“Discrete Public Goods With Incomplete Information”

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Discrete Public Goods with Incomplete Information

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Abstract

We analyze simultaneous discrete public good games with incomplete information and continuous contributions. To use the terminology of Admati and Perry (1991), we consider contribution and subscription games. In the former, contributions are not refunded if the project is not completed, while in the latter they are. For the special case where provision by a single player is possible we show the existence of an equilibrium in both contribution and subscription games where a player decides to provide the good by himself. For the case where is not feasible for a single player to provide the good by himself, we show that there exist equilibria of the subscription game where each participant pays the same amount. Moreover, using the technical apparatus from Myerson (1981) we show that neither the subscription nor the contribution games admit ex-post efficient equilibria. In addition, we provide a sufficient condition for contributing zero to be the unique equilibrium of the contribution game with $n$ players.

JEL Classification:D79, D89, H89

Key Words: private provision of public goods; contribution and subscription games; incomplete information.
1 Introduction

The literature on private provision of public goods can be divided into two broad categories.¹ The first branch of the literature focuses on the provision of continuous public goods. Papers include Warr (1982, 1983), Comes and Sandler (1984), Bergstrom, Blume and Varian (1986), Andreoni (1988), Gradstein, Nitzan and Slutsky (1994) and others. A standard result in this literature is that public goods are underprovided by voluntary contributions due to free riding behavior.² One might conjecture that the government can solve this underprovision problem by providing some of the good and financing it by imposing taxes on contributors. However, Warr (1982) and Roberts (1984), in two influential papers, show that government contributions result in a dollar to dollar reduction in private contributions if the tax on contributors does not change the set of contributing individuals. Bergstrom, Blume and Varian (1986) show that the crowding out effect is only partial if one allows for the taxes that pay for the government contribution to be also collected from non contributors. Hence the literature suggests that underprovision is a robust conclusion if the public good level is endogenously determined by voluntary contributions.³

The second strand of the literature focuses on discrete public goods where a fixed level of a public good is provided if enough contributions are collected to cover its cost c. Otherwise, the good is not provided. Typical examples include building a bridge, a library of a certain size, public radio fund raising to finance a certain program. In all of these examples, if enough money is raised to cover the cost of the public good, then the good is provided.

¹In this paper we focus on voluntary private provision. There is an extensive mechanism design literature that treats public provision. Papers in this literature develop eÁ cient mechanisms for the provision of public goods, e.g., Gradstein (1994), Maskin (1999), d'Aspremont and Varet (1979, 1982), Palfrey and Srivastava (1986) and Cornelli (1996).

²The free riding problem becomes worse in the presence of incomplete information. Gradstein (1992) considers a dynamic model of private provision for a continuous public good with incomplete information with the restriction that players either contribute zero or an exogenous positive amount. He concludes that in addition to the standard under-provision results, ineÁ ciency occurs because of a delay in contributions. This ineÁ ciency does not disappear as the population becomes large.

³Andreoni (1998) analyzes the role of seed money in the presence of nonconvexities in the production of the public good. He shows that a small amount of seed money can generate a substantial amount of voluntary contributions.
otherwise the good is not provided.

Palfrey and Rosenthal (1984) developed the first modern treatment of private provision of discrete public goods. More specifically, they analyze contribution and subscription games to use the terminology of Admati and Perry (1991) for a discrete public good under complete information where players' strategies are restricted to either contribute zero or an exogenous positive amount. In a contribution game, contributions are not refunded if the sum of the contributions does not cover the cost of the public good \( c \), while in a subscription game players get their money back if the project is not completed. An example with the main features of a subscription game can be given as follows. The Wisconsin Governor has recently pledged $27 million in state bonds to finance a new $72 million basketball arena on the condition that the rest of the money be raised by private donations (Andreoni, 1998). That is, the Governor will provide $27 million as long as the remaining $45 million is raised, otherwise the 0%er is cancelled. Other examples can be found in the fund-raising literature. Examples of contribution games include public radio and TV fund-raising efforts where contributors do not get their money back if the program is not provided. Other examples of contributions are situations where contributions take the form of physical labor, in this case volunteers cannot recover their effort if the project is not completed.

In contrast with the standard underprovision result for continuous public goods, Palfrey and Rosenthal (1984) obtain the startling conclusion that efficient provision of a discrete public good is a possible outcome of both contribution and subscription games even though inefficient equilibria also exist. Bagnoli and Lipman (1989) extend Palfrey and Rosenthal model by allowing individuals to make continuous contributions. They show that the set of undominated perfect equilibrium outcomes of the subscription game is not only efficient but coincides with the core of the economy. They also show that with a dynamic version of the subscription game, it is possible to obtain efficient outcomes even if the level of the public good is not binary as long as the number of units of the public good is countable. It follows

4 Admati and Perry analyzed the role of commitment in the provision of public goods. More specifically, they examined two-player contribution and subscription games with complete and perfect information where players alternate in making pledges or contributions until the public project is completed. They show that with linear costs, a necessary and sufficient condition for completion of the project under a contribution game is that each player would complete the project immediately if he was the only player. For the subscription game, they show that all socially desirable projects are completed in equilib-
that the general conclusion from the theoretical literature is that private
provision of discrete public goods with complete information can be efficient.
Moreover, efficient provision is a possible outcome for both subscription and
contribution games.

An important question that has not been addressed in the literature on
voluntary provision of discrete public goods is to what extent these results
generalize to a model where players have incomplete information about other
players' valuation of the public good. Moreover, there is both casual evidence
from the fund-raising literature of the superiority of subscription games
and experimental evidence from Bagnoli and McKee (1991) and Cadsby and
Maynes (1999) that a refund increases the chance of providing the good. More
specifically, Cadsby and Maynes consider a discrete public good experiment.
They provide experimental evidence showing that provision is encouraged in
a subscription game vis-a-vis a contribution game. They also provide evi­
dence that a high cost discourages provision in the contribution game but not
in the subscription game. Our results will confirm these findings.

In this paper, we examine contribution games and subscription games for
a discrete public good in the presence of incomplete information. We first
analyze provision games where it is feasible for a player to provide the public
good by himself. Common examples include opening a window in a hot room,
rescuing an injured person in a traffic accident, etc. We show that when the
cost of provision is not prohibitively high as to prevent a single player from
providing the good, there always exists an equilibrium where a player with
a sufficiently large valuation provides the good in both the subscription and
the contribution game.

We also examine contribution and subscription games when contributions
or pledges from more than one individual are necessary in order for the good
to be provided. We will show that there exist equal sharing equilibria in
the subscription game, i.e., equilibria in which each player contributes the
same share of the total cost of providing the public good. We also apply the
mechanisms design techniques developed in Myerson (1981) to show
that neither the contribution game nor the subscription game admit ex-post
efficient equilibria. Finally we will provide a sufficient condition for 'con­
tributing zero' to be the unique equilibrium of the contribution game. This
confirms the evidence from the experimental literature of the superiority of
subscription games over contribution games.
This remainder of the paper provides more details on all of these issues. In Section 2, the specifics of the model are presented. Section 3 provides an example that illustrates the main results of the paper. In Section 4, the case where a single individual can provide the good by himself is analyzed. A cost-sharing type equilibrium for the subscription game is identified. Section 6 examines the ex-post efficiency of contribution and subscription games while in Section 7 we demonstrate that the contribution game can be extremely inefficient. Section 8 illustrates a differential equations approach to equilibrium strategies in games with two players. Our main results are summarized in Section 9.

2 The Model

We now present the formal model. There are \( N \geq 2 \) individuals with private independent values for a discrete public good. That is, each individual \( i, i = 1, \ldots, N \) knows his own value \( v_i \) but only the distribution of his opponents' values. Values are assumed to be independently drawn from a continuous distribution \( F \). The cost of providing the public good is \( c \).

With the exception of Section 4, we assume throughout the paper that an individual cannot provide the good by himself. In this case, \( F(c) = 1 \) and thus the distribution has a bounded support. As a normalization we suppose, without loss of generality, the support to be contained in the interval \([0, 1]\).

We will examine both subscription and contribution games. In a subscription game the money contributed by players is returned if the sum of the pledges is less than \( c \). In a contribution game the money is not returned even if the good is not provided. Whatever the method to elicit donations is, if donations add up to more than \( c \), the additional money is not returned to the contributors.

More specifically, we analyze one-shot simultaneous move contribution and subscription games for a discrete public good in the presence of incomplete information about preferences. Our model also relaxes the binary con-
tribution restriction imposed in the literature. While there are important instances where binary contributions are relevant due to transaction costs, in general players can give any amount of money they desire (alumni donation, donations to a library, etc.). Moreover, in a continuous contributions framework, individuals make contributions that best match their preferences as opposed to a discrete contribution model. Cadsby and Maynes (1999) provide experimental evidence showing that allowing continuous rather than binary all-or-nothing contributions facilitates provision.

As it is standard in many games of incomplete information with ex-ante symmetric players such as in industrial organization and auctions our focus is mainly on symmetric equilibrium. Nevertheless, we provide inefficiency results that hold for both symmetric and asymmetric equilibria; we will appeal to the revenue equivalence theorem to show that for any equilibrium the probability of provision of the public good given that it is efficient to do so is less than one in both contribution and subscription games. We also provide sufficient conditions under which contributing zero is the unique equilibrium of the contribution game.

The multiplicity of equilibria that characterizes these games has two sources. Firstly, this is an instance of a coordination game. With two players who cannot provide the good by themselves, the coordination nature of the game is clear. With more than two players, then we have to account for all possible combinations of players who can provide the good as a group. This adds a new layer of complexity to standard coordination games. Secondly, the incompleteness of information and the assumption that player can contribute or pledge any values they like also creates additional coordination issues beyond the standard coordination problem. It is the combination of these two features that makes it difficult to obtain uniqueness of equilibrium. In particular, as we will explain in the context of the example below, well-known arguments to obtain uniqueness, such as the iterated deletion of dominated strategies as in Bagnoli and Lipman (1989) (with complete information) and Milgrom and Roberts (1990) (with incomplete information) for the case of supermodular games, do not apply in this context.

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Bagnoli and Lipman (1989) also considers continuous contributions. However their model is with complete information.
3 An Example

Before turning to an analysis of the model, it is instructive to consider a simple example where it is possible to make the point that the efficient results in the discrete public goods literature are not robust to the introduction of incomplete information. This example incorporates the simplest structure for which this lack of robustness is evident and for which we obtain a striking distinction between predicted outcomes in subscription and contribution games. We begin with the complete information case.

3.1 Complete Information

Assume that players 1 and 2 have valuations for the public good $v_1$ and $v_2$ in $[0,1]$ and that are common knowledge. Let the cost of provision of the public good $c$ be such that $1 < c < 2$ and $v_1 + v_2 \geq c$. That is, a single player cannot provide the good by himself and it is socially efficient to provide the good. If we denote by $b_i, i = 1, 2$, player $i$’s pledging strategy in a subscription game, then we can characterize the set of (pure strategy) equilibria as follows:

(i) $\{(b_1, b_2) : b_1 + b_2 = c; b_1 \leq v_1; b_2 \leq v_2\}$;

(ii) $(0, 0)$;

(iii) $\{(b_1, b_2) : b_1 < c - v_2; b_2 < c - v_1\}$.

The equilibria described in (i) are efficient, while the equilibria described in (ii) and (iii) are not. Moreover, the equilibria described in (ii) and (iii) are not strict as best responses are not unique.

Note that the equilibria of the contribution game can be described by (i) and (ii) above. Given that individuals in a contribution game never receive their contributions back, the strategies profiles described in (iii) are no longer equilibria. Unlike in the case of the subscription game, the equilibrium strategy $(0, 0)$ is now a strict equilibrium. That is, contributing zero is the unique best response when the other player contributes zero.

This example suggests that under complete information the distinction between contribution and subscription game both in terms of efficiency and

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7The equilibrium described in (ii) is known as the strong free riding equilibrium.
8Bagnoli and Lipman (1989) show that the inefficient equilibria in (ii) and (iii) can be eliminated by undominated perfection.
in terms of expected outcomes is unclear. We now consider the incomplete information case.

3.2 Incomplete Information

We now assume that \( v_1 \) and \( v_2 \) are independent draws from the uniform \([0,1]\) distribution. Moreover, Player \( i = 1, 2 \) knows his own valuation but only the distribution of his opponent's valuation. Let us suppose that \( 1 < c < 2 \). The subscription game has many symmetric equilibria and the contribution game has only the strong free riding equilibria. Let us see some subscription game equilibria first:

i. \((0, 0)\);

ii. \( b^*(v) = \begin{cases} 
0, & \text{if } v < \frac{c}{2} \\
\frac{v}{2}, & \text{if } v \geq \frac{c}{2} 
\end{cases} \)

iii. \( b^*(v) = \begin{cases} 
\frac{2c-1}{6} + \frac{v}{2}, & \text{if } \frac{2c-1}{3} \leq v \leq 1 \\
0, & \text{otherwise.} 
\end{cases} \)

That \((0, 0)\) is still an equilibrium in the incomplete information follows trivially from the fact that a single player cannot provide the good by himself. Now let's check that the strategy profile described in (ii) is indeed an equilibrium. Suppose Player 2 follows \( b^*(\cdot) \). If Player 1's value is less than \( \frac{c}{2} \), then 1's best reply is to pledge any number less than \( \frac{c}{2} \). In particular, pledging zero is a best response. If Player 1's value is greater than equal to \( \frac{c}{2} \), then Player 1's best response is to pledge \( \frac{c}{2} \). Pledging less than \( \frac{c}{2} \), Player 1 obtains zero profits; by pledging more than \( \frac{c}{2} \), Player 1 does not increase the probability that the good is provided but decreases profits conditional on provision.

The equilibrium pledging strategies described in (iii) can be explained as follows. A player pledges the equivalent to the expected value of the other player being lower than his own, conditional on the interval \([\frac{2c-1}{3}, 1]\), that is, on the relevant interval where pledges are less than or equal to the valuations for the public good, i.e. \( b(v) \leq v \). A formal derivation is left to the appendix.

Note that the iterated elimination of dominated strategies does not bite here. In our setting, in a first round of elimination Player \( i \) eliminates strategies \( h_i(v_i) > v_i \). The elimination process stops there! The type of bounds obtained in Milgrom and Roberts (1990) are not possible here since the game
is not supermodular: Consider two strategies $b_1 = 0$ and $b_2 = c$ and note that 

$$f(b_2, b_2) - f(b_2, b_1) - [f(b_1, b_2) - f(b_1, b_2)] = (v - c) - (v - 0) = -v < 0.$$ 

Of course this multiplicity of equilibria can potentially raise difficulties in questions of welfare. Nevertheless, we will show later that any equilibria of the subscription game is inefficient. More precisely we will show that the probability of provision of the public good given that it is efficient to provide is (uniformly) less than one in any equilibrium. Moreover, we will provide a sufficient condition under which the welfare comparison between subscription and contribution games is straightforward as the only equilibrium of the contribution game is to contribute zero.

Suppose $(b_1, b_2)$ is an equilibrium of the contribution game. It is immediate that $b_2(v_2) \leq v_2$. Let us find the best response of Player 1 to $b_2$. If Player 1 bids $x \geq 0$ his expected utility is 

$$\phi(x) = vP(b_2) \geq c - x - x.$$ 

If $x \neq 0$ and $c - x \geq 1$ then $\phi(x) = -x < 0$. Now suppose $c - x < 1$. We have 

$$\phi(x) \leq vP(v_2 \geq c - x) - x = v(1 - (c - x)) - x = v(1 - c) + (v - 1)x < 0.$$ 

Thus $x = 0$ is the best response. (Note that we are not limiting ourselves to symmetric pure strategy equilibrium). Therefore, despite the multiplicity of equilibria of the subscription game, we are still able to make welfare comparisons. More precisely, the contribution game is less efficient than the subscription game in this example. This result contrasts greatly with the case where players have complete information and both games have efficient equilibria.

### 4 Provision by a Single Player

In this section, we analyze contribution and subscription games when valuations for the public good can take values larger than the cost. The following theorem shows that there always exists a symmetric equilibrium where a player with a sufficiently high valuation provides the good by himself.

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9This probability is equal to 1/2 in the equilibrium described in (ii) and equal to 2/3 in the equilibrium described in (iii).
Theorem 1 Suppose $F : [0, \infty) \to \mathbb{R}$ is a continuous distribution. Suppose there are $N \geq 2$ players for a project with cost $c > 0$ and that $F(c) < 1$. Then there exists an $\alpha > 0$ where $\alpha$ solves $\alpha F(\alpha)^{N-1} = c$ such that

$$b(v) = \begin{cases} 
0 & \text{if } v \leq \alpha \\
\alpha F(\alpha)^{N-1} & \text{if } v > \alpha
\end{cases}$$

is an equilibrium strategy for both contribution and subscription games.

Proof. Suppose players $n = 2, ..., N$ play according to $b(\cdot)$. Let us find the best response of Player 1. If his value is $v \geq 0$ and his contribution is $x \geq 0$, his expected payoff in the contribution game is given by

$$g(x) = v \Pr(x + b(v_2) + ... + b(v_N) \geq c) - x$$

Since $b(v_2) + ... + b(v_N)$ is either 0 or not less than $c$, it follows that if $x < c$ then $g(x) \leq g(0)$. Thus $\max\{g(x); x \geq 0\} = \max\{g(0), g(c)\} = \max\{v \left(1 - F(\alpha)^{N-1}\right), v - c\}$. Hence if $v < \frac{c}{F(\alpha)^{N-1}} = \alpha$, the best contribution is $x = 0$. If $v > \alpha$ the best contribution is $x = c$. And if $v = \alpha$ the player is indifferent between $x = 0$ and $x = c$. For the subscription game, the expected surplus is given by

$$f(x) = (v - x) \Pr(x + b(v_2) + ... + b(v_N) \geq c)$$

The proof is identical to the contribution game since $f(0) = g(0)$ and $f(c) = g(c)$.

Theorem 1 implies that when the cost of provision is not prohibitively high as to prevent a single player from providing the good, there always exists an equilibrium where a player with a sufficiently large valuation provides the good by himself.

5 The Subscription Game

In this section we analyze the subscription game where a single player cannot provide the good by himself. This is the case only if $F(c) = 1$. Thus the

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10The solution is not unique in general. For example if $N = 2, F(x) = x, x \in [0, 1]$, then if $0 < c < 1/e$ another equilibrium strategy for the contribution game is $b(x) = \max\{c + k \log(x), 0\}$ where $k$ is such that $k^x = e^{-c}$. There are two solutions: $k_1 < 1/e < k_2$. 

distribution $F$ has a bounded support and without loss of generality we suppose the support is the interval $[0,1]$. Players' values are determined by independent draws from a continuous distribution $F : [0,1] \rightarrow \mathbb{R}$. The cost of the public good is $c \in [m, m+1)$ where $m \geq 1$ i.e. $m+1$ is the minimum number of players needed to provide the public good.

If players $i = 2,3,\ldots,N$ follow the function $b(\cdot)$ and player 1 with valuation $v$ contributes $x \geq 0$, his expected surplus is

$$
\phi(x) = (v - x) \Pr (x + b(v_2) + \ldots + b(v_N) \geq c)
$$

(1)

We see from the above that if $N > 2$, to write $\phi(x)$ as a function of the distribution of individual valuations, $F$, detailed information on $b(\cdot)$ is needed; it does not suffice to know that $b(\cdot)$ is increasing in the private valuation. Hence the standard technique that is used to characterize equilibrium involving increasing strategies in games of incomplete information cannot be used in this problem when $N > 2$. Moreover, it would be fruitless to try to guess all the functional forms that equilibria might take. Thus, it would be extremely difficult to find all the equilibria of the subscription game.

5.1 Cost Sharing Equilibria

In this subsection, we show directly that there exists an equilibrium with cut-out strategies where the good is provided if $m+1$ players have a value greater than a certain cut-out value.

Proposition 1 Suppose that players' valuations are distributed in the $[0,1]$ interval. Suppose the cost of the public good is $c \in [m, m+1)$ and the number of players $N \geq m+1$. The following strategy is a symmetric equilibrium strategy of the subscription game:

$$
b(v) = \begin{cases} 
0 & \text{if } v \leq a; \\
\frac{c}{m+1} & \text{if } a \leq v \leq 1.
\end{cases}
$$

(2)

where $a$ is such that

$$
a \frac{C_{N-1}^m (1 - F(a))^m F(a)^{N-1-m}}{\sum_{h=m}^{N-1} C_h^N (1 - F(a))^h F(a)^{N-1-h}} = \frac{c}{m+1}.
$$

(3)

and $C_n^h = \frac{n!}{h!(n-h)!}$. 

10
Proof. We first prove the existence of \( a \). Define

\[
    h(a) = \frac{C_{N-1}^m (1 - F(a))^m F(a)^{N-1-m}}{\sum_{h=m}^{N-1} C_{N-1}^h (1 - F(a))^h F(a)^{N-1-h}}
\]

Note that

\[
    \lim_{a \to 0} h(a) = 0 \quad \text{and} \quad \lim_{a \to 1} h(a) = \frac{C_{N-1}^m}{C_{N-1}^m} = 1
\]

Since \( h \) is a continuous function, by the intermediate value theorem there exists an \( a \) such that \( h(a) = \frac{c}{m+1} \in (0, 1) \).

We now prove that \( b(\cdot) \) is an equilibrium. Suppose \( v < 1 \) and player 1 pledges \( x \geq 0 \) and players \( n = 2, \ldots, N \) follow \( b(v_n) \). The expected utility of player 1 is given by

\[
    \phi(x) = (v - x) \Pr \left( \sum_{n=2}^{N} b(v_n) \geq c - x \right).
\]

The range of \( \sum_{n=2}^{N} b(v_n) \) is \( \left\{ 0, \frac{c}{m+1}, \frac{2c}{m+1}, \ldots, \frac{(N-1)c}{m+1} \right\} \).

If \( c - x \in \left( \frac{jc}{m+1}, \frac{(j+1)c}{m+1} \right] \), then

\[
    \phi(x) = (v - x) \Pr \left( \sum_{n=2}^{N} b(v_n) \geq \frac{(j+1)c}{m+1} \right) \leq \phi \left( c - \frac{(j+1)c}{m+1} \right)
\]

Thus the optimal pledge \( x^* \leq v \) is such that

\[
    x^* \in \left\{ c - \frac{(j+1)c}{m+1} ; j \geq -1 \right\}.
\]

Thus

\[
    x^* = c - \frac{(j^*+1)c}{m+1} \in [0, 1), j^* \geq -1.
\]
Therefore \( j^* \in \{m, m-1\} \) since \( \frac{2c}{m+1} \geq 2 \cdot \frac{m}{m+1} \geq 1 \). Finally we have \( j^* = m-1 \) if and only if \( \phi \left( \frac{c}{m+1} \right) \geq \phi (0) \) or equivalently if and only if

\[
\left( v - \frac{c}{m+1} \right) \Pr \left( \sum_{n=2}^{N} b(v_n) \geq c - \frac{c}{m+1} \right) \geq v \Pr \left( \sum_{n=2}^{N} b(v_n) \geq c \right).
\]

or

\[
v \Pr \left( \sum_{n=2}^{N} b(v_n) = \frac{mc}{m+1} \right) \geq \frac{c}{m+1} \Pr \left( \sum_{n=2}^{N} b(v_n) \geq \frac{mc}{m+1} \right).
\]

Since

\[
\Pr \left( \sum_{n=2}^{N} b(v_n) = \frac{mc}{m+1} \right) = \binom{m}{N-1} (1 - F(a))^m F(a)^{N-1-m}
\]

and

\[
\Pr \left( \sum_{n=2}^{N} b(v_n) \geq \frac{mc}{m+1} \right) = \sum_{h=m}^{N-1} \binom{h}{N-1} (1 - F(a))^h F(a)^{N-1-h}
\]

we conclude that \( \phi \left( \frac{c}{m+1} \right) \geq \phi (0) \) if and only if \( v \geq a \). ■

Remark 1 If every player is pivotal, i.e. if \( N = m+1 \), then the equilibrium (2) is given by:

\[
b(v) = \begin{cases} 
0 & \text{if } v < \frac{c}{m+1} \\
\frac{c}{m+1} & \text{if } \frac{c}{m+1} \leq v \leq 1.
\end{cases}
\]

Thus each player considers the cost equally divided among the players and contributes if and only if his value is at least his share.

6 Ex-post Efficiency

A natural question is whether the subscription game admits any equilibria that are efficient. Mailath and Postlewaite (1990) consider a general class of mechanisms, which includes both subscription and contribution games as special cases, and show that as the number of agents increases, the ex-post
Efficiency (the probability of provision given that it is efficient to provide the public good) goes to zero. They also claim, although they do not provide a proof, that even for finite $n$ that there are no ex-post efficient mechanisms to provide the public good within the framework that we established above. For the case of $n = 2$, the inefficiency follows from applying the well known result of Myerson and Satterthwaite (1983) with the interpretation that player 1 is a buyer with valuation $v_1$ distributed in the interval $[0, 1]$ and player 2 is a seller with valuation $c - v_2$, $v_2 \in [0, 1]$. The idea is that player 2 will pay $c$ to provide but receives a transfer from player 1 in order to cover part of the cost of production. For $n > 2$, it is more difficult to infer the result directly. However, the same principle that is applicable in Myerson and Satterthwaite is also applicable here, namely, the revenue equivalence theorem of Myerson (1981). We use such theorem not only to show ex-post inefficiency of both contribution and subscription games but also to compute an upper bound for the inefficiency.

6.1 Subscription and contribution games in the language of mechanisms.

Suppose bidder $i$ is type $v_i \in T_i = [0, 1], 1 \leq i \leq n$. Define $T = \prod_{i=1}^{n} T_i$. If $v \in T$ we write $v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$. And the expectation operator $E_{-i}$ denotes the expectation among the variables $v_{-i}$. We define a direct mechanism as a function $(y, x): T \to \{0, 1\} \times \mathbb{R}^n$. The good is provided if $y(v) = 1$ and $x_i (v)$ is player $i$'s payment. Thus his expected utility, if his type is $v_i$ is $\phi_i (v_i) = E_{-i} [v_i y (s, v_{-i}) - x_i (s, v_{-i})], \forall s \in T_i$. Using Myerson's revelation principle we only consider mechanisms that are individually rational and incentive compatible. That is

$$\phi_i (v_i) \geq 0 \quad \text{(IR)}$$
$$\phi_i (v_i) \geq E_{-i} [v_i y (s, v_{-i}) - x_i (s, v_{-i})], \forall s \in T_i. \quad \text{(IC)}$$

We may rewrite (IC) as

$$\phi_i (a) - \phi_i (b) \geq (a - b) E_{-i} [y (b, v_{-i})] , \forall a, b \in T_i.$$

Using the same argument as in Myerson (1981) we conclude that

$$\phi_i (a) = \phi_{i0} + \int_{0}^{a} E_{-i} [y (u, v_{-i})] du, a \in T_i.$$
The voluntary participation constraint is satisfied if and only if $\phi_{i0} \geq 0$. Bidder $i$ expected payment is therefore

$$E_{-i} [x_i (v)] = v_i E_{-i} [y(v)] - \phi_{i0} - \int_0^{v_i} E_{-i} [y(u, v_{-i})] \, du.$$  

(5)

We could prove in the same way as Myerson’s the following

Lemma 1 The direct mechanism $(y, x)$ satisfy (IC) and (IR) if and only if

$$(5), \quad \phi_{i0} \geq 0, \text{ and } u \to E_{-i} [y(u, v_{-i})] \text{ is non-decreasing for every } i.$$

6.2 Ecient mechanisms are not budget balanced.

We need two deenitions:

Deenition 1 The mechanism $(y, x)$ is ecient if $y(v) = 1$ if and only if

$$\sum_{i=1}^n v_i \geq c.$$  

Deenition 2 The mechanism $(y, x)$ is budget balanced if for every $v \in T$,

$$\sum_{i=1}^n x_i (v) \geq 0 \text{ and } y(v) \left( \sum_{i=1}^n x_i (v) - c \right) \geq 0.$$  

In this subsection we prove that an ecient mechanism for public provision cannot be budget balanced. Although this is a known result, a simple and direct proof is not available. Note that subscription and contribution games are budget balanced and, therefore, we can conclude that they are not ecient.

Proposition 2 Suppose $(y, x)$ is an ecient mechanism. Then

$$\Pr \left( \sum_{i=1}^n x_i (v) \geq c \mid \sum_{i} v_i \geq c \right) \leq \frac{\int_{\sum_{i} v_i \geq c} \sum_{i=1}^n \left( c - \sum_{j \neq i} v_j \right)^+ f(v) \, dv}{c}.$$  

In particular the mechanism is not budget balanced.
Proof. We need to estimate $\Pr \left( \sum_{i=1}^{n} x_i (v) \geq c \mid \sum_i v_i \geq c \right)$. Thus

$$\Pr \left( \sum_{i=1}^{n} x_i (v) \geq c \mid \sum_i v_i \geq c \right) \leq \frac{\Pr \left( \sum_{i=1}^{n} x_i (v) \geq c \right)}{\Pr \left( \sum_i v_i \geq c \right)} \leq \frac{E \left[ \sum_{i=1}^{n} x_i (v) \right]}{c \Pr \left( \sum_i v_i \geq c \right)}.$$

We now proceed to find an upper bound for $E \left[ \sum_{i=1}^{n} x_i (v) \right]$. Since $\int_{0}^{v_i} E_{-i} [y (u, v_{-i})] \, du = E_{-i} \int_{0}^{v_i} [y (u, v_{-i})] \, du$ using (5) the expected payment is $E \left[ x_i (v) \right] = E [v_i y (v)] - \phi_0 - E \left( \int_{0}^{v_i} y (u, v_{-i}) \, du \right)$. Thus

$$E \left[ \sum_{i=1}^{n} x_i (v) \right] \leq E \left[ \left( \sum_{i=1}^{n} v_i \right) y (v) \right] - \sum_{i=1}^{n} E \left( \int_{0}^{v_i} [y (u, v_{-i})] \, du \right). \quad (6)$$

If the mechanism is efficient then $y (u, v_{-i}) = 1 \Leftrightarrow u \geq c - \sum_{j \neq i} v_j$. If $c - \sum_{j \neq i} v_j > v_i$ then $\int_{0}^{v_i} y (u, v_{-i}) \, du = 0$. If $c - \sum_{j \neq i} v_j \leq v_i$ then $\int_{0}^{v_i} y (u, v_{-i}) \, du = v_i - (c - \sum_{j \neq i} v_j)^+$. Substituting this in (6):

$$E \left[ \sum_{i=1}^{n} x_i (v) \right] \leq E \left[ \left( \sum_{i=1}^{n} v_i \right) y (v) \right] - \sum_{i=1}^{n} E \left[ y (v) \left( v_i - (c - \sum_{j \neq i} v_j)^+ \right) \right].$$

Therefore

$$E \left[ y (v) \sum_{i=1}^{n} \left( c - \sum_{j \neq i} v_j \right)^+ \right].$$

Finally applying the next lemma we finish the proof. ■

**Lemma 2** If $c > 0$ then $E \left[ \chi_{\sum_{i=1}^{n} v_i \geq c} \sum_{i=1}^{n} \left( c - \sum_{j \neq i} v_j \right)^+ \right] < c \Pr \left( \sum_{i=1}^{n} v_i \geq c \right)$.
Proof. Define $X = \mathcal{P} \left( \{1, \ldots, n\} \right)$. For every $v \in [0,1]^n$ define $H(v) = \{i \leq n; c - \sum_{j \neq i} v_j \geq 0\}$ and $h_v = \#H(v)$. If $h_v \geq 1$ we may rewrite

$$
\sum_{i=1}^{n} \left( c - \sum_{j \neq i} v_j \right) = \sum_{i \in H(v)} \left( c - \sum_{j \neq i} v_j \right) = h_v c - h_v \sum_{j \notin H(v)} v_j - (h_v - 1) \sum_{i \in H(v)} v_i = h_v c - \sum_{j \notin H(v)} v_j - (h_v - 1) \sum_{i = 1}^{n} v_i = c - \sum_{j \notin H(v)} v_j - (c - \sum_{i = 1}^{n} v_i) \leq c.
$$

The last inequality is strict with positive probability. Thus since

$$
E \left[ \chi_{\sum_{i=1}^{n} v_i \geq c} \sum_{i=1}^{n} \left( c - \sum_{j \neq i} v_j \right) \right] = \sum_{H \in X} E \left[ \chi_{\sum_{i=1}^{n} v_i \geq c \chi_{H(v)} = H} \sum_{i=1}^{n} \left( c - \sum_{j \neq i} v_j \right) \right] < 
$$

$$
\sum_{H \in X} E \left[ c \cdot \chi_{\sum_{i=1}^{n} v_i \geq c \chi_{H(v)} = H} \right] = c \Pr \left( \sum_{i=1}^{n} v_i \geq c \right)
$$

we finish the proof. ■

Remark 2 It is true that

$$
E \left[ \chi_{\sum_{i=1}^{n} v_i \geq c} \sum_{i=1}^{n} \left( c - \sum_{j \neq i} v_j \right) \right] = \int_{\sum_{i=1}^{n} v_i \geq c} \sum_{i=1}^{n} \left( v_i - \frac{1 - F(v_i)}{f(v_i)} \right) f(v) \, dv.
$$

The proof is immediate using $E \left[ \int_0^v \left[ y(u, v_{-i}) \right] \, du \right] = E \left[ \frac{1 - F(v)}{f(v)} y(v) f(v) \, dv \right]$.

6.3 The probability of provision of budget balanced mechanisms.

We now focus on budget balanced mechanisms. Since they are necessarily inneÁ cient it is important to know how inneÁ cient they are. We will measure inneÁ ciency by the probability of provision given that it is socially desirable to provide the good. More precisely, if $(y, x)$ is a budget balanced mechanism we want to obtain an upper bound for $\Pr (y(v) = 1| \sum_{i=1}^{n} v_i \geq c)$. Note that the upperbound for the eÁ ciency applies to any mechanism satisfying the conditions put forward by Myerson, including the contribution and subscription games analyzed in this paper.
Proposition 3 If \((x, y)\) is a budget balanced mechanism then

\[
\Pr \left( y(v) = 1 \mid \sum_{i=1}^{n} v_i \geq c \right) \leq \frac{E \left[ \chi_{\sum_{i=1}^{n} v_i \geq c} \left( \sum_{i=1}^{n} \left( v_i - \frac{1-F(v_i)}{f(v_i)} \right) \right)^+ f(v) \, dv \right]}{c \Pr \left( \sum_{i=1}^{n} v_i \geq c \right)}.
\]

Proof. The following chain of inequalities prove the proposition.

\[
\Pr (y(v) = 1) \leq \Pr \left( \sum_{i} x_i \geq c \right) \leq \frac{E [\sum_{i} x_i]}{c} \leq \frac{1}{c} E \left[ \sum_{i=1}^{n} \left( v_i - \frac{1-F(v_i)}{f(v_i)} \right) \right] y(v) f(v) \, dv \leq \frac{1}{c} E \left[ \chi_{\sum_{i} v_i \geq c} \left( \sum_{i=1}^{n} \left( v_i - \frac{1-F(v_i)}{f(v_i)} \right) \right)^+ f(v) \right].
\]

Example 1 Suppose \(n = 2\) and the uniform distribution. We have to calculate

\[
\int_{a+b \geq c} 2(a+b-1)^+ \, dadb.
\]

Suppose \(2 > c > 1\). Then \(\int_{a+b \geq c} 2(a+b-1)^+ \, dadb = 2 \int_{a=c-1}^{1} \int_{c-a}^{1} (a+b-1) \, dbda = 2 \int_{c-1}^{1} (-\frac{1}{2} + \frac{1}{2}a^2 - \frac{1}{2}c^2 + c) \, da = -\frac{4}{3} - 3c^2 + 4c + \frac{2}{3}c^3\). And \(\Pr (a+b \geq c) = \frac{(2-c)^2}{2}\). Thus \(\Pr (y(v) = 1 | v_1 + v_2 \geq c) \leq \frac{-\frac{4}{3} - 3c^2 + 4c + \frac{2}{3}c^3}{\frac{(2-c)^2}{2}} = \frac{2c-1}{3}\). We now find a lower bound for the supremum eA ciency. Consider the following symmetric equilibrium\(^{11}\):

\[
b^*(v) = \begin{cases} \frac{2c-1}{6} + \frac{v}{2}, & \text{if } \frac{2c-1}{3} \leq v \leq 1, \\ 0, & \text{otherwise}. \end{cases}
\]

\(^{11}\)The proof that this is an equilibrium strategy of the subscription game is included in the appendix.
We need to compute \( \frac{\Pr\left(\{(v_1, v_2); b^*(v_1) + b^*(v_2) \geq c\}\right)}{\Pr\left(\{(v_1, v_2); v_1 + v_2 \geq c\}\right)} \). Note that

\[
\Pr\left(\{(v_1, v_2); b^*(v_1) + b^*(v_2) \geq c\}\right) = \frac{1}{2} \left(\frac{4 - 2c}{3}\right)^2 + \left(\frac{2c - 1}{3} - (c - 1)\right) \left(1 - \frac{c + 1}{3}\right)
\]

\[
= \frac{4}{3} - \frac{4}{3}c + \frac{1}{3}c^2 = \frac{1}{3}(c - 2)^2
\]

Moreover \( \Pr\left(\{(v_1, v_2); v_1 + v_2 \geq c\}\right) = \frac{1}{2}(2 - c)^2 \). Therefore,

\[
\Pr\left(b^*(v_1) + b^*(v_2) \geq c|v_1 + v_2 \geq c\right) = \frac{2}{3}.
\]

From the cost sharing equilibria described in the previous section we know that the probability of provision given that is efficient to provide the good, although less than one, is positive in the subscription game. In the next section we demonstrate that the probability of provision in a contribution game is equal to zero when the cost of provision is slightly above the mean of the distribution of values.

7 The Contribution Game Strong Inefficiency

We now prove a much stronger result for the contribution game. Namely that the probability of provision is zero if the cost \( c \) is high enough. In terms of mechanisms the contribution game is a mechanism \((y, x)\) such that \(x_i(v)\) does not depend on \(v_{-i}\). An all-pay auction is a contribution mechanism in this sense.

The following theorem shows that the coordination problem in the contribution game is so severe that if the cost of the public good is slightly above the aggregate mean of the valuations, then the unique equilibrium of the contribution game is for each player to contribute zero no matter what his value is.

Define \(\mu = E[v_1]\).

**Theorem 2** The public good is never provided in any equilibrium of the contribution game if

\[
c \geq \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4n(n - 1)\mu}
\]
Proof. The contribution game is the game in which the mechanism \( x_i(v) = x_i(u_i) \) for every \( v_{-i} \). Thus from (5) we have

\[
x_i(v_i) \leq v_i E_{-i} [y(v)] - \int_0^{v_i} E_{-i} [y(u, v_{-i})] \, du \leq E_{-i} y(1, v_{-i}).
\]

Thus \( \sum_{i=1}^n x_i(v_i) \leq \sum_{i=1}^n E_{-i} y(1, v_{-i}) \). Since provision never occurs if \( \sum_i v_i < c \), \( E_{-i} y(v) \leq \Pr \left( \sum_{i \neq i} v_i \geq c - v_i \right) \). Therefore the following inequality is true:

\[
\sum_{i=1}^n x_i(v_i) \leq \sum_{i=1}^n \Pr \left( \sum_{i \neq i} v_i \geq c - v_i \right) \leq \sum_{i=1}^n \Pr \left( \sum_{i \neq i} v_i \geq c - 1 \right) = n \Pr \left( \sum_{i \neq i} v_i \geq c - 1 \right).
\]

Thus

\[
\sum_{i=1}^n x_i(v_i) \leq n \frac{E \left[ \sum_{i \neq i} v_i \right]}{c - 1} = n \frac{(n - 1) \mu}{c - 1} < c
\]

ending the proof. □

Remark 3 If \( c_n = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4n(n - 1) \mu} \) it is true that \( \frac{c_n}{n} \to \mu \) if \( n \to \infty \). Thus if cost per capita is higher than the average the strong free riding equilibrium is the unique equilibrium of the contribution game.

Remark 4 Define \( \sigma^2 \) as the distribution variance. A more precise inequality for the strong inefficiency lower bound is given by the positive root of \( (c - 1 - (n - 1) \mu)^2 c = n(n - 1) \sigma^2 \). We omit the proof.

Example 2 If \( F(x) = x \) then \( \mu = \frac{1}{2} \) and \( \sigma = \frac{1}{\sqrt{12}} \). Then if \( n = 30 \), \( c \geq 17.5 \) if \( n = 100 \), \( c \geq 54.39 \).

The above result implies that for a wide range of the cost of the public good there is extreme underprovision when refunds are not allowed.

Remark 5 For small \( n \) we can obtain sharper bounds on \( c_n \). For example if \( n = 2 \), and the distribution function satisfies \( F(x) \geq x, x \in [0, 1] \) then the free riding equilibrium is the unique equilibrium of the contribution game. We omit the proof.

Remark 6 The result mentioned in the last remark is not true for every distribution. A counterexample can be found among the \( F(x) = x^n \) distributions.
8 The differential equation approach to symmetric equilibrium strategies

The theory of single-object auctions with private independent values provides us with a method to find the (symmetric) equilibrium strategy that is very natural. Usually one supposes that bidders \(i = 2, \ldots, N\) bid accordingly to the strictly increasing strategy \(b(\cdot)\) and finds Bidder 1's best reply \(x\) by means of the first order condition for expected utility maximization. Then in equilibrium \(x = b(v)\) and we obtain a differential equation for \(b(\cdot)\). Let us try to mimic this approach to symmetric equilibria for the contribution and subscription games. Recall from the discussion following equation (1) that if \(N > 2\), detailed information on \(b(\cdot)\) is needed in order to derive the distribution of \(\sum_{j \neq i} b(v_j)\). Let us therefore suppose \(N = 2\). We suppose also that \(f = F' > 0\) exists and is continuous everywhere. First we examine the subscription game.

8.1 The subscription game differential equation.

So suppose Player 2 bids accordingly to \(b(\cdot)\) which is strictly increasing. Player 1's expected surplus given that he has a value \(v\) and contributes \(x \geq 0\) is given by

\[
\phi(x) = (v - x) \Pr(b(v_2) \geq c - x)
\]

(7)

\[
= (v - x) \Pr(v_2 \geq b^{-1}(c - x))
\]

(8)

\[
= (v - x)(1 - F(b^{-1}(c - x))).
\]

Note that we took the inverse of \(b(\cdot)\) to pass from (7) to (8). Player 1's expected payoff is simply his surplus if the project is completed times the probability of completion. The first-order condition is

\[
\phi'(x) = -(1 - F(b^{-1}(c - x))) + (v - x)f(b^{-1}(c - x))(b^{-1})'(v - x) = 0.
\]

In a symmetric equilibrium \(x = b(v)\), so

\[
(v - b(v)) f(b^{-1}(c - b(v)))(b^{-1})'(c - b(v)) = 1 - F(b^{-1}(c - b(v))).
\]

(9)
This is not an ordinary differential equation since the function \( b(v) \) appears inside the argument. We will transform (9) in a system of two ordinary differential equations. Define \( G(v) = b^{-1}(c - b(v)) \). This implies that

\[
b(G(v)) + b(v) = c \tag{10}
\]

\[
(v - b(v)) f(G(v)) (b^{-1})'(b(G(v))) = 1 - F(G(v)) \tag{11}
\]

In (11) we used that \( b(G(v)) = c - b(v) \). Since \( b^{-1}(b(\omega)) = \omega \) we have that \((b^{-1})'(b(\omega)) \omega'(\omega) = 1\). Thus choosing \( \omega = G(v) \) we get

\[
(b^{-1})'(b(G(v))) = 1/b'(G(v)).
\]

Substituting this in (11) it follows that

\[
\frac{(v - b(v)) f(G(v))}{1 - F(G(v))} = b'(G(v)). \tag{12}
\]

Now from (10) applied to \( v \) and to \( G(v) \) we have that \( b(G(v)) + b(v) = c = b(G^2(v)) + b(G(v)) \). Hence \( b(G^2(v)) = b(v) \) and therefore \( G(G(v)) = v \).

Therefore from (12) we conclude that

\[
b'(v) = \frac{(G(v) - b(G(v))) f(v)}{1 - F(v)}. \tag{13}
\]

Differentiating (10) we obtain \( b'(G(v)) G'(v) + b'(v) = 0 \). The following theorem describes the system of differential equations that an increasing equilibrium strategy to the subscription game must satisfy.

**Theorem 3** The solution of the system of differential equations below characterizes a symmetric equilibrium involving strictly increasing strategies for the subscription game with two players with values determined by independent draws from a distribution \( F : [0,1] \rightarrow \mathbb{R} \), where \( F \) is continuously differentiable and \( f = F' > 0 \) everywhere.

\[
b'(v) = \frac{(G(v) - b(G(v))) f(v)}{1 - F(v)} \tag{14}
\]

\[
G'(v) = -\frac{(1 - F(G(v))) (G(v) - b(G(v))) f(v)}{(1 - F(v)) (v - b(v)) f(G(v))}.
\]

**Remark 7** It is important to note that the system (14) gives only a candidate for an equilibrium strategy. An example in the appendix shows that some tuning may be necessary. This contrasts greatly with the solution we obtain in auction theory which is usually defined for every signal. Here it is possible to invert \( b(\cdot) \) only locally. Moreover the existence of \( G \) defined above is global in nature.
8.2 The contribution game differential equation.

An analysis similar to the analysis made above gives the system of differential equations that a strictly increasing equilibrium of the contribution game must satisfy.

Theorem 4 The solution of the system of differential equations below characterizes a symmetric equilibrium involving strictly increasing strategies for the contribution game with two players with values determined by independent draws from a distribution $F : [0, 1] \to \mathbb{R}$, where $F$ is continuously differentiable and $f = F' > 0$ everywhere.

\[
\begin{align*}
 b'(v) &= G(v) f(v), \\
 G'(v) &= -\frac{G(v)}{vf(G(v))}.
\end{align*}
\] (15)

For example if $F$ is the uniform distribution,

\[
\begin{align*}
 b'(v) &= G(v), \\
 G'(v) &= -\frac{G(v)}{v}.
\end{align*}
\]

This gives $G(v) = \frac{1}{v}$ and $b(v) = a + k \log(v)$. Naturally $b(\cdot)$ cannot be an equilibrium for all $v$ since it is negative for small $v$. If $0 < c < 1/e$, $b(v) = \max\{0, c + k \log(v)\}$ where $k$ is such that $ke^{-k} = e^{-c}$, is an equilibrium.

9 Conclusion

We analyzed two mechanisms for private provision of discrete public goods with incomplete information and continuous contributions. The subscription mechanism refunds the money to contributors if the public good is not provided whereas the contribution mechanism does not allow for refunds.

Our analysis showed that, unlike the model with complete information, the coordination problem becomes more complex and efficient provision is no longer possible. Moreover, the two mechanisms lead to completely different outcomes. We showed that for the contribution game, for a wide range of costs of the public good, the good will never be provided. This confirms the evidence from the experimental literature of the superiority of subscription games over contribution games.
References


Appendix

The subscription game with two players: The uniform distribution case

In what follows, we provide some intuition how to solve this problem when we take into account the boundary conditions. For the uniform distribution on the interval $[0,1]$, the unconstrained solution to the system of differential equations stated in Section 7.1 is given by the following pledging function:

$$b(v) = \frac{2c - 1}{6} + \frac{v}{2}$$

Note, however, that in equilibrium a player may not follow $b(v)$ for any $v$ in $[0,1]$ as this may lead to pledging more than his value or more than the cost of the public good. Hence we need to impose the following boundary conditions

$$b(v) \leq c \quad \text{(b1)}$$
$$b(v) \leq v \quad \text{(b2)}$$
$$b(v) \geq 0 \quad \text{(b3)}$$

It turns out that (b1) and (b3) are not binding since $c > 1$. Condition (b2) is binding as $b(v) > v$ for $v < \frac{2c-1}{3}$. We now show formally that the following is a symmetric equilibrium pledging strategy for the subscription game with two players whose values are uniformly distributed on $[0,1]$ and $1 < c < 2$:

$$b^*(v) = \begin{cases} 
\frac{2c-1}{6} + \frac{v}{2}, & \text{if } \frac{2c-1}{3} \leq v \leq 1, \\
0, & \text{otherwise.}
\end{cases}$$

To verify that this is an equilibrium, we assume that Player 2 is following the proposed equilibrium pledging strategy, we have to find the best response of player 1. We first show that for $\frac{2c-1}{3} \leq v \leq 1$, $b_1(v) = \frac{2c-1}{6} + \frac{v}{2}$ is a best response to $b^*(v_2)$. Player 1's expected surplus, if he pledges $b$, is given by

$$\phi(b) = (v - b) \Pr (b + b^*(v_2) \geq c).$$

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To find the maximum of $\phi$, first note that if $0 < c - b < b^* \left( \frac{2c - 1}{3} \right) = \frac{2c - 1}{3}$ then

$$\phi(b) = (v - b) \left( 1 - \frac{2c - 1}{3} \right) \leq (v - \frac{c + 1}{3}) \left( 1 - \frac{2c - 1}{3} \right) = \phi \left( \frac{c + 1}{3} \right).$$

Note that if $c - b \geq b^* (1)$ then $\phi(b) = 0$. Let us consider now $c - b^* (1) < b < c - b^* \left( \frac{2c - 1}{3} \right)$. Then we have

$$\phi(b) = (v - b) \Pr \left( \left\{ v_2 \geq \frac{2c - 1}{3}; v_2 \geq \frac{4c + 1}{3} - 2b \right\} \right) = (v - b) \left( 1 - \max \left\{ \frac{2c - 1}{3}, \frac{4c + 1}{3} - 2b \right\} \right).$$

There are two cases to consider:

1) $\frac{2c - 1}{3} > \frac{4c + 1}{3} - 2b$

In this case $\phi(b) = (v - b) \left( 1 - \frac{2c - 1}{3} \right) < \phi \left( \frac{c + 1}{3} \right)$.

2) $\frac{2c - 1}{3} \leq \frac{4c + 1}{3} - 2b$

In this case $\phi(b) = (v - b) \left( 1 - \frac{4c + 1}{3} + 2b \right)$. This quadratic function has a unique maximum at $b^* = \frac{2c - 1}{3} + \frac{v}{2}$. Thus $b^*$ is the optimal pledge if $b^* \in \left[ c - b(1), c - b \left( \frac{2c - 1}{3} \right) \right]$ and $\frac{2c - 1}{3} \leq \frac{4c + 1}{3} - 2b^*$. The last inequality is valid for all $v \in [0, 1]$. The first inequality is valid if $v \in [\frac{2c - 1}{3}, 1]$. Thus $b(v) = \frac{2c - 1}{3} + \frac{v}{2}$ is the best response to $b^*(v_2)$ if $v \in [\frac{2c - 1}{3}, 1]$. To finish let us find the best response for $v \in [0, \frac{2c - 1}{3}]$. It is clear from the reasoning in (2) above that the maximum of $\phi$ is not interior. Thus we need only to compare $\phi(c - b^* (1)) = \phi \left( \frac{2c - 1}{3} \right) = (v - \frac{2c - 1}{3}) \left( 1 - \frac{4c + 1}{3} + 2\frac{2c - 1}{3} \right) = 0$ and $\phi(c - b^* \left( \frac{2c - 1}{3} \right)) = \phi \left( \frac{c + 1}{3} \right) = (v - \frac{c + 1}{3}) \left( 1 - \frac{2c - 1}{3} \right) < 0$. Thus if $v \in [0, \frac{2c - 1}{3}]$ the maximum expected surplus is zero. Hence since pledging zero and pledging $\frac{2c - 1}{3}$ give the same expected surplus, we finished the proof that $b(\cdot)$ is an equilibrium. As the equilibrium pledging strategy is strictly increasing and differentiable in the relevant range our previous analysis is justified.