Seminários de Pesquisa Econômica I (1ª parte)

"WHY MARKETS ARE INCOMPLETE AND WHICH MARKETS ARE OPEN"

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Why Markets are Incomplete and Which Markets are Open

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I. Introduction

The theory of equilibrium in incomplete markets was one of the more important advances in economic theory in the 80's, and until today it is a very active research topic. Recent surveys on the subject (Duffie [1990], Genakoplos [1990], Magill and Shaffer [1991]) testify it and give large references on the literature. The bulk of the research has been done in constructing formal descriptions of an economy with incomplete contracts, and characterizing the equilibria. However there is not much research addressing the question of why the markets are incomplete, and there is a lack of research addressing the question of which markets are open and which markets are closed. Our paper is an effort in the direction of giving formalized answers to both questions.

We build a model in two versions. In both versions there are two periods, period zero and period one. At period 0, there is uncertainty about the state of nature at period one. At period 1 all uncertainty is resolved. There is one monopolist producer and seller of an intermediate product and a continuum of competitive buyers. The production process of the intermediate product takes one period of time, so output at period 1 is determined at period 0. At period 0 buyers are better informed than the producer about the likelihood of the different states of nature at period 1. The intermediate product at period 1 is sold in two different ways: as a contingent contract at period 0; and as a spot contract at period 1. However the producer and seller decides which contingent contracts to offer at period 0.\(^1\) All participants in the markets are risk-neutral and have rational expectations, in the sense that at period 0 they correctly forecast the spot price that will occur in each state of nature at period 1 and they use the correct probability distribution conditional to their information at period 0.

In the first version of the model buyers at period 0 have perfect information about the state of nature that will occur at period 1. In the second model buyers have at period 0 imperfect information about the state of nature at period 1, but are better informed than the producer. In both models the producer in interested in open contingent markets at period 0 to acquire some information from buyers. However due to the rational expectations assumption, if the producer opens a contingent market for state she has to give up his monopoly power at that state of nature.\(^2\) Let's see why this is true in the first and simpler model where buyers know at period 0 which state of nature at period 1 will occur. Since they have rational expectations they will buy in the contingent market only if the contingent price at period 0 is smaller than the spot price that will occur at period 1. This implies that the producer has to charge at the contingent market the competitive price, i.e., the price that equates demand to marginal cost. Any price below that cannot be an equilibrium. If contingent price is above the competitive price the producer will have incentives to produce quantities over and above the quantities already committed in the contingent market. That is because he knows that if buyers

\(^1\)This is possible because there is no reselling of the product

\(^2\)Coutinho and Saldanha [1989] were the first to notice this fact.
buy the contingent product the state upon which the product is contingent will occur for sure and the price at which buyers will be willing to buy in the spot market at period 1 is above marginal cost for quantities above the quantity already sold in the contingent market. This will make the spot price in period 1 bellow the contingent price. But it is a contradiction that retailers have bought in the contract markets, have rational expectations and the spot price is bellow the contingent price.

Therefore the producer faces the following trade off: by opening a contingent market at state \(s'\) he is able to make a better informed production decision, however he has to charge the competitive price and thus give up his monopoly power at state \(s'\); if he does not open the contingent market for state \(s'\) he does not make a well-informed production decision but can charge monopolist price in the spot market at period 1 when state 1 occurs. Therefore it is not surprising that some contingent markets may be closed. However we have been able to show in both versions of the model that at least one market will be closed. The intuition behind the result is that by closing one market the monopolist does not lose information compared to the case with complete markets and can make informed monopolist profits in the state of nature for which there is no contingent market. In the first version we compute which markets are open and which markets are closed.

Section II describes the basic assumptions that will be used in the model. Section III develops some basic results that will be used in both versions of the model. Section IV develops the perfect information model. Section V develops the imperfect information model. Section VI concludes and gives directions for future research.

II. Basic Assumptions

A1. The model has two periods, called period zero and period one.

A2. There is one monopolist, producer and seller of an intermediate product, called the "producer". There is a continuum of price-taking buyers of the intermediate product, called the "retailers", indexed by \(\alpha\) in the interval \([0,1]\).

A3. The producer's production process of the intermediate good takes one period of time, so period 1's output is completely determined at period 0. The cost of throwing away output at period

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3He does not know the retailers' information but knows that retailers have and can infer from retailers' behavior in the contingent market their information. Because retailers are prefectly informed they may buy the product contingent on state \(s'\) only in the state they know that state \(s'\) will occur. Therefore if retailers buy contingent on state \(s'\) the producer can infer that the state \(s'\) will occur fore sure.
1 is zero.\textsuperscript{4} The retailer's output is made in a fixed proportion of the intermediate product used as input, so we will use the same symbol to represent both, the intermediate good and the final good.

\textbf{A4.} The producer and the retailers are risk neutral.

This assumption allows to focus on information incentives to contracting, simplifies the calculations because it focuses only on expected values, and yet, it is the natural assumption in an environment where all agents are firms.

\textbf{A5.} There is not reselling of the intermediate product.

\textbf{A6.} There is a finite set $\Omega$ of states of nature at period 1. There is a probability space $(\Omega, P)$ where the probability $P$ is public known. The state of nature at period one is the realization of a random variable $\omega: \Omega \rightarrow \Omega$. The induced probability distribution of $\omega$, $\nu = P \circ \omega^{-1}$ is public known. There is also another "informational" random variable $\theta: \Omega \rightarrow \mathcal{E}$ to some measurable space $\mathcal{E}$. $\theta$ is constant on each set $\omega^{-1}(\{\omega_b\})$ for any $\omega_b \in \Omega$.

\textbf{A7.} The intermediate product is sold in two different ways. As a contingent futures contract at period 0: the product to be delivered in period 1 only if the state of nature is $\omega \in \Omega$, is transacted at period 0. As a spot contract at period 1: The product to be delivered at period 1 is transacted at period 1. The spot contract is transacted after all information about the state of nature is made public. The contingent futures are offered by the producer at period 0. The producer decides at period zero which contingent contracts (if any) to offer. The price of the futures contract contingent to state $\omega \in \Omega$ is denoted by $q_\omega$ and the total quantity of contracts transacted is denoted by $Z_\omega$. Each contract is made for 1 unity of intermediate product. Both $q_\omega$ and $Z_\omega$ are random variables $q_\omega, Z_\omega: \Omega \rightarrow \mathbb{R}^+$.\textsuperscript{5}

\textbf{A8.} The informational scheme is the following: Before period zero the producer makes his decisions using the probability $\nu: \Omega \rightarrow [0,1]$ given by $\nu(A) = P(\omega^{-1}(A))$. Also before period zero the retailers observe the value $\theta_0$ of the random variable $\theta$ and use the conditional probability\textsuperscript{5} $\mu : \Omega \rightarrow [0,1]$, $\mu(\omega) = \mu_{\theta_0}(\omega) = P(\omega|\theta_0)$.

Immediately after period zero the producer observes the prices $q_\omega$ and the quantities $Z_\omega$ of the contingent futures contracts sold at period zero and then updates his probability distribution by the conditional probability\textsuperscript{6} $\hat{\nu}(\omega) = P(\omega|q_\omega, Z_\omega)$. At period 1 all the uncertainty is resolved and $\omega \in \Omega$ becomes public known.

\textsuperscript{4}So that the producer can decide not to sell all of his production if he finds it more profitable.

\textsuperscript{5}Where the conditional probability $P(\omega|\theta_0) = \mu_{\theta_0}$ is defined by

$$P(B) = \int_B \left(\int_{\omega \in \omega^{-1}(A)} d\mu_{\theta_0}(\omega)\right) dP(\omega).$$

\textsuperscript{6}The probability $\hat{\nu}(\omega)$ is actually $\hat{\nu}$ given that $\theta$ is consistent with $q_\omega$ and $Z_\omega$; $\hat{\nu}(\omega) = P(\omega|q_\omega(\theta) = q$ and $Z_\omega(\theta) = Z_\omega))$. 

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A9. The agents have rational expectations. The retailers believe at period 0 that the spot price of the intermediate product at period 1 is going to be $p_\omega$ if the state is $\omega$. This hypothesis says that their beliefs are actually realized: $\hat{p}_\omega = p_\omega$.

A10. The demand for the final good at period 1 state of nature $\omega$ is $D_\omega(p_\omega) = a_\omega - b_\omega p_\omega$, and where their beliefs of the agents are consistent with these demands. We will see that the aggregate derived demand for the intermediate product in the spot and futures market is $S_\omega = h_\omega - k_\omega s_\omega$, where $s_\omega := \min(q_\omega, r_\omega)$, and $h_\omega := \frac{a_\omega}{b_\omega} + 1$, $k_\omega := \frac{b_\omega}{b_\omega}$. All the results in this model are written in terms of the parameters $h_\omega$ and $k_\omega$.

A11. The producer has the cost function $c(y) = \beta y^2$, where $\beta > 0$. Retailer $\alpha$ has the cost function $c_\alpha(x) = \frac{1}{4\alpha} x^2$.7

We analyze two cases for the model. The Perfect information case, in which the retailers know with certainty at period 0 which state of nature is going to occur at period 1, but the producer doesn't know it. This corresponds to a random variable $\theta$ which gives full information, i.e. there exists an injective function $f: \Omega \rightarrow \mathcal{E}$, such that the diagram

$$
\begin{array}{ccc}
\Omega & \xrightarrow{\theta} & \Omega \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{f} & \mathcal{E}
\end{array}
$$

commutes. We also analyze the Imperfect information model, in which all the retailers observe the same random variable $\theta$.

III. Some Basic Results

III.1 Retailers' Demand

Retailer $\alpha$'s maximization problem at period 1, state $\omega$, is

$$\max_{x_\omega \geq 0} \left\{ p_\omega (x_\omega + z_\omega) - r_\omega x_\omega - \sum_{v \in \Omega} q_\omega z_\omega - c_\alpha(x_\omega + z_\omega) \right\}$$

where $x_\omega$ is the amount he buys at the spot market at period 1, and $z_\nu$ is the amount of future contracts contingent to state $\nu$, already bought at period 0. This is equivalent to the problem:

$$\max_{x_\omega \geq 0} \left\{ (p_\omega - r_\omega)x_\omega - c_\alpha(x_\omega + z_\omega) \right\}$$

whose interior solution is

$$p_\omega - r_\omega = (Dc_\alpha)(x_\omega + z_\omega) = \frac{1}{2\alpha}(x_\omega + z_\omega) \quad \text{if } x > 0,$$

$$x_\omega + z_\omega = 2\alpha(p_\omega - r_\omega) \quad \text{if } x > 0.$$

But the interior condition $x_\omega > 0$ is equivalent to $z_\omega < 2\alpha(p_\omega - r_\omega)$. Therefore

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7 This specific functional form is made only to simplify calculations.
At period 0, the retailer has the probability distribution $\mu : \Omega \rightarrow [0,1]$ and believes that the prices at period 1, state $\omega$, are going to be $\hat{p}_\omega, \hat{r}_\omega$. He maximizes his expected profits:

$$\max_{\{z_\omega|\omega\in\Omega\}} E\left[ \hat{p}(x+z) - \hat{r}x - \sum_{\omega\in\Omega} q_\omega z_\omega - c_\omega (x+z) \right].$$

But he believes that his demand at period 1, $x_\omega(z_\omega)$, will be given by (3.1) with $p_\omega = \hat{p}_\omega$, $r_\omega = \hat{r}_\omega$. Therefore his problem is

$$\max_{\{z_\omega|\omega\in\Omega\}} E\left[ \hat{p}(x(z)+z) - \hat{r}x(z) - \sum_{\omega\in\Omega} q_\omega z_\omega - c_\omega (x(z)+z) \right].$$

The maximizing choice $(z_\omega^*)_{\omega\in\Omega}$ should in particular maximize the expected profits function when restricted to a subspace $\{z_v = z_v^* \text{ for all } \nu \neq \omega\}$:

$$\max_{z_\omega \geq 0} \{Q + \mu_\omega \hat{p}_\omega (x_\omega(z_\omega)+z_\omega) - \mu_\omega \hat{r}_\omega x_\omega(z_\omega) - q_\omega z_\omega - \mu_\omega c_\omega (x_\omega(z_\omega)+z_\omega)\}$$

where $Q$ is a function depending on $z_v$, $\nu \neq \omega$ and not on $z_\omega$.

Consider the function

$$g(z) := \mu_\omega \hat{p}_\omega (x_\omega(z)+z) - \mu_\omega \hat{r}_\omega x_\omega(z) - q_\omega z - \mu_\omega c_\omega (x_\omega(z)+z).$$

If $0 \leq z \leq 2\alpha(\hat{p} - \hat{r})$, we have that

$$g(z) = \mu_\omega \hat{p}(2\alpha(\hat{p} - \hat{r})) - \mu_\omega \hat{r}(2\alpha(\hat{p} - \hat{r}) - z) - q_\omega z - \mu_\omega c_\omega (2\alpha(\hat{p} - \hat{r}))$$

$$g(z) = K + (\mu_\omega \hat{r}_\omega - q_\omega)z \quad \text{if} \quad 0 \leq z < 2\alpha(\hat{p} - \hat{r}),$$

where $K$ is a constant. This is an affine function with positive slope if $q < \mu \hat{r}$ and negative slope if $q > \mu \hat{r}$. Therefore the maximum of $g$ in the closed interval $0 \leq z < 2\alpha(\hat{p} - \hat{r})$ is attained at

$$z = \begin{cases} 2\alpha(\hat{p} - \hat{r}) & \text{if } q < \mu \hat{r} \\ [0,2\alpha(\hat{p} - \hat{r})] & \text{if } q = \mu \hat{r} \\ 0 & \text{if } q > \mu \hat{r} \end{cases}$$

If $z > 2\alpha(\hat{p} - \hat{r})$, then

$$g(z) = (\mu \hat{r} - q)z - \frac{\mu}{4\alpha} z^2 \quad \text{if } z > 2\alpha(\hat{p} - \hat{r})$$

which is a quadratic map with maximum at $z = 2\alpha(\hat{p} - \frac{q}{\mu})$. This maximum is inside the interval $z > 2\alpha(\hat{p} - \hat{r})$ only when $q < \mu \hat{r}$. Observe that the function $g(z)$ is continuous at $z = 2\alpha(\hat{p} - \hat{r})$. Moreover, $g(z)$ is $C^1$ with

$$\frac{dg}{dz} \bigg|_{z=2\alpha(\hat{p}-\hat{r})} = \mu \hat{r} - q.$$
Therefore the maximizer $z^*_w$ is

$$
z^*_w = \begin{cases} 
2\alpha(p-q) & \text{if } q < \mu \hat{\alpha} \\
[0,2\alpha(p-\hat{\alpha})] & \text{if } q = \mu \hat{\alpha} \\
0 & \text{if } q > \mu \hat{\alpha}
\end{cases}
$$

and his expectation for period 1 is

$$
\hat{x}_w = \begin{cases} 
0 & \text{if } q < \mu \hat{\alpha} \\
2\alpha(p-\hat{\alpha}) - z^*_w & \text{if } q = \mu \hat{\alpha} \\
2\alpha(p-\hat{\alpha}) & \text{if } q > \mu \hat{\alpha}
\end{cases}
$$

Actually we will assume that

$$z_w = 2\alpha(p-\hat{\alpha}) = 2\alpha\left(p - \frac{q}{\mu}\right) \text{ and } x_w = 0 \text{ when } q = \mu \hat{\alpha}.$$

Using (3.1), the actual demand of the retailer $\alpha$ in the spot market will be

$$x_w = \begin{cases} 
0, & \text{if } q \leq \mu \hat{\alpha}; \\
2\alpha(p-r), & \text{if } (p-\hat{p}) + \left(\frac{q}{\mu} - r\right) > 0 \text{ and } q \leq \mu \hat{\alpha}; \\
2\alpha(p-r), & \text{if } q > \mu \hat{\alpha}.
\end{cases}$$

If all the retailers have the same probability distribution $\mu: \Omega \to [0,1]$ and the same beliefs $\hat{p}, \hat{r}$ then the aggregate demands $Z_w = \int_0^1 z_w^* d\alpha, X_w = \int_0^1 x_w^* d\alpha$ are

$$Z_w = \begin{cases} 
\hat{p} - \frac{q}{\mu}, & \text{if } q \leq \mu \hat{\alpha}; \\
0, & \text{if } q > \mu \hat{\alpha}
\end{cases}$$

$$X_w = \begin{cases} 
0, & \text{if } (p-\hat{p}) + \left(\frac{q}{\mu} - r\right) \leq 0 \text{ and } q \leq \mu \hat{\alpha}; \\
(p-\hat{p}) + \frac{q}{\mu} - r, & \text{if } (p-\hat{p}) + \left(\frac{q}{\mu} - r\right) > 0 \text{ and } q \leq \mu \hat{\alpha}; \\
p-r, & \text{if } q > \mu \hat{\alpha}.
\end{cases}$$

In the case of perfect information the probability $\mu$ gives total probability to the true state of nature $\omega^*$. The aggregate demands in this case are

$$Z_w = \begin{cases} 
\hat{p} - q, & \text{if } \omega = \omega^* \text{ and } q_w \leq \hat{\alpha}^*_w; \\
0, & \text{otherwise.}
\end{cases}$$
\[
X_\omega = \begin{cases} 
0, & \text{if } (p-\hat{p})+(q-r) \leq 0 \text{ and } q \leq \mu; \\
(p-\hat{p})+q-r, & \text{if } (p-\hat{p})+(q-r) > 0 \text{ and } q \leq \mu; \\
p-r, & \text{if } q > \mu.
\end{cases}
\]

III.2 Final Product’s Demand, Competitive Prices and Monopoly price

The demand for the final product at period 1, state \( \omega \), is \( D_\omega (p_\omega) = a_\omega - b_\omega p_\omega \). The aggregate supply given by the retailers is

\[
X_\omega + Z_\omega = \begin{cases} 
p_\omega - q_\omega, & \text{if } q_\omega \leq r_\omega; \\
p_\omega - r_\omega, & \text{if } q_\omega > r_\omega.
\end{cases}
\]

From the equilibrium condition \( D_\omega (p_\omega) = X_\omega + Z_\omega \), we get that

\[
p_\omega = \frac{a_\omega}{b_\omega + 1} + \frac{s_\omega}{b_\omega + 1}, \quad s_\omega = \min\{q_\omega, r_\omega\},
\]

\[
X_\omega + Z_\omega = h_\omega - k_\omega s_\omega;
\]

\[
h_\omega = \frac{a_\omega}{b_\omega + 1}, \quad k_\omega = \frac{b_\omega}{b_\omega + 1} < 1.
\]

Suppose for a moment that (i) The producer acts as a price-taking agent, (ii) The producer knows at period 0 that the true state of nature is \( \omega \in \Omega \), and (iii) There are no futures contracts. Then he would produce the quantity \( y \) for which his marginal cost is equal to the spot price of the intermediate product \( \frac{dc}{dy}(y) = r \) and he would sell all of his production. The production \( y \) would be given by \( 2 \beta y = r \), and the aggregate supply at the final product market on period 1, state \( \omega \), would be

\[
\frac{r}{2\beta} = y = X_\omega = h_\omega - k_\omega r.
\]

Therefore

\[
r_c(\omega) = \frac{2\beta h_\omega}{2\beta k_\omega + 1}, \quad \text{and} \quad y_c(\omega) = \frac{h}{k + 2\beta}. \tag{3.2}
\]

Call this price the (full information) “competitive price”.

Now suppose that (i) The producer knows at period 0 that the true state of nature is \( \omega \in \Omega \), (ii) The producer acts as a monopolist and (iii) There are no futures contracts. The monopolist would produce a quantity \( y = D_\omega (p) = X_\omega = h_\omega - k_\omega r_\omega \) (where \( Z_\omega = 0 \)). He will maximize his profits:

\[
\max_r \{r(h - kr) - c(h - kr)\},
\]

This gives
\[ h - 2kr + 2k\beta(h - kr) = 0 \]

\[ r_M := \frac{h(1 + 2k\beta)}{2k(1 + k\beta)}, \quad \text{and} \quad y_M = h - kr_M = \frac{h}{2(1 + \beta)} \]  

(3.3)

Call this price \( r_M \) the (fully informed) "monopoly price".

The profits of the producer in the fully informed competitive situation are

\[ G_e = r_c y - c(y) = \frac{\beta h^2}{(1 + 2k\beta)^2}, \]  

(3.4)

and in the fully informed monopolist case are

\[ G_M = \frac{h^2}{4k(1 + k\beta)} \]  

(3.5)

IV. The Perfect Information model

In this part we will assume that the retailers know with certainty at period 0 which is the true state of nature \( \omega \in \Omega \), but the producer doesn't. Nevertheless, the producer knows that the retailers are fully informed. Before period 0, the producer has a ex-ante probability distribution \( \nu: \Omega \rightarrow [0,1] \), which he uses in order to decide which contingent futures he will sell at period 0, and at which prices. Immediately after period zero, the monopolist makes his production decisions using the information given by the futures contracts sold at period 0. At period 1 all the uncertainty is resolved.

Suppose that on period 0, retailers believe that on period 1, state \( \omega \), the spot price for the intermediate product will be \( \tilde{r}_\omega \). By Section III, if the producer decides to sell futures contingent to state \( \omega \), the future's price must be \( q_\omega \leq \tilde{r}_\omega \), and the aggregate demand for futures will be

\[ Z_\omega = \begin{cases} \tilde{p} - q = h_\omega - k_\omega q, & \text{if } \omega = \omega^*; \\ 0, & \text{if } \omega \neq \omega^*, \end{cases} \]

where \( \omega^* \) is the true state of nature.

Suppose that \( \omega \) is the true state of nature and the producer sells \( Z_\omega \) futures contingent to \( \omega \). Then the producer gets the information that \( \omega \) is the true state and decides to produce a quantity

\[ y = \max\{Z_\omega, h_\omega - k_\omega r\} \]

such that he maximizes his profits:
\[ \max_r \left \{ rX_\omega + \sum_{v=0} q_v Z_v - c(y) \right \}, \]

with \( X_\omega + Z_\omega = y = \max \{ Z_\omega, h_\omega - k_\omega r \} \). Observe that \( Z_\omega = h - kq < h - kr \) only when \( q > r \).

Since the demands \( Z_v \) and the prices \( q_v \) are already realized, this problem is equivalent to

\[ \max \left \{ r(h_\omega - k_\omega r - Z_\omega) - c\left( \max \{ h - kr, Z_\omega \} \right) \right \}. \]

Suppose that \( h_\omega - k_\omega r > Z_\omega \), i.e. \( q > r \). Moreover, suppose that \( r_c(\omega) < q \leq \hat{r} \), where \( r_c(\omega) \) is the fully informed competitive price of section III.2. Then the problem becomes:

\[ \max \ G(\hat{r}), \quad G(\hat{r}) := r(h_\omega - k_\omega r - Z_\omega) - c(h_\omega - k_\omega r). \]

The function \( G(\hat{r}) \) has a maximum at

\[ r = \frac{h - Z + 2\beta h}{2k(1 + k\beta)} \]

\[ r = \frac{q + 2\beta h}{2(1 + k\beta)} =: f(q) \]

We have that \( 0 < \frac{\partial}{\partial x} f(x) = \frac{1}{2(1 + k\beta)} < 1 \) and \( f(r_c) = r_c \). So that \( r_c < f(x) < x \) for \( x > r_c \). Since \( r_c < q \leq \hat{r} \), we have that \( r_c < r < q = \hat{r} \).

This contradicts the hypothesis of rational expectations. Therefore \( q \leq r_c \). In this case the function \( G(\hat{r}) \) has no interior maximum and \( q \leq r_c \leq r \).

The intuition behind this is that if \( r_c(\omega) < q = \hat{r} \), then the producer will have incentives to sell an additional amount at period 1 at a price \( r < q \leq \hat{r} \) if it is still profitable for him (this will be the case if \( r_c(\omega) < r < q ) \). In that case it would be better for the retailer to buy at the spot market at period 1 and not to buy futures. Therefore equilibrium under rational expectations implies \( q \leq r \leq \hat{r} \).

Thus if \( \omega \) is the true state of nature, the prices \( r_\omega, q_\omega \) have to be \( q \leq r_c \leq r \) and \( q \leq \hat{r} \). We now see that \( q = r_c \). By the rational expectations hypothesis, we have that \( r = \hat{r} \), then \( q \leq r = \hat{r} \) and \( Z_\omega = h_\omega - k_\omega q \). At period 0 the producer maximizes his profits:

\[ \max_{q \leq r_c} F(q), \quad F(q) := q(h_\omega - k_\omega q) - c(h_\omega - k_\omega q). \]

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8Where \( Y^* := \max \{ 0, Y \} \).

9Note that it is immaterial what valued takes \( \hat{r} \) and \( r \) as long as \( q \leq \hat{r} \) and \( q \leq r \). In every case all allocations and profits are the same.
The maximum of the function $F$ is attained at $q = r_M > r_c$. So that the maximization problem has a boundary solution $q = r_c$. Therefore the prices are settled$^{10}$ at $q = r_c \leq r = \hat{r}$. We will prove this in general in the next proposition, but before we need a definition.

Observe that under rational expectations $r = \hat{r}$, the total demand of the retailers at stage $\omega$ is given by

$$X_\omega + Z_\omega = \begin{cases} h_\omega - k_\omega q_\omega = p_\omega - q_\omega, & \text{if } q_\omega \leq r_\omega; \\ h_\omega - k_\omega r_\omega = p_\omega - r_\omega, & \text{if } q_\omega > r_\omega. \end{cases}$$

Note that

$$X_\omega + Z_\omega = \begin{cases} y(q_\omega), & \text{if } q_\omega \leq r_\omega; \\ y(r_\omega), & \text{if } q_\omega > r_\omega; \end{cases} \begin{cases} X_\omega = 0 \\ Z_\omega = y(q_\omega) \end{cases}$$

where $y(x) = h - kx = p(x) - x$ is a differentiable function. We will use this function in the following proposition. Recalling the calculations in section III.2, we have that the general definition of $y(r)$ is given by

$$x_\alpha + z_\alpha = (c'_\alpha)^{-1}(p-r) = \varepsilon_\alpha(p-r)$$

$$y(r) = X + Z = \int_{[0,1]} (x_\alpha + z_\alpha) d\alpha$$

$$y(r) = \int_{[0,1]} \varepsilon_\alpha(p(r)-r) d\alpha$$

where $\varepsilon_\alpha(x) = \left( \frac{de_\alpha}{d\alpha} \right)^{-1}(x)$ is the inverse function of the retailer $\alpha$'s marginal cost function, and where we use $p = p(r)$ by the rational expectations assumption.

**Proposition.** Suppose that

(i) The retailers are fully informed.

(ii) The contingent futures contract for the true state $\omega$ is sold at period $0$.

(iii) The aggregate total demand at state $\omega$, $y(r_\omega) = X_\omega + Z_\omega$ is a strictly decreasing differentiable function of $r$. The monopolist's cost function $c(y)$ is convex: $c''(y) \geq 0$. Retailer's cost function is strictly convex $c''(y) > 0$, where $r_c(\omega)$ is the fully informed competitive price: $c'(y(r_c)) = r_c$.

**Proof:** The futures contract is only sold at a price $q \leq \hat{r}$. In that case the retailers beliefs imply $X(\hat{x}) = 0$ and $Z(q) = y(q)$. The maximization problem for the producer at period 1 is

$$\max_r \left\{ r(y(r) - Z(q))^\dagger - c(y(r)) \right\}.$$
The producer can only increase his profits at period 1 if \( r \leq q \) because only in this case
\[
(y(r) - Z(q))' = y(r) - y(q) \geq 0.
\]

Suppose that \( r_c < q \leq \hat{r} \). For \( r \leq q \), the first order condition implies that
\[
y - Z + ry' = c'(y)y'.
\]
Since \( y' < 0 \) and \( y(r) - Z(q) = y(r) - y(q) \geq 0 \), we have that
\[
c'(y(r)) = r + \frac{y(r) - Z(q)}{y'(r)} \geq 0, \quad \text{if } r \leq q. \quad (4.1)
\]
Since \( c'(y(r_c)) = r_c \) and \( \frac{d}{dr} c'(y(r)) = c''(y)y' \leq 0 \), we have that \( r \geq r_c \). But \( r \neq q \) because if \( r = q \) then from section II.1 \( c'(y(q)) = q \) and then \( r = r_c \), contrary to our hypothesis. Thus \( r_c \leq r < q \leq \hat{r} \). This contradicts the rational expectations hypothesis \( r = \hat{r} \). Therefore \( q \leq r_c \leq r \).

Since \( q \leq r_c \leq r \), we have that \( X = 0 \) and \( Z = y(q) \). The maximization problem of the producer at period 0 is
\[
\max_{q \in \Omega} \{ qy(q) - c(y(q)) \}.
\]
This problem has a boundary solution at \( r = q \). \( \blacksquare \)

We state now the main result of this section:

**Proposition.** It is never optimal for the monopolist to offer all the contingent futures contracts. Therefore this market is always incomplete.

**Proof:** If the monopolist offers all the contingent futures he will obtain the fully informed competitive profits \( G_c \) in any state of nature. If the monopolist does not offer the contingent contract for state \( \hat{\omega} \) and does offer all the rest, then he obtains \( G_c \) (at period zero and no more at period 1) on any state \( \omega \neq \hat{\omega} \). But if the state of nature is \( \hat{\omega} \) then he will know it at period 0 and will be able to produce the exact quantity in order to charge the monopoly price \( r_M \) and obtain the fully informed monopoly profits \( G_M > G_c \). \( \blacksquare \)

Indeed, if the monopolist does not offer exactly one contingent future contract \( \hat{\omega} \), then he
gets the same information as opening all the future contracts, but does not have any commitment.

**Solution of the producer's choice**
Now we solve the producer's problem: which future contracts to offer, how much to produce, $y$, and how much to charge: $r_o$.

Let $\Omega = E \cup F$, $E \cap F \neq \emptyset$ be a partition of $\Omega$. Suppose that the producer decides to offer contracts for all states in $E$ and not to offer futures for states in $F$. Then for all $\omega \in E$ the contract price will be $q = r_c(\omega)$, the competitive price, and the producer will obtain the competitive profits $G_c(\omega)$ if $\omega \in E$ is realized.

Suppose that some $\omega \in F$ is the true state of nature. Then the only information that the producer gets is that the true state is in $F$. We assume that he updates his probability distribution using $\mu: \Omega \rightarrow [0,1]$ with $\mu(\omega) = 0$ for $\omega \in E$ and

$$\mu(\omega) = \frac{\nu(\omega)}{\nu(F)} \quad \text{for} \quad \omega \in F,$$

i.e. $\mu = \text{the conditional probability of } \nu \text{ given } F$, where $\nu = P \circ \omega^{-1}$ is his ex-ante probability distribution. Given this probability $\mu$, he decides to produce a quantity $y$ which maximizes his expected profits.

Rename the states in $F$ so that $F = \{1,2,\ldots,m\}$ and $h_1 < h_2 < \cdots < h_m$. Then the producer will solve

$$\max_{y, \{\omega \in F\}} \mathbb{E}_\nu \{rX - c(y)\} \quad \text{subject to} \quad X_\omega = h_\omega - k_\omega r_\omega \leq y,$$

where $\mathbb{E}_\nu$ is the $\mu$-expectation. This is equivalent to

$$\max_{\eta, A \subseteq [\frac{h}{2}] \times \mathbb{R}} \sum_{\omega \in F} \mu(\omega) (h_\omega - k_\omega r_\omega) - \beta y^2 \quad \text{(4.2)}$$

Suppose we have already chosen $0 < y < \frac{h_m}{2}$. Let $\alpha \in F$ be such that $\frac{h}{2} < y < \frac{h + 1}{2}$ and $A = \{1, \cdots, m\}$, $B = \{\alpha + 1, \cdots, m\}$.

Case A: If $i \in A$, i.e., $y \geq \frac{h_i}{2}$, i.e. (over-production).

Suppose that we have an interior solution $y > h_i - k_i r_i$, where $y$, $r_i$ are the solutions of the problem. Then the first order conditions $\frac{\partial}{\partial \eta_i}$ on (4.2) give

$$r_i = \frac{h_i}{2k_i}, \quad X_i = h_i - k_i r_i = \frac{h_i}{2}.$$

Since $y > \frac{h_i}{2}$, this is indeed the case. The producer's profits in this state $i$ are

$$G_i = \frac{h_i^2}{4k_i} - \beta y^2 \quad \text{for} \quad i \in A.$$

11For simplicity we make the generic assumption that $h_i \neq h_j$ for all $i \neq j$ in $\Omega$. 12
Case B: If \( j \in B \), i.e., \( y \leq \frac{h_j}{2} \)

By case A we are in a border solution, so that \( y_j = h_j - k_j r_j \). Then

\[
r_j = \frac{h_j - y}{k_j}, \quad X_j = y
\]

and the producer's profits are

\[
G_j = \frac{h_j}{k_j} y - \frac{y^2}{k_j} - \beta y^2 \quad \text{if } j \in B.
\]

Therefore the expected profits, given that the production is \( y \), are

\[
\mathbb{E}G(y) := \frac{1}{4} \sum_{i \in A} h_i^2 k_i + \sum_{j \in B} \frac{h_j}{k_j} y - \sum_{j \in B} \mu_j \frac{1}{k_j} y^2 - \beta y^2
\]

\[
= \frac{1}{4} \mathbb{E}_A \left( \frac{h^2}{k} \right) + \mathbb{E}_B \left( \frac{h}{k} \right) y - \left( \mathbb{E}_B \left( \frac{1}{k} \right) + \beta \right) y^2
\]

\[
= \mathcal{A} + \mathcal{B} y + C y^2
\]

where \( \mathbb{E}_A (\ell) = \sum_{i \in A} \mu_i \ell_i \) and \( \mathbb{E}_B (\ell) = \sum_{j \in B} \mu_j \ell_j \) and \( \mathcal{A}, \mathcal{B} \) and \( C \) are defined by comparing coefficients. Observe that \( \mathbb{E}_A \) and \( \mathbb{E}_B \) are not the conditional expectations because for example \( \mathbb{E}_B (1) = \mu(B) \neq 1 \).

The "production" \( y \) which maximizes \( \mathbb{E}G(y) \) in this formula is \( y^* = \frac{B}{2C} \) and \( \mathbb{E}G(y^*) = \mathcal{A} + \frac{B^2}{4C} \), so that

\[
y = \frac{\mathbb{E}_B \left( \frac{h}{k} \right)}{2 \left( \mathbb{E}_B \left( \frac{1}{k} \right) + \beta \right)}, \quad \text{if } \frac{h_a}{2} < y < \frac{h_{a+1}}{2} \quad (4.3)
\]

\[
\mathbb{E}G = \frac{1}{4} \mathbb{E}_A \left( \frac{h^2}{k} \right) + \frac{1}{4} \mathbb{E}_B \left( \frac{h}{k} \right) + \frac{1}{4} \left( \mathbb{E}_A \left( \frac{1}{k} \right) + \beta \right), \quad \text{if } \frac{h_a}{2} < y < \frac{h_{a+1}}{2} \quad (4.4)
\]

It is interesting to compare this production \( y \) in (4.3) with the monopoly production \( y_M \) from (3.3). \( \frac{h_a}{2} < y < \frac{h_{a+1}}{2} \) is then equivalent to

\[
\frac{h_a \mathbb{E}_B \left( \frac{1}{k} \right) + \beta h_a}{\mathbb{E}_B \left( \frac{h}{k} \right)} \leq \left( \frac{h_{a+1}}{h_a} \right) \mathbb{E}_B \left( \frac{1}{k} \right) + \beta \frac{h_{a+1}}{h_a}
\]

\[
\frac{1}{h_{a+1}} \mathbb{E}_B \left( \frac{h}{k} \right) - \mathbb{E}_B \left( \frac{1}{k} \right) \leq \beta \leq \frac{1}{h_a} \mathbb{E}_B \left( \frac{h}{k} \right) - \mathbb{E}_B \left( \frac{1}{k} \right).
\]
Let \( q : [0, h_{m}] \rightarrow \mathbb{R} \) be defined by \( q(x) = \frac{1}{x} E_{F} \left( \frac{h}{k} \right) - E_{F} \left( \frac{1}{k} \right) \) if \( 0 < x \leq h \) and \( q(x) = \frac{1}{x} E_{B} \left( \frac{h}{k} \right) - E_{B} \left( \frac{1}{k} \right) \) if \( h_{a} < x \leq h_{a+1} \), where \( B = \{ \alpha + 1, \alpha + 2, \cdots, m \} \).

Claim 1.

(i) \( \lim_{x \to 0^+} q(x) = +\infty \) and \( \lim_{x \to h_{m}^-} q(x) = 0 \).

(ii) \( q : [0, h_{m}] \rightarrow \mathbb{R} \) is continuous, strictly decreasing and surjective.

**Proof:** We only prove (ii). Let \( B = \{ \alpha + 1, \alpha + 2, \cdots, m \} \) and \( D = \{ \alpha, \alpha + 1, \cdots, m \} \). Let \( q_{a} : [h_{a}, h_{a+1}] \rightarrow \mathbb{R} \) be that restriction of \( q(x) \) to \( h_{a} < x \leq h_{a+1} \). It is clear that \( q_{a}(x) \) is strictly decreasing. We only need to prove that \( \lim_{x \to h_{a}^+} q_{a}(x) = \lim_{x \to h_{a}^-} q_{a-1}(x) \). But

\[
\lim_{x \to h_{a}^+} q_{a}(x) = \frac{1}{h_{a}} E_{B} \left( \frac{h}{k} \right) - E_{B} \left( \frac{1}{k} \right) = \frac{1}{h_{a}} E_{B} \left( \frac{h}{k} \right) - \frac{\mu_{a}}{k} - \frac{E_{B} \left( \frac{1}{k} \right)}{k} + \lim_{x \to h_{a}^-} q_{a-1}(x).
\]

This implies that there exists exactly one \( \alpha \in F \) such that \( q(h_{a+1}) \leq \beta \leq q(h_{a}) \). Now consider the function \( E \ G : [0, h_{m}] \rightarrow \mathbb{R} \).

Claim 2. \( E \ G : [0, h_{m}^+] \rightarrow \mathbb{R} \) is continuous.

**Proof:** Let \( E \ G : [h_{m}^+, h_{m}^+] \rightarrow \mathbb{R} \) be the restriction of \( E \ G(y) \) to the interval \( \frac{h_{m}}{2} < y < \frac{h_{m}}{2} \). Let \( A = \{ 1, \cdots, \alpha \} \), \( B = \{ \alpha + 1, \alpha + 2, \cdots, m \} \) and \( C = \{ 1, \cdots, \alpha - 1 \} \), \( D = \{ \alpha, \alpha + 1, \cdots, m \} \); then

\[
\lim_{y \to \left( \frac{h_{m}}{2} \right)^+} E \ G_{a}(y) = \lim_{y \to \left( \frac{h_{m}}{2} \right)^-} \left\{ \frac{1}{4} E_{A} \left( \frac{h}{k} \right) - E_{B} \left( \frac{h}{k} \right) y - \left( E_{B} \left( \frac{1}{k} \right) + \beta \right) y^{2} \right\}
\]

\[
= \frac{1}{4} E_{C} \left( \frac{h}{k} \right) + \frac{1}{4} \mu_{a} \frac{h_{a}^{2}}{k_{a}} + E_{D} \left( \frac{h}{k} \right) h_{a} - \frac{1}{2} \mu_{a} \frac{h_{a}^{2}}{k_{a}} \left( E_{B} \left( \frac{1}{k} \right) + \beta \right) \frac{h_{a}}{4} + \frac{1}{4} \mu_{a} \frac{h_{a}^{2}}{k_{a}}.
\]

We have that \( E \ G(y) \) is a quadratic function on each interval \( \frac{h_{m}}{2} < y < \frac{h_{m}}{2} \), it is continuous and there is only one of these intervals - the one with \( q(h_{a+1}) \leq \beta < q(h_{a}) \) - where it has a local maximum, which has to be its global maximum, given by (4.3), (4.4).
The producer’s problem is then solved as follows: For each choice \( E \subseteq \Omega \) with \( \#(\Omega - E) \geq 2 \) of contingent futures to be offered, do the following:

1. Let \( F = \Omega - E \) and order the elements of \( F \) by \( F = \{1, \ldots, m\} \) with \( h_1 < h_2 < \cdots < h_m \).

2. Let \( \mu(\omega) := \frac{\nu(\omega)}{\nu(F)} \), for all \( \omega \in F \).

3. For \( \alpha \in F \), let
   \[
   q(h_\alpha) := \sum_{\alpha \leq j < cm} \frac{\mu_j}{k_j} \left( \frac{h_j}{h_\alpha} - 1 \right)
   \]
   and choose \( \alpha \in F \) such that \( q(h_{\alpha+1}) \leq \beta < q(h_\alpha) \).

4. Let \( A = \{1, \ldots, \alpha\} \), \( B = \{\alpha+1, \ldots, m\} \) and
   \[
   y_B = \frac{\mathbb{E}_B \left( \frac{h}{k} \right)}{2 \left( \mathbb{E}_B \left( \frac{1}{k} \right) + \beta \right)}
   \]
   \[
   \mathbb{E} G(E) = \frac{1}{4} \mathbb{E}_A \left( \frac{h^2}{k} \right) + \frac{\mathbb{E}_B \left( \frac{h}{k} \right)^2}{4 \left( \mathbb{E}_B \left( \frac{h}{k} \right) + \beta \right)}
   \]

5. Compare the expected profits \( \mathbb{E} G(E) \) for all possible choices of \( E \subseteq \Omega \), com \( \#(\Omega - E) \geq 2 \) and choose the one \( E^* \) with higher \( \mathbb{E} G \).

6. Compare \( \mathbb{E} G(E^*) \) with all the choices \( E = \Omega - \{\omega\} \) whose expected profits are
   \[
   \mathbb{E} G(\omega) = \sum_{v \in \Omega - \{\omega\}} \nu_G(v) + \nu_G(\omega)
   \]
   where \( G_e \) and \( G_M \) are from § III.2. Choose the (proper) subset \( E \subseteq \Omega \), but \( E \neq \Omega \) with higher expected profits.

7. If the choice was \( E = \Omega - \{\omega^*\} \), then the producer has full information after offering the futures and can solve his maximization problem without uncertainty. Under rational expectations this will give
   \[
   \begin{cases}
   \{ y_\omega = Z_\omega = y_c = \frac{h_\omega}{2\beta k_\omega + 1} \} & \text{for } \omega \neq \omega^* \\
   q_\omega = r_\omega = r_c(\omega) = \frac{2\beta h_\omega}{2\beta k + 1} \end{cases}
   \]
\[
\begin{align*}
\begin{cases}
    y_\omega^* = y_M = \frac{h_\omega}{2(1+\beta k_\omega)} & \text{for } \omega = \omega^* \\
    r_\omega^* = r_M(\omega) = \frac{h_\omega(1+2\beta k_\omega)}{2k_\omega(1+\beta k_\omega)}
\end{cases}
\end{align*}
\]

8. If the choice was \( E \subset \Omega \), \( \text{com } \#(\Omega - E) \geq 2 \), then

\[
\begin{align*}
\begin{cases}
    y_\omega = Z_\omega = y_E \\
    r_\omega = r_E(\omega)
\end{cases} & \text{for } \omega \in E, \\
\begin{cases}
    y = y_E \text{ from 4.} \\
    r_\omega = \begin{cases}
        \frac{h_\omega}{2k_\omega} & \text{se } h_\omega < y_E \\
        \frac{h_\omega - y_E}{k_\omega} & \text{se } h_\omega \geq y_E
    \end{cases} & \text{for } \omega \not\in E
\end{cases}
\end{align*}
\]
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