"TOWARD A THEORY OF INTERNATIONAL CURRENCY: A STEP FURTHER"

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Toward a Theory of International Currency:
A Step Further

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Abstract

We generalize the two-country, two-currency model of Matsuyama, Kiyotaki and Matsui to resolve two "shortcomings" in their approach. First, we endogenize prices and exchange rates. Second, we introduce monetary policy. We then use the model to address the following new questions: How does the fact that a currency circulates internationally affect its purchasing power? Where does an international currency purchase more? What are the effects on seignorage and welfare when a currency becomes international? How is policy affected by concerns of currency substitution? How are national monetary policies connected, and what is the scope for international cooperation?
1 Introduction

In "Toward a Theory of International Currency," Matsuyama, Kiyotaki, and Matsui (1993) extend the search-theoretic approach to monetary economics to develop a two-country, two-currency model of the world economy. Their model allows one to answer several questions in monetary theory and policy that could not even be raised in the context of earlier models. Examples include the following: What features of a country make it possible, or likely, for its currency to circulate internationally? How and when can local currencies survive in the presence of a universally accepted international currency? Does an international currency emerge naturally as economies become integrated? What are the costs and benefits to a country of having its currency serve as an international medium of exchange?

The analysis of Matsuyama et al. provides a key first step toward developing a theoretical foundation for international monetary economics (see Zhou [1994] for further applications of their model). The authors themselves acknowledge, however, two serious "shortcomings" in the framework they construct. One, in their model, as in all of the first-generation search-based models of money, every exchange is assumed to be a one-for-one trade, "which makes it impossible to talk about the determination of prices and exchange rates" (p. 304). Two, national governments or monetary authorities are not explicitly incorporated, "which makes it impossible to discuss any policy issue" (p. 303).
The purpose of this paper is to remedy these two shortcomings. First, in order to determine nominal prices and exchange rates endogenously, we adopt a version of the bargaining approach introduced into search models of money by Shi (1995) and Trejos and Wright (1995). Second, we incorporate national governments in such a way that they face nontrivial dynamic seignorage opportunities, depending not only on their own policy but also on that of the other country.

These extensions allow us to raise a whole range of new issues, including the following: How does the fact that a currency circulates internationally affect its purchasing power at home? Where does an international currency purchase more? What are the effects on seignorage and welfare, in each country, when one money becomes an international currency, and how are policies designed to maximize either seignorage or welfare affected by concerns of currency substitution? How are national monetary policies connected, and what is the scope for international cooperation?

The model in which we address these questions can be described as follows. There are two countries, each of which continuously issues its own currency. Although we generalize somewhat the matching technology in Matsuyama et al., we maintain their basic notion that you interact with individuals from your own country more often than a foreigner interacts with these same individuals. Parameter values determine the realm of circulation for each currency, and potentially generate three distinct types of equilibrium — or three distinct regimes — where either none, one, or both of the monies circulate internationally. Although for some parameter values there exist multiple equilibrium regimes, in any given regime there is a unique set
of prices (at least, in the simplest version of the model; we also sketch an extension that leads to multiple equilibrium prices within a regime).

In terms of results, in all of the regimes we find that prices in one country are increasing in the rate of monetary expansion by that country. Prices in one country are also increasing in the rate of monetary expansion by the other country in all regimes except the one where both currencies circulate only locally (in which case the two economies are completely independent). Comparing across regimes, other things being equal, we find that prices in a country are higher when a foreign currency circulates within that country. Also, prices in a country are lower when it’s currency circulates abroad.

When one money circulates internationally and the other circulates only locally, the former has greater purchasing power than the latter. Also, the international currency commands higher purchasing power at home than abroad (excluding exceptional cases where the technologies in the two countries are very different). The host country may benefit from the international currency, since it serves as a medium of exchange and helps to facilitate trade; but it may also suffer, since using the international currency increases prices denominated in the local currency. For a given rate of money creation, seignorage in a country increases when its money is used abroad. Also, when a country’s money is used abroad, welfare at home may increase but may also decrease if liquidity is very scarce.

We then endogenize policy by assuming that governments are interested either in maximizing the welfare of their citizens or, alternatively, in maximizing seignorage. If governments choose policies independently the outcome is not efficient. For instance, if at least one money circulates internationally
and governments are interested in maximizing seignorage, both could raise more revenue if they cooperate. Private agents may be worse off when the governments cooperate, however, since this enables governments to better extract resources from them. When policies are endogenous, we also find that, other things being equal, if only one money circulates internationally then citizens from the country that issues it are unambiguously better off than their foreign counterparts.

The rest of the paper is organized as follows. In Section 2 we present the basic model. In section 3 we describe the set of equilibria for fixed monetary policies. In Section 4 we present the results on policy. We provide some brief concluding remarks in Section 5.

2 The Model

The model consists of two countries, labeled \( i = 1, 2 \). One’s country is important in that it determines the frequency with which one interacts with other agents. Individuals interact (or “meet”) bilaterally according to a random matching process in continuous time, and \( \hat{\alpha}_{ij} \) denotes the Poisson arrival rate at which a citizen of country \( i \) meets citizens of country \( j \). We assume that \( \hat{\alpha}_{ii} \geq \hat{\alpha}_{ji} \) for \( j \neq i \). This simply says that, for example, a Mexican meets Mexicans more frequently than an American meets Mexicans.\(^1\)

\(^1\)We do not need to assume \( \hat{\alpha}_{ii} \geq \hat{\alpha}_{ij} \). Thus, although a Mexican meets Mexicans more frequently than an American meets Mexicans, we do not necessarily rule out the possibility that a Mexican meets Mexicans less frequently than he meets Americans (which might be the case if, for example, the U.S. is bigger than Mexico).
Each country starts with a continuum of citizens, and the fraction of individuals from country $i$ is $N_i$, with $N_1 + N_2 = 1$. Thereafter, new agents are continuously born in each country at the same rate $\gamma \geq 0$. Once an agent is born, he lives forever. The population sizes and the meeting technology parameters are not independent, since we have the identity $N_1 \hat{\alpha}_{12} = N_2 \hat{\alpha}_{21}$ (both sides of the equality give the total number of meetings between citizens of countries 1 and 2 per unit time). For now we take $\hat{\alpha}_{ij}$ as primitive and let the populations be free to satisfy this identity. Note that the specification in Matsuyama et al. (1993) is the special case with $\alpha_{ii} = N_i$ and $\alpha_{ij} = \beta N_j$, where $\beta \leq 1$ measures the degree of economic integration.

Agents are distinguished not only by their country and date of birth, but also by their tastes and technologies. We assume there are $K \geq 3$ goods, and the population of each country contains equal numbers of $K$ types, where each type $k$ consumes only good $k$ and produces only good $k + 1 \pmod{K}$. If we assume that meetings are random with respect to consumption-production type, then $\alpha_{ij} = \hat{\alpha}_{ij}/K$ is the rate at which a citizen of country $i$ meets a citizen of country $j$ that consumes the good he produces and also the rate at which he meets a citizen of country $j$ that produces the good he consumes.

There is no centralized market or auctioneer in this model: all trade is bilateral and quid pro quo. The large number of agents rules out private credit, since the probability of meeting a particular individual a second time is zero. Our specification for tastes and technologies rules out direct barter (but see below). We also make the assumption that goods are nonstorable, which rules out commodity money. Hence, trade requires the use of some form of fiat money as a medium of exchange — although we do not impose
that any particular currency plays this role in any particular transactions.\footnote{That is, which monies are used in which transactions will be determined as part of the equilibrium. This is what distinguishes this class of models from ones with particular "cash-in-advance" constraints imposed exogenously.} Fiat monies are introduced by the government of country $j$ issuing one unit of currency $j$ to some fraction $M_j \in (0,1)$ of its newborn citizens at each point in time, in exchange for real output at the going price. Issuing money in this way yields a continuous flow of seignorage.

To keep things tractable, we make the following assumptions. First, we assume that an agent holding a unit of currency always spend it all at once (which could obviously be guaranteed if we simply say that the monetary object is indivisible); this implies that no one holding currency ever holds less than one unit of the stuff. Second, we assume that (except for the newborn) no agent can produce until after he consumes; this implies that two agents with currency do not trade with each other, and no one ever holds more than one unit of money. Hence, at each point in time there will be some agents with one unit of money each, looking to buy goods, and a disjoint group with no money, looking to sell goods. The fraction of agents from country $i$ with currency $j$ and trying to buy is denoted $m_{ij}$, and the fraction of agents from country $i$ with no money and trying to sell is denoted $m_{i0} = 1 - m_{i1} - m_{i2}$.

Suppose that a type $k$ seller meets a type $k+1$ buyer (i.e., a buyer that consumes the seller's type of output). If the seller produces $q$ units of output for the buyer, the latter enjoys utility $u(q)$ while the former suffers disutility $c(q)$. With no loss in generality, we can normalize $c(q) = q$, as long as we
also renormalize $u(q)$. We assume $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$ for all $q > 0$, $u''(q) < 0$ for all $q > 0$, and there exists $\hat{q} > 0$ such that $u(\hat{q}) = \hat{q}$.

If a buyer trades a unit of currency $j$ for $q_j$ units of output then the price level in terms of that currency is $p_j = 1/q_j$. As in Shi (1995) and Trejos and Wright (1995), the quantity $q_j$ will be determined here using bilateral bargaining theory. For simplicity, in this paper, we assume that the buyer makes a "take it or leave it" offer (see below for a discussion of how this simplifies the analysis and how things can be generalized). This implies that, if they trade, the buyer extracts the entire surplus. It is possible, however, that a buyer may prefer to not trade, rather than to agree to any terms that a particular seller is willing to accept. For example, although an American seller will always take pesos at some price, a buyer with pesos may prefer to wait until he meets a Mexican seller if the latter is expected to agree to a sufficiently better price.

Determining when trade occurs, and at what prices, when the variables $M_i$ are fixed exogenously is the task of the next section. In the section after that we endogenize policy by letting the governments choose $M_i$.

3 Equilibrium

We are interested here in steady state equilibria, where the fractions of agents holding different assets and prices are constant with respect to time. We distinguish different types of equilibria — or different regimes — as follows. Let $\Lambda(h, i, j) = 1$ if a seller from country $h$ and a buyer from country $i$ with currency from country $j$ trade, and $\Lambda(h, i, j) = 0$ otherwise (note that we
only consider pure-strategy equilibria). Then a regime is a list of values for \( \Lambda(h, i, j) \) for all \( h, i, j \).

In principle, a large number of different regimes are possible. However, we ignore perverse or uninteresting cases by only considering equilibria where \( \Lambda(i, i, i) = \Lambda(i, j, i) = 1, i = 1, 2, j \neq i \). That is, we adopt as a maintained hypothesis that currency \( i \) is used to buy from sellers from country \( i \) (of course, we still have to check that this is incentive compatible). The following lemma, which says that Americans sell to Mexicans with pesos if and only if Americans sell to other Americans with pesos, reduces the number of possible cases further.

**Lemma 1** In any equilibrium where \( \Lambda(i, i, i) = 1 \) we have \( \Lambda(i, j, j) = \Lambda(i, i, j) \) for \( j \neq i \).

**Proof:** See the Appendix. □

Based on the maintained hypothesis \( \Lambda(i, i, i) = \Lambda(i, j, i) = 1 \) and Lemma 1, a regime is completely determined by two numbers: \( \lambda_1 = \Lambda(1, 2, 2) \) and \( \lambda_2 = \Lambda(2, 1, 1) \). When \( \lambda_1 = 1 \), sellers from country \( i \) trade with buyers holding money \( j \neq i \), in which case we call money \( j \) an international currency. The possible regimes are \( \lambda \equiv (\lambda_1, \lambda_2) = (0, 0), (1, 1), (0, 1), \) and \( (1, 0) \). In the first case there is no international currency, in the second case both currencies are international, and in the final two cases only one currency is international. Since the two regimes with one international currency are mirror images, we focus for now only on the latter, \( \lambda = (1, 0) \).

Hence, we must analyze three distinct regimes, with either 0, 1, or 2 international monies. These are the same as the three types of equilibria
emphasized in Matsuyama et al., although the logic here is quite different. In their model, with indivisible output, buyers always want to trade; thus, whether trade occurs depends exclusively on the seller, who may or may not be willing to accept a particular currency. Here, sellers can always be induced to produce some output for either money — that is, they can always be induced to trade at some price — but the buyer may or may not be willing to trade at that price.³

The next step is to determine prices. Begin by letting $V_{ij}$ denote expected lifetime utility for a buyer from country $i$ with currency $j$, and $V_{i0}$ expected lifetime utility for a seller from country $i$. We call these the value functions. Two things follow immediately from the assumption that the bargaining problem is resolved by a "take it or leave it" offer from the buyer. First, assuming that the buyer wants to trade, he offers to exchange his money for the quantity that makes the seller indifferent between accepting and rejecting. Second, this implies that the seller never gets any of the gains from trade, and therefore $V_{i0} = 0$.

Given $V_{i0} = 0$ and the normalization $c(q) = q$, the offer that makes a seller from country $i$ indifferent between accepting and rejecting currency $j$ satisfies

$$q_{ij} = V_{ij}, \quad (1)$$

³The result that sellers are always willing to provide some amount of output for a unit of any currency depends on assumptions that buyers make "take it or leave it" offers and that there is no direct barter. In the generalized version of the model presented in Section 3.5 it is possible that a seller may not be willing to accept a foreign currency at any positive price.
for $i, j = 1, 2$. The price of output in country $i$ in terms of currency $j$ is therefore $p_{ij} = 1/q_{ij}$. Notice that not only is it possible for a dollar and a peso to purchase different amounts from the same seller, it is also possible for the same money to purchase different amounts from American and Mexican sellers. However, the price in terms of a given currency does not depend on the nationality of the buyer because of the "take it or leave it offer" assumption.⁴

A buyer is willing to trade on the terms given by (1) if and only if the utility he derives from consuming $q_{ij}$ exceeds the value to holding onto his money in order to spend it elsewhere. Hence, the following condition must be satisfied if regime $\lambda$ is to constitute an equilibrium:

$$
\lambda_i = 1 \Rightarrow u(q_{ij}) \geq V_{jj} \quad \text{and} \quad \lambda_i = 0 \Rightarrow u(q_{ij}) \leq V_{jj}.
$$

(2)

The condition $u(q_{ij}) \geq V_{jj}$ means that it is incentive compatible for buyers from country $j$ with money $j$ to trade with sellers from country $i$ (by Lemma 1, this implies that it is also incentive compatible for buyers from country $i$).

⁴This and some other simplifying features of the model would be lost if we adopted a different bargaining solution. For example, the generalized Nash solution says that when a seller from country $h$ and a buyer from country $i$ with money $j$ trade, the quantity $q$ solves

$$
\max [V_{hk} - q - V_{h0}]^\theta[u(q) + V_{j0} - V_{ij}]^{1-\theta}
$$

subject to $V_{hk} - q - V_{h0} \geq 0$ and $u(q) + V_{j0} - V_{jk} \geq 0$, assuming that the threat points are $V_{h0}$ and $V_{ij}$ (see Trejos and Wright 1995 for an extended discussion of Nash bargaining and of strategic bargaining in search-theoretic models of money). The parameter $\theta \in [0, 1]$ measures the "bargaining power" of the seller, and the model analyzed in the text is the special case where $\theta = 0$. 
with money $j$ to trade with sellers from country $i$). In principle, we also need to check the maintained hypothesis $\Lambda(i, i, i) = \Lambda(i, j, i) = 1$, which requires $u(q_{ii}) \geq V_{ii}$ and $u(q_{ji}) \geq V_{ji}$; but, with the exception of one possible case discussed below, these constraints will never bind.

Next we describe the value functions for buyers. These satisfy the standard dynamic programming equations from search theory,

$$rV_{ii} = \alpha_{ii}m_{i0}[u(q_{ii}) - V_{ii}] + \alpha_{jj}m_{j0}\lambda_j[u(q_{ji}) - V_{ji}] + \dot{V}_{ii}$$

$$rV_{ij} = \alpha_{ii}m_{i0}\lambda_i[u(q_{ij}) - V_{ij}] + \alpha_{jj}m_{j0}[u(q_{jj}) - V_{ij}] + \dot{V}_{ij}$$

for $i = 1, 2$ and $j \neq i$, where $r$ is the rate of time preference (see, e.g., Trejos and Wright 1995). In steady state, $\dot{V}_{ii} = \dot{V}_{ij} = 0$. Then the first equation says the flow payoff for a buyer from country $i$ with money $i$, $rV_{ii}$, is the sum of two terms. The first term is the rate at which he meets citizens of his own country who are sellers, $\alpha_{ii}m_{i0}$, times the gain from trade $u(q_{ii}) - V_{ii}$. The second term is the rate at which he meets foreigners who are sellers, $\alpha_{jj}m_{j0}$, times the probability that they trade, $\lambda_j$, times the gain from trade $u(q_{ji}) - V_{ji}$. The second equation has a similar interpretation.

Finally, we need to determine the steady state values of $m_{ij}$. First, note that $\lambda_i = 0$ implies $m_{ij} = 0$ (if Americans never trade for pesos then they never hold pesos). Second, note that trades between a buyer and seller of the same nationality cannot alter $m_{ij}$, since the two compatriots simply switch states and leave the aggregate distribution unchanged. These considerations
imply that the steady state equations are

\[ \dot{m}_{ii} = \alpha_{ii}m_{i0}m_{ji} - \alpha_{ij}m_{ii}m_{j0}\lambda_j + \gamma(M_i - m_{ii}) = 0 \]  

(4)

\[ \dot{m}_{ij} = \alpha_{ij}m_{0i}m_{jj}\lambda_i - \alpha_{ij}m_{ij}m_{j0} - \gamma m_{ij} = 0 \]

where \( m_{i0} = 1 - m_{ii} - m_{ij} \), for \( i, j = 1, 2, i \neq j \).

To explain (4), consider the first equation, which decomposes \( m_{ii} \) into three terms. The first term says that \( m_{ii} \) increases when a seller from country \( i \) meets a buyer from country \( j \) with currency \( i \) (given the maintained assumption that when a seller from country \( i \) meets a buyer with currency \( i \) they always trade). The second term says that \( m_{ii} \) decreases when a buyer from country \( i \) with currency \( i \) meets a seller from country \( j \) if they trade (that is, if \( \lambda_j = 1 \)). The final terms says that \( m_{ii} \) increases when the fraction of the newborn in country \( i \) who receive currency, \( M_i \), exceeds \( m_{ii} \). The second equation has a similar interpretation.

Define the vectors \( q = (q_{ij}) \), \( V = (V_{ij}) \), and \( m = (m_{ij}) \), where \( i, j = 1, 2 \). Then define a steady state equilibrium as a list \((\lambda, q, V, m)\) satisfying (1)-(4). Henceforth we ignore the value functions, since they are redundant by virtue of the equilibrium condition \( V_{ij} = q_{ij} \). Therefore, a steady state equilibrium is completely characterized by the regime \( \lambda \), the prices \( q \), and the asset distribution \( m \). We now describe the conditions that must hold in order for each of the regimes to constitute an equilibrium. The idea is to

\footnote{More generally, an equilibrium (not necessarily a steady state) is a list \((\lambda_t, q_t, V_t, m_t)\), where all variables potentially vary through time, satisfying (1)-(4), except that we do not set \( \dot{m}_{ij} = 0 \) or \( \dot{V}_{ij} = 0 \), plus some initial conditions for date 0 asset holdings.}
choose \( \lambda \), solve for \( m \) and \( q \), and determine the parameter restrictions under which the incentive conditions are satisfied.

### 3.1 No International Money

Consider first the regime with no international money, and, therefore, no international trade: \( \lambda = (0, 0) \). In this case (4) implies \( m_{ii} = M_i \), \( m_{ij} = 0 \), and \( m_{0i} = 1 - M_i \), for \( i = 1, 2, j \neq i \). Also, (3) can be rearranged to show that prices satisfy

\[
q_{ii} = \frac{\alpha_{ii} m_{0i}}{r + \alpha_{ii} m_{0i}} u(q_{ii})
\]

\[
q_{ij} = \frac{\alpha_{ij} m_{0j}}{r + \alpha_{ij} m_{0j}} u(q_{ij})
\]

for \( i = 1, 2, j \neq i \). In this regime, no transactions actually occur at \( q_{ij} \) for \( j \neq i \), but \( q_{ij} \) tells us how much one could get with currency \( j \) from country \( i \) sellers.

As long as this equilibrium exists, \( q_{ii} \) and \( q_{ji} \) are both decreasing in \( M_i \), and independent of \( M_j \) for \( j \neq i \). Also, \( q_{11} > q_{22} \) if and only if \( \alpha_{11}(1 - M_1) > \alpha_{22}(1 - M_2) \). Since one unit of currency \( i \) buys \( q_{ii} \) units of real output, we can say that 1 unit of currency \( 1 \) is worth \( e = q_{11}/q_{22} = p_2/p_1 \) units of currency \( 2 \); this is the exchange rate that would be implied by purchasing power parity. Monetary policy affects the exchange rate in the sense that \( e \)

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Suppose one peso buys \( q_{11} \) units of output in Mexico and one dollar buys \( q_{22} \) units of output in America, and suppose that there is a market in which currencies can be traded at the nominal exchange rate \( e \) (i.e., one peso buys \( e \) dollars). Then one peso could be used to buy \( eq_{22} \) units of output in America using dollars. Purchasing power parity holds if a peso buys the same amount directly at home and using dollars abroad: \( q_{11} = eq_{22} \).
falls with $M_1$ and rises with $M_2$. Real factors also affect the exchange rate in the sense that $e$ rises with $\alpha_{11}$ and falls with $\alpha_{22}$. Of course, international differences in utility and production functions would also influence $e$; we have kept these the same across countries purely for notational simplicity.

To see when this regime actually constitutes an equilibrium, we must check the maintained hypothesis $\Lambda(i,i,i) = \Lambda(i,j,i) = 1$ and the incentive condition (2). The former can easily be seen to hold for all parameter values. The latter, which says that individuals with money $j$ do not want to buy from sellers from country $i$, holds if and only if $u(q_{ij}) \leq q_{jj}$, for $i = 1, 2$ and $i \neq j$. Using (5), we can rewrite this inequality as

$$u \frac{\alpha_{ij}(r + \alpha_{jj}m_{j0})}{\alpha_{jj}(r + \alpha_{jj}m_{j0})} q_{jj} \leq q_{jj},$$

which is satisfied if and only if $\alpha_{ij}$ is small compared to $\alpha_{jj}$. Intuitively, when agents from country $i$ do not meet agents from country $j$ very frequently, sellers from country $i$ do not value money $j$ very much, and buyers with money $j$ prefer to wait until they meet a seller from country $j$.

### 3.2 Two International Monies

Now turn to the regime with two international currencies, $\lambda = (1, 1)$. The first thing to do is to solve (4) for $m_{ij}$. Routine algebra yields the steady state for citizens of country 1,

$$m_{10} = 1 - \frac{(\gamma + \alpha_{21})M_1 + \alpha_{12}M_2}{\gamma + \alpha_{12} + \alpha_{21}}$$

$$m_{11} = \frac{\gamma(\gamma + \alpha_{12} + \alpha_{21}) + \alpha_{21}(\gamma + \alpha_{21})(1 - M_1) + \alpha_{12}(1 - M_2)}{(\gamma + \alpha_{12} + \alpha_{21})[\gamma + \alpha_{21}(1 - M_1) + \alpha_{12}(1 - M_2)]}$$

(7)
and $m_{12} = 1 - m_{10} - m_{12}$. The steady state for citizens of country 2 is described by reversing the subscripts. Notice that $m_{10}$ is decreasing in both $M_1$ and $M_2$.

We claim that in this regime there is a unique nonzero solution to (3), and it has the property that $q_{ii} = q_{ij} = Q_{ii}$; that is, the two monies are perfect substitutes in the sense that they purchase the same amount from a given seller. To see this, first note that when $\lambda = (1, 1)$ the system of equations defined by (3) implies that for $V_{11} = q_{11}$ and $V_{21} = q_{21}$ we have:

$$
\begin{align*}
    r_{11} &= \alpha_{11} m_{10} [u(q_{11}) - q_{11}] + \alpha_{12} m_{20} [u(q_{21}) - q_{11}] \\
    r_{21} &= \alpha_{21} m_{10} [u(q_{11}) - q_{21}] + \alpha_{22} m_{20} [u(q_{21}) - q_{21}].
\end{align*}
$$

(8)

This subsystem can be solved for $q_{11}$ and $q_{21}$ independent of $q_{12}$ and $q_{22}$. Similarly, there is a subsystem that can be solved for $q_{12}$ and $q_{22}$ independent of $q_{11}$ and $q_{21}$. Inspection of the two subsystems reveals that they are identical, and so the set of solutions for $q_{11}$ and $q_{21}$ is the same as the set of solutions for $q_{12}$ and $q_{22}$.

Now use the first equation of (8) to define a function $q_{21} = g(q_{11})$ and the second to define a function $q_{21} = h(q_{11})$. Solutions to this subsystem are given by intersections in the $(q_{11}, q_{21})$ plane of the graphs of $g$ and $h$. It is straightforward to verify that $g(q) = 0$ for some $q > 0$, that $g' > 0$, and that $g(\hat{q}) > \hat{q}$. Similarly, $h(0) > 0$, $h' > 0$, and $h(\hat{q}) < \hat{q}$. Hence, $g$ and $h$ intersect in $(0, \hat{q})^2$. Moreover, one can show that $g' > h'$ whenever $g = h$, and so $g$ and $h$ intersect no more than once. As shown in Figure 1, the unique nonzero solution is $(q_{11}, q_{21}) = (Q_1, Q_2)$. Since the set of solutions for
\( q_{12} \) and \( q_{22} \) is the same as the set of solutions for \( q_{12} \) and \( q_{22} \), we also have

\[ (q_{12}, q_{22}) = (Q_1, Q_2), \]

as claimed.

In this regime, \( q_{ij} = Q_i \) does not depend on which currency the buyer is using, so the two monies are perfect substitutes. Hence, if there were a market in which agents can trade the two monies, the market clearing price would have to be 1. But the purchasing power parity exchange rate, as defined in Section 3.1, is given by \( e = Q_1/Q_2 \); and in general \( e \) differs from unity in this regime because even though quantities do not depend on which currency the buyer is using they do depend on the nationality of the seller.

For future reference, rearrange (8) to write \((Q_1, Q_2)\) as the solution to

\[ Q_1 = \frac{\alpha_{11}m_{10}u(Q_1) + \alpha_{12}m_{20}u(Q_2)}{r + \alpha_{11}m_{10} + \alpha_{12}m_{20}} \]

\[ Q_2 = \frac{\alpha_{21}m_{10}u(Q_1) + \alpha_{22}m_{20}u(Q_2)}{r + \alpha_{21}m_{10} + \alpha_{22}m_{20}}. \]

It is shown in the Appendix that \( Q_1 \) and \( Q_2 \) are both decreasing in \( M_1 \) and \( M_2 \). Thus, an increase in either money supply increases the price level in both countries. One can also show that, as long as the two countries are not too different in terms of \( \alpha_{ij} \) and \( M_j \), the effect of a change in \( M_1 \) is stronger in country 1 than in country 2 and vice-versa. This means that the purchasing power parity exchange rate \( e = Q_1/Q_2 \) is decreasing in \( M_1 \) and increasing in \( M_2 \).

To see when this regime constitutes an equilibrium, we need to check the maintained hypothesis \( \Lambda(i, i, i) = \Lambda(i, j, i) = 1 \) and the incentive condition (2). All of these conditions hold in this regime if and only if \( u(Q_i) \geq Q_j \), \( i, j = 1, 2, j \neq i \). In Figure 1, \( u(Q_1) \geq Q_2 \) and \( u(Q_2) \geq Q_1 \) define a
region that contains the 45° line. If the countries are symmetric, in the sense that $M_1 = M_2$, $\alpha_{ii} = \alpha_{jj}$ and $\alpha_{ij} = \alpha_{ji}$, the functions $g$ and $h$ intersect on the 45° line and $u(Q_i) \geq Q_j$ holds. By continuity, the equilibrium with two international currencies exists if the economies are similar; but if they are sufficiently dissimilar then the incentive conditions will be violated and $\lambda = (1,1)$ is not an equilibrium.

The model is very different in this regard from the model in Matsuyama et al. In their model, an equilibrium with two international currencies exists for all parameter values, since if agents regard the two monies as perfect substitutes then buyers will always spend their money on sellers from either country (because they always get an indivisible unit of output). By contrast, in our model, even if the two countries' monies are perfect substitutes, the two nationalities of sellers are not. For example, if $\alpha_{jj}$ is low relative to $\alpha_{ii}$ then country $j$ sellers do not give much for (either) money compared to country $i$ sellers, and country $i$ buyers may prefer to hold on to their cash rather than spend it in country $j$. Hence, $\lambda = (1,1)$ will not be an equilibrium when there are big differences in technologies across countries.

In the symmetric case with $\alpha_{ij}$ small, this equilibrium coexists with the $\lambda = (0,0)$ equilibrium. If $M_1$ and $M_2$ are similar, then $\lambda = (1,1)$ Pareto dominates $\lambda = (0,0)$, because internationally accepted money allows international trade, which is impossible without a suitable payment mechanism. However, if $M_1$ is much larger than $M_2$, then country 2 can be better off in the $\lambda = (0,0)$ regime. We will discuss this in more detail in the section on policy.
3.3 One International Money

We now turn to the regime where citizens of country 1 accept money 2, but not vice-versa, \( \lambda = (1, 0) \). The first thing to do is solve (4) for \( m_{ij} \). Since money 1 is not held by citizens of country 2, \( m_{11} = M_1 \) and \( m_{21} = 0 \). The distribution of money 2 holdings is given by

\[
m_{12} = \frac{\alpha_{12} (1 - M_1) M_2}{\gamma + \alpha_{12} + \alpha_{21}(1 - M_1)}
\]

\[
m_{22} = \frac{(\gamma + \alpha_{12})M_2}{\gamma + \alpha_{12} + \alpha_{21}(1 - M_1)}.
\]

These results imply that the fraction of sellers in each country is

\[
m_{10} = (1 - M_1) \frac{\gamma + \alpha_{21}(1 - M_1) + \alpha_{12}(1 - M_2)}{\gamma + \alpha_{12} + \alpha_{21}(1 - M_1)}
\]

\[
m_{20} = \frac{\alpha_{21}(1 - M_1) + (\gamma + \alpha_{12})(1 - M_2)}{\gamma + \alpha_{12} + \alpha_{21}(1 - M_1)}.
\]

For given values of \( M_1, m_{10} \) is lower and \( m_{20} \) is higher in this regime than in the other two regimes. Also, \( m_{40} \) is decreasing in both \( M_1 \) and \( M_2 \).

The purchasing power of the national currency is given by

\[
q_{11} = \frac{\alpha_{11}m_{10}}{r + \alpha_{11}m_{10}} u(q_{11})
\]

\[
q_{21} = \frac{\alpha_{21}m_{10}}{r + \alpha_{21}m_{10}} u(q_{11}).
\]

These equations are the same as (5) in the \( \lambda = (0, 0) \) regime, and, as in that case, although money 1 is never spent in country 2, \( q_{21} \) tells how much one could buy with money 1 from a country 2 seller. Since \( m_{10} \) is lower in this
regime than in $\lambda = (0, 0)$, $q_{11}$ and $q_{21}$ are lower here. Intuitively, the influx of foreign money inflates prices denominated in the domestic currency. Also, $q_{11}$ and $q_{21}$ are decreasing in both $M_1$ and $M_2$ because $m_{10}$ is.

The purchasing power of the international currency depends on the seller, as given by

$$q_{12} = \frac{\alpha_{11} m_{10} u(q_{12}) + \alpha_{12} m_{20} u(q_{22})}{r + \alpha_{11} m_{10} + \alpha_{12} m_{20}}$$

$$q_{22} = \frac{\alpha_{21} m_{10} u(q_{12}) + \alpha_{22} m_{20} u(q_{22})}{r + \alpha_{21} m_{10} + \alpha_{22} m_{20}}.$$ 

These equations are the same as (9) in the $\lambda = (1, 1)$ regime, and the same analysis implies that there exists a unique nonzero solution for $(q_{12}, q_{22})$ and both are decreasing in $M_1$ and $M_2$. Since $m_{10}$ is lower and $m_{20}$ higher in this regime, $q_{12}$ is lower here than in the $\lambda = (1, 1)$ regime. Moreover, $q_{12} > q_{11}$, with the difference increasing in $M_1 - M_2$. Hence, citizens of country 1 value the international currency more than their own domestic currency. In general, $q_{12}$ can be greater or less than $q_{22}$, depending on $M_i$ and $\alpha_{ij}$; in the symmetric case, however, one can show unambiguously that $q_{22} > q_{12}$.

We still need to check when this equilibrium exists. The maintained hypothesis $\Lambda(i, i, i) = \Lambda(i, j, i) = 1$ holds if and only if $u(q_{22}) \geq q_{12}$. Then (2) requires $u(q_{12}) \geq q_{22}$, so that country 2 buyers with money 2 buy from country 1 sellers, and $u(q_{21}) \leq q_{11}$, so that country 1 buyers with money 1 do not

\footnote{Note that this condition, which says that agents from country 1 with money 2 buy from country 2 sellers, can bind for some parameter values in this regime; it is the one exceptional case referred to earlier where the maintained hypotheses may bind. If this condition were violated, then Mexicans would sell goods for dollars and then would spend these dollars on Mexican sellers but not American sellers.}
buy from country 2 sellers. Each of these conditions looks qualitatively like one that has been encountered earlier and holds under similar circumstances; however, since $q_{ij}$ differs across regimes, the parameter values for which the equilibria exist are quantitatively different. We take up the issue of which equilibria exist for which parameter values in the next subsection.

3.4 Summary and Illustration

Here we summarize what has been learned from the above analysis and present some examples to illustrate results that we have discovered numerically. The proof of the following analytic results consists of arguments already given in previous subsections.

**Proposition 1** Given our maintained assumptions, there are three qualitatively different regimes, or types of equilibria. There is at most one equilibrium of each type (i.e., each $\lambda$ implies a unique $m$ and $q$). Other things being equal, the value of a money is higher if it circulates internationally, and lower if the other money circulates internationally. Other properties of the different regimes include:

1. $\lambda = (0,0)$ (dollars circulate only in America, pesos only in Mexico) \(\Rightarrow\)
   
   - $q_{11} > q_{22}$ if $\alpha_{11}(1 - M_1) > \alpha_{22}(1 - M_2)$;
   - $\alpha_{ii}$ decreasing in $M_i$ and independent of $M_j$;
   - this regime is an equilibrium iff $\alpha_{ij}$ is small relative to $\alpha_{ii}$.

2. $\lambda = (1,1)$ (dollars and pesos both circulate in both countries) \(\Rightarrow\)
   
   - $q_{11} = q_{12} = Q_1$ and $q_{21} = q_{22} = Q_2$ (dollars and pesos perfect substitutes);
$Q_i$ is decreasing in $M_i$ and $M_j$; also, $e = Q_1/Q_2$ is decreasing in $M_1$ and increasing in $M_2$ at least if the two countries are not too dissimilar;

this regime is an equilibrium if and only if the two countries are not too dissimilar.

3. $\lambda = (1,0)$ (dollars circulate in both countries, pesos only in Mexico) $\implies$
$q_{ij}$ decreasing in $M_i$ and $M_j$ for all $i,j$;
$q_{12} > q_{11}$ (Mexicans value dollars more than Mexicans value pesos);
$q_{22} > q_{12}$ (Americans value dollars more than Mexicans value dollars) at least if the two countries are not too dissimilar.

In order to say more, we now assume that $u(q) = \sqrt{q}$. With this utility function, one can solve (5) or (12) explicitly for the purchasing power of a currency that does not circulate internationally in either the $\lambda = (0,0)$ or $\lambda = (1,0)$ regime,

$$q_{ii} = \left[ \frac{\alpha_{ii}m_{i0}}{r + \alpha_{ii}m_{i0}} \right]^2.$$  

The incentive condition for currency $i$ to not circulate internationally reduces to

$$\frac{\alpha_{ij}}{\alpha_{ii}} \leq \left[ \left( \frac{r}{\alpha_{jj}m_{j0}} \right)^2 + 3 \left( \frac{r}{\alpha_{jj}m_{j0}} \right) + 3 \right]^{-1}.$$  

Hence, currency $i$ will not circulate internationally if and only if $\alpha_{ij}/\alpha_{ii}$ is below some cutoff that is decreasing in $r/\alpha_{jj}m_{j0}$.\(^\text{6}\)

Even with this utility function it is not possible to solve explicitly for the purchasing power of a currency that circulates internationally in either the

\(^6\)This implies the following somewhat counterintuitive result: increasing $M_j$ makes it more difficult to sustain an equilibrium where money $j$ does not circulate internationally. We will argue below, however, that this is an artifact of the simplifying assumptions that buyers make "take it or leave it" offers and that there are no direct barter trades.
\( \lambda = (1, 1) \) or \( \lambda = (1, 0) \) regime. Neither do the incentive conditions reduce to anything manageable. Hence, we illustrate the regions of parameter space where the different equilibria exist based on some numerical calculations. The baseline parameters are \( \alpha_{11} = \alpha_{22} = 1, \alpha_{12} = \alpha_{21} = 0.15, r = 0.1, \gamma = 0.02, \text{ and } M_1 = M_2 = 0.75, \) from which we vary the values and check when each of the regimes constitutes an equilibrium. We explicitly consider both of the cases \( \lambda = (1, 0) \) and \( \lambda = (0, 1) \), even though they are mirror images for symmetric values of \( \alpha_{ij} \) and \( M_i \), because we want to consider some nonsymmetric values of \( \alpha_{ij} \) and \( M_i \).

In Figure 2a we examine different values of \( M_1 \) and \( M_2 \) holding the other parameters at their baseline values. If a region in the figure contains the label \( \lambda = (\lambda_1, \lambda_2) \), then equilibrium \( (\lambda_1, \lambda_2) \) exists in this region. Notice that equilibrium \( \lambda = (1, 1) \) exists for all \( M_1 \) and \( M_2 \) in this example. Also, since \( \alpha_{ij} \) is relatively small, equilibrium \( \lambda = (0, 0) \) exists for all but very high values of \( M_1 \) and \( M_2 \). Equilibria where only one money circulates internationally exist only if the other currency is not too abundant; e.g., \( \lambda = (1, 0) \) exists only if \( M_1 \) is not too big.\(^9\)

In Figure 2b we examine variation in \( \alpha_{11} \) and \( \alpha_{12} \). Now equilibrium \( \lambda = (1, 1) \) exists in only a small region of the figure. As we know from the above analysis, equilibrium \( \lambda = (0, 0) \) exists if and only if \( \alpha_{12} \) is below some

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\(^9\)Intuitively, Americans holding dollars buy from Mexican sellers only if the latter value dollars highly, which requires that they have ample opportunity to spend them, which requires that the ratio of buyers to sellers in Mexico not be too high. Hence, the peso supply cannot be too high if dollars are to circulate in Mexico. This result may seem somewhat counterintuitive, but the comments in the previous footnote also apply here.
cutoff. Also notice that as either \( \alpha_{11} \) or \( \alpha_{12} \) increases it becomes more likely that equilibrium \( \lambda = (1, 0) \) exists. This is because as country 1 increases either its internal economic activity or its openness, sellers in country 1 value money more highly, and this makes country 2 buyers more willing to deal with them.

3.5 A Digression

One possibly troublesome feature of the discussion in the last subsection is that a large value of \( M_i \) can prevent the existence of an equilibrium where currency \( i \) circulates only at home. In other words, by printing more money a government increases the likelihood that its currency becomes international. This is counterintuitive because one might expect that printing more of a money should decrease its realm of circulation (perhaps because it decreases its value). We now argue that the result is an artifact of our simplifying assumptions that there is no direct barter and that buyers get to make "take it or leave it" offers.

An implication of these two assumptions is that buyers extract all of the surplus from sellers in every transaction; consequently, in equilibrium a seller's value function is \( V_{i0} = 0 \). This makes the analysis more tractable. However, it also implies that no matter how low the purchasing power of a currency there is always some amount a seller is willing to give for it, and so exchange is conditioned only on the incentives of the buyer. For example, Mexicans are willing to use pesos to buy from Americans when the prospects of trade in Mexico are sufficiently bad, and by construction American sellers
always take pesos at the terms implied by a "take it or leave it" offer.

Changing the assumption that sellers have no bargaining power or the assumption that sellers cannot barter directly means that their payoffs will be positive in equilibrium, and this puts additional incentive constraints on trade. For instance, if barter is possible, a seller may prefer to hold out for a direct exchange rather than trade at any terms to which a buyer with a particular currency would agree. Thus, if pesos are too abundant then trading with pesos is very unattractive, and the most an American seller would give for a peso is less than the value a Mexican gets by keeping it. Generally, a currency can circulate locally but not internationally as long as it is either sufficiently abundant, so that foreign sellers do not want it, or sufficiently scarce, so that local buyers prefer to spend it locally.

To pursue this, we can consider a more general production-consumption specification where now there is some probability of a double coincidence of wants, so that it is possible that two sellers sometimes meet and barter directly. Conditional on meeting someone who produces a good you consume, let $x$ be the probability that he also consumes a good you produce. For a seller from country $i$, then, the arrival rate of barter transactions is $x(\alpha_{i1}m_{10} + \alpha_{i2}m_{20})$. Also, although this is not at all essential for the results, suppose for the sake of illustration that when barter occurs both agents produce the same quantity $q^* \in (0, \bar{q})$.

---

10If we adopt the symmetric Nash bargaining solution with zero threat points, for example, then $q^*$ solves $u'(q) = c'(q)$, which means that the gains from trade for both agents are maximized and that $u(q^*) - c(q^*) > 0$. All that matters, however, is that sellers get a strictly positive surplus with strictly positive probability.
As always, a "take it or leave it" offer from a buyer with currency $j$ to a seller from country $i$ satisfies $q_{ij} = V_{ij} - V_{i0}$, but now we cannot say that $V_{i0} = 0$. In fact, the value function for a seller from country $i$ satisfies

$$rV_{i0} = (\alpha_{h1}m_{10} + \alpha_{h2}m_{20})[u(q^*) - q^*].$$

Hence, $V_{i0} = V_{i0}^* > 0$, where (in any given regime) $V_{i0}^*$ depends only on exogenous variables. The formulae for the other value functions, the steady state conditions, and the incentive conditions are unchanged by the introduction of barter.

It is worth pointing out that there is typically not a unique equilibrium of each type in the model with barter, and that sometimes monetary equilibria do not exist at all. To see this, consider regime $\lambda = (0,0)$. Then $V_{ii}$ satisfies

$$rV_{ii} = \alpha_{ii}[u(V_{ii} - V_{i0}^*) + V_{i0}^* - V_{ii}].$$

It is easy to show that if (the exogenous) $V_{i0}^*$ is too big then there is no positive solution to this equation, and that if $V_{i0}^*$ is not too big then there are exactly two positive solutions. Hence, there are two distinct equilibrium values of $q_{ii}$. Moreover, one can easily show that the equilibrium with the higher $q_{ii}$ is Pareto superior. 11

11As $V_{i0}^* \to 0$ the lower equilibrium value of $q_{ii}$ converges to 0, which is why there is a unique equilibrium value of money without barter. The multiplicity of monetary equilibria in the one-country, one-currency version of the model with barter is discussed in Shi (1995) and Trejos and Wright (1995). Wallace et al. (1995) discuss a different type of multiplicity in a one-country, two-currency version of this model, which arises even without barter because agents with different currencies trade; this cannot happen here.
We now return to the conditions for a currency to circulate at home but not internationally. Money 1, for example, will not circulate abroad as long as either $V_{11} > u(q_{21})$ (Mexicans prefer to keep their pesos rather than buy from Americans) or $V_{20} > V_{21}$ (American sellers will not take pesos). We found by studying examples that the first condition holds if $M_1$ is sufficiently low and the second holds if $M_1$ is sufficiently high. In fact, in the model with barter, if $M_1$ gets too high then not only do country 2 sellers reject money 1, so do country 1 sellers (even Mexicans prefer to barter rather than to accept pesos).

The essential point is this: when $V_{10} > 0$ one cannot conclude that a country can get its currency to circulate internationally by printing lots of it. In this sense, the model with barter or with a more general bargaining solution, may seem preferable. Nonetheless, we return in the next section to the basic model, with no barter and with “take it or leave it” offers, because of its relative simplicity and because the main results we wish to emphasize there do not depend critically on these assumptions.

4 Policy

In this section, we first analyze the effects of exogenous changes in the money supplies, as measured by $(M_1, M_2)$, on prices, seignorage and welfare. We then endogenize $(M_1, M_2)$ by modeling the objective functions for the two governments and the rules for their strategic interaction. Since for some questions analytical results would be difficult, at best, we also analyze numerically the example discussed in Section 3.4. The features that we highlight
are ones that we can either prove analytically or those that seem to be robust to alternative parameterizations in numerical examples.

4.1 Exogenous Policy Changes

Figure 3 depicts prices in country 1, as measured (inversely) by $q_{1j}$, as functions of $M_1$ and $M_2$ in each of the different regimes. The curve labeled $q_{ij}(\lambda_1, \lambda_2)$ depicts $q_{ij}$ in regime $(\lambda_1, \lambda_2)$\footnote{Although we only explicitly represent prices in country 1, the behavior of prices in country 2 can be inferred because both regimes $\lambda = (1, 0)$ and $\lambda = (0, 1)$ are shown and the $\alpha_{ij}$ are symmetric.}. Note that the curves are drawn only for values of $M_j$ for which the regime in question is an equilibrium. Several things that we already know from Proposition 1 are illustrated here, such as the result that all $q_{ij}$ decrease and therefore all prices increase with either money supply, except in the $\lambda = (0, 0)$ regime. Also, $q_{12}$ is necessarily bigger than $q_{11}$ in regime $\lambda = (1, 0)$. Also, the purchasing power of currency 1 is higher when it circulates internationally, and the purchasing power of currency 1 is lower when currency 2 circulates internationally.

Some interesting results that have not been proved analytically for the general case but can be seen in the example are the following: First, prices are more sensitive to local than foreign currency changes; this makes sense intuitively because governments introduce new money by spending it on their own citizens, which implies that in every regime the $m$ distribution is skewed toward country $i$ agents holding more of currency $i$. Second, when money 1 circulates internationally its purchasing power is less responsive to $M_1$;
intuitively, this is because country 1 is able to “export inflation” when its currency is accepted abroad.

Figure 4 depicts the purchasing power parity exchange rate, $e = q_{11}/q_{22}$, in each of the regimes as a function of $M_1$. One can see that, even though both currencies lose value as $M_1$ increases, currency 1 loses value faster so that $e$ decreases with $M_1$. This effect is particularly strong in regime $\lambda = (0, 0)$, where country 1 is not able to “export inflation” because its currency is not accepted in country 2, and particularly weak in regime $\lambda = (1, 1)$, where it is easiest to “export inflation” abroad.

We are also interested in the way that seignorage depends on $M_1$ and $M_2$. Since government $i$ is assumed to purchase goods from a fraction $M_i$ of its newborn citizens, per capita seignorage revenue in real terms is given by $S_i = q_i M_i \gamma$. In Figure 5, the curve labeled $S_1(\lambda_1, \lambda_2)$ depicts $S_1$ in regime $(\lambda_1, \lambda_2)$. Notice that $S_1$ first increases and then decreases with $M_1$. Also, given that multiple equilibria exist, $S_1$ is greater when currency 1 is international than when it is not, and lower when currency 2 is international than when it is not. Finally, given $M_2$, $S_1$ is maximized at different levels of $M_1$ in the different regimes, depending on the extent to which country 1 is able to “export inflation” abroad.

We are also interested in welfare, which we define as the average (steady state) utility of private citizens:

$$W_i = m_{i0} V_{i0} + m_{i1} V_{i1} + m_{i2} V_{i2}.$$ 

Figure 6 shows welfare in country 1 in regime $(\lambda_1, \lambda_2)$, denoted $W_1(\lambda_1, \lambda_2)$, as a function of $M_1$ and $M_2$. Notice that, for given values of $M_1$ and $M_2$, it is
not unambiguous which regime is best. For instance, if $M_1$ is very low then $W_1$ is highest in regime $\lambda = (1, 0)$, where currency 2 is accepted in country 1 but not vice-versa, because this makes trade easier within the host country. Also notice that $\lambda = (1, 1)$ may or may not dominate $\lambda = (0, 0)$; for instance, if $M_2$ is very low or very high then $W_1$ is highest in regime $\lambda = (0, 0)$. Finally, we point out that, although it may not be obvious from Figures 5 and 6, the value of $M_1$ that maximizes $W_1$ is lower than the value that maximizes $S_1$ taking as given $M_2$.

4.2 Endogenous Policy

The next step is to endogenize $(M_1, M_2)$. Throughout this section we assume that governments take the regime as given, and restrict their choices to policies that allow for the existence of that regime as an equilibrium (i.e., we and ignore policies aimed at changing a currency’s realm of circulation). We also restrict attention to policies that do not change over time, and to steady state comparisons.

One thing we do is look for Nash equilibria when each government seeks to maximize the welfare of its own citizens; we denote this policy pair by $(M_1^W, M_2^W)$.\footnote{In other words, $W_1$ is maximized at $M_1^W$ when $M_2^W$ is taken as given, and vice-versa. Note that in the cases we considered the Nash equilibrium is always unique.} We also look for Nash equilibria when each government seeks to maximize seignorage; we denote this by $(M_1^S, M_2^S)$. Given that they seek to maximize seignorage, we also consider the possibility of international policy coordination, by letting the governments choose policies jointly. One way
to do this is to assume that seignorage is freely transferable across countries, in which case they maximize \( S_1 + S_2 \); we denote this outcome by \((M^T_1, M^T_2)\).

Or, we can assume seignorage is nontransferable, in which case we use Nash's cooperative bargaining solution with threat points given by the noncooperative solution; we denote this by \((M^N_1, M^N_2)\).\(^{14}\)

One result is that if foreign money circulates in country \(i\), independently of whether money \(i\) circulates abroad, then the welfare maximizing level of \(M_i\) is lower than the level that maximizes seignorage. However, when foreign money does not circulate in country \(i\), welfare and seignorage are maximized at the same value of \(M_i\).\(^{15}\) Another result is that, starting from the Nash equilibrium where governments maximize seignorage, reducing the amount of money in both countries increases welfare in both countries. Furthermore, and more interestingly, reducing the amount of money in both countries increases seignorage for both governments. As neither government takes into account the effect it has on the other, the noncooperative equilibrium is characterized by too much money. The cooperative equilibrium \((M^N_1, M^N_2)\) involves less money and more seignorage.

The above results hold in general. We now describe the results from some numerical examples. We depict the frontier and the outcomes in \((S_1, S_2)\) space assuming the regime is \(\lambda = (1, 0)\) in Figure 7a and \(\lambda = (1, 1)\) in

\(^{14}\)In particular, \((M^N_1, M^N_2)\) solves \(\max[S_1 - S^S_1][S_2 - S^S_2]\), where \(S^S_j\) is seignorage in country \(j\) when policy is given by \((M^S_1, M^S_2)\).

\(^{15}\)This can be explained as follows. Welfare in country 1, for example, is \(W_1 = m_{11}V_{11} + m_{12}V_{12}\), while seignorage is \(S_1 = \gamma M_1 V_{11}\). When no foreign money is held at home \((m_{12} = 0)\), \(m_{11}\) is proportional to \(M_1\) and maximizing seignorage is the same as maximizing welfare.
Figure 7b (the regime with no international currency is uninteresting for this exercise). The points labeled $W$, $S$, $T$ and $N$ are the payoffs in the four scenarios: the noncooperative equilibrium between welfare maximizers; the noncooperative equilibrium between seigniorage maximizers; the cooperative solution when revenue is transferable; and the cooperative solution when revenue is nontransferable. In the transferable revenue case, the point $T$ depicts the seignorage raised in each country, and not the final division of the revenue between the governments. Figures 8a and 8b show the same points in $(W_1, W_2)$ space. Since the graphs are drawn using the same scales, one thing they illustrate is how the possible values of seigniorage and welfare vary across regimes.

The results we wish to highlight are as follows. First, while the cooperative solutions are on the frontier in $(S_1, S_2)$ space, the noncooperative solutions are inside the frontier. This is especially so in regime $\lambda = (1, 1)$, where it is easiest to "export inflation." Also, notice that in regime $\lambda = (1, 0)$, when transfers are allowed the cooperative solution is to concentrate seigniorage collection in country 2, where it can be done more efficiently; that is, the governments print more of the international currency and less of the national currency, in addition to printing less money in total.

Figure 8 shows how the possible value of $(W_1, W_2)$ are much higher in regime $\lambda = (1, 1)$. Also, note that each of the outcomes is inside the frontier. Also, when both governments try to maximize seignorage, one of them may actually end up with less seignorage than when both are trying to maximize welfare. And, symmetrically, citizens in one country may be worse off in terms of welfare when both governments are trying to maximize welfare than
when both governments are concerned with seigniorage, as can be seen in the figure for country 1 in regime $\lambda = (1,0)$. Finally, in regime $\lambda = (1,0)$, the country that issues the international money is better off.

5 Final Remarks

This paper has developed an extended version of the international currency model in Matsuyama, Kiyotaki and Matsui (1993). The key extensions are to endogenize prices and exchange rates, which the earlier model took to be fixed, and to introduce governments and policy considerations. Some of the findings with fixed policies include the following: other things being equal, an international currency has a higher purchasing power than a national one; a money purchases more at home than abroad; and monetary expansion in one country produces inflationary pressure in the other. When governments choose policy endogenously in a noncooperative fashion, we found a tendency to print too much money, especially in the regime where both currencies circulate internationally. There is, then, a role for cooperation between governments: by coordinating policies they can increase seignorage and/or improve welfare. When governments cooperate to maximize seignorage, they issue more of an international currency and less of a national one. Finally, with a single international money, the country that issues it is unambiguously better off than the other country.

We think that models of this class will eventually prove useful for studying important issues such as European monetary union. The analysis does illustrate the potential welfare gains from having one currency (or two cur-
rencies that serve as perfect substitutes); but it also indicates that there can be a welfare loss in one country from having a unified currency if another country is pursuing a particularly bad policy from the former's point of view. It may be interesting to pursue versions of the model where the two countries have different preferences or production technologies, or are subject to different shocks, in order to analyze the tradeoffs between having one currency (and/or one central bank), which may help to facilitate trade, and having independent monetary policies tailored to conditions in the individual economies. We leave this to future research.
Appendices

A. Proof of Lemma 1

First note that $\Lambda(i, j, j) = 0$ implies that in steady state $m_{ij} = 0$, since the only way for agent $j$ to acquire currency $i$ is to get it from agent $i$. But if $m_{ij} = 0$ then obviously $\Lambda(i, i, j) = 0$. Now we claim that $\Lambda(i, j, j) = 1$ implies $\Lambda(i, i, j) = 1$. Suppose not, and with loss of generality set $i = 1$ and $j = 2$. Then we have $\Lambda(1, 2, 2) = 1$ and $\Lambda(1, 1, 2) = 0$. Now $\Lambda(1, 2, 2) = 1$ means $u(V_{12}) \geq V_{22}$. And $\Lambda(1, 1, 2) = 0$ means either $V_{12} > u(V_{12})$ or $m_{12} = 0$, but since $\Lambda(1, 2, 2) = 1$ implies $m_{12} > 0$, we must have $V_{12} > u(V_{12})$. Hence, $V_{12} > V_{22}$. We will show this leads to a contradiction.

The value functions in general can be written

$$rV_{22} = \alpha_{22}m_{20}\Lambda(2, 2, 2)[u(V_{22}) - V_{22}] + \alpha_{21}m_{10}\Lambda(1, 2, 2)[u(V_{12}) - V_{22}]$$

$$rV_{12} = \alpha_{11}m_{10}\Lambda(1, 1, 2)[u(V_{12}) - V_{12}] + \alpha_{12}m_{20}\Lambda(2, 1, 2)[u(V_{22}) - V_{12}]$$

The $\Lambda(h, i, j)$ terms appear in these expressions, but not in (3), because in writing (3) we used Lemma 1, which is what we are now trying to prove. Inserting $\Lambda(2, 2, 2) = 1$ (which always holds) and $\Lambda(1, 2, 2) = 1$ and $\Lambda(1, 1, 2) = 0$ (which hold by supposition) and then subtracting the value functions, we have

$$r(V_{22} - V_{12}) = \alpha_{22}m_{20}[u(V_{22}) - V_{22}] + \alpha_{21}m_{10}[u(V_{12}) - V_{22}]$$

$$-\alpha_{12}m_{20}\Lambda(2, 1, 2)[u(V_{22}) - V_{12}]$$

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If $A(2, 1, 2) = 0$, then $V_{22} \geq V_{12}$ and we have our contradiction. If $A(2, 1, 2) = 1$, then use the assumption $\alpha_{22} \geq \alpha_{12}$ and simplify to arrive at

$$(r + \alpha_{12}m_{20})(V_{22} - V_{12}) \geq \alpha_{21}m_{10}[u(V_{12}) - V_{22}] \geq 0.$$ Again we have our contradiction. $\square$

**B. Proof that $\partial Q_i/\partial M_j \leq 0$**

Differentiating (9) and simplifying, we get

\[
\begin{bmatrix}
A_{11} & -\alpha_{12}m_{20}u'_1 \\
-\alpha_{21}m_{10}u'_1 & A_{22}
\end{bmatrix}
\begin{bmatrix}
dQ_1 \\
dQ_2
\end{bmatrix}
= \begin{bmatrix}
\alpha_{11}(u_1 - Q_1) \\
\alpha_{21}(u_1 - Q_2)
\end{bmatrix}
dm_{10},
\]

where $u_i = u(Q_i)$ and $A_{ii} = \alpha_{ii}m_{10}(\frac{w_i}{Q_i} - u'_i) + \alpha_{ij}m_{20}\frac{w_j}{Q_j}$. The determinant of the square matrix is

\[
\Delta = \alpha_{11}\alpha_{22}m_{10}m_{20}(\frac{w_1}{Q_1} - u'_1)(\frac{w_2}{Q_2} - u'_2) + \alpha_{11}\alpha_{21}m_{10}^2(\frac{w_1}{Q_1} - u'_1)\frac{w_2}{Q_2}
\]

\[+ \alpha_{22}\alpha_{12}m_{20}^2(\frac{w_2}{Q_2} - u'_2)\frac{w_1}{Q_1} + \alpha_{12}\alpha_{21}m_{10}m_{20}(\frac{w_1}{Q_1} - u'_1)\frac{w_2}{Q_2} - u'_1u'_2),
\]

which is positive since $\frac{w_i}{Q_i} > u'_i$ by concavity. Hence,

\[
\Delta \frac{\partial Q_i}{\partial m_{10}} = \alpha_{11}\alpha_{22}m_{20}(u_1 - Q_1)(\frac{w_2}{Q_2} - u'_2) + \alpha_{11}\alpha_{21}m_{10}(u_1 - Q_1)\frac{w_2}{Q_2}
\]

\[+ \alpha_{21}\alpha_{12}m_{20}(u_1 - Q_2)u'_2 \geq 0
\]

\[
\Delta \frac{\partial Q_i}{\partial m_{10}} = \alpha_{11}\alpha_{21}m_{10}(u_1 - Q_2)(\frac{w_1}{Q_1} - u'_1) + \alpha_{12}\alpha_{21}m_{20}(u_1 - Q_2)\frac{w_1}{Q_1}
\]

\[+ \alpha_{11}\alpha_{21}m_{10}(u_1 - Q_1)u'_1 \geq 0
\]

since $u_i \geq Q_j$, $i, j = 1, 2$, by the incentive conditions. We conclude that $Q_1$ and $Q_2$ are both increasing in $m_{10}$. Similarly, $Q_1$ and $Q_2$ are both increasing in $m_{20}$. Since $m_{10}$ and $m_{20}$ are decreasing in $M_1$ and $M_2$, $Q_1$ and $Q_2$ are decreasing in $M_1$ and $M_2$. $\square$
References


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