion measure of \( v \) at event \( A \) when \( v \) is a convex capacity.

**Proposition 2.2:** Let \( v \in V(\Omega, \Sigma) \) and \( A, B \in \Sigma \). The uncertainty aversion measure of \( v \) at event \( A \) satisfies the following properties:

i) \( c(v, \emptyset) = c(v, \Omega) = 0 \);

ii) For all \( A \in \Sigma \): \( c(v, A) = c(v, (\Omega \setminus A)) \);

iii) For all \( A, B \in \Sigma \): \( c(v, (A \cup B)) + c(v, (A \cap B)) \leq c(v, A) + c(v, B) \).

\( a^1(\Omega) \in \Sigma \), where \( a^1(\Omega):= \{ \omega \in \Omega : a(\omega) \neq 0 \} \). I denote by \( B(\Omega, \Sigma) \) the class of all real-valued function, bounded on \( \Omega \) which are \( \Sigma \)-measurable. Note that \( \mathcal{F} (\subset B(\Omega, \Sigma)) \).

\( a^2(\Omega) \in \Sigma \), where \( a^2(\Omega):= \{ \omega \in \Omega : a(\omega) \neq 0 \} \). I denote by \( B(\Omega, \Sigma) \) the class of all real-valued function, bounded on \( \Omega \) which are \( \Sigma \)-measurable. Note that \( \mathcal{F} (\subset B(\Omega, \Sigma)) \).

\( a^3(\Omega) \in \Sigma \), where \( a^3(\Omega):= \{ \omega \in \Omega : a(\omega) \neq 0 \} \). I denote by \( B(\Omega, \Sigma) \) the class of all real-valued function, bounded on \( \Omega \) which are \( \Sigma \)-measurable. Note that \( \mathcal{F} (\subset B(\Omega, \Sigma)) \).

7 It is easy to prove the following:

**Lemma 2.2:** If \( v \in V(\Omega, \Sigma) \) then, for all \( A \in \Sigma \): \( v(A) \in [0, 1] \).

**Proof:** Omitted.

8 **Proposition 2.1:** Let \( v : \Sigma \rightarrow \mathbb{R} \) be a set-function such that, for all \( A, B \in \Sigma \):

\[
v(A \cup B) + v(\Omega \setminus B) \geq v(A) + v(B)
\]

Then \( v \) is monotone.

**Proof:** Let \( A, B \in \Sigma \) be such that \( A \subseteq B \). Define \( C = B \setminus A \). Then \( A \cup C = B \) and \( A \cap C = \emptyset \). From (iii) we now that: \( v(A \cup C) + v(\Omega \setminus C) = v(B) + v(\Omega) \geq v(A) + v(C) \). Using (i): \( v(B) \geq v(A) + v(C) \). We now that \( v(C) \geq v(A) \).

Q. E. D.

9 It is easy to prove the following:

**Lemma 2.3:** Let \( v \in V(\Omega, \Sigma) \)

If for all \( A \in \Sigma \): \( c(v, A) = 0 \) then \( v \in \Delta \).

**Proof:** Omitted.
Proof: For all $A \in \Sigma$: $c(v, A) = 1 - v(A) - v(\Omega \setminus A)$

We now that for all $A \subseteq \Omega$: $v(\emptyset) = 0$ and $v(\Omega) = 1$ so: $c(v, \emptyset) = c(v, \Omega) = 0$.

For all $A \in \Sigma$: $c(v, A) = 1 - v(A) - v(\Omega \setminus A) = c(v, (\Omega \setminus A))$

To check (iii) note first that:

$c(v, A) = 1 - v(A) - v(\Omega \setminus A)$
$c(v, B) = 1 - v(B) - v(\Omega \setminus B)$
$c(v, (A \cup B)) = 1 - v(A \cup B) - v(\Omega \setminus (A \cup B))$

Morgan's law imply that:

$= 1 - v(A \cup B) - v((\Omega \setminus A) \cap (\Omega \setminus B))$
$c(v, (A \cap B)) = 1 - v(A \cap B) - v(\Omega \setminus (A \cap B))$

Morgan's law imply that:

$= 1 - v(A \cap B) - v((\Omega \setminus A) \cup (\Omega \setminus B))$

From the definition of convex capacity we know that, for all $A, B \in \Sigma$:

$v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$
$v((\Omega \setminus A) \cup (\Omega \setminus B)) + v((\Omega \setminus A) \cap (\Omega \setminus B)) \leq v(\Omega \setminus A) + v(\Omega \setminus B)$

Re-writing these expressions:

$1 - v(A \cup B) - v(A \cap B) \leq 1 - v(A) - v(B)$
$1 - v((\Omega \setminus A) \cup (\Omega \setminus B)) - v((\Omega \setminus A) \cap (\Omega \setminus B)) \leq 1 - v(\Omega \setminus A) - v(\Omega \setminus B)$

Sum:

$(1 - v(A \cup B) - v(A \cap B) + (1 - v((\Omega \setminus A) \cup (\Omega \setminus B)) - v((\Omega \setminus A) \cap (\Omega \setminus B))) \leq 1 - v(A) - v(\Omega \setminus A) - v(B)$

Re-writing and using Morgan's law:

$(1 - v(A \cup B) - v(A \cap B) + (1 - v((\Omega \setminus A) \cup (\Omega \setminus B)) - v((\Omega \setminus A) \cap (\Omega \setminus B))) \leq 1 - v(A) - v(\Omega \setminus A) - v(B))$

Finally, using the definition of uncertainty aversion measure:

$c(v, (A \cup B)) + c(v, (A \cap B)) \leq c(v, A) + c(v, B)$.

Q. E. D.

2.2. Choquet Integral

Since that capacities can be a non-additive measure I can't use an integral in the sense of Lebesgue. The appropriate notion of integral is due to Choquet [1953]. For any given real-valued function, bounded on $\Omega$, $a \in B(\Omega, \Sigma)$, the Choquet integral of $a$ with respect to a capacity $v \in V(\Omega, \Sigma)$ is defined as follows:

$$\int_{\omega \in \Omega} a(\omega) \, dA = \int_{-\infty}^{0} [v(\{\omega \in \Omega : a(\omega) \geq \alpha\}) - 1] d\alpha + \int_{0}^{\infty} [v(\{\omega \in \Omega : a(\omega) \geq \alpha\}) - v(\{\omega \in \Omega : a(\omega) \geq \alpha\}] d\alpha \tag{10}$$

From now on I will use the following simplification: $v(a \geq \alpha) = v(\omega \in \Omega : a(\omega) \geq \alpha)$. so the Choquet integral can be re-written as:
where the right hand side is a well defined integral in the sense of Riemann (because $a$ is bounded and $v$ is monotone)\(^{11}\).

All the real-valued function considered in this paper will be simple acts. Besides this, every simple act $f \in \mathcal{F}$ will be defined such that: $f(\omega_1) \leq f(\omega_2) \leq \ldots \leq f(\omega_n)$. I am particularly interested in a special subclass of the class of simple acts, which are constituted by the co-monotonic acts. Two acts, $f, g \in \mathcal{F}$ are said to be co-monotonic if, for all $\omega, \omega' \in \Omega$: $(f(\omega)-f(\omega'))(g(\omega)-g(\omega')) \geq 0$. I denote by $\mathcal{F}$ the class of co-monotonic acts which is constituted by simple acts that are pairwise co-monotonics.

For all $f, g \in \mathcal{F}$ and $u, v' \in \mathcal{A}$, the list below present some properties of Choquet integral (proved in Simonsen-Werlang [1991]):

a) **Positive homogeneous:** $\int c f dv = c \int f dv$, for $c \in \mathbb{R}_+$;

b) **Monotonic:** $f \geq g \Rightarrow \int f dv \geq \int g dv$;

c) $\int (f + c) dv = \int f dv + c$, for $c \in \mathbb{R}_+$;

d) **Co-monotonic additive:** If $f, g \in \mathcal{F}$ are non-negative then: $\int (f + g) dv = \int f dv + \int g dv$;

e) **Monotonic in the capacity:** If $v \geq v'$ then $\int f dv \geq \int f dv'$.

f) $\int f dv + \int (-f) dv \leq 0$;

g) **Jensen Inequality:** For all concave functions, $u: \mathbb{R} \rightarrow \mathbb{R}$:

$$\int u(f) dv \leq u \left( \int f dv \right).$$

In this paper I will use the Gilboa [1987]'s axiomatization of the use of the Choquet integral as a choice criterion, which in turn is based in a Savage-style. Wakker [1989] and Sarin and Wakker [1992] have an axiomatization which are very similar to Gilboa [1987]. Schmeidler [1989] present an axiomatization in an Anscombe-Aumann-style (see Anscombe-Aumann [1963]).

### 3. Convex Capacities that are Squeeze of (Additive) Probability Measures

In this section I will present a special class of convex capacities that will enables us to do comparative static exercises in a easy way. First I will discuss the support of a capacity (subsections 3.1), then I will present a function that has the same properties of an uncertainty aversion measure (subsection 3.2). The next step is present the condition on which convex capacities are squeeze of (additive) probability measures (subsection 3.3). Finally I can present the Choquet integral with convex capacities that are squeeze of (additive) probability measures.

#### 3.1. Support of a capacity

The notion of support of a capacity is the first step necessary to study the conditions in which a convex capacity can be understood as a squeeze of a(n) (additive) probability measure.

**Definition 3.1 Support of a Capacity**

Let $v \in V(\Omega, \Sigma)$ and $A, B \in \Sigma$.

The support of the capacity $v$ is an event $B$ such that:

i) $v(\Omega \setminus B) = 0$;

ii) For all $A, B \in \Sigma, A \subset B$: $v(\Omega A) > 0$.

$$\int f dv = \int (f(\alpha) - 1) d\alpha + \int f(\alpha) d\alpha$$

\(^{11}\) If $v$ is a(n) (additive) probability measure then the integral is equal to a standard (additive) integral.
The Ellisberg's urn, presented at the introduction of this paper, give us an example of a capacity that does not have an unique support (if $k = 0$).

**Example 3.1 Ellisberg's Urn**

Let $E_r$ be the event "a red ball is drawn from the urn". The events $E_b$ and $E_y$ are similarly defined.

Let $(\Omega, \Sigma, \nu)$ be a capacity space that reflects the Ellisberg's urn, i.e.:

- $\Omega = \{E_r, E_b, E_y\}$
- $\Sigma = 2^\Omega$
- $\nu(\emptyset) = 0; \nu(\Omega) = 1; \nu(E_r) = 1/3; \nu(E_b) = \nu(E_y) = k$
- $\nu(E_r, E_b) = \nu(E_r, E_y) = (1/3 + k); \nu(E_b, E_y) = 2/3$

If $k \in [0, 1/3]$ then $\nu$ is a convex capacity.

If $k \in (0, 1/3]$ then the (unique) support is the event $\{E_r, E_b, E_y\}$.

If $k = 0$ then the supports are the events $\{E_r\}$ and $\{E_b, E_y\}$.

The following example, due to Eichberger-Kelsey [2000], shows that the supports may all get a zero mass:

**Example 3.2 (Eichberger-Keysey [2000])**

Let $(\Omega, \Sigma, \nu)$ be a capacity space defined by:

- $\Omega = \{\omega_1, \omega_2, \omega_3\}$
- $\Sigma = 2^\Omega$
- $\nu(\emptyset) = 0; \nu(\Omega) = 1; \nu(\omega_1) = \nu(\omega_2) = \nu(\omega_3) = 0$
- $\nu(\omega_1, \omega_2) = 1; \nu(\omega_1, \omega_3) = \nu(\omega_2, \omega_3) = 0$

The supports are the events $\{\omega_1\}$ and $\{\omega_2\}$, both with zero mass.

The following proposition, due to Marinacci [2000], shows when there is an unique support.

**Proposition 3.1: (Marinacci [2000])**

Let $\nu \in V(\Omega, \Sigma)$.

The following two conditions are equivalent:

1. $\text{supp } \nu$ is unique
2. For all $A \in \Sigma$ and all $\omega \in \Omega$ such that $\nu(A) = \nu(\{\omega\}) = 0$:
   \[ \nu(A \cup \{\omega\}) = \nu(A) + \nu(\{\omega\}) \]

**Proof:** Marinacci, M. [2000], Games and Economic Behavior.

This suggest the following definition:

**Definition 3.2 Convex Capacity with an Unique Support**

A convex capacity has an unique support if, besides (i) to (iii) it satisfies the following property:

1. For all $A \in \Sigma$ and all $\omega \in \Omega$ such that $\nu(A) = \nu(\{\omega\}) = 0$:
   \[ \nu(A \cup \{\omega\}) = \nu(A) + \nu(\{\omega\}) \]

We denote by $A(S) (\subset A)$ the class of all convex capacities which has an unique support.

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12 Marinacci [2000] presents a very similar example.
13 This example is very interesting. The smallest event with capacity 1 is the event $\{\omega_1, \omega_2\}$, however the events that are supports have all zero mass, which means that the agent has full confidence that $\{\omega_1, \omega_2\}$ will happen but is extreme uncertain about which will be the true state of the world (between $\omega_1$ and $\omega_2$).
3.2. Uncertainty Aversion Function

In the section 2.1 I discuss about the uncertainty aversion measure and its properties when \( v \) is a convex capacity. Now I will define a set-function that has the same properties.

**Definition 3.3 Uncertainty Aversion Function**

A set-function \( \psi: \Sigma \to [0,1] \) is an uncertainty aversion function if it satisfies the following properties:

i) \( \psi(\emptyset) = \psi(\Omega) = 0 \);

ii) For all \( A \in \Sigma \): \( \psi(A) = \psi(\Omega \setminus A) \);

iii) For all \( A, B \in \Sigma \): \( \psi(A \cup B) + \psi(A \cap B) \leq \psi(A) + \psi(B) \).

I denote by \( \Psi(\Omega, \Sigma) \) the class of all uncertainty aversion function on \( (\Omega, \Sigma) \). Associate with each uncertainty aversion function \( \psi \in \Psi(\Omega, \Sigma) \) we have a subclass of convex capacities \( v \in \Lambda \) with the property that, for all \( A \in \Sigma \), the uncertain aversion measure is such that: \( c(v, A) = \psi(A) \). So, for each \( \psi \in \Psi(\Omega, \Sigma) \), I denote by \( \Lambda(\psi) \) the class of convex capacities which are such that for all \( A \in \Sigma \), \( c(v, A) = \psi(A) \).

**Example 3.3 Ellsberg’s Urn (Continuation)**

For each \( k \in [0,1/3] \) we now that the uncertainty aversion measure associate to the Ellsberg’s urn has the following characteristic:

\[
\begin{align*}
c(v, \{E_b\}) &= c(v, \{E_b, E_y\}) = 0; \\
c(v, \{E_y\}) &= c(v, \{E_b, E_y\}) = 2/3 - 2k; \\
c(v, \{E_b\}) &= c(v, \{E_b, E_y\}) = 2/3 - 2k.
\end{align*}
\]

So, for each \( k \in [0,1/3] \), we can define an uncertainty aversion function \( \psi \in \Psi(\Omega, \Sigma) \) in the following way: For all \( A \in \Sigma \), \( \psi(A) = c(v, A) \), where \( v \) is the convex capacity associate with the Ellsberg urn. That is:

\[
\begin{align*}
\psi(\{E_b\}) &= \psi(\{E_b, E_y\}) = 0; \\
\psi(\{E_y\}) &= \psi(\{E_b, E_y\}) = 2/3 - 2k; \\
\psi(\{E_b\}) &= \psi(\{E_b, E_y\}) = 2/3 - 2k.
\end{align*}
\]

Of course if \( v \) is the convex capacity associate to the Ellsberg’s urn then, for each \( k \in [0,1/3] \), \( \psi \in \Psi(\Omega, \Sigma) \) is the above uncertainty aversion function.

The following example shows that there exist at least a \( k \in [0,1/3] \) such that \( \Lambda(\psi, k) \) contains more than one element:

**Example 3.4 If \( k = 1/3 \) Then \( \Lambda(\psi, k=1/3) \) is a Non-Empty, Non-Unitary Class**

Let \( (\Omega, \Sigma) \) be a state space defined by:

\[
\begin{align*}
\Omega &= \{E_n, E_b, E_y\}; \\
\Sigma &= 2^\Omega.
\end{align*}
\]

Let \( \psi \in \Psi(\Omega, \Sigma) \) be defined as in example 3.3.

If \( k = 1/3 \) then, for all \( A \in \Sigma \): \( \psi(A) = 0 \).

Let \( p \in A \) be an additive probability measure on \( (\Omega, \Sigma) \). Then, for any \( p \) the uncertainty aversion measure is such that, for all \( A \in \Sigma \), \( c(v, A) = 0 \).

So, \( \Lambda(\psi, k=1/3) \) defined in example 3.3 is a non-empty, non-unitary class:

\( \Lambda(\psi, k=1/3) = \{p \in A; A \) is the set of probability measures on \( (\Omega, \Sigma) \} \).

3.3. Squeeze of (Additive) Probability Measures

In this subsection I will summarize the results that enables us to characterize convex capacity that are squeeze of (additive) probability measures associate to uncertainty aversion functions. I begin with the definition of a squeeze:
Definition 3.4 Squeeze of a(n) (additive) probability measure (Squeeze).

Let \( \psi \in \Psi(\Omega, \Sigma) \).

The capacity \( \nu \in V(\Omega, \Sigma) \) is a squeeze of a(n) (additive) probability measure \( p \in \Delta \) associate to an uncertainty aversion function if:

\[
\nu(A) = \begin{cases} 
(1 - \psi(A)) p(A) & \text{se } A \neq \Omega \\
1 & \text{se } A = \Omega 
\end{cases}
\]

Example 3.5 Uniform Squeeze

Let \( \psi \in \Psi(\Omega, \Sigma) \) be such that, for all \( A \in \Sigma \), \( \emptyset \neq A \neq \Omega \): \( \psi(A) = c \) (and, as usual, \( \psi(\emptyset) = \psi(\Omega) \) where \( c \in [0, 1] \)). Then, for all \( p \in \Delta \) defined on \( (\Omega, \Sigma) \) there exists a convex capacity \( \nu \in \Lambda(\Psi) \) that is an uniform squeeze of \( p \) defined by:

\[
\nu(A) = \begin{cases} 
(1 - c) p(A) & \text{se } A \neq \Omega \\
1 & \text{se } A = \Omega 
\end{cases}
\]

To check it, define a set-function \( q : \Sigma \rightarrow [0, 1] \) by:

\[
q(A) = \begin{cases} 
\nu(A) & \text{se } A \neq \Omega \\
(1 - c) & \text{se } A = \Omega 
\end{cases}
\]

where \( \nu \) is defined as above.

It's easy to verify that, for all \( c \in [0, 1] \), \( q \in \Delta \).

The following definition extend the properties of a convex capacity that are necessary to be satisfied to be represented as a squeeze of some \( p \in \Delta \) defined on \( (\Omega, \Sigma) \).

Definition 3.5 Properties of a Convex Capacity that are Squeeze.

Let \( \nu \in \Lambda(\Sigma) \), with \( \emptyset \neq D = \text{supp } \nu \subseteq \Omega \), and \( C, C', D, E \in \Sigma \).

I can say that \( \nu \) is a squeeze of a(n) (additive) probability measure \( p \in \Delta \) associate to some \( \psi \in \Psi(\Omega, \Sigma) \) if it also satisfy the following properties:

\( \nu \) is a squeeze of a(n) (additive) probability measure \( p \in \Delta \) associate to some \( \psi \in \Psi(\Omega, \Sigma) \):

\begin{align*}
&v) \text{ For all } \emptyset \neq C, C' \subset D, \text{ with } C \cup C' \subset D \text{ and } C \cap C' = \emptyset:
&\quad \nu(C \cup C') = \nu(C) + \nu(C') \\
&vi) \text{ For all } \emptyset \neq C \subset D \text{ and all } E \subset (\Omega \setminus D):
&\quad \nu(C \cap E) = \nu(C)
\end{align*}

Proposition 3.2 Let \( \psi \in \Psi(\Omega, \Sigma) \).

If \( \psi \in \Lambda(\Psi) \) satisfy the properties (i) to (vi) then \( \nu \) is a squeeze of some \( p \in \Delta \) defined on \( (\Omega, \Sigma) \) associate to some \( \psi \).

Proof: Coimbra-Lisboa, P. [2003]. \(^{14}\)

I denote by \( \Theta(\Delta, \Psi) \) (\( \subset \Lambda(\Sigma) \)) the class of all convex capacities that are squeeze of (additive) probability measures \( p \in \Delta \) defined on \( (\Omega, \Sigma) \) associate to some \( \psi \in \Psi(\Omega, \Sigma) \).

Proposition 3.3 Let \( \nu \in \Theta(\Delta, \Psi) \), with \( \emptyset \neq D = \text{supp } \nu \subseteq \Omega \) and \( C, C', D, E \in \Sigma \).

The uncertainty aversion measure of \( \nu \) also satisfy the following properties:

\begin{align*}
&iv) \text{ For all } \emptyset \neq C, C' \subset D, \text{ with } C \cup C' \subset D \text{ and } C \cap C' = \emptyset:
&\quad c(\nu, C) = c(\nu, C') \geq c(\nu, D)
\end{align*}

v) For all $\emptyset \neq C \subseteq D$ and all $E \subseteq (\Omega \setminus D)$:
\[ c(v, (C, E)) = c(v, C) \]

**Proof:** Coimbra-Lisboa, P. [2003].

**Proposition 3.4** Let $v \in \mathcal{V}(\Omega, \Sigma)$, $p \in \Delta$, with $\emptyset \neq D = \text{supp } p \subseteq \Omega$ and $C, C', D, E \in \Sigma$.

If $\psi$ also satisfy the following properties:

iv) For all $\emptyset \neq C, C' \subseteq D$, with $C \cup C' \subseteq D$ and $C \cap C' = \emptyset$:
\[ \psi(C) = \psi(C') \geq \psi(D) \]

v) For all $\emptyset \neq C \subseteq D$ and all $E \subseteq (\Omega \setminus D)$:
\[ \psi(C \cup E) = \psi(C) \]

And if $v \in \mathcal{V}(\Omega, \Sigma)$ is defined by:
\[ v(A) = \begin{cases} (1 - \psi(A))p(A) & \text{if } A \neq \Omega \\ 1 & \text{if } A = \Omega \end{cases} \]

Then $v \in \Lambda(\psi)$.

**Proof:** Coimbra-Lisboa, P. [2003].

**Proposition 3.5** Let $v \in \mathcal{V}(\Omega, \Sigma)$, $p \in \Delta$, with $\emptyset \neq D = \text{supp } p \subseteq \Omega$ and $C, C', D, E \in \Sigma$.

If $\psi$ also satisfy the following properties:

iv) For all $\emptyset \neq C, C' \subseteq D$, with $C \cup C' \subseteq D$ and $C \cap C' = \emptyset$:
\[ \psi(C) = \psi(C') \geq \psi(D) \]

v) For all $\emptyset \neq C \subseteq D$ and all $E \subseteq (\Omega \setminus D)$:
\[ \psi(C \cup E) = \psi(C) \]

And if $v \in \Lambda(\psi)$ is defined by:
\[ v(A) = \begin{cases} (1 - \psi(A))p(A) & \text{if } A \neq \Omega \\ 1 & \text{if } A = \Omega \end{cases} \]

Then $v$ also satisfy the following properties:

v) For all $\emptyset \neq C, C' \subseteq D$, with $C \cup C' \subseteq D$ and $C \cap C' = \emptyset$:
\[ v(C \cup C') = v(C) + v(C') \]

vi) For all $\emptyset \neq C \subseteq D$ and all $E \subseteq (\Omega \setminus D)$:
\[ v(C \cup E) = v(C) \]

**Proof:** Coimbra-Lisboa, P. [2003].

**Remark 3.1** If $\text{supp } v = \{a\}$, for any $a \in \Omega$ then properties (v) and (vi) are innocuous.

**Remark 3.2** If $\text{supp } v = \Omega$ then for all $A \in \Sigma$, $\text{supp } a \in \Omega$. $\psi(A) = c \in [0, 1]$. So, (v) and (vi) are trivially satisfied.

**Example 3.6 Convex Capacity that is Squeeze of a Probability Measure**

Let $(\Omega, \Sigma, v)$ be a capacity space defined by:
\[ \Omega = \{a, b, c\}; \]
\[ \Sigma = 2^\Omega; \]
\[ v(\emptyset) = 0; v(\Omega) = 1; v(\{a\}) = 0; v(\{b\}) = a; v(\{c\}) = b; \]
\[ v(\{a, b\}) = a; v(\{a, c\}) = b; v(\{b, c\}) = a + b + c. \]

The (unique) support is the event $\{a, b\}$.

If $a + b + c \leq 1$ then $v \in \Lambda$ and satisfies (v) and (vi), so $v \in \Theta(\Delta, \Sigma)$, i.e., $v$ is a squeeze for some $p \in \Delta$ defined on $(\Omega, \Sigma)$ and some $\psi \in \mathcal{V}(\Omega, \Sigma)$ satisfying (iv) and (v) such that, for all $A \in \Sigma$: $\psi(A) = c(v, A)$, where $c(v, A)$ is the uncertainty aversion measure of $v$. 

In fact, the following uncertainty aversion function on \((\Omega, \Sigma)\) satisfies these requirements:

\[
\begin{align*}
\psi(\{\omega_1\}) &= \psi(\{\omega_2, \omega_3\}) = 1 - a - b - c; \\
\psi(\{\omega_2\}) &= \psi(\{\omega_1, \omega_3\}) = 1 - a - b; \\
\psi(\{\omega_3\}) &= \psi(\{\omega_1, \omega_2\}) = 1 - a - b.
\end{align*}
\]

So, \(v \in \mathcal{N}(\psi)\). Note also that \(\psi\) satisfies (iv) and (v), which means that there exists some \(p \in \Delta\) such that \(v\) can be defined as:

\[
v(A) = \begin{cases} 
(1 - \psi(A)) p(A) & \text{if } A \neq \Omega \\
1 & \text{if } A = \Omega
\end{cases}
\]

The set-function \(q : \Sigma \to \mathbb{R}\) is defined on \((\Omega, \Sigma)\) by:

\[
q(A) = \begin{cases} 
\psi(A) & \text{if } A \neq \Omega \\
1 & \text{if } A = \Omega
\end{cases}
\]

It is easy to check that \(q \in \Delta\). So, \(v\) is a squeeze of \(q \in \Delta\) associate to \(\psi \in \mathcal{N}(\Omega, \Sigma)\).

**Example 3.7 Convex Capacity that is Squeeze of a Probability Measure**

Let \((\Omega, \Sigma, v)\) be a capacity space defined by:

\[
\begin{align*}
\Omega &= \{\omega_1, \omega_2, \omega_3, \omega_4\}; \\
\Sigma &= \mathcal{P}(\Omega); v(\emptyset) = 0; v(\Omega) = 1; \\
v(\{\omega_1\}) &= 0; v(\{\omega_2\}) = a; v(\{\omega_3\}) = b; v(\{\omega_4\}) = c; \\
v(\{\omega_1, \omega_2\}) &= a; v(\{\omega_1, \omega_3\}) = b; v(\{\omega_1, \omega_4\}) = c; \\
v(\{\omega_2, \omega_3\}) &= a + b; v(\{\omega_2, \omega_4\}) = a + c; v(\{\omega_3, \omega_4\}) = b + c; \\
v(\{\omega_1, \omega_2, \omega_3\}) &= a + b + c; v(\{\omega_1, \omega_2, \omega_4\}) = a + b + c + d;
\end{align*}
\]

The (unique) support is the event \{\(\omega_2, \omega_3, \omega_4\)\}.

If \(a + b + c + d \leq 1\) then \(v \in \Delta\) and satisfies (vi) and (vii), so \(v \in \mathcal{N}(\Delta, \Psi)\), i.e., \(v\) is a squeeze of some \(p \in \Delta\) defined on \((\Omega, \Sigma)\), and some \(\psi \in \mathcal{N}(\Omega, \Sigma)\) satisfying (iv) and (v) such that, for all \(A \in \Sigma\):

\[c(v, A) = \psi(A) = c(v, A)\]

The following uncertainty aversion function on \((\Omega, \Sigma)\) satisfies these requirements:

\[
\begin{align*}
\psi(\{\omega_1\}) &= \psi(\{\omega_2, \omega_3, \omega_4\}) = 1 - a - b - c - d; \\
\psi(\{\omega_2\}) &= \psi(\{\omega_1, \omega_3, \omega_4\}) = 1 - a - b - c; \\
\psi(\{\omega_3\}) &= \psi(\{\omega_1, \omega_2, \omega_4\}) = 1 - a - b - c; \\
\psi(\{\omega_4\}) &= \psi(\{\omega_1, \omega_2, \omega_3\}) = 1 - a - b - c; \\
\psi(\{\omega_1, \omega_2\}) &= \psi(\{\omega_1, \omega_3, \omega_4\}) = 1 - a - b - c - d;
\end{align*}
\]

So \(v \in \mathcal{N}(\psi)\). Note also that \(\psi\) satisfies (iv) and (v), which means that there exist some \(p \in \Delta\) such that \(v\) can be defined as:

\[
v(A) = \begin{cases} 
(1 - \psi(A)) p(A) & \text{if } A \neq \Omega \\
1 & \text{if } A = \Omega
\end{cases}
\]

The set-function \(q : \Sigma \to \mathbb{R}\) is defined on \((\Omega, \Sigma)\) by:

\[
q(A) = \begin{cases} 
\psi(A) & \text{if } A \neq \Omega \\
1 & \text{if } A = \Omega
\end{cases}
\]

It is easy to check that \(q \in \Delta\). So, \(v\) is a squeeze of \(q \in \Delta\) associate to \(\psi \in \mathcal{N}(\Omega, \Sigma)\).
3.4. Choquet Integral with Convex Capacities that are Squeeze

Theorem 3.1 Let $p \in \Delta$. If $v \in \Theta(\Delta, \Psi)$ then the following conditions are equivalents:

i) $\int fdv \equiv \int \left( v(f \geq \alpha) - 1 \right) d\alpha + \int v(f \geq \alpha) d\alpha$

ii) $\int fdv \equiv \psi_i f(\{\omega_i\}) + (1 - \psi_i) \int fdp + \sum_{j \neq i} (\psi_i - \psi_{i,j}) \left( \sum_{\omega_j} p(\{\omega_j\}) \right) (f(\{\omega_j\}) - f(\{\omega_{j-1}\}))$

where: $\psi_i = \psi(\{\omega_i\})$ and $\psi_{i,j} = \psi(\{\omega_i, ..., \omega_j\})$; $\int fdp$ is the integral of $f$ with respect to the probability measure $p \in \Delta$.

Proof: $\int fdv \equiv \int \left( v(f \geq \alpha) - 1 \right) d\alpha + \int v(f \geq \alpha) d\alpha$

If $\alpha < v(\{\omega_i\})$ then $v(\{\omega_i\}) = v(\{\omega_i, \omega_{i+1}, ..., \omega_k\}) = v(\Omega) = 1$;
If $\alpha \in [v(\{\omega_i\}), v(\{\omega_{i+1}\})]$ then $v(\{\omega_i\}) = (1 - \psi_{i,j}) p(\{\omega_i, \omega_{i+1}, ..., \omega_k\})$;
If $\alpha \in [v(\{\omega_i\}), v(\{\omega_{i+1}\})]$ then $v(\{\omega_i\}) = (1 - \psi_{i,j}) p(\{\omega_i, \omega_{i+1}, ..., \omega_k\})$;

And so on...

If $\alpha \in [v(\{\omega_i, \omega_{i+1}\}), v(\{\omega_{i+2}\})]$ then $v(\{\omega_i, \omega_{i+1}\}) = 0$;
If $\alpha > v(\{\omega_k\})$ then $v(\{\omega_k\}) = 0$.

$\int fdv \equiv \int \left( v(f \geq \alpha) - 1 \right) d\alpha + \int v(f \geq \alpha) d\alpha$

$\equiv \int \left( v(f \geq \alpha) - 1 \right) d\alpha + \int \left( v(f \geq \alpha) - 1 \right) d\alpha + \int \left( v(f \geq \alpha) - 1 \right) d\alpha + ... + \int \left( v(f \geq \alpha) - 1 \right) d\alpha$

$+ \int v(f \geq \alpha) d\alpha + \int v(f \geq \alpha) d\alpha + \int v(f \geq \alpha) d\alpha + ... + \int v(f \geq \alpha) d\alpha$

$\equiv \int \left( (1 - \psi_{i,j}) p(\{\omega_i, \omega_{i+1}, ..., \omega_k\}) - 1 \right) d\alpha + ... + \int \left( (1 - \psi_{i,j}) p(\{\omega_i, \omega_{i+1}, ..., \omega_k\}) - 1 \right) d\alpha$
After a little algebra:

\[
= \psi(f(\omega_1)) + (1 - \psi) \int f dp + \\
+ \sum_{j=1}^{n} (\psi_1 - \psi_{j-1}) \left( \sum_{\omega_j} p(\omega_j) \right) (f(\omega_j)) - f(\omega_{j-1})
\]

Q. E. D.

These following examples, which extends, respectively examples 3.6 and 3.7, intends to show the equivalence between these two formulas of Choquet integral if \( \varphi \in C(\Delta, \psi) \).

Example 3.8 (Continuation of Example 3.6)
Let \( (\Omega, \Sigma) \) be a state space defined as:
\[
\Omega = \{\omega_1, \omega_2, \omega_3\}; \\
\Sigma = 2^\Omega.
\]
Let \( \psi \in \Psi(\Omega, \Sigma) \) be an uncertainty aversion function defined by:
\[
\psi(\{\omega_1\}) = \psi(\{\omega_2, \omega_3\}) = 1 - a - b - c; \\
\psi(\{\omega_2\}) = \psi(\{\omega_1, \omega_3\}) = 1 - a - b; \\
\psi(\{\omega_3\}) = \psi(\{\omega_1, \omega_2\}) = 1 - a - b.
\]
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Let \( p \in A \) be defined by:
\[
p(\{\omega_j\}) = \frac{a}{a + b}; \quad p(\{\omega_d\}) = \frac{b}{a + b}.
\]

So \( \nu \in \Theta(A, \Psi) \) can be defined as:
\[
\nu(A) = \begin{cases} 
(1 - \Psi(A)) p(A) & \text{se } A \neq \Omega \\
1 & \text{se } A = \Omega
\end{cases}
\]

Let \( f : \Omega \to \mathbb{R} \) be a real-valued function defined by:
\( f(\omega_i) = i \), for \( i \in \{1, 2, 3\} \).

It is easy to check that the "usual" Choquet integral is \( 1 + a + 2b + c \).

Using (ii):
\[
\int \! \! \! \! d\psi_f = \psi_f(\omega_i) + \int \! \! \! \! d\psi_f + (\omega_i - \psi_f(\omega_i)) p(\omega_i)(f(\omega_i) - f(\omega_i)) = \]
\[
=(1-a-b-c), \quad 1, \quad (2a + 3b) + (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c), \quad (1-a-b-c).
\]

Q.E.D.

Example 3.9 (Continuation of Example 3.7)

Let \( (\Omega, \Sigma) \) be a state space defined as:
\[
\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}; \\
\Sigma = 2^\Omega.
\]

Let \( \psi \in \Psi(\Omega, \Sigma) \) be an uncertainty aversion function defined by:
\[
\psi(\{\omega_1\}) = \psi(\{\omega_1, \omega_2, \omega_3\}) = 1 + a + b + c - d; \\
\psi(\{\omega_4\}) = \psi(\{\omega_1, \omega_2, \omega_3\}) = 1 + a - b + c.
\]

Let \( p \in A \) be defined by:
\[
p(\{\omega_1\}) = 0; \quad p(\{\omega_2\}) = \frac{a}{a + b + c}; \quad p(\{\omega_3\}) = \frac{b}{a + b + c}; \quad p(\{\omega_4\}) = \frac{c}{a + b + c}.
\]

So \( \nu \in \Theta(A, \Psi) \) can be defined as:
\[
\nu(A) = \begin{cases} 
(1 - \Psi(A)) p(A) & \text{se } A \neq \Omega \\
1 & \text{se } A = \Omega
\end{cases}
\]

Let \( f : \Omega \to \mathbb{R} \) be a real-valued function defined by:
\( f(\omega_i) = i \), for \( i \in \{1, 2, 3\} \).

It is easy to check that the "usual" Choquet integral is \( 1 + a + 2b + 3c + d \).
Using (ii):

\[
\int f dv = \psi_1 f(\omega_1) + (1-\psi_1) \int fdp + (\psi_1-\psi_3) p(\omega_1+\omega_3)(f(\omega_1)-f(\omega_3)) + \\
+ (\psi_1-\psi_3) p(\omega_3)(f(\omega_3)-f(\omega_3)) + \\
= (1-a-b-c-d).1 + (1-(1-a-b-c-d)) \left( \frac{2a+3b+4c}{a+b+c} \right) + \\
+((1-a-b-c-d)-(1-a-b-c)) \left( \frac{c+d}{a+b+c} \right) (32)+ \\
+((1-a-b-c-d)-(1-a-b-c)) \left( \frac{d}{a+b+c} \right) (43)= \\
=1+a+2b+3c+d
\]

Q. E. D.

4. NASH EQUILIBRIUM UNDER UNCERTAINTY

I can describe a two-person finite normal form game as:

\[\Gamma = (S_1, S_2; U_1(S_1, S_2), U_2(S_1, S_2))\]

where for each player \(i \in \{1,2\}\) the \(S_i\) is a finite set of pure strategies and the \(u(s_1, s_2)\) are payoffs (utilities) functions depending on the pure strategy combination \((s_1, s_2)\) ∈ \((S_1, S_2)\) played.

To ensure the existence of Nash equilibrium it is necessary to allow the possibility of playing mixed strategies. A mixed strategy can be understood as a(n) (additive) probability measure over the pure strategies. To complete the description, payoffs (utilities) are constituted by von Neumann-Morgenstern expected utility. The modify game can be described as:

\[\Gamma' = (\Delta(S_1), \Delta(S_2); U_1(p_1, p_2), U_2(p_1, p_2))\]

where, for each player \(i \in \{1,2\}\), the \(\Delta(S_i)\) is a set of mixed strategies over the finite pure strategies\(^{15}\) and the \(U_i(p_1, p_2)\) are von Neumann Morgenstern expected utilities functions depending on the mixed strategy combination \((p_1, p_2) \in (\Delta(S_1), \Delta(S_2))\) played.

Nash [1950] proved the existence of equilibrium in n-person finite normal form games on which mixed strategies are possible. A mixed strategy Nash equilibrium is defined as follows:

**Definition 4.1 Mixed Strategy Nash Equilibrium**

Let \((p_1, p_2)\) ∈ \((\Delta(S_1), \Delta(S_2))\) be a mixed strategy combination in a game \(\Gamma\).

The mixed strategy combination \((p_1', p_2')\) represents a Nash equilibrium in mixed strategies if, for each player \(i \in \{1,2\}\):

\[U_i(p_1', p_2) \geq U_i(p_1, p_2), \text{ for all } p, \in \Delta(S_i).\]

For each player \(i \in \{1,2\}\) let \(\text{supp} p\), denote the support of \(p\).

\(^{15}\)So, if player \(i\) has \(k_i\) pure strategies then the mixed strategies sets, \(p_i\), will be a list:

\[p_i = (p_{i1}, p_{i2}, ..., p_{ik})\]
In a Nash equilibrium, each $s_i \in \text{supp} p_i$ is a best response to $p_i$, i.e., for each player $i \in \{1, 2\}$, $s_i$ maximizes the von Neumann Morgenstern expected utility of player $i$ given that player -$i$ is playing the mixed strategy $p_i$.

I will adopt the same subjective interpretation as Dow-Werlang [1994]:

"A subjective interpretation (to mixed strategies Nash equilibrium) can be given (...) the mixed strategy of player $i$, $p_i$, may be viewed as the belief that player -$i$ has about the pure strategy play of player $i$."[17]

4.1. The Main Theorem

If uncertainty are considered then, for each player $i \in \{1, 2\}$, beliefs of player $i$ about player -$i$ behavior are represented by convex capacities. To generalize Dow-Werlang [1994]'s Nash equilibrium under uncertainty I will consider convex capacities that are squeeze of (additive) probability measures, i.e. for each player $i \in \{1, 2\}$: $\nu_i \in \Theta(\Delta, \Psi_i)$.

Definition 4.2 Nash Equilibrium under Uncertainty

For each player $i \in \{1, 2\}$: let $\psi \in \Psi_i^1 (S, 2^S)$, where $\psi: 2^S \rightarrow [0, 1]$ is the uncertainty aversion function of $\nu_i \in \Theta(\Delta, \Psi_i)$.[18]

A pair $(\nu_1, \nu_2)$ of convex capacities that are squeeze of (additive) probability measures $(\nu_i \in \Theta(\Delta, \Psi_i))$, $\nu_i$ over $S_i$ (for each player $i \in \{1, 2\}$) is a Nash equilibrium under uncertainty if there exists a support of $\nu_i$ and a support of $\nu_2$ such that, for each player $i \in \{1, 2\}$:

For all $s_i$ in the support of $\nu_i$, $s_i$ maximizes the Choquet integral of player $i$, given that $\nu_i$ represents player $i$'s beliefs about the strategies of player -$i$.[19]

Now I will present the main result that generalize Dow-Werlang [1994]'s result, with the use of a uncertainty aversion function of each player $i \in \{1, 2\}$ as a parameter in the game.

Theorem 4.1 Existence of Nash Equilibrium Uncertainty

Let $\Gamma = (S_1, S_2; u_1(s_1, s_2), u_2(s_1, s_2))$ be a two-person finite normal form game.

For each $i \in \{1, 2\}$: $\psi \in \Psi_i^1 (S, 2^S)$, where $\psi: 2^S \rightarrow [0, 1]$ is the uncertainty aversion function of $\nu_i \in \Theta(\Delta, \Psi_i)$.

For all $(\psi_1, \psi_2)$ there exists a Nash equilibrium under uncertainty.

Proof: My proof is the same, in spirit, to Dow-Werlang [1994]'s existence proof.

We now that if $\nu_i \in \Theta(\Delta, \Psi_i)$ then for some $p \in \Delta$ it's true that:

$$\nu(A) = \begin{cases} (1 - \psi_i (A)) p(A) & \text{if } A \neq \Omega \\ 1 & \text{if } A = \Omega \end{cases}$$

So, Choquet integral has the form:

[16] Notation 4.1 If player $i = 1$ then player -$i = 2$ and, conversely, if player $i = 2$ then -$i = 1$.


[18] Note carefully that $\nu_i$ represents player $i$'s beliefs about what player -$i$ will do, so that the uncertainty aversion function of $\nu_i$ is a characteristic of player $i$.

[19] Note that if there is no uncertainty then the pair $(\nu_1, \nu_2)$ reduces to a pair of (additive) probability measures and the Choquet integral reduces to a von Neumann Morgenstern expected utility. This implies that there exists at least one Nash equilibrium under uncertainty (if there is no uncertainty!), as presented in the following:

Lemma 4.1 Every mixed strategy Nash equilibrium is a Nash equilibrium under uncertainty

Proof: Omitted
Suppose, without loss of generality, that player \( i \) has \( n \) pure strategies. So we can order the payoffs of player \( i \in \{1,2\} \) to each pure strategy, \( s \in S_i \), as:

\[
u^i_1(s) \leq u^i_2(s) \leq \ldots \leq u^i_n(s)
\]

where \( u^i_j(s_j) = u_i(s_j, s_j) \) is the \( j \)-position \((j=1, \ldots, n)\).

We modify the original game \( \Gamma \) to \( \Gamma_{(\psi_1, \psi_2)} = (S_1, S_2; w_1(s_1, s_2), w_2(s_1, s_2)) \), where:

\[
w_j(s_j, s_j) = \psi^j_1 u^j_1(s_j) + (1-\psi^j_1) u^j_2(s_j) \quad \text{if} \quad j = 1, 2.
\]

And, if \( j = 3, \ldots, n \),

\[
w_j(s_j, s_j) = \psi^j_1 u^j_1(s_j) + (1-\psi^j_1) u^j_2(s_j) + \sum_{k=3}^n (\psi^j_k - \psi^j_{k-1})(u^k_i(s_j) - u^{k-1}_i(s_j))
\]

Note that \( \psi^j_i \) is the uncertainty aversion function associate to the strategy of player \( i \) who gives the worst payoff to player \( i \in \{1,2\} \) when his choice is the pure strategy \( s_i \).

Let \((p_1, p_2)\) be a standard mixed strategy Nash equilibrium of the modified game. We will show that the pair \((v_1, v_2)\), where

\[
v_1(A) = \begin{cases} (1-\psi^1_1(A)) p_1(A) & \text{se } A \neq S_1 \\ 1 & \text{se } A = S_1 \end{cases}\quad \text{and} \quad v_2(A) = \begin{cases} (1-\psi^2_1(A)) p_2(A) & \text{se } A \neq S_2 \\ 1 & \text{se } A = S_2 \end{cases}
\]

is a Nash equilibrium under uncertainty for the original game, with the specified uncertainty aversion functions associated.

To check that this is a Nash equilibrium under uncertainty, note that (except in case where, for each \( i \in \{1,2\} \) and for all \( A \in \mathbb{R}^\infty, A \in \mathbb{R}^\infty, \psi(A) = 1 \) the support of \( v_i \) is unique (by definition!), and coincides with the support of \( p_i \) for each player \( i \in \{1,2\} \). Since \((p_1, p_2)\) is a standard mixed strategy Nash equilibrium for the modified game, it follows that any \( s \in \text{supp} p_i \) is a best response to \( p_i \) (for the modified utility \( w_i \)). In other words, \( s \) maximizes the following expression over \( s \in S_i \):

\[
\int w_i(s_i) dp_{-i} = p^1_i (\psi^i_1 u^i_1(s_i) + (1-\psi^i_1) u^i_2(s_i)) + \ldots + \sum_{j=3}^n (\psi^i_j - \psi^i_{j-1})(u^j_i(s_i) - u^{j-1}_i(s_i)) + \ldots + \sum_{j=3}^n (\psi^i_j - \psi^i_{j-1})(u^j_i(s_i) - u^{j-1}_i(s_i)) = \psi^i_1 u^i_1(s_i) + (1-\psi^i_1) \int u_i(s_i, \cdot) dp_{-i} + \sum_{j=3}^n (\psi^i_j - \psi^i_{j-1})(\sum_{j=3}^n p^j_i (u^j_i(s_i) - u^{j-1}_i(s_i)) = \int u_i(s_i, \cdot) dv_{-i}
\]
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Thus, $s_i$ is also a best response in the original game. In case where, for each player $i \in \{1,2\}$ and for all $A \subseteq 2^S_i$, $\mathbb{E} \sigma(A) = 1$ any singleton $\{s_i\}$ is a support of $v_i$. Therefore any best response for player $i$ is in a support.

Thus $(v_1, v_2)$ is a Nash equilibrium under uncertainty for the original game.

Q. E. D.

The following proposition show that this theorem generalize Dow-Werlang [1994]'s existence theorem:

**Proposition 4.1** Generalize Dow-Werlang [1994]'s Existence Theorem.

Every Dow-Werlang [1994]'s Nash equilibrium under uncertainty is a Nash equilibrium under uncertainty as in our Definition 4.2.

Proof: Consider, for each player $i \in \{1,2\}$: $\psi \in \psi(S, 2^{S_i})$, where $\psi: 2^{S_i} \rightarrow [0,1]$ is the uncertainty aversion function defined such that, for all $A \subseteq 2^{S_i}$, $\mathbb{E} \sigma(A) = c$, $c \in [0,1]$.

Thus defined, $\psi$ exhibits constant uncertainty aversion. So, for each player $i \in \{1,2\}$, $v_i$ is an uniform squeeze of $\psi$. These beliefs $(v_1, v_2)$ form a Nash equilibrium under uncertainty as in our Definition 4.2.

Q. E. D.

5. RELATED RESULTS AND CONCLUSION

The definition of Nash equilibrium under uncertainty provided here, that extends the one presented in Dow-Werlang [1994]'s paper is related with the extension presented in Marinacci [2000]'s paper which proved the existence of Nash equilibrium under uncertainty for any given uncertainty aversion function.

The most important result of this paper is to present the uncertainty aversion function as an explicit parameter in the description of the game, which enables us to do static comparative static exercises in an easy way.

It remains an extension to $n$ players.

REFERENCES


