"Asymmetric Smiles, Leverage Effects and Structural Parameters"

Prof. René Garcia
(Université de Montréal, CIRANO E CRDE)

LOCAL
Fundação Getulio Vargas
Praia de Botafogo, 190 - 10º andar – Auditório- Eugênio Gudin

DATA
27/07/2000 (5ª feira)

HORÁRIO
16:00h

Coordenação: Prof. Pedro Cavalcanti Gomes Ferreira
Email: ferreira@fgv.br - ☏ (021) 559-5834
Asymmetric Smiles, Leverage Effects and Structural Parameters

René Garcia
*Université de Montréal, CIRANO and CRDE*

Richard Luger
*Banks of Canada and CIRANO*

Éric Renault
*Université de Montréal, CIRANO, CRDE, and CREST-Insee*

First version: June 1999
This version: July 2000

**Keywords:** Equilibrium Option Pricing, Recursive Utility, Black-Scholes Implicit Volatility, Smile effect.  **JEL Classification:** C1, C5, G1

---

Address for correspondence: Département de Sciences Économiques et C.R.D.E., Université de Montréal. C.P. 6128. Succ. Centre-Ville, Montréal, Québec, H3C 3J7, Canada.

The first two authors gratefully acknowledge financial support from the Fonds de la Formation de Chercheurs et l’Aide à la Recherche du Québec (FCAR), the Social Sciences and Humanities Research Council of Canada (SSHRC) and the MITACS Network of Centres of Excellence. The third author thanks CIRANO and C.R.D.E. for financial support.
1. Introduction

In the empirical option pricing literature, departures from the Black-Scholes (1973) (BS) model are often characterized by an implied volatility curve, whereby the volatility extracted from the BS option pricing formula given the observed option price is graphed against the moneyness of the option. The empirical biases of the BS model have been dubbed the smile effect in reference to a symmetric implied volatility curve, but numerous distorted smiles in the shape of smirks or frowns are inferred more frequently from market data. In Figure 1, we graphed several volatility curves for options on the S&P 500 index on selected dates to reflect the types of shapes that can be observed most frequently. A stochastic volatility model as in Hull and White (1987) produces a symmetric smile when the returns innovations and the volatility are uncorrelated. With stochastic volatility, the price of the option is expressed as an expectation of the BS price, where the expectation is taken with respect to the distribution of the heterogeneous stochastic volatility factor. The symmetric volatility smile is created by a related Jensen effect (see Renault, 1997 and Renault and Touzi, 1996).

Asymmetric smiles can therefore be potentially explained by an instantaneous correlation between returns and volatility. In Black (1976), an inverse relationship between the level of equity prices and the instantaneous conditional volatility is put forward for individual firms. This inverse relationship is explained by financial leverage. A drop in the price of the stock increases the debt-to-equity ratio and therefore the risk of the firm, which translates into a higher volatility of the stock. Nelson (1991) shows also that such a negative correlation exists for broad market indices. The correlation is still called a leverage effect, but explanations are given in terms of time-varying risk premia and volatility feedback (see Campbell and Hentschel, 1996, among others). If volatility risk is priced, an anticipated increase in volatility raises the discount rate of future expected dividends and lowers the present equity price. From a theoretical perspective, Platen and Schweizer (1998)

\footnote{Based on portfolios constructed from Nikkei 225 stocks, Bekaert and Wu (2000) find support}
explain the asymmetric shape of the smile by developing a model in which the diffusion process of the stock price incorporates the technical demand induced by hedging strategies. David and Veronesi (1999) propose an incomplete information model where investors' uncertainty explains the intertemporal variation in the slope and curvature of implied volatility curves.

In this paper, we characterize the asymmetries of the smile through multiple leverage effects in a stochastic dynamic asset pricing framework. The dependence between price movements and future volatility is introduced through a set of latent state variables. These latent variables can capture not only the volatility risk and the interest rate risk which potentially affect option prices, but also any kind of correlation risk and jump risk. The standard financial leverage effect is produced by a cross-correlation effect between the state variables which enter into the stochastic volatility process of the stock price and the stock price process itself. However, we provide a more general framework where asymmetric implied volatility curves result from any source of instantaneous correlation between the state variables and, either the return on the stock or the stochastic discount factor.

To be able to put forward the asymmetric deformations of the smile, we first state necessary and sufficient conditions for symmetry in a general setting. These characterizations are given in terms of the option pricing function or, alternatively, through the skewness of the pricing probability measure. We capture the aforementioned skewness of the pricing probability measure through a generalized option pricing formula based on a stochastic discount factor containing state variables. As special cases of our formula we obtain of course the BS formula, but also the Hull and White (1987) and Bailey and Stulz (1989) stochastic volatility option pricing formulas and the Merton (1973), Turnbull and Milne (1991), and Amin and Jarrow (1992) stochastic interest rate option pricing formulas for equity options.

This extended option pricing formula is motivated by the vast empirical literature for volatility feedback effects.

\(^2\)Grossman and Zhou (1996) propose also an equilibrium model of risk sharing between portfolio insurers and investors which generates a negatively skewed smile. In Renault (1997), a very small discrepancy between the option markets assessment of the stock price and the actual price can capture sensible asymmetries of the smile.
ture aimed at finding option pricing models that will reproduce the cross-sectional patterns and the dynamics of implied volatilities. A class of generalized deterministic volatility models has been proposed to overcome the empirical biases of the BS model. In this class of models, the local volatility of the underlying asset is a known function of time and of the path and level of the underlying asset price. However, Dumas, Fleming and Whaley (1998) show clearly that deterministic volatility models overfit the smile in sample and lose any predictive power out of sample. Buraschi and Jackwerth (1997) bring further evidence that deterministic volatility models are not consistent with observed option prices and that stochastic volatility models are more likely to explain the smile. Other evidence against the one-dimensional diffusion model can be found in Bakshi, Cao, Chen (1999). They show that call prices often go down when the underlying price goes up and that call prices are not perfectly correlated with each other and the underlying asset.

The main stochastic volatility models that have been submitted to empirical testing are variants of the two-dimensional (stock returns and volatility) diffusion model. Although these models produce patterns qualitatively similar to some violations, quantitatively, they only bring marginal improvement to the fit of option-price changes. Bakshi, Cao and Chen (1997), Bates (1996) and Chernov and Ghysels (1999) provide evidence against the stochastic volatility model of Heston (1993). Multi-factor volatility models as in Bates (1997) and Gallant, Hsu, Tauchen (1998) do not bring a significant improvement. The main conclusion is that an extremely high volatility of volatility is necessary to generate leptokurtosis of the magnitude consistent with the volatility smirks. Das and Sundaram (1999) explain that stochastic volatility models are not capable of generating high levels of skewness and kurtosis at short maturities under reasonable parameterizations.


---

3 These models include the constant elasticity of variance model of Cox and Ross (1976), the implied binomial tree approach of Rubinstein (1994), the deterministic volatility models of Dupire (1994) and Derman and Kani (1994), and the Kernel approach of Ait-Sahalia and Lo (1998).
patterns, the degree of skewness and excess kurtosis declines as maturity increases in a jump-diffusion model, but that for reasonable values of the parameters the decline is far more rapid than would be suggested by the data. Therefore, jump-diffusion models cannot reproduce the implied volatility smile at long horizons since it flattens out too quickly. Pan (1999) examines joint time series data on spot and option prices on the S&P 500 and provides evidence of a jump risk premium that responds quickly to market volatility and is important in explaining the volatility smirks. There remains however some misspecifications and some suggestions regarding the inclusion of jumps in the volatility are made.

The conclusion of this empirical literature is that important features for reproducing the cross-sectional patterns and the dynamics of implied volatilities are jumps in the returns, correlation between jumps and volatility, and jumps in volatility. To illustrate how our option pricing formula can incorporate these features, we specialize our latent state variables to a discrete-state Markov process. A discrete change of state will affect simultaneously the mean and variance of equity returns and of the stochastic discount factor, creating jump-like effects also in volatility. In this setting we characterize analytically both the skewness of the returns and the equity leverage effect. We show that the formulas for conditional skewness and leverage effect in the stock are very similar. We establish the conditions for negative skewness or leverage in terms of transition probabilities between states.

To be able to draw the shapes of the implied volatility smiles, we need to further specify the stochastic discount factor. We choose an example an equilibrium-based stochastic discount factor in the spirit of Rubinstein (1976), Brennan (1979) and Amin and Ng (1993). We set our equilibrium model in a recursive utility framework with time non-separable preferences (Epstein and Zin [1989]). Our latent variable framework makes it possible to parameterize parsi-
moniously the dynamic evolution of the consumption and dividend processes in this equilibrium model. We use a two-state bivariate Markov switching model as in Cecchetti, Lam and Mark (1993) and Bonomo and Garcia (1993, 1996).

When we calibrate this model to reasonable values of the parameters, we are able to reproduce the various types of implied volatility curves inferred from option market data. In other words, not only our model reproduces asymmetric smiles but it allows for reversals in the smile skewness. All the classic extensions of the basic Black-Scholes model with stochastic volatility, stochastic interest rates or jumps reviewed in Bakshi, Cao and Chen (1997) cannot produce such reversals. Indeed, as Bates (1996) emphasized, it is such changing skewness in the smile that poses a challenge to current option pricing models. In our setting, reversals in skewness can occur when the source of risk changes.

The rest of the paper is organized as follows. Section 2 provides general conditions under which the smile is symmetric. Section 3 develops a generalized option pricing formula with latent state variables. Based on this formula, Section 3 shows that the asymmetric distortions of the smile are due to leverage effects. A comparison with usual stochastic volatility models with leverage is developed in Section 5 as well as an analytical characterization of leverage when state variables follow a discrete-state Markov process. Section 6 provides a simulation exercise using an equilibrium-based stochastic discount factor in order to illustrate the various shapes of the smile that the model can produce. Section 7 concludes and announces further empirical assessment of the proposed option pricing model.

2. The symmetry of the volatility smile

To characterize the asymmetry of the implied volatility curves, one needs a benchmark model which will produce a symmetric curve. When the volatility is stochastic as in the Hull and White (1987) model, Renault (1997) and Renault and Touzi (1996) have shown that the shape of the volatility structure with respect European call options written on aggregate equity under Kreps-Porteus preferences. The regime-switching model introduced by Hamilton (1989) has recently enjoyed some popularity in the option pricing literature. See in particular Campbell and Li (1998), David and Veronesi (1998), Duan, Popova and Richken (1998). All these models can be embedded in our framework. A precursor paper in regime-switching option pricing is Naik (1993).
to the moneyness of the option is symmetric when the returns innovations and the volatility are uncorrelated. Moneyness $x_t$ is defined as the logarithm of the ratio of the forward price over the strike price, $x_t = \log \frac{S_t}{KB(t,T)}$ (with $S_t$ the price of the underlying asset, $K$ the strike price and $B(t,T)$ the price of a pure discount bond maturing at $T$). Since both the BS and the Hull and White option pricing formulas are homogeneous functions of degree one with respect to the pair $(S_t, K)$, we will provide new characterizations of the symmetry of the volatility smile in the context of a general homogeneous option pricing formula, both in terms of the option pricing function and of the pricing probability measure.

The theory for pricing contingent claims in the absence of arbitrage introduces a pricing probability measure $Q_{t,T}$ under which the price $\Pi_t$ at time $t$ of any contingent claim maturing at time $T$ is the discounted expectation of its terminal payoff. In the case of a European call option with strike price $K$, it is given by:

$$\Pi_t = B(t,T)E_t^*(S_T - K)^+,$$  
(2.1)

where $E_t^*$ denotes the expectation operator with respect to $Q_{t,T}$.

We will therefore compare a general but homogeneous option pricing formula $\Pi_t(S_t, K)$ as defined in (2.1) with the BS option pricing formula defined itself by a homogeneous function $BS(\cdot, \cdot, \sigma)$, for a given volatility parameter $\sigma$, with:

$$\begin{cases}
BS(S_t, K, \sigma) = S_t \phi(d_1) - KB(t,T)\phi(d_2), \\
d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ x_t + \frac{1}{2} \sigma^2 (T-t) \right], \\
d_2 = d_1 - \sigma \sqrt{T-t}.
\end{cases}$$  
(2.2)

The BS implied volatility is defined as a function $\sigma_t^*(x_t)$ of the moneyness $x_t$ only, and not of $S_t$ and $K$ separately:

$$\Pi_t(S_t, K) = BS(S_t, K, \sigma_t^*(x_t)),$$  
(2.3)

since a direct application of the homogeneity of degree one of $\Pi_t(\cdot, \cdot)$ and $BS(\cdot, \cdot, \sigma)$ with respect to the pair $(S_t, K)$ allows one to divide each side of (2.3) by $S_t$ and

---

6Existence and unicity of $Q_{t,T}$ were studied by several authors since the seminal paper of Harrison and Kreps (1979). In this paper, we are only interested in the existence of a pricing probability measure $Q_{t,T}$ which is well-defined and given to us, whether it is unique or not.
conclude that $\sigma_t^*(x_t)$ is well-defined as a function of $K/S_t$ or (equiv­
ally) of $x_t$ by:

$$\pi_t(x_t) = bs(x_t, \sigma_t^*(x_t))$$

(2.4)

with the obvious change of notation.

In this setting, we can investigate the slope of the BS implied volatility $\sigma_t^*(x)$ as a function of its distance $x$ to the money, remembering that moneyness $x$ is equal to 0 at the money, that is when the strike price coincides with the forward price $\frac{S_t}{B(t,T)}$. In particular, two strike prices $K_1$ and $K_2$ are said symmetric with respect to the money if the corresponding $x_1$ and $x_2$ are symmetric with respect to zero, since in this case $K_1$ and $K_2$ are on each side of the forward price but their geometric average coincides with the forward price. Therefore, the relevant symmetry property of the volatility smile is the following:

$$\sigma_t^*(x) = \sigma_t^*(-x) \text{ for any } x.$$  

(2.5)

In Proposition 2.1 below, we extend a result first stated in Renault and Touzi (1996), which characterizes the symmetry of the smile in terms of the option pricing function.

**Proposition 2.1.** If option prices are conformable to a homogeneous option pricing formula $x \to \pi(x)$, the volatility smile is symmetric ($\sigma^*(x) = \sigma^*(-x)$ for any $x$) if and only if, for any $x$:

$$\pi(-x) = e^x \pi(x) + 1 - e^x$$

Proof: See Appendix 1.

This characterization of the symmetry of the smile admits an equivalent formulation in terms of the pricing probability measure. While the pricing probability measure is usually characterized through the cumulative distribution function of $\frac{S_T}{S_t}$, it is convenient here to characterize it through either the cumulative distribution function $F_{\nu_T}(\cdot)$ or the probability density function $f_{\nu_T}(\cdot)$ of

---

Footnote: For sake of notational simplicity, the subscripts $t$ have been dropped.

---
\[ V_T = \log \frac{S_T B(t, T)}{S_t} \]

We are then able to prove (see Appendix 1) the following proposition:

**Proposition 2.2.** If the cumulative distribution function \( F_{VT}(.) \) of \( V_T \) under a pricing probability measure is absolutely continuous (associated with a density function \( f_{VT}(.) = F'_{VT}(.) \)) and such that \( \exp(V_T) \) is integrable, the volatility smile is symmetric if and only if one of the following three equivalent properties is fulfilled:

(i) For any \( x \):
\[
\pi(x) = F_{VT}(x) - e^{-x}[1 - F_{VT}(-x)]
\]

(ii) For any \( x \):
\[
F_{VT}(x) = E^*_t[e^{V_T}1_{V_T \geq -x}]
\]

(iii) There exists an even function \( g(.) \) such that for any \( x \):
\[
f_{VT}(x) = e^{-x/2}g(x)
\]

These characterizations offer various ways to extend the BS formula, while keeping both a homogeneous option pricing function and a symmetric smile. Characterization (i) provides a theoretical support to descriptive approaches which replace the standard normal cumulative distribution function of the BS formula by alternative distribution functions, possibly asymmetric. It shows that in order to keep a symmetric smile, the term \( 1 - F_{VT}(-x) \) should replace \( F_{VT}(x) \) in the second part of the option pricing formula. Characterization (ii) should be interpreted in terms of hedging. Indeed, Garcia and Renault (1998) have shown that \( E^*_t[e^{V_T}1_{V_T \geq -x}] \) is precisely the hedging ratio, in other words the derivative of the option pricing function with respect to the stock price (the so-called delta of the option)\(^8\). Finally, for characterization (iii), it should be noticed that if the pricing probability

---

\(^8\)Their proposition 2.1 shows that this characterization of the hedging ratio is a necessary and sufficient condition for homogeneous option pricing. Since hedging is not the primary focus of this paper, we leave to the reader the interpretation of this fairly natural relationship between \( F_{VT}(x) \) and the delta coefficient.
measure is characterized by a conditional log-normal distribution of future returns given available information at time $t$:

$$V_T = \log \frac{S_T B(t, T)}{S_t} | I_t \sim_{(q_t, T)} \mathcal{N}(\mu_t, \sigma_t^2),$$

the condition of Proposition 2.2 means that:

$$\mu_t = -\frac{\sigma_t^2}{2},$$

which is automatically fulfilled in the absence of arbitrage since, by application of (2.1) with $K = 0$, we have:

$$S_t = B(t, T) E^{*} S_T.$$

In the next subsection, we provide sufficient conditions on the pricing probability measure to ensure the homogeneity of the option pricing function in a convenient stochastic framework with state variables.

### 3. A generalized Black-Scholes and Hull-White formula with state variables

Merton (1973) stressed that the desirable homogeneity of option prices will be maintained as soon as asset returns are serially independent. This condition expressed in terms of the data generating process (DGP) of the underlying asset can be slightly generalized by expressing it in terms of the pricing probability measure. Given the general option pricing formula (2.1), the required homogeneity property amounts to the following conditional independence property (with respect to $Q_t, T$)\(^{9,10}\):

\(^9\)This condition for homogeneity is more general than the sufficient condition proposed by Merton (1973) since it is stated in terms of the pricing probability measure rather than the DGP. Indeed, we do not preclude a possible dependence of the risk premiums on the stock price $S_t$, which could violate assumption 3.1 for the DGP. Garcia and Renault (1998) offer a precise characterization of this property in the standard setting of continuous time arbitrage pricing.

\(^{10}\)Since $Q_{t,T}$ does not endow $I_t$ - measurable functions with a non degenerate probability distribution, we only mean by (3.1) that the $Q_{t,T}$ probability distribution of $\frac{S_T}{S_t}$ does not depend on past values of the stock price.
A way to generalize the BS and HW formulas while keeping the functional shape of the BS formula is to recover log-normality conditionally to the full path of a vectorial process \((U_t)_{t \in \mathbb{N}}\) of possibly unobserved state variables. Indeed, while the BS geometric Brownian motion world is obviously unrealistic due to heterogeneity factors, it is much more general to assume that log-normality is recovered after conditioning on a number of state variables. In particular, it is a standard way to capture the leptokurtic feature of financial time series\(^{11}\).

3.1. A State-Variable Framework for Homogeneous Option Pricing

State variables have two basic distinctive features: they are exogenous and they summarize the dynamics of the variables of interest (see Renault (1998) and Garcia and Renault (1999) for a general discussion). We provide below a definition in the context of a pricing probability measure \(Q_t, t = 1, \ldots, T\) without specifying its dynamics at this stage.

**Definition 3.1.** A vectorial process \((U_t)_{1 \leq t \leq T}\) (also written \(U^T_t\)) is called a state variable process\(^{12}\) with respect to the stock price process \((S_t)_{1 \leq t \leq T}\) and the family of pricing probability measures \((Q_t, T)_{1 \leq t \leq T}\) if for any \(t = 1, \ldots, T - 1\), the \(Q_t, T\) joint probability distribution of \((\frac{S_t}{S_t}, U_t, \tau > t)\) does not depend on the past of the price process \((S_t)_{1 \leq t \leq T}\).

\(^{11}\)Since Clark (1973), there is a long tradition of this approach in financial econometrics. Clark (1973) stressed that non-normality is a puzzle when one has in mind the geometric temporal averaging of the returns and a corresponding central limit theorem argument. In this respect, log normality of returns can be invoked without any significant loss of generality once it is recovered after conditioning on a sufficient number of state variables. As far as option pricing is concerned, the role of heterogeneity factors has been enhanced by Renault (1997) in a general setting which encompasses in particular Merton (1976) jump-diffusion model and Hull and White (1987) stochastic volatility model.

\(^{12}\)In many applications of the state variable concept, Markovianity is usually postulated. Then, the relevant conditioning information is summarized by a few recent lags of the state variable process. Since this Markovianity assumption is not needed at this stage, we maintain in full generality the whole past \(U^T_t\) of this process.
This definition implicitly means that $Q_{t,T}$ is a family of transition probabilities indexed by a conditioning set $I_t$ which includes the joint past $(S_t, U_t)_{t \leq T}$ of prices and state variables. By writing the usual factorization of probability density functions:

$$l \left[ \frac{S_T}{S_t}, U_{t+1}^T | I_t \right] = l \left[ U_{t+1}^T | I_t \right] \cdot l \left[ \frac{S_T}{S_t}, U_{t+1}^T | I_t \right] , \quad (3.2)$$

the above definition means that the state variable process $U_t$ is exogenous in the sense of being independent of the past of the price process conditionally to its own past and that the future returns are conditionally independent of past returns given the full path of state variables which, in this sense, summarize the dynamics of the price process. To make this point clear, in the particular case where the available information at time $t$ is described by the sigma field:

$$I_t = \sigma \left[ S_r, U_r, \tau \leq t \right], \quad (3.3)$$

the factorization (3.2) can be rewritten as follows, given the state variable concept just described above:

$$l \left[ \frac{S_T}{S_t}, U_{t+1}^T | I_t \right] = l \left[ U_{t+1}^T | I_t \right] \cdot l \left[ \frac{S_T}{S_t}, U_{t+1}^T | I_t \right] . \quad (3.4)$$

It is then clear in this context that (3.4) is a sufficient condition for the homogeneity property (3.1).

The symmetric smile condition of Proposition 2.2 is also valid in the more general setting just described. If $V_T$ follows under $Q_{t,T}$ a conditional Gaussian distribution $\mathcal{N}[\mu_t(U_t^T), \sigma^2_t(U_t^T)]$ given $I_t$ and the path $U_t^T$ (between $t$ and $T$) of some state variables $U$, the symmetry condition will be fulfilled (by integration over $U_t^T$) as soon as:

$$\mu_t(U_t^T) = -\frac{\sigma^2_t(U_t^T)}{2} .$$

This is the case for instance in an Hull and White world, which explains the main result of Renault and Touzi (1996): if option prices are conformable to the Hull and White option pricing formula, the volatility smile is symmetric. Proposition 2.2 characterizes precisely which type of symmetry of the pricing
probability measure is required for the symmetry of the smile. In particular, it shows that it is not the density of the log returns that should be symmetric (as it is commonly believed perhaps because of the usual log-normal setting), but the same density rescaled by a suitable exponential function.

3.2. An Option Pricing Formula with a General Stochastic Discount Factor

In order to generalize the Hull and White model, we will assume from now on that the pricing probability measure \( Q_{t,T} \) is defined through a stochastic discount factor (SDF) \( m_{t,T} \):

\[
\pi_t = E_t m_{t,T}(S_T - K)^+
\]  

(3.5)

where \( E_t \) denotes now the conditional expectation operator with respect to the DGP. Hansen and Richard (1987) provide some general conditions under which the existence and positivity of \( m_{t,T} \) is guaranteed. First, to ensure homogeneity of the option pricing formula, we will make two sufficient assumptions about the joint dynamics of the stochastic discount factor, the stock returns and the state variables.

**Assumption A:** The SDF \( m_{t,T} \) can be factorized as follows:

\[
m_{t,T} = \lambda_{t,T}(U_t^T) \prod_{r=t}^{T-1} m_{r+1},
\]  

(3.6)

where it is assumed that:

A1: The process \((m_{r+1}, \frac{S_{r+1}}{S_r})_{1 \leq r \leq T-1}\) does not cause the process \((U_r)_{1 \leq r \leq T};

A2: The variables \((m_{r+1}, \frac{S_{r+1}}{S_r})_{1 \leq r \leq T-1}\) are serially independent given \(U_t^T\).

The conditions (A1) and (A2) are consistent with the concept of state variables introduced in Definition 3.1. The non-causality property may be interpreted equivalently in Granger (1969) or in Sims (1972) terms. Granger causality means that, given the past \(U_t^T\) of state variables, the past observation of processes \(m_r\) and \(S_r\) does not bring any relevant information to forecast \(U_{t+1}\) which is in

\[\text{See Garcia and Renault (1999) for a detailed account of this point under both the pricing probability measure and the DGP.}\]
this sense exogenous). Sims causality means that the probability distribution of \( (m_{t+1}, S_{t+1}) \) given \( I_t \) and \( U_{t+1}^T \) does not depend upon \( U_{t+1}^T \). Jointly with the conditional independence assumption \( A2 \), assumption \( A1 \) permits to characterize the joint probability distribution of \( (m_{r+1}, S_{r+1}, U_{r+1}) \) given \( I_t \) as the following product:

\[
\ell \left[ (m_{r+1}, \frac{S_{r+1}}{S_r}, U_{r+1})_{r \geq t} \mid I_t \right] = \ell \left[ U_{r+1}^T \mid U_1^T \right] \ell \left[ (m_{r+1}, \frac{S_{r+1}}{S_r})_{r \geq t} \mid U_1^T \right] = \prod_{h=1}^{T-t} \ell \left[ U_{t+h} \mid U_1^{t+h-1} \right] \prod_{h=1}^{T-t} \ell \left[ m_{t+h}, \frac{S_{t+h}}{S_{t+h-1}} \mid U_1^{t+h} \right].
\]

**Proposition 3.2.** Under \( (A1) \) and \( (A2) \) there exists a deterministic function \( \Psi_{t,T} \) such that the option price (3.5) can be written as:

\[
\pi_t = \Psi_{t,T} \left[ U_1^T \frac{K}{S_t} \right] S_t.
\]

Proposition 3.2 establishes that the option pricing formula is homogeneous of degree one with respect to the pair \( (S_t, K) \).

To obtain a generalized BS and Hull-White option pricing formula starting from (3.5), one needs only, in addition to the previous assumptions \( (A1) \) and \( (A2) \), a joint log-normality assumption of \( m_t, S_t \) and \( S_T \) given \( I_t = \sigma[m_r, S_r, U_r, r \leq t] \) and a path \( U_{t+1}^T \) of state variables.

**Assumption A3:** The conditional probability distribution of \( (\log m_{t+1}, \log \frac{S_{t+1}}{S_t}) \) given \( U_1^{t+1} \) is, for \( t=1, \ldots, T-1 \), a bivariate normal:

\[
\mathcal{N} \left[ \begin{pmatrix} \mu_{m_{t+1}} \\ \mu_{s_{t+1}} \end{pmatrix}, \begin{pmatrix} \sigma_{m_{t+1}}^2 & \sigma_{m_{t+1}} \sigma_{s_{t+1}} \\ \sigma_{m_{t+1}} \sigma_{s_{t+1}} & \sigma_{s_{t+1}}^2 \end{pmatrix} \right].
\]

**Proposition 3.3.** Under assumptions A1, A2 and A3:

\[
\frac{\pi_t}{S_t} = \pi_t(x) = E_t \left\{ Q_m(t, T) \Phi(d_1(x)) - \frac{\bar{B}(t, T)}{B(t, T)} e^{-\frac{r}{2}} \Phi(d_2(x)) \right\}
\]

where:
\[
d_1(x) = \frac{x}{\tau_{t,T}} + \frac{1}{\tau_{t,T}} \log \left[ \frac{Q_{ms}(t,T) B(t,T)}{B(t,T)} \right]
\]
\[
d_2(x) = d_1(x) - \tau_{t,T}
\]
\[
\tau_{t,T}^2 = \sum_{t=1}^{T-1} \sigma^2_{\tau_{t+1}}.
\]

and:
\[
\tilde{B}(t,T) = \lambda_{t,T}(U^T_1) \exp \left( \sum_{t=1}^{T-1} \mu_{\tau_{t+1}} + \frac{1}{2} \sum_{t=1}^{T-1} \sigma^2_{\tau_{t+1}} \right),
\]
\[
Q_{ms}(t,T) = \tilde{B}(t,T) \exp \left( \sum_{t=1}^{T} \sigma_{\tau_{t+1}} \right) E \left[ \frac{S_T}{S_t} | U^T_1 \right].
\] (3.9)

To put this general option pricing formula in perspective, we will compare it to pricing formulas based on equilibrium or on the absence of arbitrage. Concerning the equilibrium approach, our setting is very general since it is based on a stochastic model for the SDF which does not rely on restrictive assumptions about preferences, endowments, or agent heterogeneity. Moreover, our factorization for the SDF is more general than the usual one \( m_{t,T} = \frac{\sigma^2_{\tau_{t+1}}}{\tau_{t,T}} \), with the standard interpretation as an intertemporal marginal rate of substitution. Actually, the additional factor \( \lambda_{t,T}(U^T_1) \) in our SDF allows to accommodate non-separable or state-dependent preferences. The non-separable case will be illustrated in the last section by a recursive utility setting. An example of state-dependent preferences could be habit formation based on state variables.

Of course, the benchmark option pricing formulas are based on the absence of arbitrage. Our general formula (3.8) nests a large number of preference-free extensions of the Black-Scholes formula. In particular if \( Q_{ms}(t,T) = 1 \) and \( \tilde{B}(t,T) = \prod_{\tau=t}^{T-1} B(\tau, \tau + 1) \), one can see that the option price (3.8) is nothing but the conditional expectation of the Black-Scholes price, where the expectation is computed with respect to the joint probability distribution of the rolling-over

interest rate $\bar{r}_{t,T} = - \sum_{\tau=t}^{T-1} \log B(\tau, \tau + 1)$ and the cumulated volatility $\bar{\sigma}_{t,T}$. This framework nests three well-known models. First, the most basic ones, the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic. Second, the Hull and White (1987) stochastic volatility extension, since $\bar{\sigma}_{t,T}^2 = \text{Var} \left[ \log \frac{S_{T+1}}{S_T} \right]$ corresponds to the integrated volatility $\int_t^T \sigma_u^2 du$ in the Hull and White continuous-time setting. Third, the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992). However, the usefulness of our general formula (3.8) comes above all from the fact that it offers an explicit characterization of instances where the preference-free paradigm cannot be maintained. Usually, preference-free option pricing is underpinned by the absence of arbitrage in a complete market setting. However, our SDF-based option pricing does not preclude incompleteness and points out in which cases this incompleteness will invalidate the preference-free paradigm. The only cases of incompleteness which matter in this respect occur precisely when at least one of the two following conditions:

$$Q_{ms}(t, T) = 1$$  \hspace{1cm} (3.10) \\
$$\bar{B}(t, T) = \prod_{\tau=t}^{T-1} B(\tau, \tau + 1)$$  \hspace{1cm} (3.11)

is not fulfilled.

In general, preference parameters appear explicitly in the option pricing formula through $\bar{B}(t, T)$ and $Q_{ms}(t, T)$ since these two quantities depend on the characteristics of the SDF: $\lambda_{t,T}(U_T^T), (\mu_{m_{\tau+1}}, \sigma_{m_{\tau+1}}, \sigma_{m_{\tau+1}})]_{t=t}^{T-1}$. However, in so-called preference-free formulas, it happens that these parameters are eliminated from the option pricing formula through the observation of the bond price and the stock price. Actually, the bond pricing formula and the stock pricing formula provide two dynamic restrictions relating the SDF characteristics to the bond and stock price processes. To avoid cumbersome notation, we will consider for the moment a one-period option price. In this case, the bond pricing equation is given by:
\begin{equation}
B(t, t+1) = E_t \left[ \tilde{B}(t, t+1) \right] 
\end{equation}

as shown in Appendix 2. Therefore, observing the bond price will make preference parameters in \( \tilde{B}(t, t+1) \) vanish from the option price as soon as \( \tilde{B}(t, t+1) = B(t, t+1) \), that is if and only if \( \tilde{B}(t, t+1) \) belongs to the information set \( I_t \).

A useful way of writing the stock pricing formula is:

\begin{equation}
E_t [Q_{ms}(t, t+1)] = 1,
\end{equation}

Therefore, similarly to the bond pricing, the stock pricing will make preference parameters vanish from the option price, as soon as \( Q_{ms}(t, t+1) \) is known at time \( t \) and therefore equal to one. From (3.9), we can then express the conditional expected stock return as:

\[
E \left[ \frac{S_{t+1}}{S_t} | I_t \right] = \frac{1}{B(t, t+1)} \exp[-\sigma_{ms,t+1}],
\]

which is very close to a standard conditional CAPM equation. Therefore, the fact that both \( \tilde{B}(t, t+1) \) and \( Q_{ms}(t, t+1) \) are known at time \( t \) produces both a preference-free option pricing formula and a CAPM-like stock pricing equation\(^{15}\).

To conclude, it should be stressed that even in an equilibrium framework with incomplete markets, option pricing is preference-free if and only if there is a kind of predictability property (\( \tilde{B}(t, t+1) \) and \( Q_{ms}(t, t+1) \) are known at time \( t \)) according to the terminology introduced by Amin and Ng (1993a)\(^{16}\). We will now show that the lack of such predictability corresponds in a general sense to a leverage effect.

4. The asymmetric distortions of the smile due to leverage effects

In this section, we focus on the option pricing formula provided by Proposition (3.3) to characterize the cases where the corresponding volatility smiles are sym-

\(^{15}\)A similar parallel is drawn in an unconditional two-period framework in Breeden and Litzenberger (1978).

\(^{16}\)Our characterization of preference free option pricing by a predictability property generalizes the one provided by Amin and Ng (1993a) since it does not depend upon a particular equilibrium setting with specific preferences and endowments.
metric. According to criterion i) of Proposition (2.2) for the symmetry of the smile, it means that the above option pricing formula:

$$\pi_t(x) = E_t \{ Q_{ms}(t,T) \Phi(d_1(x)) \} - e^{-x} E_t \left\{ \frac{\tilde{B}(t,T)}{B(t,T)} \Phi(d_2(x)) \right\}$$

can be written:

$$\pi(x) = F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)].$$

But it can be seen in the derivation of the option pricing formula (3.8) (see Appendix 2) that:

$$[1 - F_{V_T}(-x)] = E_t \left[ \frac{\tilde{B}(t,T)}{B(t,T)} \Phi(d_2(x)) \right]. \quad (4.1)$$

Therefore, a necessary and sufficient condition for symmetric smiles of (3.8) is that:

$$E_t \{ Q_{ms}(t,T) \Phi(d_1(x)) \} = F_{V_T}(x)$$
or equivalently:

$$E_t \{ Q_{ms}(t,T) \Phi(d_1(x)) \} = 1 - E_t \left\{ \frac{\tilde{B}(t,T)}{B(t,T)} \Phi(d_2(-x)) \right\}. \quad (4.2)$$

But from (3.8):

$$d_2(-x) = -d_1(x) + \frac{2}{\sigma_{t,T}} \text{Log} \left[ Q_{ms}(t,T) \frac{B(t,T)}{\tilde{B}(t,T)} \right].$$

Thus by taking into account that $E_t[\frac{\tilde{B}(t,T)}{B(t,T)}] = 1$, the symmetry criterion can be rewritten:

$$E_t \{ Q_{ms}(t,T) \Phi(d_1(x)) \} = E_t \left\{ \frac{\tilde{B}(t,T)}{B(t,T)} \Phi(d_1(x)) - \frac{2}{\sigma_{t,T}} \text{Log} \left[ Q_{ms}(t,T) \frac{B(t,T)}{\tilde{B}(t,T)} \right] \right\}.$$

We have therefore proven the following proposition:

**Proposition 4.1.** A necessary and sufficient condition for a symmetric volatility smile is the following identity:
\[ E_t \{ Q_{ms}(t, T) \Phi(d_1(x)) \} = E_t \left\{ \frac{\tilde{B}(t, T)}{B(t, T)} \Phi(d_1(x)) - \frac{2}{\sigma_{t,T}} \log \left[ Q_{ms}(t, T) \frac{B(t, T)}{\tilde{B}(t, T)} \right] \right\} \]  

(4.3)

A sufficient condition for a symmetric volatility smile is:

\[ Q_{ms}(t, T) = \frac{\tilde{B}(t, T)}{B(t, T)}, \]  

(4.4)

that is:

\[ E_t[\frac{S_{t}^{T}}{S_{t}^{T}|U_{t}^{T}}] = \frac{1}{B(t, T)} \left[ E_t[\exp\left( - \sum_{\tau=t}^{T-1} \sigma_{msr+1}\right)] \right] \]  

(4.5)

It should be stressed that the sufficient condition (4.4) is always fulfilled in expectation because, as shown in Appendix 2, the bond and stock pricing formulae are respectively given by:

\[ B(t, T) = E_t \left[ \tilde{B}(t, T) \right], \quad \text{and} \]  

(4.6)

\[ E_t[Q_{ms}(t, T)] = 1. \]  

(4.7)

Moreover, taking into account the highly nonlinear features of the two sides of the necessary and sufficient identity (4.3), Jensen effects are likely to violate it if (4.4) is not fulfilled. In other words, condition (4.6) (or equivalently (4.7)) appears at first sight not too far from being necessary.

In terms of expectations with respect to the available information \( I_t \) at time \( t \), a consequence of condition (4.7) is a CAPM-type pricing equation:

\[ S_t = B(t, T)E_t \left[ \frac{E[S_T|U_{t}^{T}]}{\exp\left( \sum_{\tau=t}^{T-1} \sigma_{msr+1}\right)} \right] \]  

(4.8)

This is something like a martingale restriction (see Longstaff, ) on the stock price at time \( t \) which, if violated, is likely to introduce skewness in the volatility smile. In the particular setting of an Hull and White model, Renault (1997)
provides simulation evidence to show that a very small discrepancy (as low for example as 0.1 per cent) between the stock price \( S_t \) and its model-based theoretical value may induce severe skewness in the smile. In order to interpret this restriction, one may notice that the general stock pricing equation (4.7) only implies that:

\[
S_t = E_t \left[ \bar{B}(t, T) E[S_T | U_T^T] \exp\left[ \sum_{\tau=t}^{T-1} \sigma_{\tau+1} \right] \right]
\] (4.9)

In other words, by taking into account the bond pricing equation (4.6), the asymmetries of volatility smiles that we capture here are produced by a nonzero conditional covariance:

\[
\text{Cov}_t \left[ \bar{B}(t, T), E[S_T | U_T^T] \exp\left[ \sum_{\tau=t}^{T-1} \sigma_{\tau+1} \right] \right] \neq 0.
\] (4.10)

This covariance represents a first channel through which the various leverage effects described in subsection 2.4 above result in asymmetric distortions of the volatility smile. This channel is basically a correlation between the interest rate risk and the stock risk, measured as idiosyncratic risk as well as market risk.\(^{17}\) Of course, another source at the origin of the asymmetric distortions of the volatility smile stems from all the random features (given \( I_t \)) of \( Q_{ms}(t, T) \) which are not summarized by the interest rate risk in \( \bar{B}(t, T) \). Typically, according to (3.9), the cause is the lack of predictability at time \( t \) of \( E[S_T | U_T^T] \) and \( \sum_{\tau=t}^{T-1} \sigma_{\tau+1} \). However, it should be remembered that, as long as the SDF can be factorized as in the classical case of time-separable preferences \( m_{t, T} = \frac{m_T}{\eta_T} \), we have:

\[
Q_{ms}(t, T) = \prod_{\tau=t}^{T-1} Q_{ms}(\tau, \tau + 1),
\] (4.11)

and then:

\(^{17}\)As an example, Platten and Schweizer (1997) introduced asymmetry in the volatility smile by a dependence between the diffusion process of the stock price and the technical demand induced by hedging strategies. The latter should introduce a conditional correlation between the interest rate risk and the stock price risk similar to what is captured in (4.10).
\[ Q_{ms}(t, T) = 1 \text{ if } Q_{ms}(\tau, \tau + 1) = 1 \text{ for } \tau = t, t + 1, ..., T - 1. \] (4.12)

In other words, only leverage effects at the stock price level (for both idiosyncratic and market risk) can produce in this case asymmetric volatility smiles.

5. A Comparison with Stochastic Volatility Models with Leverage

The predictability property referred to in the previous sections amounts to the knowledge at time \( t \) of the quantities \( \tilde{B}(t, t + 1) \) and \( Q_{ms}(t, t + 1) \) which can be written as:

\[
\begin{align*}
\tilde{B}(t, t + 1) &= \lambda_{t,t+1}(U^{t+1}_1) E_t[m_{t+1} | U^{t+1}_1] = \lambda_{t,t+1}(U^{t+1}_1) \exp \left[ \mu_{mt+1} + \frac{\sigma^2_{mt+1}}{2} \right] \\
Q_{ms}(t, t + 1) &= \tilde{B}(t, t + 1) \exp(\sigma_{ms,t+1}) E_t \left[ \frac{S_{t+1}}{S_t} | U^{t+1}_1 \right] \\
&= \tilde{B}(t, t + 1) \exp(\sigma_{ms,t+1}) \exp \left[ \mu_{st+1} + \frac{\sigma^2_{st+1}}{2} \right] 
\end{align*}
\] (5.1)

where the second equalities follow from assumption A3. Therefore, the crucial issue for predictability, which corresponds to preference free option pricing, is to determine if the parameters \( \mu_{mt+1}, \mu_{st+1}, \sigma^2_{ms,t+1}, \sigma^2_{st+1} \) and \( \sigma_{ms,t+1} \) of the joint conditional probability distribution of \( (m_{t+1}, S_{t+1}) \) given \( U^{t+1}_1 \) depend or not upon \( U_{t+1} \), that is if there is or not an instantaneous causality relationship between the state variable process \( U \) and the process \( (m_{t+1}, S_{t+1}) \). When the factor \( \lambda_{t,t+1}(U^{t+1}_1) \), possibly introduced in the SDF by a lack of intertemporal separability, does not depend on \( U_{t+1} \), there is a perfect equivalence between the absence of instantaneous causality and preference free option pricing\(^\dagger\).

\(^\dagger\)Actually, a more complete specification of the law of motion of the economy (see examples in Section 4 below and in Garcia and Renault, 2000) shows that often such an absence of instantaneous causality will per se imply that \( \lambda_{t,t+1}(U^{t+1}_1) \) does not depend upon \( U_{t+1} \). In other words, we are generally allowed to claim that preference-free option pricing is equivalent to this absence of instantaneous causality.
5.1. Returns Volatility and Leverage Effect

We will now check whether this property also eliminates any kind of leverage effect, that is any evidence of negative correlation between stock returns and changes in returns volatility. Volatility tends to rise in response to bad news (excess returns lower than expected) and to fall in response to good news (excess returns higher than expected), as pointed out in Nelson (1991). In our setting, returns volatility can be defined in three ways. The first and most obvious one is the individual stock return volatility process:

\[
\begin{align*}
    h_t^s &= \text{Var} \left[ \log \frac{S_{t+1}}{S_t} | I_t \right] \\
    &= E \left[ \sigma^2_{m_{t+1}} | U_t^s \right] + \text{Var} \left[ \mu_{m_{t+1}} | U_t^s \right]
\end{align*}
\]

This stock volatility dynamics can give rise to a micro-level leverage effect. However, both on empirical grounds and theoretical aggregation arguments\(^\text{19}\), leverage effects may be more important at an aggregate level. One way to capture this aggregate leverage effect is to consider the SDF volatility process:

\[
\begin{align*}
    h_t^m &= \text{Var} \left[ \log m_{t+1} | I_t \right] \\
    &= E \left[ \sigma^2_{m_{t+1}} | U_t^m \right] + \text{Var} \left[ \mu_{m_{t+1}} | U_t^m \right].
\end{align*}
\]

Note that this volatility process may be interpreted itself as a return volatility process if there exists a return on a portfolio (e.g. the market return) which mimics this SDF. In the conditional setting we consider here, the existence of such a mimicking portfolio may be guaranteed under very general conditions (see Hansen and Richard, 1987, Garcia and Renault, 2000).

A third measure of volatility may be defined in terms of non diversifiable risk. Let us consider for instance the simple case where there is no aggregate leverage effect (because there is no instantaneous causality between the state variable process \(U\) and the SDF process \(m\)) and instantaneous changes in return volatility do not go through the stock variance process (\(\sigma^2_{m_{t+1}}\) does not depend

\(^{19}\text{See Bakshi and Madan}\)
upon $U_{t+1}$). Then, the stock pricing formula (2.19) can still be written in a form close to the standard conditional CAPM equation:

\[
E \left[ \frac{S_{t+1}}{S_t} \mid I_t \right] = \frac{1}{B(t, t + 1)} \frac{1}{E \left[ \exp \left( \sigma_{mt+1} \right) \mid U^*_t \right]} \left( 1 - \exp \left( \sigma^2_{st+1} \right) \frac{\text{Cov} \left[ \exp \mu_{st+1}, \exp \sigma_{mt+1} \mid U^*_t \right]}{E \left[ \exp \left( \sigma_{mt+1} \right) \mid U^*_t \right]} \right),
\]

In other words,

\[
h_{t}^{ns} = \left[ E_t \left[ \exp \left( \sigma_{mt+1} \right) \mid U^*_t \right] \right]^{-1}
\]

is the relevant measure of non-diversifiable risk. It defines the part of the stock return volatility which is compensated in equilibrium, up to a correction for what we will call “non diversifiable leverage effect”, that is a non zero conditional covariance between $\left( \exp \mu_{st+1} \right)$ and $\left( \exp \sigma_{mt+1} \right)$.

To summarize, we propose to extend the usual definition of leverage effect proposed by Nelson (1991) to any positive or negative conditional covariance, given $I_t^{20}$, between the mean returns and the three concepts of volatility just defined, that is $\text{Cov} \left[ \mu_{st+1}, h_{t+1}^{st} \mid U^*_t \right]$, $\text{Cov} \left[ \mu_{st+1}, h_{t+1}^{ms} \mid U^*_t \right]$, and $\text{Cov} \left[ \mu_{mt+1}, h_{t+1}^{ns} \mid U^*_t \right]$. In other words, leverage effect may occur if and only if one of the three following properties is fulfilled: (i) Both $\mu_{st+1}$ and $\sigma^2_{st+1}$ depend upon $U_{t+1}$ (standard leverage effect for the individual stock); (ii) Both $\mu_{st+1}$ and $\sigma_{mt+1}$ depend upon $U_{t+1}$ (leverage effect for the individual stock but only through its non-diversifiable risk); (iii) Both $\mu_{mt+1}$ and $\sigma^2_{mt+1}$ depend upon $U_{t+1}$ (standard leverage effect for the portfolio return which mimics the SDF, that is aggregate leverage effect).

Moreover, our state variable setting offers a very flexible framework for parametric models of leverage effect. In standard stochastic volatility models (see Ghysels, Harvey and Renault, 1997, for a survey), the usual leverage effect is generally captured by a (negative) constant linear conditional correlation coefficient

\[^{20}\text{Of course, some nonlinearity effects may create nontrivial relations between conditional covariances of interest like } \text{Cov} \left[ \mu_{st+1}, h_{t+1}^{ns} \mid U^*_t \right] \text{ and } \text{Cov} \left[ \exp \mu_{st+1}, \exp \sigma_{mt+1} \mid U^*_t \right]. \text{ These effects are neglected in our informal comments.} \]
between $\log \frac{S_{t+1}}{S_t}$ and $h_{t+1}^*$ (given $I_t$). In our setting, this correlation coefficient depends on the two functions $\mu_{st+1}$ and $\sigma^2_{st+1}$ of the state variables $U_{1t+1}$ and upon the exogenous dynamics of these state variables through:

$$\text{Cov}_t \left[ \log \frac{S_{t+1}}{S_t}, h_{t+1}^* \right] = \text{Cov} \left\{ \mu_{st+1}, E \left[ \sigma^2_{st+2} \mid U_{1t+1} \right] + \text{Var} \left[ \mu_{st+2} \mid U_{1t+1} \right] \mid U_t \right\}$$

and the corresponding conditional variances. In other words, the leverage effect of the return process $\log \frac{S_{t+1}}{S_t}$, which features stochastic volatility, can come from two sources$^{21}$. The conditional mean process $\mu_{st+1}$ may be a stochastic volatility process which features a leverage effect defined by the negativity of $\text{Cov} \left[ \mu_{st+1}, \text{Var} \left[ \mu_{st+2} \mid U_{1t+1} \right] \mid U_t \right]$. Or, the process $\log \frac{S_{t+1}}{S_t}$ itself may be characterized by a leverage effect and then:

$$\text{Cov} \left[ \mu_{st+1}, E \left[ \sigma^2_{st+2} \mid U_{1t+2} \right] \mid U_t \right]$$

will be negative, which means that bad news about expected return (when $\mu_{st+1}$ is smaller than its unconditional expectation) imply in average a higher expected volatility of $\log \frac{S_{t+1}}{S_t}$, that is a value of $E \left[ \sigma^2_{st+2} \mid U_{1t+2} \right]$ greater than its unconditional mean.

We have seen that assumption A3 not only allows to capture the standard features of a stochastic volatility model (in terms of heavy tails and leverage effects) but also provides for a richer set of possible dynamics. Moreover, we can certainly extend these ideas to multivariate dynamics either for the joint behavior of market and stock returns or for any portfolio consideration. For instance, the dependence of $\sigma_{msst+1}$ on the whole set of state variables offers great flexibility to model the stochastic behavior of correlation coefficients, as recently put forward empirically by Andersen et al. (1999). This last feature is clearly highly relevant for asset allocation or conditional beta pricing models. In section 4 below, we will see that a simple Markov switching model offers a versatile framework to capture the exogenous dynamics of the state variables.

It is worth noting that our results of equivalence between preference-free option pricing and no instantaneous causality between state variables and asset returns are consistent with another strand of the option pricing literature, namely

$^{21}$This decomposition of the leverage effect in two terms is the exact analogue of the decomposition discussed in Fiorentini and Sentana (1998) and Meddahi (1999) for persistence.
GARCH option pricing. Duan (1995) derived it first in an equilibrium framework, but Kallsen and Taqqu (1994) have shown that it could be obtained with an arbitrage argument. Their idea is to complete the markets by plugging the discrete-time model into a continuous time one, where conditional variance is constant between two integer dates. They show that such a continuous-time embedding makes possible arbitrage pricing which is per se preference-free. It is then clear that preference-free option pricing is incompatible with the presence of an instantaneous causality effect, since it is such an effect that prevents the embedding used by Kallsen and Taqqu (1998).

5.2. An Illustration with Discrete State Markov Latent Variables

To illustrate with analytical formulas the skewness of the returns distribution and the leverage effects, we assume that the latent variables follow a two-state Markov switching process. Therefore, we assume that the log return can be written as:

\[
\log \frac{S_{t+1}}{S_t} = Y_{t+1} = \mu_0 (1 - U_{t+1}) + \mu_1 U_{t+1} + [\sigma_0 (1 - U_{t+1}) + \sigma_1 U_{t+1}]\varepsilon_{st+1}, \quad (5.2)
\]

the variable \( U_{t+1} \) taking the values 0 or 1 with the probabilities \( \pi_0 \) and \( \pi_1 \). The transition probabilities between states \( i \) and \( j \) are defined as: \( p_{ij} = \Pr[U_{t+1} = j|U_t = i] \), with \( p_{ij} = 1 - p_{ii} \) for \( i \neq j \), and \( \pi_i = \frac{1 - p_{ii}}{2 - p_{ii} - p_{jj}} \), \( i, j = 0, 1 \).

The returns are therefore a mixture of normals with two different means and variances. It is well-known that such a mixture model will generate skewness and excess kurtosis in both the conditional and unconditional distributions of returns. We are interested in the conditional distribution of returns \( Y_{t+1} \) given \( I_t \), the information available at time \( t \). The moments of \( Y_{t+1} | I_t \) are defined as:

\[
E[Y_{t+1} | I_t] = m^{(1)}_i \text{ if } U_t = i \quad (5.3)
\]

\[
E\{[Y_{t+1} - E(Y_{t+1} | I_t)]^n | I_t\} = m^{(n)}_i \text{ if } U_t = i \quad (5.4)
\]

Given the process assumed for \( Y_{t+1} \) and \( U_t \), the first three moments of this are given by:
\[ m_i^{(1)} = \begin{cases} p_{00} \mu_0 + (1 - p_{00}) \mu_1 & \text{if } i = 0 \\ (1 - p_{11}) \mu_0 + p_{11} \mu_1 & \text{if } i = 1 \end{cases} \] (5.5)

\[ m_i^{(2)} = \begin{cases} p_{00} \sigma_0^2 + (1 - p_{00}) \sigma_1^2 + p_{00}(1 - p_{00})(\mu_1 - \mu_0)^2 & \text{if } i = 0 \\ (1 - p_{11}) \sigma_0^2 + p_{11} \sigma_1^2 + p_{11}(1 - p_{11})(\mu_1 - \mu_0)^2 & \text{if } i = 1 \end{cases} \] (5.6)

\[ m_i^{(3)} = \begin{cases} p_{00}(1 - p_{00})(\mu_0 - \mu_1)[3(\sigma_0^2 - \sigma_1^2) + (1 - 2p_{00})(\mu_1 - \mu_0)^2] & \text{if } i = 0 \\ p_{11}(1 - p_{11})(\mu_1 - \mu_0)[3(\sigma_1^2 - \sigma_0^2) + (1 - 2p_{11})(\mu_1 - \mu_0)^2] & \text{if } i = 1 \end{cases} \] (5.7)

We want to compare the skewness, defined as:

\[ sk_i = \frac{m_i^{(3)}}{[m_i^{(2)}]^2} \] (5.8)

with the leverage effect that we defined as \( \text{Cov} \left[ \mu_{Y_{t+1}}, h_{t+1}^Y \left| U_t^i \right. \right] \). For the Markov case, we have:

\[ h_{t+1}^Y = m_i^{(2)} \text{ if } U_t = i \] (5.9)

\[ \mu_{t+1}^Y = \mu_0(1 - i) + \mu_1 i \text{ if } U_t = i. \] (5.10)

Therefore, we can write:

\[ \text{Cov} \left[ \mu_{Y_{t+1}}, h_{t+1}^Y \left| U_t^i \right. \right] = E_t \left[ m_{U_t+1}^{(2)}(\mu_0(1 - U_{t+1}) + \mu_1 U_{t+1}) \right] - m_{U_t}^{(1)} E_t m_{U_{t+1}}^{(2)}. \] (5.11)

After some algebraic manipulations, we obtained the following expressions for the leverage effect:

\[ \text{Cov} \left[ \mu_{Y_{t+1}}, h_{t+1}^Y \left| U_t^i \right. \right] = \begin{cases} p_{00}(1 - p_{00})(\mu_0 - \mu_1)[(\sigma_0^2 - \sigma_1^2)(p_{00} - (1 - p_{11})) & \text{if } i = 0 \\ + (\mu_1 - \mu_0)(p_{00}(1 - p_{00}) - p_{11}(1 - p_{11})) & \text{if } i = 0 \\ p_{11}(1 - p_{11})(\mu_1 - \mu_0)[(\sigma_1^2 - \sigma_0^2)(p_{11} - (1 - p_{00})) & \text{if } i = 1 \\ + (\mu_1 - \mu_0)(p_{11}(1 - p_{11}) - p_{00}(1 - p_{00})) & \text{if } i = 1 \end{cases} \] (5.12)
First, it appears clearly that the formulas for conditional skewness and leverage effect in the stock are very similar. In both the skewness and leverage expressions, irrespective of the initial state, there is a term in \((\mu_1 - \mu_0)(\sigma_1^2 - \sigma_0^2)\) and another term in \((\mu_1 - \mu_0)^3\). We would like to characterize the bad state as the state where the volatility is high (say \(\sigma_1^2 > \sigma_0^2\)) and the mean is low (\(\mu_1 < \mu_0\)). With such a characterization, the skewness in the bad state (state 1) will be negative as long as the bad state is not too persistent \((p_{11} < \frac{1}{2})\). On the contrary, for the good state, the skewness will be negative when the good state is persistent \((p_{00} > \frac{1}{2})\). The conditions for the leverage effect are a little more complex, but it will be negative in both states if \(p_{00} + p_{11} > 1\) and \(p_{11}(1 - p_{11}) > p_{00}(1 - p_{00})\). The first condition implies some persistence in at least one of the states while the second requires more persistence in the good state. This is consistent with the conditions for skewness. These conditions for negative leverage are consistent with interpretations of the states as business cycle states or bull and bear markets, where typically the good state is more persistent. Of course these are only sufficient conditions. The skewness and the leverage effects can still be negative if they are not met provided the means and variances are of the right magnitude.

6. A characterization of the smiles with a structural option pricing model

In this section, to offer a setup that leads to a computable formula, we provide an equilibrium version of the SDF with preferences in the recursive utility class (Epstein and Zin, 1989). These preferences are richer than the usual expected utility model. In particular, the elasticity of intertemporal substitution is disentangled from the risk aversion parameter. From a practical point of view, this additional parameter might help better explain prices of long-term options such as LEAPS (Long-term Equity Anticipation Securities). In this model, the latent state variables affect the fundamentals of the economy and follow a discrete-state Markov process as in the previous section.
6.1. The equilibrium stochastic discount factor

In the recursive utility framework of Epstein and Zin (1989, 1991), the stochastic discount factor is given by:

\[ m_{t+1} = \beta^\gamma \left( \frac{C_{t+1}}{C_t} \right)^{\gamma (\alpha - 1)} M^{-1}_{t+1} \]  

(6.1)

where \( \frac{C_{t+1}}{C_t} \) is the growth rate of consumption in the economy and \( M_{t+1} \) represents the return on the market portfolio. The parameters \( \beta, \gamma \) and \( \rho \) are preference parameters. The parameter \( \beta \) is the subjective rate of time preference, while \( \alpha = \gamma \rho \) can be interpreted as a relative risk aversion parameter with the degree of risk aversion increasing as \( \alpha \) falls (\( \alpha \leq 1 \)). The parameter \( \rho \) is associated with intertemporal substitution, since the elasticity of intertemporal substitution is \( 1/(1-\rho)^2 \). The position of \( \alpha \) with respect to \( \rho \) determines whether the agent has a preference towards early resolution of uncertainty (\( \alpha < \rho \)) or late resolution of uncertainty (\( \alpha > \rho \)). When \( \gamma = 1 \), we obtain the well-known stochastic discount factor for the expected utility case \( m_{t+1} = \beta (\frac{C_{t+1}}{C_t})^{\alpha - 1} \).

Given this stochastic discount factor, the price of a European option \( \pi_t \) maturing at time \( T \) is given by:

\[ \frac{\pi_t}{S_t} = E_t \left\{ Q_{XY}(t, T) \Phi(d_1) - \frac{K \tilde{B}(t, T)}{S_t} \Phi(d_2) \right\}, \]  

(6.2)

where:

\[ d_1 = \frac{\log \left( \frac{S_t Q_{XY}(t, T)}{K B(t, T)} \right)}{\left( \sum_{\tau=t+1}^{T} \sigma_{Y,\tau}^2 \right)^{1/2}} + \frac{1}{2} \left( \sum_{\tau=t+1}^{T} \sigma_{Y,\tau}^2 \right)^{1/2}, \]  

and \( d_2 = d_1 - \left( \sum_{\tau=t+1}^{T} \sigma_{Y,\tau}^2 \right)^{1/2} \).

and:

\[ \tilde{B}(t, T) = \beta^{(T-t)} a^T_t (\gamma) \exp((\alpha - 1) \sum_{\tau=t+1}^{T} m_{X,\tau} + \frac{1}{2} (\alpha - 1)^2 \sum_{\tau=t+1}^{T} \sigma_{X,\tau}^2), \]  

(6.3)

As mentioned in Epstein and Zin (1991), the association of risk aversion with \( \alpha \) and intertemporal substitution with \( \rho \) is not fully clear, since at a given level of risk aversion, changing \( \rho \) affects not only the elasticity of intertemporal substitution but also determines whether the agent will prefer early or late resolution of uncertainty.
with: $a_t^T(\gamma) = \prod_{t=1}^{T-1} \left[ \frac{(1+\lambda_t U_t^{t+1})}{\lambda_t U_t^T} \right]^{\gamma-1}$, and

$$Q_{XY}(t, T) = \tilde{B}(t, T) b_t^T \frac{\varphi(U_t^t)}{\varphi(U_t^T)} \exp\left((\alpha - 1) \sum_{\tau=t+1}^{T} \sigma_{XY, \tau} \mathbb{E} \left| \frac{S_{\tau}^T}{S_t^T} U_t^T \right| \right). \quad (6.4)$$

with: $b_t^T = \prod^{T-1}_{t=1} \frac{(1+\varphi(U_t^{t+1}))}{\varphi(U_t^2)}$.

We define $\lambda_t = \lambda(I_t) = \frac{P_t^M}{C_t}$ and $\varphi_t = \varphi(I_t) = \frac{S_t}{D_t}$ (with $D_t$ the dividend on the stock) as the solutions to Euler equations for the price of the market portfolio $P_t^M$ and the price of the stock, and $X_t = \log \frac{C_t}{C_{t-1}}$ and $Y_t = \log \frac{D_t}{D_{t-1}}$.

Our setup provides a framework to capture the empirically documented relationship of the asymmetries with the business cycle and interest rate movements (see for instance the survey by Bates (1996)). More importantly, since the violations of the symmetry condition in Proposition 4.1 (due to interest rate risk or the occurrence of a leverage effect in the general sense above) correspond precisely to cases where preference parameters matter for option pricing, an observed asymmetric smile could be indicative of the relevance of structural parameters to price options.

### 6.2. Characterization of the Smiles

In this section, we use the equilibrium framework just described to study by simulation the various shapes of the implied volatility curves that can results from the equilibrium options pricing formula. In particular, we will analyze the sensitivity of the smile skewness to various parameters entering the formula, in particular the parameters of the stochastic process driving the fundamentals and the preference parameters.

#### 6.2.1. A Markov-Chain Setup for the State Variables

We endow the state variable with a discrete Markov chain structure. This stochastic structure has two main advantages; first it is possible to obtain the option price analytically and secondly, from this framework we can easily gain the intuition of the various smile effects. The equilibrium market portfolio and stock prices
are determined by consumption and dividend growth which in turn depend on the evolution of the state variable. The process describing the joint evolution of $X_t = \log C_t/C_{t-1}$ and $Y_t = \log D_t/D_{t-1}$ is parameterized as follows:

$$
X_t = m_X(U_t) + \sigma_X(U_t)\epsilon_{Xt} \\
Y_t = m_Y(U_t) + \sigma_Y(U_t)\epsilon_{Yt}
$$

The vector $(\epsilon_{Xt}, \epsilon_{Yt})'$ follows a standard bivariate normal distribution with correlation coefficient $\rho_{XY}$. The time-varying mean and variance parameters are a function of the state variable process $\{U_t\}$, which is assumed to be a discrete first-order Markov chain such that $U_t$ takes values in $\{1, \ldots, N\}$ with $\Pr(U_t = j) = \sum_{i=1}^{N} p_{ij} \Pr(U_{t-1} = i)$ and transition probability $p_{ij} = \Pr(U_t = j|U_{t-1} = i)$ for $i, j = 1, \ldots, N$.

For given values of the transition probabilities of the Markov chain governing the state variable process and given values of the structural parameters, it is possible to compute the price of an option according to the generalized Black-Scholes and a fortiori the generalized Hull-White formula. The steps followed to compute option prices are detailed in Appendix 3.

6.2.2. State variables and the smile

We consider the case where the state variable $U_t$ takes values in the set $\{1, 2\}$ and is governed by a first-order Markov chain with a symmetric transition probability matrix $[p_{ii} = 0.8]$. The state-contingent parameter values of the consumption and dividend processes are set as $m_{X1} = 0.0015$, $m_{X2} = -0.0009$, $\sigma_{X1} = \sigma_{X2} = 0.003$, $m_{Y1} = m_{Y2} = 0$, $\sigma_{Y1} = 0.02$, $\sigma_{Y2} = 0.12$, and $\rho_{XY} = 0.6$. With this specification, $U_t = 1$ can be interpreted as an expansionary state where consumption growth is positive and stock market volatility is low. On the other hand the recessionary state, $U_t = 2$, is characterized by negative consumption growth and a relatively more volatile stock market. The preference parameters are set as follows: $\gamma = 1$, $\rho = -10$, and hence $\alpha = -10$. This configuration is taken as the benchmark for comparison. In the following we explore the various implications that the generalized option pricing model has in terms of the volatility smile.
Stochastic volatility and stochastic interest rates  The first implications for the volatility smile that we explore are those arising from the state variable. Figures 2 through 4, illustrate these effects. In order to illustrate the effects due to maturity and stochastic volatility, consider the case where the discount factor $B(t, T)$ is deterministic and where the factor $Q(t, T)$ is unity. The interest rate risk is eliminated by holding constant $m_{X_t}$ and $\sigma_{X_t}$ which implies constant $\lambda_t$ and in turn a deterministic discount factor $B(t, T)$. The left panel of Figure 2 illustrates the smile effects arising from stochastic volatility. The three curves show the effect of decreases in the coefficient of variation of the volatility from 0.71 to 0.60 to 0.33 with the flattest representing the least variation. As expected if the coefficient of variation of the volatility goes to zero, Black-Scholes pricing results and the implied volatility curve would be completely flat.

Now holding the coefficient of variation of the volatility constant, the right panel of Figure 2 illustrates the effect of increasing the option's maturity. The most curved smiles in each panel are in fact identical representing a one-period option. The two other curves in the right panel represent options whose maturity is increased by one and two periods with the flattest being associated with the three-period option. We see that as the option’s maturity increases, the smile fattens.

Figure 3 illustrates the maturity effect when $Q(t, T) \neq 1$. For comparison, the figure illustrates also the smiles that obtain from preference-free option pricing à la Hull-White. The latter smiles are distinguished by their symmetry. The solid lines are the implied volatility curves for a one-period option whereas the dashed lines are for a two-period option. The left and right panels are associated with states 1 and 2, respectively, as the current state. It is seen that the maturity effect depends on the current state: when state 1 is operative at time $t$, an increase in maturity results in flatter yet greater implied volatilities, while when in state 2 the flatter implied volatility curves associated with longer maturities are lower. It should be noticed that the smiles are moving to the right in both states. If one considers the expressions developed in section 5.2, this is indicative of a negative skewness or leverage effect due to the persistence that we assumed for both states.

Next consider the case where $B(t, T)$ is stochastic and $Q(t, T) \neq 1$. The lines
and panels of Figure 4, similar to those of the preceding figure, illustrate this case. Comparing the respective lines of figures 3 and 4, reveals that stochastic interest rates imply greater asymmetry in the smile. This is consistent with the presence of a non-zero covariance between $\tilde{B}(t, T)$ and the stock as illustrated in equation (4.10).

An important remark is that at longer maturities, the smile is more asymmetric than at shorter maturities. This feature is apparent by noticing the point of intersection between the symmetric and asymmetric smiles at the respective maturities.

**Average duration and correlation effects** Figures 5 and 6 illustrate the effect of changes in the persistence of each state on the implied volatility curves. Figure 4 considers the case where the probability of staying within a given state is greater than its exit probability, $p_{ii} > p_{ij}$. The dashed lines represent an increase in $p_{ii}$ from 0.8 to 0.9 and, as previously the left and right panels are associated with states 1 and 2, respectively, as the current states. From each panel it is seen that an increase in the persistence, or average duration $(1 - p_{ii})^{-1}$, of each state results in a greater asymmetry of the smile. It is interesting to note from the right panel of Figure 4 that frowns obtain. In Figure 6 we consider the diametrically opposite case where $p_{ii} < p_{ij}$. Increasing $p_{ij}$ from 0.8 to 0.9 in this case has the same effects in terms of asymmetric distortions of the smile as in Figure 4 but with the roles of current state reversed: Frowns in this case arise when state 1 is the time $t$ operative state.

Figure 7 graphs the schedule of option prices across moneyness for two extreme values of $\rho_{XY}$ the correlation between consumption and dividends. The solid line represents the case $\rho_{XY} = 1$ while the dashed line is for $\rho_{XY} = 0$. Regardless of which state is operative at time $t$, an decrease in the correlation between consumption and dividends results in an upward shift of the entire schedule of option prices. Intuitively when the stock is perfectly correlated with the market portfolio, there is one less source of risk to hedge and hence option prices are lower reflecting the smaller risk premium in this case.
Leverage effects  As we have seen, the preference-free option pricing formula à la Hull-White obtains when there are no leverage effects, neither through the market risk nor through the stock risk, that is, when $l(X_t, Y_t|U_t) = l(X_t, Y_t|U_{t-1})$. Figures 8 and 9 illustrate the implications that these leverage effects separately have in terms of the volatility smile. These implications are explored in the context of a two-period option which is the shortest horizon one can consider here since for a single-period option, absence of leverage through the stock risk implies Black-Scholes pricing. We preferred to keep the maturity horizon as short as possible in order to minimize the computation time.

In Figure 8 the dashed lines show the implications that an absence of leverage through the consumption process has on the volatility smile, i.e. $l(X_t|U_t^T) = l(X_t|U_{t-1}^T)$. The solid lines are the benchmark case of leverage effects through both the consumption and dividend processes. Notice that the symmetric smiles resulting from preference-free option pricing à la Hull-White are identical whether or not there is leverage through consumption. Absence of leverage through the consumption process leads to a more asymmetric smile as is apparent from both panels of the figure that, as before, are conditional on the current operative state.

Similarly, in Figure 9 the dashed lines show the effect of an absence of leverage through the dividend process, i.e. $l(Y_t|U_t^T) = l(Y_t|U_{t-1}^T)$. Again we observe an asymmetric smile, except that in this case the asymmetric distortion is far more pronounced than in the previous figure. In other words, leverage through the divided process plays a much more important role than that through the consumption process insofar that the symmetry of the smile is concerned.

6.2.3. Preferences and smile effects

We now proceed to investigate the role played by the preference parameters. We will see that preferences, in particular intertemporal substitution, can have an effect in terms of the asymmetry of the volatility smile but that they play a secondary role compared to that played by the state variable. This is reassuring if one believes that such parameters should stay relatively stable over time. The benchmark for comparison here is the expected utility case which obtains when $\gamma = 1$. This was in fact the case up to this point. For example, recall that the
preferences underlying Figure 3 are $\gamma = 1$, $\rho = -10$, and hence $\alpha = -10$.

Consider now Figure 10 where the solid lines are the volatility curves for the following configuration of preference parameters: $\gamma = 1$, $\rho = -1$, $\alpha = -1$; whereas the dashed lines are for $\gamma = 1/10$, $\rho = -10$, $\alpha = -1$. Notice that the symmetric preference-free smiles are necessarily identical under both preference configurations. Comparing the solid lines of Figure 10 with those of Figure 4, we see that reducing $\rho$ (and hence $\alpha$ since $\gamma = 1$) from $-1$ to $-10$ leads to greater asymmetry in the smile. That this asymmetry is in fact caused by the change in $\rho$ and not by that in $\alpha$ is verified by considering the dashed lines in Figure 9. Since the asymmetric dashed smile of Figure 10 is virtually identical to the asymmetric solid smile of Figure 4, we conclude that it is intertemporal substitution and not risk aversion which explains asymmetric volatility smiles.

This important point can also be made in the following way. The risk aversion parameter $\alpha$ enters the generalized pricing formula through the discounting factor $\bar{B}(t,T)$ and through the CAPM-like factor $Q(t,T)$ which accounts for the covariance risk between the stock and market portfolio. If the role of the discounting factor is held fix, then the only difference between the generalized option prices and their preference-free counterparts would be due to risk aversion as this is the only preference parameter that enters the $Q(t,T)$ factor beyond what is already embodied in the $\bar{B}(t,T)$ factor. To this end consider Figure 12. The left panel plots $E_t \{ S_t (\Phi(d_1^{HW}) - Q_X Y(t,T) \Phi(d_1^{QBS})) \}$; that is, the difference between the first parts of the preference-free and the generalized option pricing formula in which preferences matter. The difference between the second parts of these formulas, $E_t \left\{ K(B(t,T) \Phi(d_2^{HW}) - \bar{B}(t,T) \Phi(d_2^{QBS})) \right\}$, is plotted in the right panel. In each panel the solid line represents the case where $B(t,T)$ is stochastic whereas the dashed line represents the case where $B(t,T)$ is deterministic. Comparison of the left and right panels reveals no differences whatsoever.

The conclusion that emerges is that risk aversion plays no role in explaining the departures of generalized option prices in which preferences matter from their preference-free counterpart. In turn this implies that it is intertemporal substitution that explains asymmetric volatility smiles and not risk aversion as has traditionally been thought.
6.2.4. Errors from preference-free pricing

Given a world in which preferences matter for option pricing, let us consider the relative pricing error that is committed by using the preference-free formula to price options. With the values of the preference parameters and the endowment process given previously as the benchmark for comparison, we generated option prices according to the generalized pricing formula where preferences matter and its preference-free counterpart and plotted the relative difference; i.e. \( \frac{\pi_t^{HW} - \pi_t^{CBS}}{\pi_t^{CBS}} \). Figure 13 shows the relative pricing error across moneyness for options with maturities of one (solid line), two (dashed line) and three periods (short dashes). Again the left and right panels are associated with states 1 and 2, respectively, as the current operative state. In both cases we see that the greatest pricing errors are committed for out-of-the-money options and that the pricing errors are most pronounced at shorter maturities.

Recall from above that when the transition probability matrix was changed from one with persistent states \( [P_{ii} > P_{ij}] \) to one where \( [P_{ii} < P_{ij}] \), the result was a reversal of the asymmetric bias in the observed volatility smile. Such a change in the transition probabilities has of course a similar effect in terms of relative pricing errors: If \( [P_{ii} < P_{ij}] \), Figure 13 gets reversed with in-the-money options being more severely underpriced by the preference-free formula than out-of-the-money options.

This bias reversal is similar to one documented by Hull and White (1987) with respect to the Black-Scholes formula. They found that when there is a positive correlation between the stock price and its volatility, out-of-the-money options are underpriced (by the BS formula), while in-the-money options are overpriced. When the correlation is negative, the effect is reversed. We find something similar with respect to the preference-free option pricing formula à la Hull-White. For the given specification, the covariance between the stock prices and its volatility is positive when \( [P_{ii} > P_{ij}] \) and negative when \( [P_{ii} < P_{ij}] \). Hence the same pattern emerges: out-of-the-money options are underpriced by the preference-free formula when there is a positive correlation between stock prices and its volatility, and overpriced when this correlation is negative.

This result may well provide an explanation to the empirical mispricings ob-
served by Bakshi, Cao, and Chen (1997) with a pricing model that admits stochastic volatility, stochastic interest rates, and random jumps but which remains preference-free. In Luger, Garcia and Renault (2000), we assess the empirical performance of our model relative to a preference-free formula in terms of pricing and hedging errors.

7. Conclusion

In this paper, we have analyzed the symmetry of the so-called implied volatility smiles, which are often used to characterize the European option pricing biases produced by the Black-Scholes formula. We have stated conditions that an option pricing formula must obey to produce a symmetric volatility smile and translated them into conditions on the pricing probability measure. Since the stochastic volatility extension of the Black-Scholes framework does not reproduce the asymmetric smiles frequently observed, we proposed an option pricing formula with a general stochastic discount factor that generalizes the stochastic volatility option pricing formula. Such a generalization is achieved through a conditioning on state variables. We have shown that two kinds of “generalized” leverage effects may explain (besides the interest rate risk) asymmetric smiles: either a genuine leverage effect, that is an instantaneous correlation between the return on the stock and its stochastic volatility process, or a stochastic correlation between the return of the stock and the stochastic discount factor. These results provide some theoretical foundations to the observed asymmetric smiles. We have also explained how these leverage effects determine if the option pricing formula is preference-free or not.

Through an equilibrium stochastic discount factor and a Markov regime-switching process for the state variables, we have shown that the model leads itself to a computable formula that can reproduce many of the shapes observed for the implied volatility curves. The remaining task is to show that the parameters estimated from the data in such an extended framework can be used to achieve smaller pricing or hedging errors out of sample. We leave such a task for future research.
Appendix 1

Proof of Proposition 2.1:
We first check that, for any given value of \( \sigma \), the function \( \pi(.) = BS(. \sigma) \) fulfills the announced property:

\[
\pi(-x) = e^{x} \pi(x) + 1 - e^{x}.
\]

Indeed, from (2.2) and (??):

\[
BS(x, \sigma) = \Phi[d_1(x, \sigma)] - e^{-x} \Phi[d_2(x, \sigma)],
\]

with: \( d_1(x, \sigma) = \frac{x}{\sigma} + \frac{1}{2}, d_2(x, \sigma) = \frac{x}{\sigma} - \frac{1}{2} \).

But: \( \Phi[d_2(-x, \sigma)] = \Phi[-d_1(x, \sigma)] = 1 - \Phi[d_1(x, \sigma)] \), and: \( \Phi[d_1(-x, \sigma)] = \Phi[-d_2(x, \sigma)] = 1 - \Phi[d_2(x, \sigma)] \).

Therefore:

\[
BS(-x, \sigma) = \Phi[d_1(-x, \sigma)] - e^{x} \Phi[d_2(-x, \sigma)]
= e^{x} \Phi[d_1(x, \sigma)] - \Phi[d_2(x, \sigma)] + 1 - e^{x}
= e^{x} BS(x, \sigma) + 1 - e^{x}.
\]

Let us now consider another homogeneous option pricing formula \( x \rightarrow \pi(x) \).
The associated BS implied volatilities are then defined by:

\[
\pi(x) = BS[x, \sigma^*(x)],
\pi(-x) = BS[-x, \sigma^*(-x)].
\]

Therefore, for any \( x \):

\[
\sigma^*(x) = \sigma^*(-x)
\iff \pi(-x) = BS[-x, \sigma^*(x)]
\iff \pi(-x) = e^{x} BS[x, \sigma^*(x)] + 1 - e^{x}
\iff \pi(-x) = e^{x} \pi(x) + 1 - e^{x}.
\]

Proof of Proposition 2.2:

a) First, we prove that the criterion of Proposition 2.1 is equivalent to the property (i) of Proposition 2.2. We can write (??) as:

\[
\pi_t(S_t, K) = B(t, T)S_t \int_{K}^{\infty} \left( \frac{S_T}{S_t} - \frac{K}{S_t} \right) dQ_{t,T} \left( \frac{S_T}{S_t} \right)
\]

Therefore, by taking the derivative with respect to \( K \), we obtain the well-known relationship between the option pricing formula and the pricing probability measure.

36
\[
\frac{\partial \pi}{\partial K} (S_t, K) = -B(t, T) Q_t \left[ \frac{S_T}{S_t} \geq \frac{K}{S_t} \right] \\
= -B(t, T) [1 - F_{\nu_T}(-x)].
\]

Since, from (??):
\[
\frac{\partial \pi}{\partial x} (x) = \frac{\partial}{\partial x} \left[ \pi(1, \frac{K}{S_t}) \right] = -\frac{K}{S_t} \frac{\partial}{\partial K} \pi(S_t, K)
\]

we have, for any \( x \):
\[
\frac{\partial \pi}{\partial x} (x) = e^{-x} [1 - F_{\nu_T}(-x)].
\]

Therefore, the property (i) of Proposition (2.2) may be rewritten as:
\[
\pi(x) = 1 - e^{-x} \frac{\partial \pi}{\partial x} (-x) - \frac{\partial \pi}{\partial x} (x)
\]
or equivalently:
\[
-e^{-x} \frac{\partial \pi}{\partial x} (-x) = e^{\pi} [\pi(x) + \frac{\partial \pi}{\partial x} (x) - 1].
\]

This last equality is obviously a corollary of proposition (2.2) obtained by taking the derivative with respect to \( x \) of the identity in Proposition (2.2). Conversely, this equality implies that for any \( x \):
\[
- \int_x^{+\infty} \frac{\partial \pi}{\partial u} (-u) du = \int_x^{+\infty} e^u [\pi(u) + \frac{\partial \pi}{\partial u} (u) - 1] du
\]

This equation will provide the criterion of Proposition (2.2) if we are able to complete it by the following limit condition:
\[
\lim_{x \to +\infty} \pi(-x) = \lim_{x \to +\infty} [e^{\pi} \pi(x) + 1 - e^{\pi}].
\]

Therefore, the required equivalence will be proved if we show that this limit condition is always guaranteed. But, on the one hand:
\[
\lim_{x \to +\infty} \pi(-x) = \lim_{x \to +\infty} \pi(x) = \lim_{K \to +\infty} B(t, T) E_t^* M A x [0, S_T - K] = 0
\]
by virtue of the Lebesgue dominated convergence theorem since: $\max[0, S_T - K] \rightarrow_{K \rightarrow \infty} 0$ almost surely and $0 \leq \max[0, S_T - K] \leq S_T$, which is by assumption integrable with respect to the pricing probability measure. On the other hand:

\[
\lim_{x \to +\infty} e^{x}[\pi(x) - 1] + 1 = 1 + \lim_{K \to 0^+} \frac{1}{KB(t,T)}\{B(t,T)E_t^*\max[0, S_T - K] - B(t,T)E_t^*S_T\}
\]
\[
= 1 + \lim_{K \to 0^+} \frac{1}{K} E_t^*\max[-S_T, -K]
\]
\[
= 1 - \lim_{K \to 0^+} E_t^*\min\left[\frac{S_T}{K}, 1\right]
\]
\[
= - \lim_{K \to 0^+} E_t^*\min\left[\frac{S_T}{K} - 1, 0\right] = 0
\]

by virtue of the Lebesgue dominated convergence theorem since: $\min\left[\frac{S_T}{K} - 1, 0\right] \rightarrow_{K \rightarrow 0^+} 0$ almost surely and $0 \leq \min\left[\frac{S_T}{K} - 1, 0\right] \leq 1$. This proves that: $\lim_{x \to +\infty} \pi(-x) = 0 = \lim_{x \to +\infty}[e^{x}\pi(x) + 1 - e^x]$ and completes the proof of the required equivalence.

b) We now check that properties (i) and (ii) of Proposition 9 are equivalent. The general definition (2.1) of the pricing probability measure implies that:

\[
\pi_t(S_t, K) = B(t,T)E_t^*[S_T\mathbb{1}_{S_T \geq K} - B(t,T)KQ_t|S_T \geq K],
\]

that is, after dividing by $S_t$:

\[
\pi(x) = E_t^*[e^{V_T}1_{V_T \geq -x}] - e^{-x}[1 - F_{V_T}(-x)]
\]

By identification of this formula with condition (i), we see that (i) is equivalent to (ii).

c) Finally, we prove that conditions (i) and (iii) are equivalent. By taking the derivative of (i), we obtain:

\[
\frac{\partial \pi}{\partial x}(x) = f_{V_T}(x) - e^{-x}f_{V_T}(-x) + e^{-x}[1 - F_{V_T}(-x)].
\]

But, since by part a) of this proof:

\[
\frac{\partial \pi}{\partial x}(x) = e^{-x}[1 - F_{V_T}(-x)]
\]

we conclude that (i) implies:

\[
f_{V_T}(x) = e^{-x}f_{V_T}(-x)
\]

or:

38
\[ e^{\frac{x}{2}} f_{V_T}(x) = e^{-\frac{x}{2}} f_{V_T}(-x) \]

which means that the function \( x \to e^{\frac{x}{2}} f_{V_T}(x) \) is even, which is exactly condition (iii) of Proposition 9. Conversely, if this condition is fulfilled, we have, for any \( x \):

\[
\int_x^{+\infty} f_{V_T}(u) \, du = \int_x^{+\infty} e^{-u} f_{V_T}(-u) \, du.
\]

This equation will provide property (i) of proposition 9 if we complete it by the following limit condition:

\[
\lim_{x \to +\infty} \pi(x) = \lim_{x \to +\infty} [F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)]].
\]

Therefore, the required equivalence will be proved as we show that this limit condition always holds. But it is clear that:

\[
\lim_{x \to +\infty} [F_{V_T}(x) - e^{-x}[1 - F_{V_T}(-x)] = \lim_{x \to +\infty} F_{V_T}(x) = 1
\]

and that \( \lim_{x \to +\infty} \pi(x) = 1 \), since we have already shown in part a) of this proof that: \( \lim_{x \to +\infty} e^{x} [\pi(x) - 1] = -1 \). This completes the proof.

Appendix 2

A. Proof of Proposition (3.2): From (3.5) and the decomposition of \( m_t,T \) conformable to (A1) and (A2):

\[
\frac{\pi_t}{S_t} = E_t \left\{ \lambda_{t,T}(U_{I_t}^T) E \left[ \left( \prod_{r=t}^{T-1} m_{r+1} \right) \left( \prod_{r=t}^{T-1} \frac{S_{r+1}}{S_r} - \frac{K}{S_t} \right)^+ \right] | I_t, U_{I_t}^T \right\}
\]

But, by (A2), the variables \( (m_{r+1}, \frac{S_{r+1}}{S_r})_{r \geq t} \) are independent of \( (m_{r+1}, \frac{S_{r+1}}{S_r})_{r < t} \) given \( U_{I_t}^T \). Therefore:

\[
\frac{\pi_t}{S_t} = E_t \left\{ \lambda_{t,T}(U_{I_t}^T) E \left[ \left( \prod_{r=t}^{T-1} m_{r+1} \right) \left( \prod_{r=t}^{T-1} \frac{S_{r+1}}{S_r} - \frac{K}{S_t} \right)^+ \right] | U_{I_t}^T \right\}
\]

is a conditional expectation computed in the conditional probability distribution of \( U_{I_t}^T \) given \( I_t \). By (A1), this probability distribution depends on \( I_t \) only through \( U_{I_t}^T \). We are then allowed to denote this expectation by \( \Psi_{t,T}(U_{I_t}^T, \frac{K}{S_t}) \).

B. Proof of Proposition 3.3:
In what follows, we will derive a closed-form formula for $\Psi_{t,T}(U_{t}, \frac{K}{S_{t}})$ based on the log-normality assumption. We will start from the following decomposition:

$$\Psi_{t,T}(U_{t}, \frac{K}{S_{t}}) = E_{t}\left\{ \lambda_{t,T}(U_{t}^{T}) \left[ G_{t,T}(U_{t}^{T}) - \frac{K}{S_{t}} H_{t,T}(U_{t}^{T}) \right] \right\}$$

where:

$$G_{t,T}(U_{t}^{T}) = E \left[ \prod_{r=t}^{T-1} m_{r+1} \frac{S_{r+1}}{S_{r}} I_{[S_{r} \geq K]} | U_{1}^{T} \right]$$

and:

$$H_{t,T}(U_{t}^{T}) = E_{t} \left[ \prod_{r=t}^{T-1} m_{r+1} I_{[S_{r} \geq K]} | U_{1}^{T} \right].$$

a) Lemma 1: If \( \begin{pmatrix} Z_{1} \\ Z_{2} \end{pmatrix} \) is a bivariate Gaussian vector, with:

$$E \left( \begin{pmatrix} Z_{1} \\ Z_{2} \end{pmatrix} \right) = \begin{pmatrix} m_{1} \\ m_{2} \end{pmatrix}, \text{Var} \left( \begin{pmatrix} Z_{1} \\ Z_{2} \end{pmatrix} \right) = \begin{pmatrix} \omega_{1}^{2} & \rho \omega_{1} \omega_{2} \\ \rho \omega_{1} \omega_{2} & \omega_{2}^{2} \end{pmatrix}$$

$$E[\exp(Z_{1})1_{[Z_{2} \geq 0]}] = \exp[m_{1} + \frac{\omega_{1}^{2}}{2}] \Phi \left( \frac{m_{2}}{\omega_{2}} + \rho \omega_{1} \right), \text{with } \Phi \text{ the cumulative normal distribution function.}$$

Let us by \( Q \) the probability measure corresponding to the above-specified Gaussian distribution of \( (Z_{1}, Z_{2}) \) and define the probability \( \tilde{Q} \) by:

$$\frac{d\tilde{Q}}{dQ}(Z) = \exp[(Z - m_{1}) - \frac{\omega_{1}^{2}}{2}].$$

Then, with obvious notation:

$$E[(\exp Z_{1})1_{[Z_{2} \geq 0]}] = \exp(m_{1} + \frac{\omega_{1}^{2}}{2}) \tilde{Q}[Z_{2} \geq 0]$$

But by Girsanov theorem, we know that under \( \tilde{Q} \) \( Z_{2} \) is a Gaussian variable with mean $m_{2} + \rho \omega_{1} \omega_{2}$ and variance $\omega_{2}^{2}$. Therefore:

$$\tilde{Q}[Z_{2} \geq 0] = 1 - \Phi \left( \frac{-m_{2} - \rho \omega_{1} \omega_{2}}{\omega_{2}} \right) = \Phi \left( \frac{m_{2}}{\omega_{2}} + \rho \omega_{1} \right)$$

C. A closed-form formula for $H_{t,T}(U_{t}^{T})$ and bond pricing:

$$H_{t,T}(U_{t}^{T}) = E_{t}\left[ \exp \left( \sum_{r=t}^{T-1} \log m_{r+1} \right) 1_{[\sum_{r=t}^{T-1} \log \frac{S_{r+1}}{S_{r}} \geq \log \frac{K}{S_{r}}]} | U_{1}^{T} \right]$$
By virtue of assumption A, this expectation is given by lemma 1 with:

\[ Z_1 = \sum_{\tau=t}^{T-1} \log m_{\tau+1} \text{ and } Z_2 = \sum_{\tau=t}^{T-1} \log \frac{S_{\tau+1}}{S_{\tau}} - \log \frac{K}{S_t} \]

so

\[ m_1 = \sum_{\tau=t}^{T-1} \mu_{m_{\tau+1}}, \quad m_2 = \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} - \log \frac{K}{S_t} \]

\[ \omega_1^2 = \sum_{\tau=t}^{T-1} \sigma_{m_{\tau+1}}^2, \quad \omega_2^2 = \sum_{\tau=t}^{T-1} \sigma_{S_{\tau+1}}^2, \quad \rho \omega_1 \omega_2 = \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}}. \]

Therefore:

\[ H_{t,T}(U_{1T}^T) = \exp \left[ \sum_{\tau=t}^{T-1} \mu_{m_{\tau+1}} + \frac{1}{2} \sum_{\tau=t}^{T-1} \sigma_{m_{\tau+1}}^2 \right] \Phi \left( \frac{1}{\sqrt{\sum_{\tau=t}^{T-1} \sigma_{S_{\tau+1}}^2}} \left( \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} - \log \frac{K}{S_t} + \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}} \right) \right) \]

By referring to the notation introduced in proposition 3.3, we first notice that \( H_{t,T}(U_{1T}^T) \) can be written as:

\[ H_{t,T}(U_{1T}^T) = \frac{\bar{B}(t,T)}{\lambda_t(T)} \Phi (d_2(x_t)) \]

with \( x_t = \log \frac{S_t}{KB(t,T)} \) and \( d_2(x_t) \) defined in proposition 3.3 since:

\[ \frac{1}{\sqrt{\sum_{\tau=t}^{T-1} \sigma_{S_{\tau+1}}^2}} \left( \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} - \log \frac{K}{S_t} + \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}} \right) = \frac{1}{\sigma_{t,T}} (x_t + \log B(t,T) + \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} + \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}}) \]

But:

\[ E_t \left[ \frac{S_T}{S_t} | U_{1T}^T \right] = \exp \left( \sum_{\tau=t}^{T-1} \mu_{S_{\tau+1}} + \frac{1}{2} \sigma_{t,T}^2 \right) \]

Therefore, the above expression can be rewritten as:

\[ \frac{1}{\sigma_{t,T}} \left( x_t - \frac{1}{2} \sigma_{t,T}^2 + \log \left( E_t \left[ \frac{S_T}{S_t} | U_{1T}^T \right] B(t,T) \right) + \sum_{\tau=t}^{T-1} \sigma_{mS_{\tau+1}} \right) \]

\[ = \frac{x_t}{\sigma_{t,T}} - \frac{\sigma_{t,T}}{2} + \frac{1}{\sigma_{t,T}} \log \left( Q_{mS}(t,T) \frac{B(t,T)}{B(t,T)} \right) = d_1(x_t) - \sigma_{t,T} = d_2(x_t) \]
where \( d_1(x_t), d_2(x_t) \) and \( Q_{mS}(t, T) \) correspond to the expressions given in proposition 3.3.

Finally, it is worth noticing that \( B(t, T) \) can be interpreted in terms of bond pricing. Actually, the general pricing formula (3.5) implies that:

\[
B(t, T) = E_t[m_{i, T}] = E_t \left[ \lambda_t T(U^T_1)H_t,T(U^T_1) \right]
\]

when \( H_t,T(U^T_1) \) is computed in the limit case \( K = +\infty \), that is,

\[
H_t,T(U^T_1) = \frac{\tilde{B}(t, T)}{\lambda_t T(U^T_1)}, \text{ since } \lim_{K \to +\infty} d_2(x_t) = +\infty
\]

therefore the bond pricing equation is given by:

\[
B(t, T) = E_t[\tilde{B}(t, T)]
\]

D. A closed-form formula for \( G_t,T(U^T_1) \) and stock pricing:

\[
G_t,T(U^T_1) = E \left[ \exp \left( \sum_{\tau=t}^{T-1} \log m_{r+1} + \log \frac{S_{r+1}}{S_r} \right) 1_{\sum_{\tau=t}^{T-1} \log \frac{S_{r+1}}{S_r} \geq \log \frac{K}{S_t}} | U^T_1 \right]
\]

But, by virtue of assumption A, this expectation is given by lemma 1 with:

\[
Z_1 = \sum_{\tau=t}^{T-1} \log m_{r+1} + \log \frac{S_{r+1}}{S_r} \text{ and } Z_2 = \sum_{\tau=t}^{T-1} \log \frac{S_{r+1}}{S_r} - \log \frac{K}{S_t}
\]

In other words, with respect to part C above, \( m_2 \) and \( \omega^2 \) are unchanged while now:

\[
m_1 = \sum_{\tau=t}^{T-1} (\mu_{r+1} + \mu_{S_{r+1}}), \omega^2_1 = \sum_{\tau=t}^{T-1} (\sigma_{m_{r+1}}^2 + \sigma_{S_{r+1}}^2 + 2\sigma_{mS_{r+1}}), \rho \omega_1 \omega_2 = \sum_{\tau=t}^{T-1} (\sigma_{mS_{r+1}} + \sigma_{S_{r+1}}^2)
\]

Therefore:

\[
G_t,T(U^T_1) = \exp \left[ \sum_{\tau=t}^{T-1} (\mu_{r+1} + \mu_{S_{r+1}}) + \frac{1}{2} \sum_{\tau=t}^{T-1} (\sigma_{m_{r+1}}^2 + \sigma_{S_{r+1}}^2 + 2\sigma_{mS_{r+1}}) \right] \times \Phi \left( \frac{1}{\sqrt{\sum_{\tau=t}^{T-1} \sigma_{S_{r+1}}^2}} \sum_{\tau=t}^{T-1} \mu_{S_{r+1}} - \log \frac{K}{S_t} + \sum_{\tau=t}^{T-1} (\sigma_{mS_{r+1}} + \sigma_{S_{r+1}}^2) \right)
\]

42
But comparison with the above expressions of $H_{t,T}(U_1^T)$ and $E_t \left[ \frac{S_T}{S_t} | U_1^T \right]$ we see that:

$$G_{t,T}(U_1^T) = \frac{\bar{B}(t,T)}{\lambda_{t,T}(U_1^T)} \exp \left[ \sum_{\tau=t}^{T-1} (\mu_{S_\tau+1} + \mu_{S_\tau}) + \frac{1}{2} \sum_{\tau=t}^{T-1} (\sigma_{S_\tau+1}^2 + 2\sigma_{S_\tau+1} \sigma_{S_\tau}) \right] \Phi (d_2(x_\tau))$$

that is,

$$G_{t,T}(U_1^T) = \frac{Q_mS(t,T)}{\lambda_{t,T}(U_1^T)} \Phi (d_1(x_\tau))$$

Finally, it is worth noticing that $Q_mS(t,T)$ can be interpreted in terms of stock pricing. Actually the stock pricing equation corresponds to the general pricing formula (3.5) in the limit case $K = 0$, that is:

$$S_t = E_t \left[ \lambda_{t,T}(U_1^T) S_t G_{t,T}(U_1^T) \right]$$

where

$$G_{t,T}(U_1^T) = \frac{Q_mS(t,T)}{\lambda_{t,T}(U_1^T)}$$

since $\lim_{K \to 0} d_1(x_\tau) = +\infty$

In other words, the stock pricing equation can be written:

$$1 = E_t [Q_mS(t,T)]$$

E. Option pricing formula:

We conclude from parts A, B and C above that the option pricing formula $\pi_t$ is given by:

$$\frac{\pi_t}{S_t} = E_t \left[ \lambda_{t,T}(U_1^T) G_{t,T}(U_1^T) - \frac{K}{S_t} \lambda_{t,T}(U_1^T) H_{t,T}(U_1^T) \right] = E_t \left[ Q_mS(t,T) \Phi (d_1(x_\tau)) - \frac{K\bar{B}(t,T)}{S_t} \Phi (d_2(x_\tau)) \right]$$

which coincides with the announced formula of proposition 3.3 since:

$$\frac{K\bar{B}(t,T)}{S_t} = \frac{\bar{B}(t,T) K B(t,T)}{B(t,T) S_t} = \frac{\bar{B}(t,T)}{B(t,T)} \exp(-x_\tau)$$

43
Appendix 3

Computation of the option prices with a Markov state variable

To compute option prices, we first compute the equilibrium values of $\lambda_t = \lambda(U_t) = P_t^M/C_t$ and $\varphi_t = \varphi(U_t) = S_t/D_t$ as solution of:

$$
\lambda(U_t = i) = \sum_{j=1}^{N} p_{ij} \left[ \beta^t \exp \{ \alpha m_X(U_t = i) + \frac{1}{2} \alpha^2 \sigma_X^2(U_t = i) \} (\lambda(U_t = j) + 1)^t \right]
$$

and

$$
\varphi(U_t = i) = \sum_{j=1}^{N} p_{ij} \left[ \beta^t A_j \left( \frac{\lambda(U_t = j) + 1}{\lambda(U_t = i)} \right)^{t-1} (\varphi(U_t = j) + 1) \right]
$$

for $i = 1, \ldots, N$ and where

$$A_j = \exp \{ (\alpha - 1) m_X(U_t = j) + m_Y(U_t = j) + \frac{1}{2} ((\alpha - 1)^2 \sigma_X^2(U_t = j) + \sigma_Y^2(U_t = j) + 2(\alpha - 1) \rho_{XY} \sigma_X(U_t = j) \sigma_Y(U_t = j)) \}
$$

Conditional on state $i$ being operative at time $t$, the generalized Black-Scholes price of a $(T - t)$ period call option written on $S_t = \lambda(U_t = i) D_t$ at strike price $K$ is given by:

$$
\pi_t^{GBS} = \sum_{i_{t+1}=1}^{N} \ldots \sum_{i_T=1}^{N} p_{i_t, \ldots, i_T} \left\{ S_t Q_{XY}(i_t, \ldots, i_T) \Phi(d_1(i_t, \ldots, i_T)) - K \tilde{B}(i_t, \ldots, i_T) \Phi(d_2(i_t, \ldots, i_T)) \right\}
$$

where $p_{i_t, \ldots, i_T} = p_{i_{t+1} \ldots i_{t+T}} p_{i_{t+1} \ldots i_{T-1}}$ and

$$d_1(i_t, \ldots, i_T) = \frac{\log \frac{S_t Q_{XY}(i_t, \ldots, i_T)}{K B(i_t, \ldots, i_T)}} {\left( \sum_{r=t+1}^{T} \sigma_Y^2(U_r = i_r) \right)^{1/2}} + \frac{1}{2} \left( \sum_{r=t+1}^{T} \sigma_Y^2(U_r = i_r) \right)^{1/2},
$$

$$d_2(i_t, \ldots, i_T) = d_1(i_t, \ldots, i_T) - \left( \sum_{r=t+1}^{T} \sigma_Y^2(U_r = i_r) \right)^{1/2}
$$

and

$$Q_{XY}(i_t, \ldots, i_T) =
$$

$$\tilde{B}(i_t, \ldots, i_T) b(i_t, \ldots, i_T) \frac{\varphi(U_t = i_t)}{\varphi(U_T = i_T)} \exp \left\{ (\alpha - 1) \sum_{r=t+1}^{T} \sigma_{XY}(U_r = i_r) \right\} E_t \left[ \frac{S_T}{S_t} \mid U_1 \right].
$$
with
\[ b(i_t, \ldots, i_T) = \prod_{\tau=t}^{T-1} \left( \frac{1 + \varphi(U_{\tau+1} = i_{\tau+1})}{\varphi(U_{\tau} = i_{\tau})} \right), \]

and
\[ E_t \left[ \frac{S_T}{S_t} \mid U_1^T \right] = \frac{\varphi(U_t = i_T)}{\varphi(U_t = i_t)} \exp \left\{ \sum_{\tau=t+1}^{T} m_Y(U_{\tau} = i_{\tau}) + \frac{1}{2} \sum_{\tau=t+1}^{T} \sigma_Y^2(U_{\tau} = i_{\tau}) \right\} \]

Finally,
\[ \tilde{B}(i_t, \ldots, i_T) = \beta^{(T-t)} a(i_t, \ldots, i_T) \exp \left\{ (\alpha - 1) \sum_{\tau=t+1}^{T} m_X(U_{\tau} = i_{\tau}) + \frac{1}{2} (\alpha - 1)^2 \sum_{\tau=t+1}^{T} \sigma_X^2(U_{\tau} = i_{\tau}) \right\} \]

with
\[ a(i_t, \ldots, i_T) = \prod_{\tau=t}^{T-1} \left( \frac{1 + \lambda(U_{\tau+1} = i_{\tau+1})}{\lambda(U_{\tau} = i_{\tau})} \right)^{T-1} \]
Examples of implied volatility curves inferred from S&P 500 call option Prices

15 day S&P 500 call option: 89-01-05

18 day S&P 500 call option: 94-01-02

49 day S&P 500 call option: 92-05-01

79 day S&P 500 call option: 92-07-01

11 day S&P 500 call option: 89-01-09

30 day S&P 500 call option: 90-05-16

Figure 1
46
Figure 2: Maturity and stochastic volatility effects when $B(t,T)$ is deterministic and $Q_{XY}(t,T) = 1$. Left panel: The effect of a decrease in the coefficient of variation of the volatility results in flatter smiles. Right panel: As the option's maturity increases, the smile flattens. The most curved smiles in each panel are in fact identical.

Figure 3: Maturity effects when $B(t,T)$ is deterministic and $Q_{XY}(t,T) \neq 1$. The solid lines are the implied volatility curves for a one-period option whereas the dashed lines are those for an option with a two-period maturity. In each case the symmetric smiles are those resulting from preference-free option pricing à la Hull-White. The preference-free smile is easily recognizable as the one centered on zero. The left and right panels are associated with states 1 and 2, respectively, as the current state.
Figure 4: Maturity effects when \( B(t, T) \) is stochastic and \( Q_{XY}(t, T) \neq 1 \). The solid lines are the implied volatility curves for a one-period option whereas the dashed lines are those for an option with a two-period maturity. In each case the symmetric smiles are those resulting from generalized option pricing à la Hull-White. The left and right panels are associated with states 1 and 2, respectively, as the current state.

Figure 5: Average duration effects when \( p_{ii} > p_{ij} \). The dashed lines represent an increase in \( p_{ii} \). The left and right panels are associated with states 1 and 2, respectively, as the current state.
Figure 6: Average duration effects when \( p_{ij} < p_{ij} \). The dashed lines represent an increase in \( p_{ij} \). The left and right panels are associated with states 1 and 2, respectively, as the current state.

Figure 7: Option prices and the correlation between consumption and dividends. The solid lines represent the schedule of option prices across moneyness when the correlation between consumption and dividends is zero. The other extreme of perfect positive correlation is represented by the dashed lines. The left and right panels are associated with states 1 and 2, respectively, as the current state.
Figure 8: Leverage effects and the volatility smile. The dashed lines show the effect on the volatility smile of an absence of a leverage effect through the consumption process. The solid lines are the benchmark case of leverage effects through both consumption and dividend processes. The left and right panels are associated with states 1 and 2, respectively, as the current state.

Figure 9: Leverage effects and the volatility smile. The dashed lines show the effect on the volatility smile of an absence of a leverage effect through the dividend process. The solid lines are the benchmark case of leverage effects through both consumption and dividend processes. The left and right panels are associated with states 1 and 2, respectively, as the current state.
Figure 10: The role of preferences. The solid lines are the volatility smiles for the following configuration of preference parameters: $\gamma = 1, \rho = -1, \alpha = -1$; whereas the dashed lines are for $\gamma = 1/10, \rho = -10, \alpha = -1$. The left and right panels are associated with states 1 and 2, respectively, as the current state.

Figure 11: The role of preferences. The solid lines are the volatility smiles for the following configuration of preference parameters: $\gamma = 1/10, \rho = -10, \alpha = -1$. The dashed lines show the volatility smiles for the same preference parameter configuration but for an option whose maturity is increased by one period. The left and right panels are associated with states 1 and 2, respectively, as the current state.
Figure 12: Intertemporal substitution and option pricing. The left panel plots \( E_t \left\{ S_t \Phi(d_{1}^{HW}) - Q_{XY}(t,T)\Phi(d_{1}^{GBS}) \right\} \); that is, the difference between the first parts of the preference-free and the generalized option pricing formula in which preferences matter. The difference between the second parts of these formulas, \( E_t \left\{ \Phi(B(t,T) - \tilde{B}(t,T)) \Phi(d_{2}^{GBS}) \right\} \), is plotted in the right panel. The solid lines represent the case where \( B(t,T) \) is stochastic whereas the dashed lines represent the case for \( B(t,T) \) deterministic.

Figure 13: Relative pricing errors. The lines plot the relative difference across moneyness between preference-free option prices and option prices in which preferences matter; that is, the graphs report the relative pricing error that is committed by using a preference-free pricing formula in a world in which preferences matter for option prices. The three lines are for options whose maturity increases by one period with the steepest representing the benchmark one-period option. The left and right panels are associated with states 1 and 2, respectively, as the current state.
References


