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**“VALUATION EQUILIBRIUM
WITH CRUBS”**

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ABSTRACT

This paper considers model worlds in which there are a continuum of individuals who form finite sized associations to undertake joint activities. We show that if there are a finite set of types and the commodity space contains lotteries, then the classical equilibrium results on convex economies can be reinterpreted to apply. Furthermore, in this lottery economy deterministic allocations (that is, degenerate lotteries) are generally not Pareto optimal, nor are they equilibria. In the interests of making the model seem more "natural," we show that the set of equilibria in a decentralization in which individuals first gamble over value transfers and then trade commodities in a deterministic competitive market economy are equivalent to those of our competitive economy with a lottery commodity space.

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Introduction

Many human interactions in which economists are interested involve agents voluntarily forming associations in order to undertake joint activities. Important examples of such organizations are households and firms. Other examples include organizations formed to provide some set of local public goods, such as a condominium association. There is an extensive literature on matching models which analyzes the equilibria of job-worker or husband-wife matching models, as well as an extensive literature on associations which provide local public goods, commonly referred to as clubs.¹ In these literatures, cooperative and noncooperative game theoretic tools have been the predominant mode of analysis. In this paper we show that general equilibrium analysis with linear price systems can be extended to quite general environments in which associations are being endogenously formed, and hence that all the standard competitive results apply.

The associations in our model can undertake a broad range of possible activities, encompassing both consumption and production activities. The technology available to these associations can depend on the characteristics of its membership and can deal with the jointness of activities within the association. We do require that any possible association be small relative to the large number of people in the economy, however. We will refer to these associations as clubs because they fall within the spirit of the club literature. We use the term *club type* to refer not only to the relevant characteristics of the membership, but also to the nature of the economic activities undertaken.

There are some complications with this approach, though. One complication is that the set of possible club types is large and membership is discrete. We deal with the first problem by

¹For the cooperative and noncooperative literatures on matching models standard references are Roth and Sotomayor (1990) and Osborne and Rubinstein (1990), respectively. In the club literature, the standard model is one in which there is an excludable local public good to be consumed with congestion and a private consumption good. Standard references for this literature are Cornes and Sandler (1986) or Sandler and Tschirhart (1980).

considering each separate club type as a separate commodity, with, possibly, a continuum of differentiated commodities. To deal with the problem of discrete membership, we follow Mas-Colell (1975), Hart (1979), and Jones (1984) in using the space of signed measures as our commodity space. Another complication with our environment is that in club environments where there is a deterministic commodity space and people maximize expected utility, there typically exist gains from trade in randomized contracts, due to the indivisibility in agents' consumption sets. In order to deal with this problem, we follow Prescott and Townsend (1984a, 1984b) and Rogerson (1984, 1988) and consider consumption possibility sets that consist of probability distributions over final consumption bundles.

Our choice of commodity space greatly simplifies a competitive analysis of our environment. While agents' possible choices over clubs are not convex, their possible choices over random contracts that specify the probability of membership are. Since the membership in any one club is small relative to the size of the population, the aggregate production technology is a convex cone. Additionally, since we will restrict ourselves to economies with a continuum of agents, but only a finite number of types, the convexity of our economies permits us to consider only type-identical allocations: Each agent of a given type chooses the same point in the commodity space. As a result, virtually all the classical general equilibrium results for convex economies with a finite set of agents can be extended to our club economies. The standard results on the equivalence of core and competitive allocations with a continuum of agents can also be shown to hold.

The previous literature on club economies has considered the same questions that we address here, but with some important differences.² Wooders (1978, 1980, 1989) analyzes club economies

²There is also literature on clubs in which it is assumed that the club technology does not depend on the characteristics of the members of the club. (See Scotchmer and Wooders (1987).) This is commonly referred to in the club literature as the case of *anonymous crowding*. Unlike this literature, we will use the term *anonymous* to refer to situations in which the price system is type—

with standard competitive prices for private goods and Lindahl, or person-specific, prices for club goods and shows that in large economies the core is not empty, and the set of core and the “market” equilibrium allocations are equivalent. We think that there is a problem with market equilibrium with individual-specific prices. The assumption of price taking is just not a reasonable one when prices are personalized. (See Chari and Jones (1991).) Scotchmer (1993) shows the equivalence of equal-utility core allocations (where all agents of the same type receive equal utility levels in the core allocation) and market allocations with type-specific prices. Even the work on the relationship between equal utility core allocations and competitive equilibria suffers from the deficiency that these allocations are often not core or equilibrium allocations when agents have access to a randomizing device.

Our work improves on this earlier literature in the following respects: First, our switch to a lottery commodity space allows us to model the outcomes that will arise if people have access to a randomizing device. Second, with our commodity space all core allocations are equal utility allocations. Third, the issue of the existence of a competitive equilibrium with a type-specific price system depends only on the standard conditions required in the classical existence proofs. Indeed our structure is in a sense just the standard Arrow-Debreu structure.³ And finally, the set of competitive equilibrium allocations and core allocations are equivalent if these classical conditions are satisfied.

We should also mention the related work of Cole, Mailath, and Postlewaite (1992, 1994). The example that we consider in the first section of this paper is related to the model in their more

but *not individual*—specific. Additionally, Scotchmer (1985) has looked at decentralizing club economies in which the owners of the club have market power due to entry barriers.

³With Arrow-Debreu, commodities are indexed by events, and these events can be random. Thus the lottery equivalent exists in the Arrow-Debreu world, provided there is some random variable with a continuous density.

recent paper, which they analyzed from both a cooperative and noncooperative perspective. In their two period model, matching takes place in the second period and agents make an investment decision in the first period which increases the set of second period payoffs to the pair. Because there are close substitutes with regard to matching, the strong presumption of inefficiency with regard to investment and resource division with the pair is removed if agents can commit when they match over resource division within the pair. This so-called “weak” commitment technology is shown to be equivalent to a “strong” commitment technology in which agents can commit prior to making their investment decisions. They also show that these core allocations are decentralizable as market outcomes for the specific example they consider.

The remainder of the paper is divided into two sections. In the first we analyze a simple two sex matching model in order to illustrate how to represent the model environment as a economy in the classical general equilibrium sense—that is, one with no externalities. We show that a commodity space that allows for trade in randomized contracts can and typically does lead to substantially different equilibrium allocations relative to the set of equilibria in which the commodity space only allows for deterministic exchanges. In the second we consider a much more general matching model and show how the standard results in general equilibrium theory can be readily extended. Since one seems to observe a great preponderance of methods of gambling over wealth transfers relative to lotteries over final consumption allocations, we were motivated in this section to also consider an alternative decentralization of our environment. In this alternative decentralization there are gambles over value transfers in the first stage and a standard competitive economy in the second, where all the trades are deterministic exchanges. We show that the set of equilibrium allocations for the two decentralizations are equivalent.

1. A Prototype Environment With Clubs

We begin by considering a simple model of household formation. We then show, through a suitable definition of the commodity space, how this structure can be studied using valuation equilibrium theory with its single budget constraint and linear valuation or pricing function. We use this example to explain why within matching models a commodity space which includes randomized allocations is appropriate. This is done by considering a simple example and showing that under a deterministic decentralization there are mutually advantageous gains from entering into trade in randomized contracts. We then formally present a valuation equilibrium model of this simple matching environment, and show that the welfare theorems apply and that a competitive equilibrium exists when the commodity space allows for trade in randomized contracts. We then develop the implications of the first welfare theorem and the connection between planner's problems and equilibrium allocations in convex economies to develop an algorithm for computing all competitive equilibria of the prototype economy. Finally, we show that the properties of this equilibrium allocation differ substantially from those in an economy without random contracts.

The Environment

There is a continuum of people. They are of two types, which we will refer to as male or female, and which we will denote by $s \in \{f, m\}$. When referring to an agent of type s , we will use s' to denote the agent's opposite sex. The measure of persons having type s is given by $\lambda_s > 0$. There are two types of goods in the model: a personal consumption good and a club consumption good. A matched pair of the opposite sex can consume both a pair-specific public good or club good, and individually, the personal consumption good. People can be matched with just one person of the opposite sex. The unmatched can consume only the personal consumption good. People are endowed with the personal consumption good, and ω_s denotes his or her endowment level. We

assume that there is a production technology for converting one unit of the personal consumption good into half a unit of the club good.

The underlying consumption possibility set is $A \times B$, where A is the set of feasible personal consumptions and B the set of feasible pair or club consumptions. In this section, A will be $\mathbb{R}_+$ and B will be either the interval $[0, b_{\max}]$ or a finite subset of this interval that includes zero. Preferences are identical for all people and are given by

$$(1) \quad E\{u(a) + v(b)\}$$

where u and v are defined on the positive real numbers and are assumed to be strictly concave, differentiable functions. Further, $u'(0) = \infty$, v bounded, $v(0) = 0$, and $v'(0)$ finite.

Valuation Equilibrium Without Random Contracts

To apply valuation equilibrium analysis, the balance condition for feasibility must depend only on the sum of the individuals' commodity vectors. Feasibility requires that the number of males consuming any given nonzero level of the club good must be equal to the number of females consuming that level. Thus, we index the club good by sex in order to distinguish between the sex of the person consuming a particular level of the club good. Further, since this equality must hold for every level of the club good, we treat every level of b for each sex as a different commodity. These considerations, following Mas-Colell (1975), lead us to use the commodity space $L = \mathbb{R} \times M(B) \times M(B)$, where $M(B)$ denotes the set of signed measures on the Borel sigma algebra of B . An element of L is the triplet (x_0, x_f, x_m) . The element x_0 is the personal consumption good while the elements x_s are signed measures of consumptions of the club goods of which there can be a continuum. In the case that the set of club consumptions is finite, the x_s are finite dimensional vectors.

With this commodity space there is a fundamental indivisibility in the consumption possibility set. An individual can consume one and only one level of the club good. A person of type s has the consumption possibility set

$$(2) \quad X_s = \{x \in L_+ : x_s(\cdot) \in \{0,1\}, x_{s'} = 0\},$$

where s' denotes the opposite sex, utility function

$$(3) \quad U_s(x) = u(x_0) + \int v(b)dx_s,$$

and endowment $e^s = (\omega_s, 0, 0) \in L$.

Our numeraire good is the personal consumption good. A *pricing system* p is a pair of bounded measurable functions p_f and p_m which map from B to $\mathbb{R}$, where $p \cdot x = x_0 + \int p_f(b)x_f(db) + \int p_m(b)x_m(db)$. A type s 's budget constraint under this pricing system is $p \cdot x \leq p \cdot e^s$. Clearly given our linear production technology if there are to be zero profits in equilibrium $p_f(b) + p_m(b) \leq 2b$, and this condition must hold with equality at nonzero club consumption levels which are supplied in strictly positive amounts (or more precisely for b in the support of any x_s).

Given the nature of the consumption possibility sets, all x belonging to these sets can be characterized by a pair of real numbers (a,b) where a is personal consumption and b is club consumption. An allocation specifies the distributions of the (a,b) for each type. As we are not keeping track of names, a deterministic allocation will be a set of measures which give the distribution of consumption choices of each type.

An equilibrium consists of a price system and a male and a female distribution of consumption choices which satisfy the following conditions: (i) the total measure of the distribution of consumption choices for each sex s is λ_s , (ii) the measure of males and females consuming any nonzero

level of club consumption are equal,⁴ (iii) total personal consumption plus two times the total consumption of b over all clubs must be less than or equal to the total endowment, and (iv) the supports of the distributions of consumption choices must consist only of points which are in $\text{argmax}\{u(a) + v(b) \mid a + p_s(b) \leq e^s\}$, and (v) $p_f(b) + p_m(b) \leq 2b$; with equality if club level b is consumed in strictly positive amounts in equilibrium.

Need for Random Contracts

To see that a deterministic competitive equilibrium is an equilibrium only if people *do not* have access to a randomization mechanism, consider the case where there are more males than females, that is, $\lambda_m > \lambda_f$. Thus males will be the marginal type, since not all males can be matched. Therefore a deterministic equilibrium allocation, when one exists, maximizes the utility of females subject to the resource constraint and the constraint that a male's utility in a match is at least as high as it would be if a male consumed his endowment. Let a_f^* , a_m^* , and b^* be the solution to the program

$$\max\{u(a_f) + v(b)\}$$

subject to

$$(4) \quad a_f + a_m + b \leq \omega_f + \omega_m$$

$$(5) \quad u(a_m) + v(b) \geq u(\omega_m)$$

$$(6) \quad a_f, a_m, b \geq 0.$$

This program has a unique solution which determines the core allocation for this economy. A deterministic competitive equilibrium exists if and only if constraint (5) is binding at the optimum. Constraint (5) will be binding at the optimum if and only if ω_f is below some critical value, which,

⁴To be exact, the measure of males and females with club consumption levels which lie in any subset of B not containing 0 must be equal.

depending on u and v , could be plus infinity. If this constraint is binding, the equilibrium distribution of consumption choices for females has measure λ_f on the point (a_f^*, b^*) and zero measure elsewhere. The equilibrium distribution of consumption choices for males has measure λ_f on the point set $\{(a_m^*, b^*)\}$, measure $\lambda_m - \lambda_f$ on the point set $\{(\omega_m, 0)\}$, and zero measure elsewhere. The equilibrium prices of b^* are $p_m(b^*) = v(b^*)/u'(a_m^*)$ and $p_f(b^*) = 2b^* - p_m(b^*)$. For other b commodities for which markets clear at zero, we pick the demand reservation prices for males, $p_m(b) = v(b)p_m(b^*)/v(b^*)$ and set $p_f(b) = 2b - p_m(b)$.⁵

The allocation implied by the solution to this programming problem is a Pareto optimum in a world without the technology to trade in randomized contracts. If people have access to a randomizing mechanism, however, this allocation is neither a competitive equilibrium nor a Pareto optimal allocation. We now establish this point.

To establish that this allocation is not a competitive equilibrium, we show that two males can enter into a mutually beneficial gamble. One mutually beneficial gamble is the fair one in which the winner receives $p_m(b^*)/2$ units of the consumption good from the loser. The winner and loser both have personal consumption $\omega_m - p_m(b^*)$. The loser has level 0 club consumption while the winner has level b^* club consumption. Both these males have expected utility

$$(7) \quad u(a_m^* + p_m(b^*)/2) + v(b^*)/2$$

which exceeds the utility that they would have realized if they had not entered into the gamble; that is, it exceeds $u(\omega_m) = u(a_m^*) + v(b^*)$. This last point follows from the strict concavity of the u

⁵An arbitrary matching of males and females in which each female gets matched with one and only one male, and with the allocation being that a matched pair's consumption is determined by the previous programming problem, while unmatched males simply consume their endowment, is a core allocation. In this core allocation, the welfare levels of males will differ if (5) does not bind, and hence it cannot be supported as a competitive equilibrium. This core allocation could be supported as a quasi-competitive equilibrium (see later discussion) but quasi-competitive equilibria need not be competitive equilibria even though there exists a cheaper point in each agent's budget set since the implicit consumption sets are not convex due to the discrete nature of matching.

function and the fact that $p_m(b^*) = v(b^*)/u'(c_m^*)$. *The existence of a mutually beneficial gamble establishes that the deterministic competitive equilibrium is not an equilibrium if people have access to a randomizing mechanism.*⁶

To show that the deterministic competitive equilibrium is not a Pareto optimum we extend this argument to show that the welfare level of a positive measure of males can be improved relative to the deterministic equilibrium allocation, while leaving the utility of other people unchanged. To do this, simply take an equal positive measure of matched and unmatched males and repeat the above argument leaving others with the same commodity vectors. This establishes the existence of a Pareto superior feasible allocation. We now develop a competitive equilibrium model with randomized contracts and show that a competitive equilibrium always exists and that competitive equilibrium allocations are Pareto optimal.

Valuation Equilibria With Lotteries

Here we will present a formal description of our simple example in which we allow for trade in randomized contracts. We will show that both welfare theorems apply and that a competitive equilibrium always exists.

The form of the example that we will consider is slightly more general in that we will allow for a finite set of possible types of persons, indexed by $i = (s, \omega)$, where agents can differ in terms of their sex, $s \in \{m, f\}$, and their endowment, $\omega > 0$. Once again we will use s to denote an individual's own sex and s' the opposite sex.

⁶If $v(a)$ is not strictly concave beneficial gambles may not exist. In particular, if $v(a) = a$, gambles are not needed. (See Mas-Colell 1975.)

The commodity space is once again given by $L = \mathbb{R} \times M(B) \times M(B)$ with element $x = (x_0, x_f, x_m)$. The commodity space norm is the sum of the absolute value of x_0 and the total variation norms of x_f and x_m ; that is,

$$(8) \quad \|x\|_L = |x_0| + \|x_f\|_{M(B)} + \|x_m\|_{M(B)}.$$

The consumption possibility set of a type i person is

$$(9) \quad X_i = \{x \in L_+ : x_s(B) = 0, x_s(B) = 1, |x_0| \leq a_{\max}\}$$

where the number $a_{\max}$ exceeds $\sum_i \lambda_i \omega_i / \min \lambda_i$, which is an upper bound for feasible type-identical consumptions of the private good. The type i utility function $U_i: X_i \rightarrow \mathbb{R}$, is

$$(10) \quad U_i(x) = u(x_0) + \int v(b) x_s(db).$$

The endowment of a type i is $e^i = (\omega_i, 0, 0) \in L_+$. The aggregate production possibility set is

$$(11) \quad Y = \left\{ y \in L : y_f(Q) = y_m(Q) \text{ for any measurable } Q \subseteq B_{++} \text{ and } \int b y_f(db) + \int b y_m(db) + y_0 \leq 0 \right\}^7$$

The only major change from our deterministic economy is that we have replaced the restriction in the consumption sets that $x_s(\cdot) \in \{0,1\}$ by the restriction $x_s(\cdot) \in [0,1]$. The consumption sets are now convex. Here $x_s(Q)$ has the interpretation of being the probability measure that is placed on outcomes in the measurable set $Q \subset B$.

We restrict attention to allocations with the property that all people of the same type have the same commodity point. Given that preferences are convex, this is an innocuous restriction that dramatically simplifies notation.⁸ Given this restriction, the resource balance constraint is simply

⁷By measurable we mean an element of the borel sigma algebra of B which we denote by $B(B)$.

⁸More formally, we say competitive equilibria are in the same equivalence class if the p is the same, and the type-mean commodity vectors are identical for all types. Within every equivalence class of equilibria there is one type-identical equilibrium.

$$(12) \quad \sum_I \lambda_i(x^i - e^i) = y.$$

This constraint along with the requirements that $x^i \in X_i$ all i and $y \in Y$ define feasible allocations.

A *price system* for our economy is a continuous linear functional $p: L \rightarrow \mathbb{R}$ with representation

$$p \cdot x = x_0 + \sum_S \int p_s(b) x_s(db)$$

where the $p_s(\cdot)$ functions are integrable. Note x_0 is the numeraire. A *(type-identical) competitive equilibrium* for our economy is a price system p and an allocation $[y^*, \{x^{i*}\}]$ such that:

- (i) x^{i*} maximizes $U_i(x)$ subject to $x \in X_i$ and $p \cdot (x - e^i) \leq 0$,
- (ii) y^* maximizes $p \cdot y$ subject to $y \in Y$, and
- (iii) $\sum_I \lambda_i(x^{i*} - e^i) = y^*$.

PROPOSITION 1. Any competitive equilibrium allocation of this economy is a Pareto optimum.

Proof. The first welfare theorem states that competitive equilibrium allocation are Pareto optimal given local nonsatiation, which holds here because more personal consumption is always preferred to less. (See, e.g., Stokey and Lucas 1989, Theorem 15.3.) $\square$

PROPOSITION 2. Any Pareto optimum allocation with $x_0^i > 0$ for all i can be supported as a competitive equilibrium with transfers.

Proof. If B is finite, the commodity space is finite dimensional and Theorem 15.7 in Stokey and Lucas (1989) implies (type-identical) Pareto optimum can be supported as a (type-homogenous) quasi-competitive equilibrium with transfers.⁹

⁹A quasi-competitive equilibrium is the same as a competitive equilibrium except that condition (i) is replaced by (i') $U_i(x) \geq U_i(x^{i*})$ implies $p \cdot x \geq p \cdot x^{i*}$.

To establish the second welfare theorem for our economy when the set of club consumptions is not finite, we first establish the second welfare theorem for an economy with a larger production possibility set. This larger aggregate production possibility set has a nonempty interior and has the same sets of (type-identical) Pareto optima and competitive equilibrium as our economy.

Consider the economy with aggregate production possibility set

$$Y' = Y + \left\{ y \in L: y_f, y_m \geq 0, \sigma \int b \, dy_f + \sigma \int b \, dy_m + y_0 \leq 0 \right\}.$$

This aggregate technology set is the sum of Y and another technology set which permits the production of the goods $y_s(b)$ without matching. Let σ be sufficiently large that for type-identical Pareto optimal allocations this technology will not be operated. The set Y' has an interior and consequently there is a quasi-competitive equilibrium with a price system in the dual of our commodity space. Given that y^* belongs to Y , y^* maximizes value over the smaller set Y given that it maximizes value over the larger set Y' . Thus there is a quasi-competitive equilibrium for the prototype economy with a price system in the dual of our commodity space.

Not all points in this dual have the dot product representation that we require for our definition of competitive equilibrium. Given that people discount and truncation is possible (that is Assumptions 15.1–15.7, 15.8', and 15.9' in Stokey and Lucas (1989) are satisfied), a quasi-competitive equilibrium with transfers exists with a dot product representation if one exists with a price system that does not have a dot product representation.¹⁰

For any Pareto optima with $x_0^i > 0$ for all i , and its associated quasi-competitive equilibrium, there will be a cheaper point in the consumption possibility set for every type i . This condition,

¹⁰Before this theorem can be applied the X_i must be modified so that truncation is possible. This is done as follows. The point zero is removed from B , and the requirement that $x_s(B) = 1$ is replaced by the requirement that $x_s(B) \leq 1$. Note that $v(0) = 0$.

along with convexity of preferences and consumption sets and continuity of preferences, guarantees that a quasi-competitive equilibrium is a competitive equilibrium. $\square$

PROPOSITION 3. A competitive equilibrium always exists for these economies in our class.

Proof. If B is finite, and as a result the commodity space is finite dimensional, existence of an equilibrium is guaranteed by what McKenzie (1981) calls the classical theorem on the existence of competitive equilibrium. For a detailed discussion of McKenzie's conditions, see the later discussion of existence in the more general case.

For B an interval, we follow Bewley (1972) and take the limit of a sequence of equilibria for economies with finite B sets. For the n^{th} economy with a restricted commodity space the set of possible club consumptions is

$$B_n = \{0, b_{\max}/n, \dots, (n-1)b_{\max}/n, b_{\max}\}.$$

We extend the n^{th} equilibrium in the restricted commodity space to our general commodity space B as follows. We set $Y_n = Y \cap B_n$, so for the n^{th} economy only measures with support in B_n are feasible. The corresponding allocations are defined in the more general commodity space in the obvious manner; that is, by requiring that they agree on B_n and put zero weight on subsets of B that do not contain points in B_n . We extend the price system as follows, where throughout the numeraire is personal consumption which has a price of one. For the n^{th} economy, for b 's which males consume with positive probability, we use equilibrium prices for the restricted finite dimensional economy. For other b 's, we set $p_m^{(n)}(b)$ equal to the demand reservation prices of the males. We set $p_f^{(n)}(b) = 2b - p_m^{(n)}(b)$; that is, equal to the supply reservation price given $p_m^{(n)}(b)$. Given that $v'(b)$ is bounded by assumption, the slopes of the $p_m^{(n)}(b)$ functions are bounded uniformly in b and n .

For each economy in our general commodity space with restricted production set Y_n , we have constructed a competitive equilibrium. There will be a subsequence of equilibria for which the allocations converge weak star and the pricing functions p_s converge uniformly. The space of feasible allocations is closed and bounded. This implies that the set of feasible allocations is weak star compact which in turn implies the existence of a weak star convergent subsequence. The existence of a price subsequence which converges uniformly in the L_∞ norm is guaranteed given that the values and the slopes of the $p_s(b)$ functions are bounded in absolute value uniformly in b and n . The limit of the sequence of competitive equilibrium for the finite economies is an equilibrium for the limiting economy. $\square$

Properties of Pareto Optimal and Equilibrium Allocations

In the previous section we found that competitive equilibria allocations are Pareto optima. Consequently, any properties that the Pareto optima have are shared by the competitive equilibria. In this section, we establish properties of competitive equilibrium allocations by establishing the properties of the Pareto optima.

The prototype economy of the previous section is convex. Consequently, the set of Pareto optima is the closure of the set of solutions to weighted social planner's problems with positive weights. The θ weighted anonymous social planner problem, where the sum of the θ_i is one and the $\theta_i > 0$, is

$$(13) \quad \sum_i \theta_i \lambda_i U_i(x^i)$$

subject to feasibility. These programs are weak star continuous concave programming problems. Given that the constraint set is weak star compact and nonempty, the set of maxima is nonempty and convex.

PROPOSITION 4. For Pareto optima, all people of the same type have the same personal consumption.

Proof. This result follows from the strict concavity of u , the underlying utility function on personal consumption. $\square$

A *match type* is defined by the endowments of the two parties to the match. There is a finite number of possible match types.

PROPOSITION 5. For Pareto optima, all people in a given match type consume the same level of the club good.

Proof. This result follows immediately from the strict concavity of v , the underlying utility function on club consumption. If this were not the case, having all clubs of the same type have the mean b for the clubs of that type would increase the value of the social planner's objective function without violating feasibility. $\square$

PROPOSITION 6. If $\theta_{\omega s} > \theta_{\omega' s}$, then the corresponding Pareto optima satisfy (i) $a_{\omega s} > a_{\omega' s}$, (ii) $b_{\omega s} > b_{\omega' s}$ with probability one, and (iii) $U_i(x^{\omega s}) > U_i(x^{\omega' s})$.

Proof. If (i) were not true, the objective could be increased by shifting consumption from (ω', s) types to (ω, s) types while maintaining feasibility. If (ii) did not hold, then the objective could be increased by the following switch. Switch a positive measure of type (ω, s) in a club with club consumption $b_{\omega s}$ with an equal measure type (ω', s) who are in clubs with club consumption $b_{\omega' s}$. Given (i), (ii), and the fact that the underlying utility functions are increasing, (iii) follows. $\square$

We now use these properties of Pareto optima to deduce properties of competitive equilibria.

Necessarily, competitive equilibrium utility is increasing in endowment. Therefore, a competitive equilibrium is a solution to some weighted social planner's problem with weights weakly increasing in endowments for each sex.¹¹ This leads to

PROPOSITION 7. Competitive equilibria are characterized by all individuals of a given type having the same personal consumption. Further, equilibria are characterized by richer people of the same sex having either (i) larger personal and club consumptions or (ii) the same personal consumption.

Proof. This follows from the first welfare theorem, Propositions 4–6, and the fact that people with larger endowments must have higher equilibrium utilities.

Computing Lottery Equilibria for the Simple Example

We now return to our original two type example. As we have already established, a lottery valuation equilibrium is efficient. Therefore, all males have an identical personal consumption level, as do all females. We denote these levels as a_m and a_f respectively. Efficiency also requires that all the females be paired and that club good consumption be identical for all matches. Letting p_m and p_f be the male and female prices of b level club consumption, any set of (a_f, a_m, b, p_f, p_m) satisfying the following five equations determines a lottery equilibrium in the case where $\lambda_m > \lambda_f$:

$$(14) \quad \text{Male budget constraint: } a_m + (\lambda_f/\lambda_m)p_m = \omega_m$$

$$(15) \quad \text{A male marginal condition: } u'(a_m)p_m = v(b)$$

$$(16) \quad \text{Female budget constraint: } a_f + p_f = \omega_f$$

$$(17) \quad \text{Profit maximization requirement: } p_m + p_f = 2b$$

¹¹Note the utility function is not strictly concave and therefore, the weights in the corresponding planner's problem need not be strictly increasing.

$$(18) \quad \text{Pair efficiency condition: } v'(b)/u'(a_m) + v'(b)/u'(a_f) = 2.$$

This system of equations has a unique solution whenever the measures of males exceeds the measure of females. To see this consider the following: We can use (14) and (15) to derive a functional relationship $a_m(b)$ which must hold for any solution to this system of equations. The function $a_m(b)$ has a negative slope. Equations (15)–(17) can be solved for a_f as a function of b and a_m . Using the $a_m(b)$ function and this function, yields $a_f(b)$, which has a negative slope. Using these functions to substitute for a_f and a_m in (18) yields a function $h(b) = 2$. The fact that the $a_f(b)$ and $a_m(b)$ have negative slope and function $v(b)$ is strictly concave implies that the continuous function $h(b)$ has negative slope. Consequently, there is at most one solution to this system of equations.

Let π denote the fraction of females and consider the case where $\omega_f = \omega_m = 1$. The symmetry of the model implies that equilibrium utilities satisfy $U_m^*(\pi) = U_f^*(1 - \pi)$, and that the club good consumption level of matched pairs along with the sex-specific prices, are symmetric around $1/2$. We plot the equilibrium utility function of males in Figure 1, along with the utility level in the deterministic economy. Note the equilibrium utility level in the lottery does not exhibit the knife edge property of the deterministic economy. Instead, a sex benefits at the margin if their relative number is smaller. Secondly, the band of indeterminacy at $1/2$ is smaller in the lottery economy than in the deterministic economy because in the lottery economy some of the surplus from matching begins to accrue to the more numerous sex prior to the equalization of relative numbers. This occurs because in the lottery economy people can reduce their probability of being matched, and this extra margin reduces the “market power” of the scarce sex relative to the deterministic economy where matching is all or nothing. As a result, in the lottery economy the females do relatively worse as π approaches $1/2$ from below, since in equilibrium they are not able to force the males down to their no trade welfare levels.

In the lottery economy, and unlike the deterministic economy except at $\pi = 1/2$, the equilibrium allocation varies with π . The level of club good consumption declines monotonically as π approaches $1/2$ from above or below. This is because the implicit transfer from unmatched agents to matched agents is going to zero. The level of male personal consumption is 1 at $\pi = 0$, somewhat above the female level. Male personal consumption monotonically falls as π rises to $1/2$, while the female level rises monotonically and becomes larger than the male level prior to π reaching $1/2$.

In the limiting case, as $\pi \rightarrow 1$, the probability of a female getting matched is going to zero. Note that since $v(\cdot)$ is bounded, this implies that $\pi p_f \rightarrow 0$, otherwise the females would be forced below their no-trade welfare level in the limit. Since in equilibrium the number of probability units of being matched that each female is purchasing must also go to zero, the female's marginal price of being matched is converging to $v(b)/u'(1)$. The equilibrium welfare level of the males in the limit is therefore the solution to the following maximization problem:

$$U_m^*(1) = \max\{u(a_m) + v(b)\}$$

subject to $a_m + 2b \leq 1 + v(b)/u'(1)$. The females welfare level is just their no-trade welfare level, $u(1)$.

If $\pi \neq 1/2$, all types identical Pareto optimal and the competitive equilibrium allocations involve gambling. If $\pi = 1/2$, however, all Pareto optimal and competitive equilibrium allocations are deterministic.

Alternative Decentralizations

Consider our simple two type example when there are twice as many males as females. The equilibrium allocation for our lottery economy could be supported by different decentralizations. One way is for two male agents to get together, pool their resources, and enter into an arrangement

with a female agent in which they agreed to contribute $p_m(b^*)$, while setting aside a personal consumption level of $a_m^* = \omega_m - p_m(b^*)/2$ for each of them. They then flip a coin to see which one of them gets to be matched with the female, and thereby consumes the club good along with his personal consumption good. In this scenario, the males are gambling directly over final allocations. An alternatively way is that they first entered into the following wealth gamble: they flip a coin, and the loser pays the winner $p_m(b^*)/2$; after which they face a standard deterministic market in final consumption, which here includes the club good. The first scenario is essentially the decentralization that we have consider thus far, though the second may seem more appealing on positive grounds. However, this discussion suggests that the set of equilibrium allocations are equivalent under either decentralization. Later, we show for the general class of club economies that this is in fact the case.

2. A More General Class of Economies With Clubs

The method of decentralization that we have considered is applicable to a richer class of environments than the simple example that we have thus far considered might suggest. Here we indicate how one can accommodate a much wider range of matching models, with a greater variety of types and a more extensive set of joint activities being undertaken. We can also allow for multiple club membership; for example an individual can be a member of both a household and a production club.

Set of People: There are a *finite* number of types of people indexed by $i \in I$. The measure of type i is $\lambda_i > 0$. We will refer to a distinction between two types of people as being *basic* if some club technology does not treat them as perfect substitutes. In the previous section the sex distinction is basic. Type i 's basic type is $s_i \in S$, where S is a finite set.

Commodities: The set of possible personal consumptions belong to A , a compact subset of some Euclidean space. There are a finite set of categories of clubs indexed by $j = 1, \dots, J$. Each

person must be a member of one and only one club in each category. Forcing membership is not a restriction since one can allow for a single member club which does nothing in each category. We denote the set of possible club consumption vectors in category j by B_j , which like A is a compact subset of a Euclidean space. We define $C \equiv A \times B_1 \times \dots \times B_J$. The commodity space is $L \equiv M(C)^{\#S}$, where $M(C)$ denotes the set of signed measures on the Borel sigma algebra of C and $\#S$ denotes the cardinality of S .

Preferences and Endowments: A person of type i has preferences over probability measures on the Borel sigma algebra of C_i , a closed measurable subset of C , ordered by the expected value of the continuous function $u_i(c)$ defined on C_i . Type i 's consumption possibility set is

$$X_i = \{x \in L_+ : x_s(C_i) = x_s(C) = 1, x_{s'} = 0 \text{ for } s' \neq s\}.$$

The utility functions $U_i: X_i \rightarrow \mathbb{R}$ are

$$U_i(x) = \int u_i dx_s.$$

(Here and subsequently the subscript i on s will be implicit.) The endowment point of a type i is a point $e^i \in L$ with $e_{s'}^i = 0$ for $s' \neq s$ and with e_s^i being the measure that assigns measure 1 to sets containing $\omega_i \in C$, and 0 to all other sets.

Club Technologies: A club type of category $j = 1, \dots, J$ is defined by its vector of exchangeable commodities a , the vector of club goods b_j , and its membership n . Its membership vector n specifies the integer number of each basic type s in the club. The set of possible memberships is the finite set N . Thus, a club type in category j is denoted $t_j \in A \times B_j \times N$. The set of possible category j club types is $T_j \subseteq A \times B_j \times N$, for $j = 1, \dots, J$.

Aggregate Production Possibility Set: In order to determine whether or not a point $y \in L$ is in the aggregate production set Y , there must be a distribution of types of clubs in each category which is consistent both with y and T_j for all $j = 1, \dots, J$. Since any number of clubs of a given

type can be operated and the output of any club has measure zero, the aggregate production technology is a cone. The specification of Y requires that we first specify the set of possible measure of club types that are operated, given the club technologies. To do this we define Δ_j to be the set of measures on $M_+(A \times B_j \times N)$ with support in T_j . The measure $\delta_j \in \Delta_j$ specifies the number of clubs of category j belonging to the Borel measurable subsets of T_j .

The aggregate production possibility set is

$$Y = \{y \in L: \exists \text{ a set of measures } \{\delta_j\} \text{ with } \delta_j \in \Delta_j \text{ for which}$$

$$(i) \sum_j \int a \delta_j(dt_j) = \sum_s \int a y_s(dc)$$

$$(ii) \int n_s \delta_j(dt_j | b_j \in Q) = \int y_s(dc | b_j \in Q) \text{ all } j, s, \text{ and measurable } Q\}.$$

Constraint (i) is that the net supplies of commodities a summed over all clubs is equal to aggregate net supply of a . Constraints (ii) are that the measure of people of type s receiving club consumption $b_j \in Q$ in the output measure be consistent with the distribution of clubs in category j .

Remark: Given that both the consumption possibility sets and the preference ordering are convex, then for any equilibrium there is an equivalent type-identical equilibrium. The equivalence is in terms of the price system p , the aggregate supply y , the expected utilities, and for each type the distribution of c over people of that type.

Price System and Competitive Equilibrium: A *price system* for our economy is a continuous

linear functional $p: L \rightarrow \mathbb{R}$ with representation

$$p \cdot x = \sum_s \int p_s(c) x_s(dc)$$

where the $p_s: A \times B \rightarrow \mathbb{R}$ are measurable..

A *type identical competitive equilibrium* for our economy is a price system p and an allocation $\{x^i, y^*\}$ such that:

- (i) x^i maximizes $U_i(x)$ subject to $x \in X_i$ and $p \cdot x \leq p \cdot e^i$,
- (ii) y^* maximizes $p \cdot y$ subject to $y \in Y$, and
- (iii) $\sum_i \lambda_i (x^i - e^i) = y^*$.

The Welfare Theorems and Existence

As in the simple economy, the first welfare theorem follows from local nonsatiation. (See Theorem 15.3 in Stokey and Lucas (1989).) Given the convexity of consumption sets, continuity of preferences, and our restriction to type identical allocations, if there exists a point in each type's consumption set that is strictly preferred to all points which are that type's consumption for some feasible allocation, then local nonsatiation will hold.

Just as in our two-sex matching model establishing the second welfare theorem is somewhat more complicated. If L were finite dimensional and/or Y had an interior point, Theorem 15.4 in Stokey and Lucas (1989) could be applied to conclude that all Pareto optimum can be supported as a quasi-competitive equilibrium with transfers. If this is not the case, then one may be able to proceed, as in the two-sex matching model, to establish the second welfare theorem for an economy with a larger production possibility set that has an interior, but with same set of Pareto optima and competitive equilibrium as our economy.

The second welfare theorem guarantees the existence of a competitive equilibrium in which the price system is a continuous linear functional on the commodity space. This does not imply that this price system has a dot product representation when the commodity space is infinite dimensional. To establish the existence of a competitive equilibrium with transfers and with a price system that has a dot product representation, we verify that assumptions 15.8' and 15.9' in Stokey and Lucas

(1989, p. 468) hold and apply their Theorem 15.9. Assumption 15.8' requires that if $x \in X_i$, the modification of x obtained by setting the x_s equal to zero on some measurable subset of B is also in X_i . If $\omega_i \in C_i$, the $x(\{\omega_i\})$ correspond to the zero point. More generally (see Prescott and Rios-Rull 1992) for the case $\omega_i \notin C_i$. Assumption 15.9' is essentially that preferences are weak star continuous which is the case given expected utility maximization and the fact that the function p is bounded and continuous.

If C is finite, the existence proof discussed in McKenzie (1981) will generally apply. Note that (i) the consumption sets are closed and convex, (ii) the preference ordering is continuous and quasi concave, and (iii) the aggregate production set is a closed convex cone. The additional assumptions that need to be satisfied are:

- (iv) $X_i + \{-e^i\} \cap Y$ is not empty.
- (v) $Y \cap R_+ = \{0\}$.
- (vi) There is a common point in the relative interiors of Y and $X = \sum_I \lambda_i (X_i + \{-e^i\})$.
- (vii) For any two nonempty, disjoint partitions of I , I' , and I'' , and $x' = y - x''$, where $x' \in \sum_{I'} \lambda_i (X_i + \{-e^i\})$, $x'' \in \sum_{I''} \lambda_i (X_i + \{-e^i\})$ and $y \in Y$, there exists a $w \in \sum_I \lambda_i (X_i + \{-e^i\})$, such that $x' + x'' + w \in Y$, and $x' + w$ can be decomposed into an allocation for I' which is weakly preferred by all members of I' and strictly by at least one.

There are obviously many ways to satisfy (iv). For example, in our simple two-type matching model autarky is feasible. If we assume that the projection of T_j on $A \times B_j$ only intersects the positive orthant at $\{0\}$ for all j , then the finite dimensional version of our general model satisfies (v). There are many ways to satisfy (vi); the requirement that there is a feasible set of net trades and associated output which lie in the interior of the smallest hyperplane containing Y and the smallest hyperplane containing the sum of the sets of net trades consistent with both individual consumption sets and

endowments. There are also many ways to satisfy (vii) which ensures that all agents have positive income. In the simple example the existence of a good which is desired by all and with which every type is positively endowed guarantees (vii).

If C is not finite, the commodity space is infinite dimensional and establishing the existence of a competitive equilibrium is more difficult. A standard method of proof is show the limit of a sequence of equilibria for finite dimensional economies exists and is an equilibrium for the infinite dimensional limit economy. This was the approach we followed in the simple example. An outline of this approach as applied to our more general class of economies is as follows. Restrict the aggregate production possibility set to elements whose support belong to finite set $C_n \subset C$ where all points in C lie in a $1/n$ -neighborhood of at least one some point in C_n . For economy $[\{X_i, u_i, e^i\}, Y_n]$ the commodity space $L_n = M(C_n)^{\#S}$ is finite dimensional. Given competitive equilibrium $[\{x_n^i\}, y_n, p_n]$ for the n^{th} economy, the problem is to find a convergent subsequence with the property that the limit is an equilibrium for our economy with Y_n unrestricted.

The key part of the proof is to find a uniformly convergent subsequence of the p_n . This insures that $\lim x_n^i$ is maximal given price system $p = \lim p_n$ where n now indexes elements of the sub subsequence for which allocations converge weak star and prices uniformly. For our economy, if the $u_i(c)$ have uniformly bounded slopes, a sequence of competitive equilibria exists with the desired property. First we normalize prices so that the $\sup |p_{ns}(c)| = 1$. Next we let $r_{ns}(c)$ be the maximal reservation price over all $i \in I(s)$. Note the r_{ns} functions are defined on all of $\cup_{I(s)} C_i$. Note also that $[\{x_n^i\}, y_n, r_n]$ is a competitive equilibrium for economy with commodity space $L = M(C)^{\#S}$ and aggregate technology set Y_n . The functions $r_{ns}(c)$ will have uniformly bounded slopes given the underlying utility function $u_i(c)$ have uniformly bounded slope. This completes the outline of the extension to economies with C being a compact metric space.

Equivalence of Core and Equilibrium Allocations

In this section we show that the standard arguments with regard to the equivalence of the set of core and equilibrium allocations hold with a lottery commodity space. (See Hildenbrand and Kirman (1988).) That this was not the case in our prototype economy with a deterministic commodity space was pointed out in footnote 5. In order to specify a core allocation we will need to allow for allocations in which people of a given type not only do not receive the same commodity point, but may in fact enjoy different welfare levels. We then show that any core allocation must satisfy equal welfare among people of the same type. Next we show that any equilibrium allocation is in the core. Finally we show that given a core allocation there exists a price system which will support it as a quasi competitive equilibrium, and discuss the conditions under which a quasi competitive equilibrium is a competitive equilibrium.

A general definition of an *allocation* is a set of measures $z = \{z_i\}$, where z_i is a map from the Borel sigma algebra of L to $\mathbf{R}_+$ such that $z_i(X_i) = \lambda_i$ all i . A *suballocation* is similar to an allocation except that we weaken the membership restriction to be $\lambda_i(z) \equiv z_i(X_i) \leq \lambda_i$ for all i . An (sub) allocation z is *feasible* if: (i) the support of z_i is a subset of X_i for all i , and (ii) $y(z) \equiv \sum_I \int (x - z_i) z_i(dx) \in Y$.

We say (sub) allocation $\bar{z}$ *Pareto dominates* (sub) allocation $\hat{z}$ if

- (i) $\lambda(\bar{z}) = \lambda(\hat{z})$ and
- (ii) $\bar{z}_i(\{x \in X_i : U_i(x) \geq u\}) \geq \hat{z}_i(\{x \in X_i : U_i(x) \geq u\})$

all $i \in I$, $u \in \mathbf{R}$ with strict inequality for some (i, u) .

Feasible suballocation $\bar{z}$ *blocks* feasible allocation z if there is a suballocation $\hat{z} \leq z$ with the properties (i) $\lambda(\hat{z}) = \lambda(\bar{z})$ and (ii) $\bar{z}$ Pareto dominate $\hat{z}$.

Note $\bar{z}$ is the distribution of consumptions under allocation z of the blocking coalition while $\bar{z}$ is their distribution under the blocking allocation. An allocation z is a *core allocation* if no suballocation blocks it.

PROPOSITION 8. If z is a core allocation, all people of the same type receive the same welfare level.

*Proof.*¹² If utilities were not equal for almost all of each type, a feasible allocation could be blocked as follows: Select the measures $\epsilon\lambda_i$, where $0 < \epsilon < 1$, of people of type i who receive the lowest utility levels under z . For ϵ sufficiently small all members of this coalition z receive utility levels weakly less than their type average level for z . Further for this coalition a positive measure of some type receive utility levels strictly below the average for z . A suballocation which is feasible relative to this coalition is for all of type i to receive the average $\bar{x}^i$ for the z allocation. This follows given feasibility of z and the fact that Y is a cone. This feasible suballocation would block z . This establishes the proposition. $\square$

A *competitive equilibrium* which is consistent with our general definition of an allocation is a triplet $[z, y, p]$ such that: (i) the support of $z^i \subseteq \operatorname{argmax}\{U_i(x) : x \in X_i \text{ and } p \cdot x \leq p \cdot e^i\}$ for each i and (ii) $y \in \operatorname{argmax}_Y p \cdot y$, and (iii) $\sum_I \int (x - e^i) z^i(dx) = y$. Given a type-identical competitive equilibrium $[\{x_i\}, y, p]$, we can construct its associated general competitive equilibrium $[z, y, p]$ in the obvious manner: have z^i put weight λ^i on sets containing x^i and 0 otherwise. For a quasi competitive equilibrium condition (i) is replaced by the following:

(i') If x' is in the support of z^i , then for any $x \in X_i$, $U_i(x) \geq U_i(x')$ implies $p \cdot x \geq p \cdot x'$.

PROPOSITION 9. If $[z, y, p]$ is a competitive equilibrium, then its allocation z is a core allocation.

¹²This proposition was established in Kaneko and Wooders (1986). We include a proof for completeness.

Proof. We note first that for a competitive equilibrium all people of the same type have the same utility level. If suballocation $\bar{z}$ blocked z , for all people in $\bar{z}$

$$p \cdot \bar{x}^i \geq p \cdot x^i$$

with strict inequality for those in $\bar{z}$ whose utility is strictly higher than under the z allocation. This implies that

$$p \cdot y(\bar{z}) = \sum_I \int p \cdot (x - e^i) \bar{z}^i(dx) > 0.$$

Feasibility of $\bar{z}$ requires $y(\bar{z}) \in Y$. But, given Y is a cone, the maximum equilibrium profits must be zero. This contradiction establishes the proposition. $\square$

In order to show the existence of a price system which supports a core allocation as a quasi-competitive equilibrium we will need to apply the Hahn-Banach theorem. However, just as with the second welfare theorem, the potential emptiness of the interior of the production set when the commodity space is infinite dimensional precludes this theorem's direct application. There are two obvious resolutions to this problem: (i) assume that C is finite and hence L is finite dimensional, or (ii) construct an alternative production set Y' which includes Y and for which the set of Pareto optimal allocations is the same regardless of the distribution of the population (the λ 's). The condition in (ii) is stronger than with the second welfare theorem since we need to ensure that given any blocking sub allocation which is feasible for production set Y' , there is at least as good a suballocation which is feasible for Y , and hence the expansion in the set of possible blocking suballocations under Y' has not shrunk the core. We pursue approach (i) below. Approach (ii) can be applied in our prototype economy.

PROPOSITION 10. If C is finite, then for any core allocation z , there exist a p and y such that $[z, y, p]$ is a quasi-competitive equilibrium.

Proof. The proof is by construction. Since z is a feasible allocation, with $y = y(z)$ condition (iii) is satisfied. Now, we need only find a supporting price system wherein conditions (i') and (ii) are satisfied. The proof is in two parts. First we establish the existence of a supporting hyperplane which satisfies certain conditions from which we derive a price system. Then we establish that this price system along with our allocation and $y(z)$ is a quasi-competitive equilibrium.

First, from Proposition 6, for any core allocation there is a unique utility level associated with each type which we denote by μ_i . This utility level implies a preferred set of net consumptions for each type $X_i^+ = \{x \in X_i: U_i(x+e^i) > \mu_i\}$. Let X^+ denote the convex hull of $\cup_i X_i^+$. Now, we want to establish that $X^+ \cap Y$ is empty. Assume otherwise; that is that there exists a set of weights $\{\theta_i\}$ and points $\{x^i \in X_i^+\}$ such that $\sum_i \theta_i x^i \in Y$. But then we could construct a blocking suballocation where the relative measures of each type corresponded to those implied by these weights, and where each type i in the blocking coalition receives x^i . Hence, by contradiction the intersection is empty. Given that C is finite, and hence L is finite dimensional, the Hahn-Banach theorem guarantees that there exists a separating hyperplane, which since L is finite dimensional, we can represent as a point p in L .

Now, in order to establish that z , p , and $y(z)$ is a quasi-competitive equilibrium we need only show that conditions (i') and (ii) are satisfied. To show that (ii) is satisfied, note that from the continuity of preferences $y(z)$ is in the closure of the set $\sum_i X_i^+$. Since $\sum_i X_i^+$ is a subset of the convex hull of $\cup_i X_i^+$ and our price system is continuous, this implies that $p \cdot y(z) \geq p \cdot y$ for all $y \in Y$. By an analogous argument, where we make use of the fact that the continuity of preferences implies that the support of z^i is in the closure of X_i^+ , we can establish (i'). $\square$

Just as in the second welfare theorem, the existence of a cheaper point than the support of z^i in i 's budget set for each i , along with the convexity of consumption sets implies that a quasi-

competitive equilibrium is a competitive equilibrium. In the case of deterministic competitive equilibria, consumption sets are not convex, and this argument does not hold. This is why this equivalence failed to hold in our simple two type example discussed earlier, even though all agents had positive endowments of the numeraire good.

As a final point, note that for any equal welfare Pareto optimal allocation, which includes any core or general competitive equilibrium allocation, there is an equivalent type-identical allocation. By equivalent, we mean that the associated utilities are the same, and further the implied distribution over the underlying consumption space C by type is the same. To construct the equivalent type-identical allocation of an equal welfare allocation z , simply put mass $z^i(X_i)$ on sets containing the point $x^i = \int x \, dz^i$ for each $i \in I$. To see that this new allocation is equivalent to our original allocation, note first that since preferences are convex, welfare levels cannot be lower. They also cannot be higher since the initial allocation was a Pareto optimum and the new allocation is certainly feasible. Finally, note that by construction the distribution over the underlying commodity space by type is unchanged. These fairly obvious insights are what motivated our restricting ourselves to type homogeneous allocations in the previous sections of the paper.

Equivalence of Lottery and Value-Gamble Equilibria

A gambling decentralization is a two-stage process with gambles over value transfers in the first stage and a deterministic competitive equilibrium in the second. We will show that the set of equilibria for the two-stage gambling decentralization is equivalent to the set of lottery equilibria.

In the first stage of our gambling economy agents can enter into actuarially fair gambles over value transfers. The second stage of our gambling economy is just a deterministic version of our general lottery economy. The only change in switching to a deterministic economy from a lottery economy is that condition (i) in the definition of a competitive equilibrium becomes

$$(i') \quad \text{support of } x^{i*} \subseteq \{c \in \text{argmax}\{u^i(c): p_{s_i}(c) \leq p \cdot e^i\}.$$

Note in our definition of a deterministic competitive equilibrium we are implicitly allowing agents of the same type to make different choices of c only if they are indifferent over these choices ex-post.

The set of possible value transfers in the first stage is denoted by $W = \mathbf{R}$. A gambling allocation is a set $g = \{g^i\}$, where $g^i \in M_+(W \times C)^{\#S}$, and $y \in L$. We can define gambling consumption possibility sets as

$$G_i \equiv \{g \in M_+(W \times C)^{\#S}: g_{s_i}^i(W \times C) = g_{s_i}^i(W \times C_i) = 1 \text{ for } s = s_i \text{ and } g_{s_i}^i(W \times C) = 0 \text{ for } s \neq s_i\}.$$

The preferences of agent i over G_i are given by

$$U_{ig}(g) = \int u_i(c) dg_{s_i}.$$

A *gambling equilibrium* is defined as an allocation $[\{g^i\}, y]$, and a price system p in $L_1(C)^{\#S}$ such that

- (i) g^i maximizes $\int u_i(c) dg_{s_i}$ over G_i subject to
 - (1) $\int w dg \leq 0$ and
 - (2) $c \in \text{argmax}\{u^i(c): p_{s_i}(c) \leq p \cdot e^i + w\}$ if (w, c) in the support of g_{s_i} ,
- (ii) y maximizes $p \cdot y$ subject to $y \in Y$, and
- (iii) $\sum_i \lambda_i [\int g^i(dw \times \cdot) - e^i(\cdot)] = y(\cdot).$

We will refer to an equilibrium in our lottery economy being *equivalent* to an equilibrium in the gambling economy if the price functions and output measure, and if the distributions over consumption by type are equal. The equality of distributions over consumption by type implies that

ex-ante welfare levels by type are also the same. The following proposition implies that the sets of lottery and gambling equilibria are equivalent:

PROPOSITION 11. Any lottery equilibrium allocation can be supported as a gambling equilibrium and vice versa with common price system.

Proof. We will show that if $[\{g^i\}, y]$ and price system p is a gambling equilibrium, then allocation $[\{\theta_i(g^i)\}, y]$ and price system p is a lottery equilibrium where

$$\theta_i(g^i)(\cdot) = \int g^i(dw \times \cdot).$$

We also will show that if $[\{x^i\}, y]$ and price system p is a lottery equilibrium, then $[\{\phi_i(x^i, p)\}, y]$ and p is a gambling equilibrium where

$$\theta_{is}(x^i, p)(\cdot) = x_s^i \left(\{c: (p_s(c) - p \cdot e^i, c) \in \cdot\} \right) \text{ all } s.$$

It is immediate that these mappings preserves the feasibility and profit maximization conditions of the respective definitions of equilibrium. Further utility levels are the same given that the distribution of consumptions are the same for each type i . What remains is to show that if x^i is a solution to type i 's problem in the lottery economy, then its associated element under the mapping ϕ is a solution to i 's problem in the gambling economy, and vice versa. The nature of the proof of this claim is to show that if g^i satisfies the type i 's gambling budget constraints then $\theta^i(g^i)$ satisfies the type i 's lottery budget constraint and that if x^i satisfies the type i 's lottery budget constraint then $\phi_i(x^i, p)$ satisfies the type i 's gambling equilibrium budget constraints. These results imply $\theta_i(g^i)$ is maximal given p , if g^i is maximal given p by the following argument. Element g^i being maximal for a type i and $\theta_i(g^i)$ being lottery budget feasible implies the maximal utility that a type i can achieve in lottery system is at least $U_i(\theta_i(g^i)) = U_{ig}(g^i)$. If there were some other point x'^i that satisfied the budget constraint and yielded higher utility, then $\phi_i(x'^i, p)$ would be gambling budget feasible and

yield higher utility than g^i . Thus, $\theta_i(g^i)$ must be maximal given p . The analogous argument establishes that given p , if x^i is maximal, then $\phi_i(x^i, p)$ is maximal.

The type i budget constraint for the lottery equilibrium is

$$(19) \quad p \cdot x^i - p \cdot e^i \leq 0$$

while the type i budget constraints for the gambling equilibrium are

$$(20) \quad \sum_s \int w \, dg_s^i \leq 0$$

and

$$(21) \quad p_s(c) \leq w + p \cdot e^i \quad \text{for all } (w, c) \text{ in the support of the } g_s^i.$$

Condition (21) is satisfied by $\phi_{is}(x^i, p)$ by construction. We use (19) and the definition of ϕ to establish (20):

$$\sum_s \int w \, d\phi_{is}(x^i, p) = \int (p(c) - p \cdot e^i) dx^i = p \cdot x^i - p \cdot e^i \leq 0.$$

Thus the corresponding gambling elements satisfy their budget constraints if the set of lottery elements satisfies their budget constraints with equality. The reverse follows by a similar argument.

This establishes the proposition. $\square$

Remark. If local nonsatiation holds for all equilibria, the budget constraints hold with equality, and the map between equilibria of the lottery and gambling decentralizations is a bijection.

The equivalence between the set of equilibrium allocations of the gambling economy and the lottery economy makes clear that there are no gains to allowing for additional trading opportunities after the markets have shut in either of our economies. This point seems obvious with regard to our gambling economy since the final allocation is a deterministic competitive equilibrium, given the allocation of wealth that emerges from the first stage. In private information lottery economies, such

as Prescott and Townsend (1984b), this need not be the case (see Kahn and Mookherjee 1992 for a discussion of this issue), but absent private information it is.

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Figure 1

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Título: Valuation equilibrium with crubs.

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