

Endogenous Borrowing Constraints and Default when Markets are Incomplete

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September 17, 2001

Abstract

Incomplete markets and non-default borrowing constraints increase the volatility of pricing kernels and are helpful when addressing asset-pricing puzzles. However, ruling out default when markets are incomplete is suboptimal. This paper endogenizes borrowing constraints as an intertemporal incentive structure to default. It models an infinite-horizon economy, where agents are allowed not to pay their liabilities and face borrowing constraints that depend on the individual history of default. Those constraints trade off the economy's risk-sharing possibilities and incentives to prevent default. The equilibrium presents stationary properties, such as an invariant distribution for the assets' solvency rate.

Keywords: borrowing constraint, general equilibrium, incomplete markets, risk, default, Markov, stationary. JEL Classification: D52, D90, G10.

*I thank Cristian Huse and seminar participants at the University of Chicago for helpful comments. I am also indebted to Aloísio Araújo for introducing me to the literature on default with incomplete markets. Financial support from CNPq-Brazil is gratefully acknowledged. URL: <http://home.uchicago.edu/~lhbraido>; e-mail: l-braido@uchicago.edu.

1 Introduction

A general equilibrium economy is a world representation intending to explain how markets coordinate egoistic individuals acting independently. This school of economic thought was founded by Leon Walras in 1874, and its most elegant synthesis is due to Kenneth J. Arrow, Gerard Debreu, and Lionel W. Mackenzie, who independently developed their work in the fifties. Among many other areas, the theory of finance has largely benefited from the general equilibrium methodology. Dynamic models with incomplete markets and exogenous borrowing constraints have proven to be useful when addressing many asset-pricing puzzles (Scheinkman and Weiss (1986), Telmer (1993), and Heaton and Lucas (1996)).

One crucial assumption behind that methodology is the existence of a strong enforcement power that makes the individuals always honor their contracts; i.e., each agent faces a budget constraint in which all of the liabilities must be paid. However, the widespread occurrence of default makes that assumption rather counterfactual, and some recent research has incorporated the possibility of default into the economic general equilibrium models.

A group of articles, pioneered by Kehoe and Levine (1993), address the fact that agents could not pay their liabilities and run away from the economy. In that spirit, Zhang (1996) endogenizes borrowing constraints by a system of non-default conditions and shows that, in association with incomplete financial markets, those constraints increase the volatility of pricing kernels. However, as noted by Dubey, Geanakoplos and Shubik (1988), and Zame (1993), ruling out default in economies with incomplete markets is suboptimal (default may expand the risk-sharing possibilities).

This paper endogenizes borrowing constraints as an intertemporal penalty to default. The model contributes to the literature on default, and points to a new set of possibilities to be explored by the finance agenda. It consists of a general equilibrium economy with incomplete markets and infinite periods. Agents are allowed to default on financial assets and face a borrowing-constraint structure that depends on the individual history of default. Instead of being a constant bound for short sales, borrowing constraints are presented as a history-dependent incentive mechanism to prevent default. Those constraints replace the ad hoc utility penalty, commonly found in the default literature. They explore intertemporal tie-ins to penalize default and impose recursive features on the default decision.

In addition to proposing an alternative penalty to default, the paper presents borrowing constraints as an optimal social structure that trades off the economy's risk-sharing possibilities and incentives to prevent default.

The remainder of this paper is organized as follows. Section 2 briefly reviews the literature. Section 3 characterizes the basic model, and shows how borrowing constraints would emerge endogenously as an optimal structure in economies with incomplete markets and default. Section 4 presents a stationary Markov version of the model and proves the existence of an ergodic Markov equilibrium, in the sense of Duffie, Geanakoplos, Mas-Colell, and McLennan (1994). Section 5 concludes.

2 Review of the Literature

The economic models treat default in two distinct ways. Kehoe and Levine (1993), in a first perspective, present an infinite-horizon economy with complete markets where agents are excluded forever from the financial market when defaulting. The rational individuals anticipate potential defaulters so that the default is ruled out in equilibrium.

Alvarez and Jermann (2000) explore the *exclusion penalty* to endogenize borrowing constraints in an economy with complete markets. In that model, since agents transact contingent claims (complete markets), rationality implies that lenders lend up to the point at which borrowers have no incentive to default. Therefore, borrowing constraints are endogenously obtained by a system of non-default equations. A similar idea was used by Zhang (1997) in an incomplete markets context.

Non-default borrowing constraints increase the volatility of pricing kernels, which makes them helpful to explain some of the asset-pricing puzzles. That approach, however, presents some theoretical inconveniences. First, excluding one agent forever from the financial market may be non-credible in small economies, i.e., such a penalty could increase the aggregate risk so that, *ex post*, the society would have an interest in forbidding defaulters. Contra intuitively, the non-default constraints are tighter for richer agents (those with higher incentives to run away). Also, default is ruled out by rationality, which makes those models unable to explain the economic data. Finally, the *exclusion penalty* is not optimal regardless of whether the asset structure is complete or incomplete.¹

¹The complete markets equilibrium without default is Pareto optimal, so that the society would be interested in applying stronger penalties to prevent default. In the incomplete markets setup, default expands the risk-sharing possibilities, and agents would be interested in transacting assets even if the buyer defaults in some states of the world. As an example, nobody would be interested in buying the *contingent claim* $(1, 0, \dots, 0)$ if the seller has incentives to default in the first state of the world. The same is not true about the *asset* $(1, 1, \dots, 1)$ in an incomplete markets scenario.

In a second perspective, Dubey et al. (1988) present default as an institution that improves risk sharing in economies with incomplete markets. In their work, agents are allowed not to pay the assets sold and face a utility penalty proportional to the amount of default. The greatest result obtained is that default effectively happens in equilibrium. The biggest inconvenience is that the unique incentive to prevent default is an ad hoc utility penalty.

Araújo, Monteiro, and Páscoa (1996) extend that model to an infinite-horizon economy. According to them, each agent receives a utility penalty proportional to the amount of liabilities not paid in the period. The within-period penalty does not explore intertemporal tie-ins that may appear in dynamic economies (default is penalized in the period in which it happens, and the history of default is not considered).

This paper connects those two perspectives by introducing borrowing constraints as a mechanism that solves the trade off between risk sharing and incentives to prevent default.

3 An Economy with Default

Consider an infinite-horizon economy with uncertainty. Define \mathbb{I} as the set of agents, and let \mathbb{S} be the set of period shocks. As usual, both sets are nonempty and finite, with cardinality I and S , respectively. The realization of the period-shock (s_t) affects the amount of the L consumption goods received as endowment by each agent. In each period $t \in \mathbb{T} = \{0\} \cup \mathbb{N}$, agents receive endowments according to a function $w^i : \mathbb{S} \rightarrow \mathbb{R}_{++}^L, \forall i \in \mathbb{I}$. It is assumed that $s \in \mathbb{S}$ follows a first-order Markov chain.

Assumption 1. *The random variable $s \in \mathbb{S}$ follows a first-order Markov chain (\mathcal{P}).*

In each period, there are spot markets for the L existing goods and $J < S$ financial assets. The assets are one-period securities that pay in units of good 1, according to the function $a : \mathbb{S} \rightarrow \mathbb{R}_+^J$.

Although only the period realization of $s_t \in \mathbb{S}$ affects the endowments and assets' payoff, the agents make their choices conditional on the entire state history, $s^t = (s_0, \dots, s_t) \in \mathbb{S}^t$. In each state of the world (s^t), they choose their consumption bundle ($x_{s^t}^i \in \mathbb{R}^L$) and the number of shares held in each of the J assets ($q_{s^t}^i \in \mathbb{R}^J$). Moreover, in the spirit of Dubey et al. (1988), agents do not necessarily honor all of the assets sold in the previous period, so that they must also choose the amount of payment done on each asset ($\eta_{s^t}^i \in \mathbb{R}_+^J$).

As usual, define $[y]^+ = \max(0, y)$ and $[y]^- = |\min(0, y)|$ for any $y \in \mathbb{R}$. It is said that agent i defaults whenever $\eta_{s^t, j}^i < p_{s^t, 1} a_{s^t, j} [q_{s^t-1, j}^i]^-$ for some $j \in \mathbb{J} = \{1, \dots, J\}$.

Definition 1. Agent i 's default rule is $\mathbf{d}^i = \{d_{s^t}^i \in \mathbb{R}^J, \forall (t, s^t) \in \mathbb{T} \times \mathbb{S}^t\}$, where $d_{s^t, j}^i = \left[p_{s^t, 1} a_{s^t, j} [q_{s^t-1, j}^i]^- - \eta_{s^t, j}^i \right]^+$, $\forall j \in \mathbb{J}$.

Remark 1. Throughout the paper, bold letters represent the entire sequence of the respective variable, e.g., $(\mathbf{x}^i, \mathbf{q}^i, \eta^i) = \{(x_{s^t}^i, q_{s^t}^i, \eta_{s^t}^i) \in \mathbb{R}_+^L \times \mathbb{R}^J \times \mathbb{R}_+^J, \forall (t, s^t) \in \mathbb{T} \times \mathbb{S}^t\}$.

Agents face a borrowing constraint (b_j) in each of the J assets. They are punished for defaulting by facing narrow borrowing possibilities in the following periods. Since this paper intends to develop a recursive approach, all of the rules (including the borrowing-constraint structure) must be time independent. For that reason, it is assumed that the information about agents' history of default is recorded for a finite number of periods. Such an assumption must be interpreted as a reduced form for the existence of recording costs.

Assumption 2. The information about agents' default is recorded for a finite number of periods ($\tau > 0$), i.e., $h_t^i = (d_{s^t-\tau}^i, \dots, d_{s^t-1}^i) \in \mathbb{H} = \mathbb{R}_+^{J\tau}$.

The variable $h_t^i \in \mathbb{H}$ summarizes the agent i 's history of default. The borrowing-constraint structure is then represented by a bounded function $b : \mathbb{S} \times \mathbb{H} \rightarrow \mathbb{R}^J$. Agents' borrowing possibilities depend on the period realization of s , and on the individual history of default. Such a structure works as an incentive mechanism to prevent default. In section 4, it is shown how $b(s_t, h_t^i)$ would be optimally designed.

Each agent has a preference relation \succsim_i , which is numerically represented by a time-separable expected utility. This assumption simplifies the proofs in this paper and could be relaxed in many ways (see Ma, 1993 for a recursive-utility approach).

Assumption 3. For any $i \in \mathbb{I}$, \succsim_i is numerically represented by $V^i(\mathbf{x}^i) = E \left\{ \sum_{t=0}^{\infty} \beta^t u^i(x_{s^t}) \right\}$, where $\beta \in (0, 1)$, and $u^i : \mathbb{R}_+^L \rightarrow \mathbb{R}$ is continuous, quasiconcave, and strictly monotone.

Briefly, this economy is described by $\xi = \{\mathbb{I}, \mathbb{J}, \mathbb{T}, \mathbb{S}, \mathcal{P}, (\succsim_i, w^i)_{i \in \mathbb{I}}, a, b\}$. Its price system is composed of a sequence of prices for goods and assets

$(\mathbf{p}, \boldsymbol{\pi})$ and a sequence of solvency rates $(\boldsymbol{\gamma})$; where $(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma}) = \{(p_{st}, \pi_{st}, \gamma_{st}) \in \mathbb{R}^{L+J} \times [0, 1]^J, \forall (t, s^t) \in \mathbb{T} \times \mathbb{S}^t\}$.

As usual, agents take the price system $(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma})$ as given, and choose a \succeq_i -maximal bundle in the set of affordable bundles, $BC^i(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma})$.

Definition 2. Agent i 's set of affordable bundles is $BC^i(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma}) = \{\mathbf{x}^i \in \mathbb{R}_+^\infty; \exists(\mathbf{q}^i, \boldsymbol{\eta}^i) \text{ s.t. (1)-(2) hold } \forall (t, s^t) \in \mathbb{T} \times \mathbb{S}^t\}$, where:

$$p_{st} \cdot (x_{st}^i - w_{st}^i) + \pi_{st} \cdot q_{st}^i \leq \sum_{j=1}^J \left(\gamma_{st,j} p_{st,1} a_{st,j} [q_{st-1,j}^i]^+ - \eta_{st,j}^i \right); \quad (1)$$

$$q_{st}^i \geq b(s_t, h_t^i). \quad (2)$$

Equation (1) states that agents can spend the value of their endowments plus the amount received among the assets owned, in consumption goods, a new portfolio, and financial payments. Since trades are anonymous, agents receive their payments according to the market solvency rate $(\gamma_{st,j})$. Inequality (2) represents the borrowing constraints.

Definition 3. A competitive equilibrium for the economy ξ is a price system $(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma})$ and an allocation $(\mathbf{x}^i, \mathbf{q}^i, \boldsymbol{\eta}^i)_{i \in \mathbb{I}}$ such that:

- (i) $(\mathbf{x}^i, \mathbf{q}^i, \boldsymbol{\eta}^i)$ is \succeq_i -maximal in $BC^i(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma})$, $\forall i \in \mathbb{I}$;
- (ii) all markets clear, i.e., for any $(t, s^t) \in \mathbb{T} \times \mathbb{S}^t$:

$$\sum_{i \in \mathbb{I}} (x_{st}^i - w_{st}^i, q_{st}^i) = \mathbf{0} \in \mathbb{R}^{L+J}; \quad (3)$$

$$\sum_{i \in \mathbb{I}} \left(\gamma_{st,j} p_{st,1} a_{st,j} [q_{st-1,j}^i]^+ - \eta_{st,j}^i \right) = 0, \quad \forall j \in \mathbb{J}. \quad (4)$$

Except for the last market-clearing condition, the equilibrium concept is very standard. Equation (4) states that the amount of credits received by the agents must equal the amount of payments done by them.

In order to prove the existence of an equilibrium one should first prove the existence of equilibrium for the T -horizon economy (see Lemma 1 in the appendix), and then explore the limit properties. Instead of following that procedure, this paper will explore a stronger equilibrium concept. Section 4 proves the existence of an ergodic Markov equilibrium, which implies the existence of equilibrium in the sense of Definition 3.

Notice that, except for being bounded, no particular assumption was imposed on the borrowing constraints. However, if the borrowing-constraint rule is either too weak or too tight, the equilibrium will present no financial trade (no transaction would be repaid in the former case, and they cannot happen in the latter). Clearly, those cases tend to be suboptimal since the agents end up with no risk sharing. The next subsection shows how an optimal borrowing-constraint scheme would be optimally chosen.

3.1 Optimal Borrowing Constraints

The assumption that agents face exogenous borrowing constraints has been largely used in the finance literature, and such an ad hoc institution is frequently associated with the possibility of default. This subsection shows how an optimal borrowing-constraint structure would be designed. The procedure follows the idea behind the classical Ramsey-taxation problem. For each fixed borrowing-constraint function (b) , agents play a competitive equilibrium. Anticipating that, a market designer chooses a borrowing-constraint rule that maximizes social welfare.

Initially, define \mathbb{B} as the set of bounded functions mapping $\mathbb{S} \times \mathbb{H}$ into \mathbb{R}^J and $\mathbf{x}(b) = \{x_{s^t}^i \in \mathbb{R}_+^L, \forall (i, t, s^t) \in \mathbb{I} \times \mathbb{T} \times \mathbb{S}^t\}$ as an equilibrium allocation associated with each borrowing-constraint function $b \in \mathbb{B}$.² Thus, for fixed weights $\lambda^i > 0 \forall i \in \mathbb{I}$, an optimal borrowing-constraint structure is given by:

$$b^* \in \arg \sup_{b \in \mathbb{B}} \sum_{i \in \mathbb{I}} \lambda^i V^i(\mathbf{x}^i(b)).$$

Since \succeq_i is time consistent, the optimal borrowing-constraint rule (b^*) induces a subgame perfect equilibrium. It means that the planner has no interest in reviewing such a rule after any possible history.

Notice that the optimal borrowing-constraint penalty does not necessarily exclude the defaulters from the financial market. In fact, it trades off the economy's risk-sharing possibilities and incentives to prevent default. This result has a strong empirical counterfactual: in real economies, the history of default affects the liquidity faced by the agent, but the defaulters are usually not excluded from the financial market.

²From Theorem 1, such an equilibrium allocation exists, for any $b \in \mathbb{B}$. However, it may not be unique. In that case, the choice of which equilibrium to consider must reflect the planner's expectation. Simon and Zame (1990) and Braidó (2001) discuss this same problem in different contexts.

4 Ergodic Markov Equilibrium

Although elegant, the setup explored in section 3 presents a very unpleasant characteristic: the history dependence of the optimal rules. The infinite-horizon literature usually skips this feature by developing an alternative world representation that eliminates all the unnecessary dependence on the past. This section explores such a time-independent representation.

THE STATE SPACE

In this setup, the state of the world is summarized by a state variable (z) that completely describes the current endogenous and exogenous variables and is a sufficient statistic for the future evolution of the model. The exogenous state variable is $s \in \mathbb{S}$. The endogenous state variable is composed of each agent's previous-period portfolio, history of default, and current choices, i.e., $(q_{-1}, h, x, q, \eta) \in \mathbb{R}^{JI} \times \mathbb{H}^I \times \mathbb{R}_+^{LI} \times \mathbb{R}^{JI} \times \mathbb{R}_+^{JI}$, as well as the current prices, $(p, \pi, \gamma) \in \mathbb{R}^{L+J} \times [0, 1]^J$.

The state variable must then be: $z = (s, q_{-1}, h, x, q, \eta, p, \pi, \gamma) \in Z \subset \mathbb{S} \times \mathbb{Y}$, where $\mathbb{Y} = \mathbb{R}^{JI} \times \mathbb{H}^I \times \mathbb{R}_+^{LI} \times \mathbb{R}^{JI} \times \mathbb{R}_+^{JI} \times \mathbb{R}^{L+J} \times [0, 1]^J$. As usual, the feasibility conditions are imposed on the definition of Z . The state space is then defined by $Z = \{z \in \mathbb{S} \times \mathbb{Y} \text{ s.t. (5)-(6) hold}\}$, where:

$$\sum_{i \in \mathbb{I}} (x^i - w_s^i, q^i) = \mathbf{0} \in \mathbb{R}^{L+J}; \quad (5)$$

$$\sum_{i \in \mathbb{I}} (\gamma_j p_1 a_{s,j} [q_{-1,j}^i]^+ - \eta_j^i) = 0, \quad \forall j \in \mathbb{J}. \quad (6)$$

THE EXPECTATION CORRESPONDENCE

For any generic set Ω , define \mathcal{J} as the Borel σ -algebra for Ω and Λ_Ω as the space of probability measures on (Ω, \mathcal{J}) . The economy ξ is represented by a state space, Z , and an expectation correspondence, $G : Z \rightarrow \Lambda_Z$. Following Duffie et al. (1994), define a set-valued function $g : Z \rightarrow \Lambda_Z$ by letting $\mu \in g(\hat{z})$, where $\hat{z} = (\hat{s}, \hat{q}_{-1}, \hat{h}, \hat{x}, \hat{q}, \hat{\eta}, \hat{p}, \hat{\pi}, \hat{\gamma})$, if:

- the support of μ is the graph of some function $f : \mathbb{S} \rightarrow \mathbb{Y}$;
- the marginal of μ on \mathbb{S} is $\mathcal{P}_{\hat{s}}$ and the marginal of μ on $\mathbb{R}_+^{JI} \times \mathbb{H}^I$ is degenerated on (\hat{q}, \hat{h}') almost surely (where \hat{h}' is obtained from \hat{z});

- for any random variable $z = (s, q_{-1}, h, x, q, \eta, p, \pi, \gamma)$ with distribution μ , the plan (x^i, q^i, η^i) is optimal given (s, q_{-1}^i, h^i, μ) , $\forall i \in \mathbb{I}$.

The expectation correspondence, G , is the closure of the convex hull of g . It represents the agents' beliefs about the states in the next period given the current period state (z) . Such an expectation is consistent with the exogenous-shock probability and agents' optimal decisions.

Definition 4. An equilibrium transition for G is a pair (E, Π) , where E is a measurable subset of Z and $\Pi : E \rightarrow \Lambda_E$ is a law of motion, such that:

- (i) each time-homogeneous Markov process $\{z_t\}_{t \in \mathbb{T}}$ s.t. $z_t \in E \forall t$ with transition Π is such that $\{x_t^i, q_t^i, \eta_t^i\}_{t \in \mathbb{T}}$ is optimal for agent i given $\{z_t\}_{t \in \mathbb{T}}$;
- (ii) there exists $(\bar{s}, \bar{q}_{-1}, \bar{h}, x, q, \eta, p, \pi, \gamma) \in E$, $\forall (\bar{s}, \bar{q}_{-1}, \bar{h}) \in \mathbb{S} \times \mathbb{R}_+^{J_I} \times \mathbb{H}^I$.

Condition (i) says the allocation associated with each state $z \in E$ must be optimal for the agents. Condition (ii) states that any starting parameters $(\bar{s}, \bar{q}_{-1}, \bar{h})$ are admissible. (Recall that the market-clearing conditions were already imposed in the definition of the set Z .)

Definition 5. An ergodic Markov equilibrium for G is an equilibrium transition (E, Π) and an invariant ergodic measure $\mu \in \Lambda_E$ such that $\mu = \Pi \cdot \mu$.

An equilibrium transition (Definition 4) is a set of equilibrium states and a law of motion, such that the current realization of z determines the future stochastic equilibrium path. An ergodic Markov equilibrium (Definition 5) adds to the previous concept the fact that the distribution of future realization of the system is invariant and ergodic. This is the analogue of the deterministic notion of steady state.

Theorem 1. There exists an ergodic Markov equilibrium (E, Π, μ) for the economy ξ .

Proof. See Appendix. ■

Theorem 1 guarantees the existence of equilibrium for a class of economies with incomplete markets and default, $\xi = \{\mathbb{I}, \mathbb{J}, \mathbb{T}, \mathbb{S}, \mathcal{P}, (\bar{\omega}_i, w^i)_{i \in \mathbb{I}}, a, b\}$. The distribution for the assets' solvency rate is stationary and ergodic, which implies that the expected solvency rate of each asset is constant over time. Thus, one can define a constant coefficient for the risk of default,

$$risk_j = \int_E (1 - \gamma_j) d\mu, \quad \forall j \in \mathbb{J}. \quad (7)$$

That property would justify some empirical approaches used to estimate the one-period risk of default.

5 Conclusion

In order to understand the role of borrowing constraints and default in modern societies, this paper presents an alternative model to those usually found in the literature. The model consists of an infinite-horizon economy with heterogeneous agents choosing among goods, assets, and default level. Agents face a borrowing-constraint structure, which depends on the individual history of default.

In this economy, default actually happens in equilibrium, and the results do not depend on any exogenous utility penalty (as usual in the default literature). The model avoids the utility penalty by using borrowing constraints as an incentive structure to prevent default. It is shown how the borrowing-constraint penalty would be optimally designed. Such a design explores intertemporal tie-ins to solve the trade off between risk sharing and incentives to prevent default. It is a much more flexible structure than the *exclusion penalty* found in Zhang (1997), since it restricts the participation of defaulters in the financial market, but does not necessarily exclude them.

The economy presents recursive properties as the existence of an ergodic Markov equilibrium, an invariant distribution for the assets' solvency rate, and a constant measure for the risk of default.

A Appendix: Proof of Theorem 1

Initially, define the bounds, $\bar{x} = 2 \sum_{i \in \mathbb{I}} \max_s (w_s^i)$, $\bar{b} = 2 \sup_{i,j,s,h} |b_j^i(s, h)|$, and $\bar{\eta} = 2\bar{b} \max_{s,j} |a_{s,j}|$, which will be used throughout the proof.

Next, define a T -horizon economy as $\xi_T = \{\mathbb{I}, \mathbb{J}, \mathbb{T}_T, \mathbb{S}, \mathcal{P}, (V_T^i, w^i)_{i \in \mathbb{I}}, a, b\}$, where $\mathbb{T}_T = \{0, \dots, T\}$ and $V_T^i(\cdot) = E \left\{ \sum_{t=0}^T \beta^t u^i(x_{s^t}^i) \right\}$. As usual, impose the initial condition $(q_{-1}, h_0) = \mathbf{0} \in \mathbb{R}^{JI(1+\tau)}$, and the terminal condition $q_{s^T} = \mathbf{0} \in \mathbb{R}^J$, $\forall s^T \in \mathbb{S}^T$. Moreover, since the default occurred in periods $T-1$ and T would not be penalized, impose $d_{s^{T-1}} = d_{s^T} = \mathbf{0} \in \mathbb{R}^{JI}$, for any s^{T-1} and s^T .

Notice that the number of nodes in ξ_T is $S^* = (S - S^{T+1})/(1 - S)$ and, thanks to Assumption 3, V_T^i is continuous, quasiconcave, and strictly monotone, $\forall i \in \mathbb{I}$.

Lemma 1. *There exists a competitive equilibrium for ξ_T .*

Proof. A few definitions will be useful in the construction of an equilibrium. Initially, consider the compact $K = \{(\mathbf{x}^i, \mathbf{q}^i, \boldsymbol{\eta}^i) \in \mathbb{R}_+^{LS^*} \times \mathbb{R}^{JS^*} \times \mathbb{R}_+^{JS^*} \text{ s.t. } x_{st}^i \leq \bar{x}, -\bar{b} \leq q_{st,j}^i \leq I\bar{b} \text{ and } \eta_{st,j}^i \leq \bar{\eta}, \forall j \in \mathbb{J} \text{ and } \forall (t, s^t) \in \mathbb{T}_T \times \mathbb{S}^t\}$. Also, for any $\delta > 0$, let $\Delta_\delta = \{(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma}) \in \mathbb{R}_+^{(L+J)S^*} \times [0, 1]^{JS^*} \text{ s.t. } \min(\mathbf{p}) \geq \delta \text{ and } \sum_{l=1}^L p_{st,l} + \sum_{j=1}^J \pi_{st,j} = 1, \forall (t, s^t) \in \mathbb{T}_T \times \mathbb{S}^t\}$.

Next, let the truncated demand correspondence be given by $m^i(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma}, K) = \{(\mathbf{x}^i, \mathbf{q}^i, \boldsymbol{\eta}^i) \in \arg \max_{(\mathbf{x}^i, \mathbf{q}^i, \boldsymbol{\eta}^i) \in K} V_T^i(\mathbf{x}^i) \text{ s.t. (1)-(2) hold and } q_{sT,j}^i = d_{sT,j}^i = d_{sT-1,j}^i = 0, \forall j \in \mathbb{J} \text{ and } \forall (t, s^t) \in \mathbb{T}_T \times \mathbb{S}^t\}$. Define then $\phi_\delta : \Delta_\delta \times K^I \rightarrow \Delta_\delta \times K^I$ as $\phi_\delta = (\phi_{1,\delta}, \phi_2)$, where $\phi_{1,\delta}(\mathbf{p}, \mathbf{x}, \mathbf{q}, \boldsymbol{\eta}) = \{(\hat{\mathbf{p}}, \hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\gamma}}) \in \Delta_\delta \text{ s.t. } (\hat{p}_{st}, \hat{\pi}_{st}) \in \arg \max_{\tilde{p}_{st}, \tilde{\pi}_{st}} (\tilde{p}_{st} \cdot \sum_{i \in \mathbb{I}} (x_{st}^i - w_{st}^i) + \tilde{\pi}_{st} \cdot \sum_{i \in \mathbb{I}} q_{st}^i) \text{ and } \hat{\gamma}_{st,j} = \min\left(1, \frac{\delta + \sum_{i \in \mathbb{I}} \eta_{st,j}^i}{\delta + p_{st,1} a_{st,j} + \sum_{i \in \mathbb{I}} [q_{st,j}^i]^+}\right), \forall j \in \mathbb{J} \text{ and } \forall (t, s^t) \in \mathbb{T}_T \times \mathbb{S}^t\}$ and $\phi_2(\mathbf{p}, \boldsymbol{\pi}, \boldsymbol{\gamma}) = \{m^1, \dots, m^I\}$.

By the continuity of V_T^i and compactness of K , m^i is nonempty. By the continuity of V_T^i and $(\mathbf{p}, w^i) \gg 0$, m^i is upper hemicontinuous. By the quasiconcavity of V_T^i and convexity of K , m^i is convex valued. Therefore, ϕ_δ is a nonempty, convex valued, and u.h.c. correspondence, which maps a compact into itself. It follows then from Kakutani's fixed point theorem that $\exists (\bar{\mathbf{p}}, \bar{\boldsymbol{\pi}}, \bar{\boldsymbol{\gamma}}, \bar{\mathbf{x}}, \bar{\mathbf{q}}, \bar{\boldsymbol{\eta}}) \in \phi_\delta(\bar{\mathbf{p}}, \bar{\boldsymbol{\pi}}, \bar{\boldsymbol{\gamma}}, \bar{\mathbf{x}}, \bar{\mathbf{q}}, \bar{\boldsymbol{\eta}})$. Moreover, adding equation (1) over i one gets: $\bar{p}_{st} \cdot \sum_{i \in \mathbb{I}} (\bar{x}_{st}^i - w_{st}^i) + \bar{\pi}_{st} \cdot \sum_{i \in \mathbb{I}} \bar{q}_{st}^i \leq J\delta, \forall (t, s^t) \in \mathbb{T}_T \times \mathbb{S}^t$. Now, take a sequence of positive real numbers converging to zero, $\{\delta_n\} \rightarrow 0$, and define $E_n = (\bar{\mathbf{p}}, \bar{\boldsymbol{\pi}}, \bar{\boldsymbol{\gamma}}, \bar{\mathbf{x}}, \bar{\mathbf{q}}, \bar{\boldsymbol{\eta}})_n \in \phi_{\delta_n}(E_n), \forall n \in \mathbb{N}$. Notice that $\{E_n\}$ is a bounded sequence in $\Delta \times K^I$ (where $\Delta = \Delta_{\delta=0}$). Therefore, there exists a subsequence converging to $E^* = (\mathbf{p}^*, \boldsymbol{\pi}^*, \boldsymbol{\gamma}^*, \mathbf{x}^*, \mathbf{q}^*, \boldsymbol{\eta}^*)$.

Finally, it is necessary to check if $E^* = (\mathbf{p}^*, \boldsymbol{\pi}^*, \boldsymbol{\gamma}^*, \mathbf{x}^*, \mathbf{q}^*, \boldsymbol{\eta}^*)$ is an equilibrium for ξ_T . From the above construction, it must be the case that $p_{st}^* \cdot \sum_{i \in \mathbb{I}} (x_{st}^{*i} - w_{st}^i) + \pi_{st}^* \cdot \sum_{i \in \mathbb{I}} q_{st}^{*i} \leq 0, \forall (t, s^t) \in \mathbb{T}_T \times \mathbb{S}^t$. Strict monotonicity of V_T^i implies that $\exists \bar{\delta} > 0$ s.t. $\min(\bar{\mathbf{p}}_n) > \bar{\delta}, \forall n \in \mathbb{N}$. Therefore, the definition of $\phi_{1,\delta}$ implies $\sum_{i \in \mathbb{I}} (x_{st,l}^{*i} - w_{st,l}^i) = \sum_{i \in \mathbb{I}} q_{st,j}^{*i} = 0, \forall l = 1, \dots, L, \forall j \in \mathbb{J}$, and $\forall (t, s^t) \in \mathbb{T}_T \times \mathbb{S}^t$. Continuity of B^i (for \mathbf{p} strictly positive) and V_T^i imply optimality of $(\mathbf{x}^*, \mathbf{q}^*, \boldsymbol{\eta}^*)$ for the truncated economy. Finally, continuity and convexity of preferences together with the fact that the allocation is in the interior of K imply that agents have in fact optimized on their untruncated budget sets. ■

Lemma 2. Let $G : Z \rightarrow \Lambda_Z$ be an expectation correspondence and Q be a compact subset of Z . If for every $T > 0$, there exists a T -horizon equilibrium for G such that $z_t \in Q$ almost surely for all t , then G has an ergodic Markov equilibrium.

Proof. See Duffie, et al. (1994), Proposition 1.2, page 752. ■

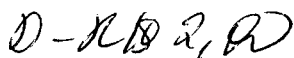
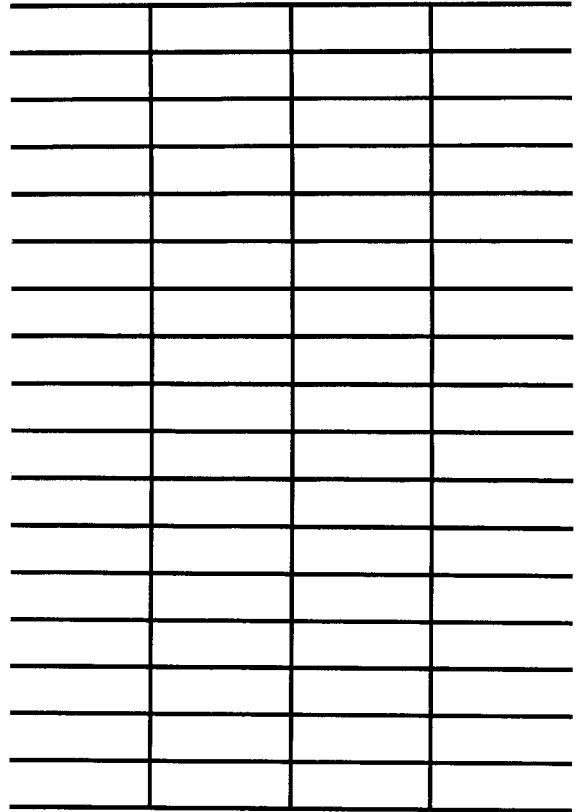
From Lemma 1, any T -horizon equilibrium $(\mathbf{p}^*, \pi^*, \gamma^*, \mathbf{x}^*, \mathbf{q}^*, \eta^*)$ belongs to $\Delta \times K$. Therefore, for any $T > 0$, $\|p_{st}^*, \pi_{st}^*, \gamma_{st}^*\| \leq 1$ and $\|x_{st}^*, q_{st}^*, \eta_{st}^*\| \leq k = \max(\bar{x}, \bar{I}, \bar{\eta})$, $\forall j, t, s^t$. Since (q^*, η^*) is bounded, there exists a compact set \mathbf{H} such that $h_{st}^* \in \mathbf{H}$, $\forall t, s^t$. Therefore, to conclude the proof, define $Q = \{z \in Z \text{ s.t. } s \in \mathbb{S}, h \in \mathbf{H}, \|q_{-1}, x, q, \eta\| \leq k \text{ and } \|p, \pi, \gamma\| \leq 1\}$. *Q.E.D.*

References

- [1] Alvarez, Fernando, and Urban J. Jermann, 2000, Efficiency, equilibrium and asset pricing with risk of default, *Econometrica* 68, 775-798.
- [2] Araújo, Aloísio P., Paulo K. Monteiro, and Mário R. Páscoa, 1996, Infinite horizon incomplete markets with a continuum of states, *Mathematical Finance* 6, 119-132.
- [3] Braido, Luís H. B., 2001, General equilibrium with endogenous financial design and moral hazard, mimeo, University of Chicago.
- [4] Dubey, Pradeep, John D. Geanakoplos, and Martin Shubik, 1988, Default and efficiency in a general equilibrium model with incomplete markets, mimeo, Cowles Foundation 879R, Yale University.
- [5] Duffie, Darrell, John D. Geanakoplos, Andreu Mas-Collé, and Andrew M. McLennan, 1994, Stationary Markov equilibria, *Econometrica* 62, 745-781.
- [6] Heaton, John, and Deborah Lucas, 1996, Evaluating the effects of incomplete markets on risk sharing and asset pricing, *Journal of Political Economy* 104, 443-487.
- [7] Kehoe, Timothy J. and David K. Levine, 1993, Debt-constrained asset markets, *Review of Economic Studies* 60, 865-888.
- [8] Ma, Chenghu, 1993, Market equilibrium with heterogeneous recursive-utility-maximizing agents, *Economic Theory* 3, 243-266.

- [9] Scheinkman, José A., and Laurence Weiss, 1986, Borrowing constraints and aggregate economic activity, *Econometrica* 54, 23-45.
- [10] Simon, Leo K. and William R. Zame, 1990, Discontinuous games and endogenous sharing rules, *Econometrica* 58, 861-872.
- [11] Telmer, Chris I., 1993, Asset pricing and incomplete markets, *Journal of Finance* 48, 1803-1832.
- [12] Zame, William R., 1993, Efficiency and the role of default when security markets are incomplete, *American Economic Review* 83, 1142-1164.
- [13] Zhang, Harold H., 1997, Endogenous borrowing constraints with incomplete markets, *Journal of Finance* 52, 2187-2209.

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