"An Empirical Comparison of Forward-Rate and Spot-Rate Models for Valuing Interest-Rate Options"

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ABSTRACT

Our main goal is to investigate the question of which interest-rate options valuation models are better suited to support the management of interest-rate risk. We use the German market to test seven spot-rate and forward-rate models with one and two factors for interest-rate warrants for the period from 1990 to 1993. We identify a one-factor forward-rate model and two spot-rate models with two factors that are not significantly outperformed by any of the other four models. Further rankings are possible if additional criteria are applied.

A VALUATION MODEL FOR INTEREST-RATE derivatives represents the core of any system designed to measure, control, and supervise interest-rate risk. This is true regardless of whether a value-at-risk methodology, sensitivity analysis, stress test, or scenario technique is applied. Unfortunately, there is no empirical evidence that evaluates the performance of the most popular competing pricing models using the same data from a risk management perspective. This paper provides such empirical evidence using data from the German market for interest-rate warrants for the period from 1990 to 1993.

The more recent valuation models are dominated by two groups of models, forward-rate and spot-rate models. The approach of the first group, pioneered by Ho and Lee (1986) and Heath, Jarrow, and Morton (HJM) (1992), directly uses the arbitrage-free dynamics of the entire zero bond price curve or, equivalently, the term structure of forward rates to price interest-rate derivatives. We refer to this approach as the forward-rate (or HJM) model. The approach of the second group (e.g., see Hull and White (1990, 1993), Black, Derman, and Toy (1990), Black and Karasinski (1991), Jamshidian (1991), and Sandmann and Sondermann (1993)) is based on the dynamics of

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the instantaneous spot interest rate. The second group's papers follow a suggestion by Cox, Ingersoll, and Ross (1985) and fit the endogenous term structure (and volatility structure) of interest rates to the observed term structure. This fitting is achieved through time-dependent parameters of the stochastic factor processes. Both approaches that model the stochastic behavior of the term structure of interest rates, as well as the subsequent valuation of derivatives, are closely related to each other. In fact, for some of the models, a mathematical equivalence is easily established.

In this comprehensive empirical study we test one-factor and two-factor spot-rate and forward-rate models. The paper's main goal is to clarify the question of whether spot- or forward-rate models are better suited to support the measurement, control, and supervision of interest-rate risk. Existing work provides no answer to this important question. Though the two classes of models are close to each other from a theoretical point of view, they can exhibit very different behaviors in application. Furthermore, it is not at all obvious whether two-factor models outperform one-factor models in each class.

The models we test have the following basic structures. The one-factor spot-rate model is characterized by a mean-reverting drift and a diffusion coefficient with constant elasticity. Within the class of two-factor spot-rate models we consider two variants: In the first model, the factors are identified with a long rate and the difference between this long rate and a short rate. In the second model, the factors we consider are the short rate and its volatility. These two-factor models represent generalizations of models developed by Schaefer and Schwartz (1984) and Longstaff and Schwartz (1992).

We test two forward-rate models with one factor. The first model represents the continuous-time version of Ho and Lee's (1986) model with Gaussian forward rates and constant (absolute) volatility. The second model has a linear proportional volatility—that is, the proportional volatility depends solely on time-to-maturity of forward rates. We also test two forward-rate models with two factors. We determine the volatility functions of both models by principal component analyses. The volatilities, therefore, depend on empirically specified factor loadings and these factors' volatilities. In one model, forward-rate volatilities are independent of forward rates. In the other, they are proportionally dependent on forward rates.

Assessing the applicability of the different models within a risk management system requires two important decisions about the test methodology. First, valuation models within risk management systems must be capable of predicting future option prices if they are to correctly measure risk exposure. This capability is best evaluated by the ex ante predictability of a model. Therefore, we use the valuation quality of a model, not its ability to identify mispriced options, as the most important assessment criterion. This implies that this study is not a test of an option market's efficiency.

Second, we estimate all parameters, including the volatility, from time series. There are two reasons to estimate parameters historically rather than implicitly. We cannot prematurely rule out that model-dependent parameters, such as implied volatilities, favor one model versus the other. More important, testing valuation models with implied parameters only repre-

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sent a "local test," in the sense that the current option price is used to value the same option one period later. Therefore, "local tests" consider only small deviations from observed prices of derivatives. Tests on the basis of historically estimated parameters are "global," in the sense that they do not use information from derivatives markets.

In addition to the valuation quality, there are other important criteria for assessing a valuation model for interest-rate options. These refer to the difficulties in estimating the model parameters, in fitting the model to the current term and volatility structures, in computing the option values numerically, and to the stability of the model's performance over different time periods.

We test the seven valuation models for interest-rate warrants from the German market for the period from 1989 to 1993. In contrast to standardized options traded on the German Futures and Options Exchange (DTB) in Frankfurt, interest-rate warrants are options issued by banks. Underlying these warrants are German government bonds, which represent the most liquid market segment in the German bond market. The market for interest-rate warrants started in 1989 and is now more liquid than the market of standardized options on the BUND-Future that are traded on the DTB. We also selected this market segment of German interest-rate options because warrants show a much wider variety of terms than do standardized options. There are warrants of both the European and American types whose maturities range up to 2.9 years, compared with nine months for the standardized futures options on the BUND-Future.

Very few papers study the empirical performance of models for the valuation of interest-rate options. Dietrich-Campbell and Schwartz (1986) value interest-rate options on U.S. government bonds and treasury bills, using the two-factor Brennan and Schwartz (1977) model. Bühler and Schulze (1995), Flesaker (1993), and Amin and Morton (1994) present empirical studies of the HJM model. The study by Bühler and Schulze analyzes the optimal call policy of callable bonds that are issued by German public authorities. Flesaker, as well as Amin and Morton, presents results for Eurodollar futures options. However, none of these studies compares spot-rate and forward-rate models.

The paper is organized as follows. In Section I we describe our selection of the empirically tested models on the basis of an extensive data analysis and the decisions made in the different implementation steps. Section II presents relevant information on the German bond and interest-rate options market. In Section III we describe the design of the empirical study and provide estimation results for the input data of the different models. The valuation results, including a multivariate error analysis and a paired comparison of the seven models, appear in Section IV. We summarize our final assessment of the models and present conclusions in Section V.

I. Selected Valuation Models and Their Implementation

In this section, we describe our selection of the empirically tested spot-rate and forward-rate models. We also specify the formal structure of the models and describe the decisions made in the different implementation steps.
A. Data Analysis and Selection of Models

The first step in testing interest-rate options valuation models is to pre-select the basic features of the models. Preselection refers to the number of factors driving the term structure of interest rates and the functional form of the stochastic processes for these factors. To do this, we perform an extensive data analysis of the factor structure of zero bond yields and the behavior of individual yields in the German bond market. We then summarize our findings, which are the basis of the models discussed in Section I.B. The details are presented in Bühler et al. (1996).

We first apply principal component analyses to determine the number of factors. Because of the high degree of correlation among yields with different maturities, two factors explain more than 95 percent of the variation in the term structure of interest rates. These findings are stable over different time periods with varying lengths between 1970 to 1993. Litterman and Scheinkman (1991) report similar results for the U.S. market. On the basis of these results, we only consider one- and two-factor models. One-factor models should be understood as a reference case against which we measure the improvement of introducing a second factor. Of course, to understand whether two factors are sufficient, the whole study should be carried out for three factors.

A.1. One-Factor Models

The distinguishing feature of a one-factor forward-rate model is the functional form of the forward-rate volatility. Amin and Morton (1994) test different parsimonious (one and two parameter) parameterizations and find that the number of parameters has a stronger effect on the behavior of the model than does the form of the models used. Two-parameter models tend to fit prices better. Indeed, the model with the best fit in-sample and out-of-sample is the two-parameter model with a linear proportional volatility function. However, the one-parameter models result in implied parameter estimates that are more stable, and the models earn larger and more consistent profits from their perceived mispricings. Amin and Morton conclude that the model with constant volatility (the absolute model) seems to be preferable among the one-parameter models.

In light of these findings, we test two one-factor HJM models, a one-parameter and a two-parameter model. The one-parameter model we choose for our investigation is the one with constant volatility, which is in fact the continuous-time version of the Ho and Lee model. The two-parameter model is the one with linear proportional volatility.

One-factor spot-rate models start with a specification of the process of the short rate, \( r \). In line with studies for other markets, we find that an increase (decrease) in short rates is more likely than a decrease (increase) if the values of the rates are historically low (high). Additionally, large interest-rate movements take place in periods of high interest rates, and moderate movements are observed in low-rate periods. Therefore, we use the standard model for the short rate, a mean-reverting process with an instantaneous volatility shown as \( \sigma r \). Contrary to the results of Chan et al. (1992), who report a value of 1.5 for the exponent \( \epsilon \), we find values between 0.5 and 1. These estimates result in a unique solution of the stochastic differential equation of \( r \).

A.2. Two-Factor Models

For a two-factor forward-rate model, two volatility functions—one for each factor—must be specified. We consider two different structural specifications. In the first case, we assume that both volatility functions are independent of the forward rate's level. In the second case, the two volatility functions are proportional to the forward rates. We empirically determine the precise functional form of the volatility functions in both two-factor models from principal component analyses.

Our choice of the state variables for the two two-factor spot-rate models we investigate is motivated by two empirical findings. First, principal component analyses in combination with regression analyses reveal that the first component can be identified with the level of the yield curve; the second is closely related to the spread between the long and the short rate. We take these findings as a guideline for the construction of our first two-factor model, which uses both a long rate and the spread between the same long rate and the short rate as factors. The basic idea for this line of approach goes back to Brennan and Schwartz (1979) and Schaefer and Schwartz (1984).

The second two-factor model is based on the observation that the short-rate volatility exhibits typical volatility clusterings. Therefore, a model with stochastic volatility of the short rate could be an appropriate description of the data. This model represents a generalization of the Longstaff and Schwartz (1992) approach.

Both two-factor spot-rate models are special cases of the affine class of term structure models. (See Duffie and Kan (1996), pp. 383-391.) Although they are mathematically equivalent, empirically, they can behave very differently.

B. Review of the Models and Basic Implementation Steps

B.1. The Forward-Rate Models

Here, we briefly review the HJM approach and the concrete implementation realized in this study. Rather than discussing the approach in general terms, we concentrate on the simplest derivation for the forward-rate models that we use in our empirical investigation.

See Chan et al. (1992) for the U.S. market, Barone at al. (1991) for the Italian market, and Walter (1996) for the German market.

See Uhrig and Walter (1999). The details of this estimation procedure are discussed later.

These results are also found for other markets. For example, see Litterman and Scheinkman (1991) for the U.S. market, Rebonato (1996) for the British market, and Böhler and Zimmermann (1996) for a recent study of the Swiss market.
This table summarizes the parametric specification of the volatility functions. \( a(t, T, f) \) denotes the volatility function for the one-factor models. \( \sigma_1(t, T, f) \) and \( \sigma_2(t, T, f) \) represent the two volatility functions for the two-factor models. \( f \) denotes the instantaneous forward rate at date \( t \) for instantaneous borrowing or lending at date \( T \) (\( T \geq t \)). \( \sigma \), \( \sigma_1 \), and \( \sigma_2 \) are positive parameters. \( \sigma_1(t, T) \) and \( \sigma_2(t, T) \) are functions of \( t \) and \( T \), which are to be empirically determined. In order to avoid an explosion of the forward-rate processes in finite time, the proportional volatility is capped by a large positive number \( M \).

<table>
<thead>
<tr>
<th>Panel A. One-Factor Models</th>
<th>Panel B. Two-Factor Models</th>
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<tbody>
<tr>
<td>( a(t, T, f) = \sigma )</td>
<td>Absolute I</td>
</tr>
<tr>
<td>( a(t, T, f) = (\sigma_0 + \sigma_1(T - t)) \text{min}(f, M) )</td>
<td>Linear proportional</td>
</tr>
<tr>
<td>( \sigma_1(t, T, f) = \sigma_1(t, T) )</td>
<td>Absolute II</td>
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<tr>
<td>( \sigma_2(t, T, f) = \sigma_2(t, T) )</td>
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<tr>
<td>( \sigma_1(t, T, f) = \sigma_1(t, T) \text{min}(f, M) )</td>
<td>Proportional II</td>
</tr>
<tr>
<td>( \sigma_2(t, T, f) = \sigma_2(t, T) \text{min}(f, M) )</td>
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The fundamental building block of this approach is the whole instantaneous forward-rate curve. HJM start with a fixed number of unspecified factors that drive the dynamics of these forward rates:

\[
df(t, T) = a(t, T; \cdot) \, dt + \sum_{i=1}^{2} \sigma_i(t, T, f) \, dz_i(t),
\]

where \( f(t, T) \) denotes the instantaneous forward interest rate at date \( t \) for borrowing or lending at date \( T \) (\( T \geq t \)), \( z_1(t) \) and \( z_2(t) \) denote independent one-dimensional Brownian motions, and \( a(t, T; \cdot) \) and \( \sigma_i(t, T, f) \) are the drift and the volatility coefficients of the forward rate of maturity \( T \). As HJM show, when a number of regularity conditions and a standard no-arbitrage condition are satisfied, then the drift of the forward rates under the risk-neutral measure is uniquely determined by the volatility functions \( \sigma_i(t, T, f) \):\(^8\)

\[
a(t, T; \cdot) = \sum_{i=1}^{2} \sigma_i(t, T, f) \int_{t}^{T} \sigma_i(t, s, f) \, ds.
\]

As we noted earlier, we focus on four specifications of this approach: two one-factor models and two two-factor models. The parametric specification of the volatility functions is shown in Table I.

\(^8\) In the following, we concentrate on two factors. In general, drift and volatilities can depend on the path of the forward-rate curve. See Heath et al. (1992), p. 80.
where $r$ represents the instantaneous risk-free rate, and $\Theta_1(x_1,x_2,t)$ and $\Theta_2(x_1,x_2,t)$ are the market prices of risk for the two factors. It follows from no-arbitrage arguments that $\Theta_1$ and $\Theta_2$ are real-valued functions of only the state variables and time. The subscripts of $F$ denote partial derivatives. We obtain the values for contingent claims by solving this parabolic partial differential equation subject to appropriate initial and boundary conditions.

In this study, we derive the three spot-rate models tested from equation (3) by specifying the nature of the stochastic processes driving the factors, the functional form of the market prices of risk, and the relation between the factors and the instantaneous spot rate, $r$. Table II summarizes the assumptions underlying the one-factor model and the models with two factors.

### 8.2.1. One-Factor Spot-Rate Model

The first model is a one-factor interest-rate model, in which we assume that the dynamics of the short-term interest rate $r(t)$ exhibit mean-reversion and that the diffusion coefficient depends on the level of the short rate. $\kappa$, $\gamma$, $\sigma$, and $\varepsilon$ are positive constants.

The basic idea behind Hull and White's (1990) procedure is to allow for time-dependent parameters in the risk-neutralized process of $r$, in order to match the solution of equation (3), in the case of zero bonds, to an exogenously given term structure of zero bond prices. If the elasticity parameter $\varepsilon$ is positive, this calibration must be carried out numerically. Generally, any of the parameters can be selected as a time-dependent function. However, for several economic and technical reasons, we select the market price of risk as a time-dependent function. In principle, if a second parameter is assumed to depend on time, the model can also be fitted to an exogenously given current volatility structure. However, this procedure has an important drawback: this second time-dependent parameter results in unstable and partially unrealistic future endogenous volatility structures. (Hull and White (1993, 1996) report similar results.)

In light of these findings, we do not calibrate the model to the whole current volatility structure with a second time-dependent parameter. Instead, we fit only two points of the endogenous volatility structure to the observable one, the volatilities of the short and the long rates. Volatilities of intermediate rates are interpolated by the model.

The advantage of this parsimonious fitting procedure is that the model results in stable future volatility structures. Technically, we achieve the two-point calibration of the model to the current volatility structure by the (constant) mean-reversion parameter $\kappa$, which determines the transmission of the instantaneous interest rate's volatility to the volatilities of long rates (cf. Uhrig and Walter (1996), pp. 87–88).

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7 The arguments refer to the economic interpretation of the calibration process, the existence of a solution for the fitting function, the consequences for the endogenous volatility structure, and the relation between risk-adjusted and original measure. Cf. Heath et al. (1992), pp. 96–97, and Uhrig and Walter (1990), pp. 84–85.
The model's implementation requires four steps. First, we determine the current yield curve of zero bonds from prices of coupon bonds. Second, we use an Euler-discretization to estimate the drift and volatility parameters of the short-rate process. We obtain the maximum likelihood estimates from time-series observations of the one-month money market rate. Third, we achieve the fitting to the current yield curve and to the volatility of the long rate with maturity $T = 9$ years by using a numerical algorithm that simultaneously determines the time-dependent function $\lambda(t)$ and the parameter $\kappa$. This requires the solution of the partial differential equation (3) with one state variable $x_1 = r$ under the following conditions:

i. maturity condition for zero bonds: $F(r, T, T) = 1$

ii. fitting condition for endogenous zero bond prices to observed prices $\hat{F}(T)$ conditional on the current instantaneous rate, $r(0)$:

$$F(r(0), 0, T) = \hat{F}(T) \quad (0 < T \leq T)$$

iii. fitting condition for the volatility of the long rate:

$$\frac{1}{T} F_1(r(0), 0, T) = 0.$$  \hspace{1cm} (5)

The left-hand side of equation (5) represents the ratio of the endogenous yield volatility of a zero bond with maturity $T$ and the short-rate volatility $\sigma r(0)^{\lambda}$. The right-hand side denotes the ratio of the historical observed volatility of the zero bond yield with maturity $T$ and the short-rate volatility. We use the inverted implicit finite difference method introduced by Uryü and Walter (1996) to solve equations (3)-(5). In the fourth and final step, we determine the values of the interest-rate warrants by solving equation (3) together with the appropriate boundary conditions for calls and puts by a fully implicit Crank-Nicolson method. We use a time step $\Delta t$ of one day for the time variable and a grid size of $1/30$ for the transformed state variable $z = 1/(1 + r(0))$. This change of the state variable, proposed by Brennan and Schwartz (1979), has the advantage that the original state space $[0, \infty)$ is transformed into the bounded interval $[0, 1]$, and that the boundary conditions are easier to handle. For $z = 0$ ($r = \infty$) we use the fact that the values of bonds and European options are zero, and for $z = 1$ ($r = 0$) we exploit numerically the special structure of the transformed partial differential equation (3). We take into account the early exercise possibilities for American options in the recursive backward procedure.

B.2.2. Two-Factor Model with Long Rate and Spread. The choice of the state variables in the second model is motivated by our findings in our data analysis process, and by the fact that the correlation among the changes of these two factors is less than 0.23. These empirical observations suggest that a two-factor model that uses a long-term rate $l$ and the spread $s$ between the long-term rate $l$ and the short-term rate $r$ as independent stochastic factors can plausibly describe the yield curve dynamics.

An examination of the time-series behavior of long rates in the German bond market reveals that there is only a slight mean-reversion tendency in long rates. The maximum likelihood estimate of the discrete version of the process $dl = \kappa(l - 1)dt + \sigma_l l^{-\kappa}dz^1$ for the mean-reversion parameter $\kappa$ is not significantly different from zero on the 1 percent level. As a proxy for the long rate, we use the yield to maturity of a nine-year zero bond.

To exclude, additionally, negative long rates $l$ and to keep the model analytically tractable, we model the long rate as a martingale with a square root representation $\sigma_r l^{\lambda}$ of the diffusion coefficient. The spread process is assumed to follow an Ornstein-Uhlenbeck process, in line with the observation that this process can take on both positive and negative values. The parameters $\sigma_l$, $\gamma$, and $\sigma_r$ are positive and constant.

We calibrate the model to the current term structure of interest rates by means of a time-dependent market price of spread risk. Because of the separability of variables and the choice of an Ornstein-Uhlenbeck process for the spread, we can solve this problem analytically. Moreover, this spread process ensures that the endogenous term structure of interest rates can be adapted to every observed one. This is not generally true for two-factor models, which use nonnegative state variables.

We use the market price of long-term interest-rate risk to overcome a problem that is typical for two-factor models in which both factors are interest rates (see also Duffie and Kan (1996), p. 383). The state variable $l$ is labeled "long rate," but it does not have this property, since the price of a zero bond depends on both state variables $s$ and $l$. Therefore, a zero bond with a maturity corresponding to $l$ has a yield to maturity that does not rely only on $l$. The internal inconsistency can be considerably reduced by an appropriate choice of the market price of risk $\theta_l$ of the long rate.\footnote{However, this modeling has the drawback that for this process, $l = 0$ is an absorbing barrier.}

Again, the implementation of the model consists of four steps. The first step coincides with the first step of the one-factor model. The second step is reduced to an estimation of the two volatilities of the long rate and the spread. These two parameters also reflect the information about the volatility structure. In the third step, the endogenous zero bond prices are analytically fitted to the observed prices by exploiting the separability of the solution $F(l, t, T) = G(l, t, T) H(s, t, T)$. Finally, we compute the option values by means of the alternating direction implicit method.\footnote{In contrast to the valuation of zero coupon bonds within the two-factor models, the valuation of options on coupon bonds makes it necessary to solve a partial differential equation in two state variables, because the terminal condition cannot be separated.}

\hspace{1cm}  

\footnote{Hall and White (1993) solve equations (3)-(5) on the basis of a trinomial tree. This method can be considered as an explicit scheme. Cf. Brennan and Schwartz (1978), p. 464.}
The grid sizes for the time and state variables are fixed identically to that of the one-factor model. Because the state variable \( s \) can take on both positive and negative values, we choose a special treatment. In the numerical procedure, we restrict this state variable on the interval \((-\infty, 0]\). We choose the upper boundary \( l(0) \) to ensure nonnegative short rates for the current level of \( l \). By an appropriate change of the state variable, we transform the original state space \((-\infty, 0]\) into the state space \([0, 1]\). For \( s = -\infty \), we use the fact that the values of bonds and European options are zero. For \( s = l(0) \) we impose a boundary condition, setting the second derivative \( F_{ss} \) equal to zero.

### B.2.3. Two-Factor Model with the Short Rate and Its Volatility

From our data analysis step, we know that the short rates exhibit volatility clusters. This behavior can be modeled approximately by stochastic volatility. Since interest-rate volatility is a key variable in option pricing, a promising approach might be to add to the one-factor model the level of interest-rate volatility as a second state variable.

The resulting model represents a generalized version of the general equilibrium model proposed by Longstaff and Schwartz (1992). Longstaff and Schwartz base their model on assumptions about the stochastic evolution of two abstract independent factors \( x \) and \( y \), described in Table II, in which \( \kappa_x, \gamma_x, \sigma_x, \kappa_y, \gamma_y, \) and \( \sigma_y \) are (positive) parameters. The short-term rate \( r \) and its instantaneous variance \( V \) are determined endogenously as part of the equilibrium:

\[
\begin{align*}
  r &= x + y \\
  V &= \sigma_x^2 x^2 + \sigma_y^2 y^2.
\end{align*}
\]

Using this system of linear equations, we can represent the fundamental valuation equation (3) for interest-rate derivatives in terms of the observable state variables \( r \) and \( V \).

To achieve consistency with the current term structure, we generalize the model by allowing for a time-dependent risk parameter. Because of the separability of the partial differential equation in the state variables \( x \) and \( y \), we can reduce the adaptation of the endogenous to the exogenous term structure of interest rates to the adaptation problem within the one-factor Cox et al. (1985) model.

Again, the implementation procedure requires four steps. As in the other models, the first step involves the estimation of the current term structure of interest rates. Compared with the other two spot-rate models, the estimation of the parameters in the second step is more complex, because the volatility of the short-term rate is not directly observable. Following Longstaff and Schwartz (1992), we use a two-phase approach. In the first phase, we estimate the volatility of the short-term rate by using a generalized autoregressive conditional heteroskedastic (GARCH) model. In the second phase, we estimate the parameters describing the movement of the short-term interest rate and its volatility. To estimate these parameters, we equate the first two moments of the long-run stationary unconditional distribution of \( r \) and \( V \) with their historical counterparts.

In addition to these four equations, we obtain two further conditions by choosing the volatility parameters \( \sigma_x^2 \) and \( \sigma_y^2 \) as the minimum and the maximum of the ratio \( V(t)/r(0) \), respectively. By using these six conditions, we calculate the six parameters of the model by solving a nonlinear system of six equations.

The calibration of the model to the current term structure in the third step follows the same procedure as the one-factor model. Again, we apply the inverted implicit difference method. The computation of the option prices in the fourth step is comparable to the procedure in the other two-factor spot-rate model.

### II. The German Fixed-Income Market

The German bond market is the third largest in the world. At year-end 1995, the nominal value of outstanding publicly issued bonds totaled more than three trillion deutsch marks (DM).

Traditionally, the bank bond sector is the largest component. However, bonds issued by the federal government are the most liquid. Typically, so-called Bundesanleihen (BUNDS) are issued with an initial maturity of ten years and Bundesobligationen (BOBLs) with an initial maturity of five years. BUNDS are termed as long term, and BOBLs as medium term. Various interest-rate derivatives have been launched in the last seven years, of which the BUND-Future (futures on ten-year government bonds) at the Deutsche Terminbörse is the most popular.

Three types of interest-rate options trade in Germany: options on the BUND-Future and on the BOBL-Future (futures on five-year government bonds), interest-rate warrants, and over-the-counter (OTC) interest-rate options of all types. Because the market for options on futures is not particularly liquid and sufficiently long-term time-series data are not available for OTC-options, our empirical study is based on the valuation of interest-rate warrants. Underlying these warrants are German government bonds. Most of them are the American type, with maturities of up to three years. Therefore, they represent a more diverse sample than would standardized options traded at options exchanges.

#### A. Government Bonds

Most German government bond issues are straight bonds with a fixed coupon size and one coupon payment per year. These bonds build a homogeneous market segment in bankruptcy risk, liquidity, and taxes. A subsample of these bonds represents the underlyings of the interest-rate warrants.
At every exchange, trading is organized as a call-auction market with a determinate rate at noon. This auction price is set so that the market-clearing price at noon each day. In this study, we use all call and put options listed in the two market segments, Amtescher Handel and Geregelter Markt, of the Frankfurt Stock Exchange. The sample period covers the period from January 1990 through November 1993. During this period, nineteen different calls and fourteen different puts traded on thirteen different German government bonds. Ten of the thirteen underlying bonds were BUNDS, the remaining three were BOBLs.

During the sample period, the time-to-maturity of the bonds ranged from 6.9 to 9.1 years for the long-term bonds, and from 3.4 to 3.8 years for the medium-term bonds. The average time-to-maturity for the options was 0.85 years, with a maximum of 2.91 years. With the exception of three European interest-rate warrants, the options under consideration were American-type options. We use weekly observations. The total number of option prices we collected amounts to 1,751. A detailed description of the interest-rate warrants' terms, including the average number of daily trades and the average daily turnover, appears in Table A1 in the Appendix.

**B. Interest-Rate Warrants**

German interest-rate warrants began trading at the end of 1989. These instruments are issued by banks and, as in the bond market, exchange trading takes place in daily noon auctions, with one single-market-clearing price each day. In this study, we use all call and put options listed in the two market segments, Amtescher Handel and Geregelter Markt, of the Frankfurt Stock Exchange. The sample period covers the period from January 1990 through November 1993. During this period, nineteen different calls and fourteen different puts traded on thirteen different German government bonds. Ten of the thirteen underlying bonds were BUNDS, the remaining three were BOBLs.

The average time-to-maturity of the bonds ranged from 6.9 to 9.1 years for the long-term bonds, and from 3.4 to 3.8 years for the medium-term bonds. The average time-to-maturity for the options was 0.85 years, with a maximum of 2.91 years. With the exception of three European interest-rate warrants, the options under consideration were American-type options. We use weekly observations. The total number of option prices we collected amounts to 1,751. A detailed description of the interest-rate warrants' terms, including the average number of daily trades and the average daily turnover, appears in Table A1 in the Appendix.

**C. Money Market Rates**

Bid and offer rates in the German money market are available for one day, as well as one month, and two, three, six, twelve, and twenty-four months. Because the daily rate fluctuates strongly and the level and changes of this rate are only loosely related to other short-, medium-, and long-term rates, the daily rate cannot reasonably be used to explain the evolution of the whole term structure of interest rates. However, these restrictions do not hold to the same extent for the second shortest rate, the monthly rate. This rate is therefore selected as "the short rate" for the empirical part of our study. In addition, we use German money market rates with a time-to-maturity of up to six months to support the yield curve estimation for short maturities.

### III. Design of the Study and Estimation Results

**A. Methodology**

The quality of different valuation models can be assessed by at least two well-known methodologies. The first compares out-of-sample differences between model and market prices; the second checks whether observed differences can be exploited by a dynamic replication strategy.

In this study, we apply the first strategy. The reason for this is that our test is not directed toward the efficiency of the German market for interest-rate warrants or toward the applicability of the models considered in a trading environment. Instead, we address the problem of which of the models are best suited to measure the exposure to interest-rate risk. This goal of the study, together with additional arguments elaborated earlier, results in our decision to estimate the parameters of the stochastic factors from time series. As the valuation quality of the different models is unavoidably assessed together with the estimation procedure of the input data, we use the same raw data and the same statistical methodology throughout:

1. For all models, we base the estimation of the term structures of interest rates on an identical set of German government bonds. The forward rates for the HJM models are determined from these term structures.
2. We estimate all time-independent parameters of the models by the maximum likelihood method.
3. Concerning the length of the historical time series, we distinguish between structural and volatility parameters. We estimate the structural parameters (those parameters that are relevant for the basic structure of the model) by using an estimation period of at least 20 years. The volatility parameters must reflect the current market information. Therefore, we estimate them by using only observations of the previous nine to twelve months. Figure 1 illustrates this procedure.

---

13 Pearson and Sun (1994), p. 1285, point out that the identification of the instantaneous rate with the one-month money market rate introduces measurement errors because the instantaneous interest rate does not depend on the market prices of risk, but the one-month rate does. This effect could be avoided by using state variables that can be observed. Since, contrary to the study of Pearson and Sun, we fit the model to the current term structure, we avoid this error. The choice of the one-month rate does, however, affect the volatility estimates.
Figure 1. Design of the study. This figure shows the basic design of our study. The valuation period from January 5, 1990, through November 16, 1993, consists of 204 weeks. One day (Friday) is taken from each of these weeks as a valuation day. For each of these valuation days, we carry out the following steps for each of the seven models: (i) estimation of the current term structure of interest rates, (ii) estimation of the structural and volatility parameters, (iii) calibration of the spot-rate models to the current term and volatility structures of interest rates, and (iv) valuation of all interest-rate warrants traded on the current day. Concerning the length of the historical time series, we distinguish between structural and volatility parameters. We estimate the structural parameters (those parameters that are relevant for the basic structure of the model) by using an estimation period of at least twenty years. The volatility parameters must reflect the current market information. Therefore, we estimate them by using observations of the previous nine to twelve months.

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1. estimation of the current term structure of interest rates
2. estimation of the structural and volatility parameters
3. calibration of the spot-rate models to the current term and volatility structures of interest rates
4. valuation of all interest-rate warrants traded on the current day.

B. Estimation of the Current Term Structure of Interest Rates

We estimate the term structure of interest rates from the homogeneous market segment of government bonds. Since the German government does not issue zero coupon bonds, and a stripping possibility comparable to the U.S. STRIPS program did not exist in Germany until July 1997, the term structure of interest rates can only be determined from traded coupon bonds. To estimate the current term structure of interest rates, we use all the straight bonds issued by the German government that have a time-to-maturity from six months to ten years. We exclude bonds with a maturity less than six months, because this market segment has a lower liquidity and its transaction costs influence short-term yields more than long-term yields. Instead, we construct synthetic short-term bonds to reflect the prevailing money market rates.

The estimation procedure for the term structure of interest rates affects the valuation of interest-rate options in two important ways: First, we value the bond underlying the option using the term structure of interest rates. Second, in models of the HJM type, we estimate the volatility of the forward rates from a time series of the term structures of interest rates. In the spot-rate models, the estimated term structure directly affects the time-dependent market price of risk $\lambda(t)$.

These two effects result in different requirements on the term structure estimates that are not in line with each other: The first effect leads to the recommendation to implement an estimation procedure that minimizes the deviations between observed and theoretical prices of the underlyings. The reason behind this is that deviations are directly transferred to differences between the observed and theoretical option values. Consequently, these differences should not be attributed to the valuation model. However, such an estimation fully transfers noise of coupon bond data to the term structure of interest rates, the noise in which results in irregular time-dependent market prices of risk in the spot-rate models and leads to unreasonably high volatility estimates of forward rates. In both cases, the ex ante predictability of the valuation models turns out to be very low. Therefore, a balance between accuracy and smoothness of the term structure of interest rates must be determined.

We achieve this compromise in two steps. In the first step, for each cash flow date of one of the bonds in the sample, we determine a discount factor by using a quadratic linear programming approach. This results in discrete term structure estimates with the highest possible accuracy in explaining observed bond prices. In the second step, we smooth out this discrete term structure by using cubic splines with ten nodes. This smoothing procedure increases the mean absolute deviation in the sample period from DM 0.071 to DM 0.148 per DM 100 nominal value. Table III shows some summary statistics of the deviations between the theoretical and the market prices of the bonds.
C. Parameter Estimates

Due to four complications, parameter estimates across models are not directly comparable. First, we use different factors (forward rates, spot rates, volatility) as basic variables. Second, even if we concentrate on models with the same factors, drift and volatility functions differ in form. Third, the number of different parameters that we need to estimate varies substantially across models. The one-factor forward-rate model with absolute volatility requires the estimation of one single parameter. In contrast, six parameters must be estimated for the spot-rate model with stochastic volatility. The fourth complication is that all forward-rate models and the two-factor spot-rate model with long rate and spread as factors have only volatility parameters, but the two remaining spot-rate models depend on both structural and volatility parameters.\(^{14}\)

C.1. Estimation of Structural Parameters

In the one-factor spot-rate model, we interpret two parameters, the long-term mean \(\gamma\) and the elasticity parameter \(\varepsilon\), as structural parameters. We estimate these parameters by using the discrete process obtained by applying the Euler scheme. We calculate both parameters for each valuation day from the time series of one-month money market rates, starting in January 1970 and running up to (but excluding) the current valuation day.\(^{15}\) The maximum likelihood estimates of \(\gamma\) vary between 0.062 and 0.067 and those of \(\varepsilon\) between 0.77 and 0.90.

The stationary mean \(\gamma\) of the spread process is the only candidate for classification as a structural parameter in the two-factor spot-rate model with long rate and spread. However, there is no need to estimate this parameter separately in the risk-neutral process, because it appears only in combination with the market price of spread risk. Since we use the latter parameter to calibrate the model to the initial term structure of interest rates, we can use an arbitrary value for \(\gamma\).

The first requirement for a historical parameter estimation for the two-factor model is that we interpret two parameters, the long-rate process \(\delta\) and \(\sigma\), as structural parameters. We estimate these parameters by using the discrete process obtained by applying the Euler scheme. We calculate both parameters for each valuation day from the time series of long-term interest rates, starting in January 1970 and running up to (but excluding) the current valuation day.\(^{16}\) The maximum likelihood estimates of \(\delta\) vary between 0.062 and 0.067 and those of \(\sigma\) between 0.77 and 0.90.

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The first requirement for a historical parameter estimation for the two-factor model is that we interpret two parameters, the long-rate process \(\delta\) and \(\sigma\), as structural parameters. We estimate these parameters by using the discrete process obtained by applying the Euler scheme. We calculate both parameters for each valuation day from the time series of long-term interest rates, starting in January 1970 and running up to (but excluding) the current valuation day.\(^{16}\) The maximum likelihood estimates of \(\delta\) vary between 0.062 and 0.067 and those of \(\sigma\) between 0.77 and 0.90.

C.2. Estimation of Volatility Parameters

The second type of parameters are related to interest-rate volatilities. Empirical results for a wide variety of markets show that volatilities vary significantly over time. Thus, we estimate the volatility parameters from short time series with a length of nine to twelve months.

As we have noted for the HJM models, only volatility parameters need to be estimated. We base all estimates on time series of weekly changes in instantaneous forward rates \(f(t, T)\) with a fixed maturity date \(T\). The time series covers a period of nine months. For the one-factor models, we use these forward-rate changes \(f(t, T) - f(t - \Delta t, T)\) directly to estimate the volatility parameters \(\sigma\) and \(\sigma_0\), respectively. For the two-factor models, on every valuation date \(t\), we conduct a principal component analysis to extract the first two principal components (factors) that explain the co-movement of the forward-rate changes. The volatilities of these two factors and the corresponding factor loadings determine the two required volatility functions \(\sigma_1(t, T)\) and \(\sigma_2(t, T)\) of Table 1. We summarize the results of the parameter estimation for the HJM models in Table AIII in the Appendix.

The spot-rate models require the estimation of widely differing volatility parameters. Therefore, we give a brief description of the estimation procedure and the obtained results for each of the models. For the one-factor model, we need the volatility \(\sigma\) of the short-rate process and the volatility of a long-rate process. As described earlier, we use the latter estimate to determine implicitly the mean-reversion parameter \(\kappa\) so that the endogenous volatility of the long rate equals the estimated volatility. As a by-product, this implicit estimation of \(\kappa\) allows us to cope with the problem that the maximum likelihood estimates of \(\kappa\) prove to be considerably upward biased. We estimate the two volatility parameters from changes of the one-month money market rates and the nine-year zero bond yields derived from the market.
term structures of interest rates, respectively. Again, for each valuation day, we base our estimates on the weekly observations of the previous nine months. Panel A of Table AIV in the Appendix presents summary statistics for the two parameters.

The two-factor model with the long-term rate and spread requires the volatilities of these two factors. Once again, we determine the parameters from the weekly changes of the nine-year yield as well as the changes of the spread, using observations of the nine months preceding the valuation day. We summarize the estimates in Panel B, Table AIV, in the Appendix.

For the two-factor model with stochastic volatility, we need to estimate the first two moments of the volatility's long-run stationary unconditional distribution, $E(V_\omega)$ and $\text{Var}(V_\omega)$, and the maximum and minimum values for the ratio $V(t)/r(t)$. We determine these four parameters from the weekly changes of estimated volatilities and the weekly changes of the one-month money market rate, based on our observations of the preceding twelve months. The summary statistics of these results appear in Panel C, Table AIV, in the Appendix. We use these results and the estimates of the structural parameters $E(r_\omega), \text{Var}(r_\omega)$ to obtain the six parameters, $\alpha_u, \kappa_u, \sigma_u, \kappa_y, \sigma_y$, and $\sigma_{u/y}$, that describe the dynamics of the two unspecified factors, $x$ and $y$. As in Longstaff and Schwartz (1993), we compute the parameters from the nonlinear system of equations represented in Table AIV in the Appendix. Panel C, Table AIV, in the Appendix also displays the estimated values for these six parameters.

The volatility estimates we obtain for the seven models are not directly comparable. To facilitate the comparison of the results across models, we compute the volatilities for two selected rates from these estimates. More precisely, for the forward-rate models, we determine the implied instantaneous standard deviation of changes in instantaneous spot rates $f(t,t)$, and also in instantaneous forward rates $f(t,t+9)$ that mature nine years from the current day. Accordingly, for the spot-rate models, we compute the instantaneous standard deviation of changes in instantaneous spot rates $r(t)$ as well as in the instantaneous nine-year zero yields $r(9)$ implied by the models.

To compute these volatilities for the forward-rate one-factor models, we only need the estimated volatility parameters and, for the linear proportional model, the forward rates $f(t,t)$ and $f(t,t+9)$ for each day. For the two-factor forward-rate models, we also require the corresponding factor loadings.

The computation of the long rate's volatilities is much more cumbersome for these two spot-rate models in which $l$ is not a factor of the model. The models' endogenous volatility depends, in general, not only on the constant parameters of the models and the current value of the nine-year yield, but also on the time-dependent market price of risk, which itself is influenced through the calibration process by the entire term structure of interest rates on the day under consideration.

In Tables IV and V we report the estimation results for the spot-rate and forward-rate models, respectively. Before discussing the results, we point out that the estimates for the nine-year forward-rate volatilities reported in Table IV are not comparable to the estimates for the nine-year zero-rate volatilities of Table V. The nine-year forward-rate volatility refers to an instantaneous forward rate maturing nine years from the current day. The latter refers to a rate for a period that is nine years long. Only the instantaneous volatility of the spot rate $r(t) = f(t,t)$ is comparable across all models.

Table IV and V are not comparable to the estimates for the nine-year zero-rate volatilities of Table V. The nine-year forward-rate volatility refers to an instantaneous forward rate maturing nine years from the current day. The latter refers to a rate for a period that is nine years long. Only the instantaneous volatility of the spot rate $r(t) = f(t,t)$ is comparable across all models.

If $\sigma_{spot}$ and $\sigma_l$ are multiplied by 100, we obtain the (absolute) volatilities in percentage per annum (p.a.). A division of $\sigma_{spot}$, as estimated for the absolute model by an average short rate of 0.088 in the valuation period, results in an approximation for the mean relative volatility of about 15 percent p.a.

On average, the volatilities of the forward rates increase with maturity. The only exception is the absolute one-factor model in which constant volatilities are assumed. The (constant) volatility of this model is approximately equal to the arithmetic mean of $\sigma_{spot}$ and $\sigma_l$ in the linear proportional model.

Additionally, we find that volatilities are very similar across the two-factor HJM models. In comparison to the one-factor models, both the mean volatility of the spot and the nine-year forward rate are lower. However, the standard deviations are higher for the two-factor models. This bias of the
short-term forward-rate changes. Estimates relative to the one-factor models and the high variability of those estimates might be one reason for the surprising valuation results that we present in Section IV.

Implied volatility functions obtained from cap data often exhibit a humped volatility structure (see, e.g., Amin and Morton (1994), p. 160, and Hull and White (1996), p. 33). However, this structure is not reflected in historical forward-rate changes. To the contrary, we find that for the majority of valuation days in our research period, historical volatilities of forward-rate changes increase with maturity. One reason for this result could be that term structures estimated by cubic spline techniques result in highly volatile long-term forward rates. Furthermore, we note that implied forward-rate volatilities are typically calculated using Black's (1976) model. They therefore represent volatilities of relative forward-rate changes.

The volatilities of the spot rates decrease with time-to-maturity for all spot-rate models. On average, the volatility of the instantaneous spot rate is twice as high as the volatility of the nine-year rate. The volatilities for the short rate are very similar for the one-factor model and the two-factor model with long rate and spread, even though they are determined very differently. The short-rate volatility is highest for the two-factor model with stochastic volatility. This difference can be explained by the fact that for this model, the short-rate volatility is not a model parameter, but a state variable. The realizations of this state variable are determined by a GARCH procedure; the volatility estimates for the other two models are smoothed by taking an average of squared differences in the time series of short rates.

The endogenous volatility of the nine-year zero bond yield differs considerably across the three models. The mean volatility of the nine-year rate in the generalized version of the Longstaff and Schwartz (1992) model lies between the estimates of the other two models. This represents a positive result insofar as only the short-rate volatility is given exogenously and the model is not calibrated to the current volatility structure.

The differences between the nine-year volatilities for the one-factor model and the two-factor model with the long rate and spread as factors happen because we choose the mean-reversion parameter $\alpha$ in the one-factor model so that the model-endogenous volatility structure $\sigma_T/\sigma_{spot}$ fits the empirically determined volatility structure. Because the historical and model endogenous volatilities $\sigma_T^*$ for the short-rate are different, the historical and endogenous nine-year volatilities must also differ.

A comparison of the short-rate volatilities in Tables IV and V shows that, on average, the forward-rate models result in lower short-rate volatilities than the spot-rate models. The reason for this difference is that the HJM volatility functions are not only estimated using short-rate changes, but also must reflect the behavior of all other forward rates.

A cautious comparison of the parameter estimates of the one-factor HJM models in Table AIII with the results by Amin and Morton (1994) shows that, for the absolute model, the mean estimates do not differ greatly. On the other hand, the standard deviation of our historical estimate is lower than their implied values. For the linear proportional model, Amin and Morton report a negative effect ($\alpha_T < 0$) of an increasing maturity $T$. As they estimate volatilities for the upward-sloping part of the humped volatility structure, they expect a positive $\sigma_T$. Our result is that the volatility structure increases with $T$ ($\sigma_T > 0$) for the reasons we have discussed above.

### IV. Valuation Results

Here we examine the empirical quality of the models by directly comparing model prices to market prices. Table VI presents a first impression on the performance of the seven models. This table gives some summary statistics for the deviations between theoretical values and market prices of the warrants.

The mean option price of the sample is DM 3.13. The average absolute pricing errors range from DM 0.30 for the best models to DM 0.37 for the worst model. The third column indicates that with the exception of the one-

<table>
<thead>
<tr>
<th>Model</th>
<th>Volatilities</th>
<th>Mean</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short rate</td>
<td>$\sigma_{spot}$</td>
<td>0.0149</td>
<td>0.0090</td>
<td>0.0312</td>
<td>0.0031</td>
</tr>
<tr>
<td></td>
<td>$\sigma_T$</td>
<td>0.0103</td>
<td>0.0050</td>
<td>0.0190</td>
<td>0.0038</td>
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<tr>
<td></td>
<td>$\sigma_{spot}/\sigma_T$</td>
<td>0.6840</td>
<td>0.4326</td>
<td>1.0580</td>
<td>0.1450</td>
</tr>
<tr>
<td>Long rate and spread</td>
<td>$\sigma_T$</td>
<td>0.0146</td>
<td>0.0094</td>
<td>0.0214</td>
<td>0.0032</td>
</tr>
<tr>
<td></td>
<td>$\sigma_T/\sigma_{spot}$</td>
<td>0.0075</td>
<td>0.0044</td>
<td>0.0130</td>
<td>0.0025</td>
</tr>
<tr>
<td></td>
<td>$\sigma_T/\sigma_{spot}$</td>
<td>0.5028</td>
<td>0.3739</td>
<td>0.6354</td>
<td>0.0682</td>
</tr>
<tr>
<td>Short rate and volatility</td>
<td>$\sigma_{spot}$</td>
<td>0.0185</td>
<td>0.0127</td>
<td>0.0351</td>
<td>0.0038</td>
</tr>
<tr>
<td></td>
<td>$\sigma_T$</td>
<td>0.0090</td>
<td>0.0047</td>
<td>0.0112</td>
<td>0.0013</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{spot}/\sigma_T$</td>
<td>0.5112</td>
<td>0.1353</td>
<td>0.7288</td>
<td>0.1316</td>
</tr>
</tbody>
</table>

16 The short-term Eurodollar data used by Amin and Morton only show the upward sloping part of the hump. See Amin and Morton (1994), p. 160.
Without any error in the prices of the underlyings. In our study, we must
errors. Second, they are able to value the option on Eurodollar futures
between Arnin and Morton’s (1994) study and ours. First, because they use
average fractional absolute deviations of 15.2 percent for the linear
sample’s puts’ average price of DM 2.56. Amin and Morton (1994) report
this might be that the calls’ average price of DM 3.51 is higher than our
228, of which 81 are calls and 147 are puts. Absolute I is the one-
rate and spread as stochastic factors. The stochastic volatility model is denoted as
The stochastic volatility model is denoted as
Proportional II, both volatility functions are proportional to the
level of the forward rate. In the two-factor forward-rate model Proportional II, both volatility functions are proportional to the
level of the forward rate. Short rate is the one-factor spot-rate model. Long rate and spread
denotes the two-factor spot-rate model with the long rate and the spread as stochastic factors.

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean Absolute Deviation (DM)</th>
<th>Mean Percentage Absolute Deviation</th>
<th>Standard Deviation (DM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute I</td>
<td>0.31</td>
<td>-0.15</td>
<td>0.24</td>
</tr>
<tr>
<td>Linear proportional</td>
<td>0.30</td>
<td>-0.17</td>
<td>0.24</td>
</tr>
<tr>
<td>Absolute II</td>
<td>0.36</td>
<td>-0.27</td>
<td>0.31</td>
</tr>
<tr>
<td>Proportional II</td>
<td>0.37</td>
<td>-0.28</td>
<td>0.32</td>
</tr>
<tr>
<td>Short rate</td>
<td>0.35</td>
<td>-0.13</td>
<td>0.43</td>
</tr>
<tr>
<td>Long rate and spread</td>
<td>0.30</td>
<td>-0.17</td>
<td>0.21</td>
</tr>
<tr>
<td>Short rate and volatility</td>
<td>0.30</td>
<td>-0.09</td>
<td>0.23</td>
</tr>
</tbody>
</table>

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Table VI Deviation between Model and Market Values
This table presents summary statistics for the deviations between theoretical values and market
prices of the warrants. Columns 1 to 5 report the average absolute pricing errors, the
average pricing errors (defined as model value minus market value), and the average percent-
ages absolute pricing errors. Column 6 shows the standard deviation of the absolute pricing
error. The sample period is January 1990, to November 1993. There are 1,037 call prices and
744 put prices in the sample. To calculate the relative percentage errors, we remove all observ-
ations in which the market price of the option is less than DM 0.10. The total number of observ-
ations eliminated is 228, of which 81 are calls and 147 are puts. Absolute I is the one-
factor spot-rate model, on average, all models underprice the options. The one-factor spot-rate model results in an average overpricing of DM 0.13. The underpricing of the other models ranges from the best value of DM −0.09 to the worst value of DM −0.28.
The average absolute percentage pricing errors vary between 21 percent and 37 percent. A comparison of calls and puts shows that the absolute percent-
age pricing error is uniformly lower for calls than for puts. A reason for this might be that the calls’ average price of DM 2.56. Amin and Morton (1994) report
average fractional absolute deviations of 15.2 percent for the linear propor-
tional model and 21.1 percent for the absolute model, and therefore lower errors.
To put these figures into perspective, we note two important differences between Amin and Morton’s (1994) study and ours. First, because they use implied volatilities, Amin and Morton carry out a “local” test of the HJM model. Second, they are unable to value the option on Eurodollar futures without any error in the prices of the underlyings. In our study, we must
accept an absolute valuation error in the underlying bond averaging 0.15 percent. Since interest-rate warrants have a high elasticity compared with stock options, a correction for this error in the underlying reduces the mispricing.
The two one-factor forward-rate models show similar patterns of mispricing. The observation also holds true for the two-factor forward-rate models but, surprisingly, the two-factor models perform uniformly worse than the one-factor models. This unexpected bad performance can be attributed to two reasons. The first is the low volatility estimates compared with the one-factor forward-rate models (see Table IV). These result in systemati-
cally lower option values compared with the other models (column three of Table VI) and in higher absolute deviations (column two). This interpreta-
tion of the aggregated deviations proves very accurate if we analyze the deviations for each interest-rate warrant individually. Second, the higher variability of the volatility estimates in the two-factor models presented in Table IV could indicate an overfitting problem related to our use of the
principal component analyses to estimate the input data for the HJM two-
factor models.

For the spot-rate models, the relation between the one- and two-factor models is as expected. If measured by the mean absolute deviation, the two-
factor models uniformly outperform the one-factor model. Furthermore, the one-factor spot-rate model is the only model that on average overvalues the
warrants.

The fundamental reason for this comparatively bad performance comes from fitting the volatility curve by using the mean-reversion parameter \( \lambda \). First, Panel A of Table AIV in the Appendix shows 0.06 as average value for
\( \lambda \). This value is very low if compared with an unbiased estimate for \( \lambda \) of 0.25 from a time-series of one-month rates for the period from 1970 to 1993. Since the endogenous volatilities of medium- and long-term yields increase if
\( \lambda \) decreases, the low \( \lambda \)-values explain the relatively high option values for the one-factor spot-rate model. Second, for some out-of-the-money options, the model prices are close to zero even though the market prices are greater than one DM. This breakdown of the model occurs in periods with a sharp
decline in the ratio of the long- and short-term volatility. To capture a sharp decline of the relative volatility, \( \lambda \) must take on relatively high values that
result in low time values for the options.

Table VII reports correlations between pricing errors across models. We
find the highest correlation, 0.99, between the two-factor HJM models, the second highest, 0.97, between the one-factor HJM models. In contrast, the one-factor spot-rate model has a very low correlation with all other models. Also, the spot-rate model with stochastic volatility shows a comparably low correlation to other models. Surprisingly, the HJM one-factor absolute model has a relatively high correlation with all other models, apart from the one-
factor spot-rate model. Compared with the correlations reported by Amin and Morton (1994) for one-factor HJM models, we find considerably higher
correlations because we estimate volatilities historically.
To study the model performance in more detail, we analyze pricing errors for the different models. We first regress absolute pricing errors on moneyness (measured in DM) and on the maturity of an option. In addition to these two fundamental option characteristics, we include dummy variables for the years 1990, 1991, and 1992, and a dummy variable for a call. We introduce the calendar dummy variables to test whether the market for German interest-rate warrants, which started in 1989, shows maturity effects similar to those that have been reported for other markets. The dummy variable for calls allows us to test whether pricing errors are systematically different for calls and puts.

The results of the regressions are summarized in Table VIII. These show that the calendar dummy variables $a_1$, $a_2$, and $a_3$ are significant at a 1 percent level for almost all models. In addition, the dummy variable for 1990, $a_1$, is always higher than the dummy variable for the years 1991 and 1992, $a_2$ and $a_3$; often, it is twice as high as the dummy variable for the year 1992, $a_3$. This illustrates that absolute pricing errors are obviously very high in 1990 and that the errors reduce significantly over time for all the models. Since these results are very similar across models, there does not seem to be a model-specific effect, but rather a common component in pricing errors due to the presence of market imperfections.

The sign of the moneyness variable, $a_4$, differs across models. In addition, it is insignificant at the five percent level for all but one of the models. From this result we conclude that absolute pricing errors are not significantly influenced by the moneyness of the options. Contrary to the moneyness, the influence of the time-to-maturity of an option on the absolute pricing error is significant and positive for all models.

The estimate of the dummy variable for calls, $a_6$, is negative for all models. This means that on average, calls result in lower absolute errors than puts. However, the results are only significant for the two-factor model with stochastic volatility.

To study possible systematic biases of each model, we regress the pricing error, defined as the difference between market and model prices, on the absolute pricing error, defined as the difference between model and market prices, on the maturity of the option in years, m is an error term. The sample period is from January 1990 to November 1993. There are 1,751 observations for each model. The t-statistics on the regression coefficients, adjusted for heteroskedasticity and 5th degree autocorrelation in residuals based on Newey and West (1987), are given below the coefficient values in parentheses.
underpricing fell over time. In addition, we find that in 1993, on average the pricing error was close to zero for all models. Thus, we can conjecture that at the beginning of the new market segment, which is covered by our sample period, German interest-rate warrants were considerably overpriced and some (presumably inexperienced) market participants were willing to accept these prices. This conjecture is reinforced by discussions with traders in the Frankfurt market who were able to hedge their short positions in London at better prices.

The estimate of the moneyness variable, $\beta_4$, is significantly negative for all forward-rate models. For the spot-rate models, the $\beta_4$ estimate is also negative, but much lower in absolute terms and insignificant. Consequently, all models seem to underprice out-of-the-money options and to overprice in-the-money options. A more detailed analysis of the moneyness effect reveals that it is stronger for puts than for calls. These results are similar to those of Amin and Morton (1994).

The reason for the differences between the $\beta_4$-estimates in Table IX is related to the different volatilities implicitly used in the models. Lower volatilities result in lower values for out-of-the-money options. For in-the-money options, which are close to the exercise boundary, a reduction of volatility has only a small effect on the option price. Therefore, those models that implicitly use the lowest volatilities undervalue out-of-the-money options most and should show the (absolute) maximum moneyness effect.

Table IX demonstrates exactly this effect.

The regression coefficient $\beta_5$, which measures the influence of maturity on mispricing, has a positive sign whenever it is significant. All models except the one-factor spot-rate model underprice options more if maturity increases. This effect could be attributed to the lower liquidity of warrants with longer maturities. We find particularly high estimates for the HJM two-factor models, again a consequence of the low volatility estimates described in Table IV.

Finally, the coefficient $\beta_6$ for the call dummy variable indicates that undervaluation is stronger for puts in general. However, this effect is significant for only two of the seven models.

The regression results show that the mispricing of the different models can be explained to some extent by common factors, but it is also evident that some models are more susceptible than others to certain influencing factors. Because our main objective is to compare the ex ante predictabilities of different models, we try to separate model effects from common factors in pricing errors. Therefore, our focus now is on a paired comparison of absolute pricing errors. For this purpose, we compute the differences of absolute mispricing for each paired model and test whether the mean of these paired differences is equal to zero. Panel A, Table X, summarizes the results of this comparison.

The model with the lowest absolute deviation is the linear proportional one-factor HJM model. For this model, we can reject the hypothesis with a probability of at least 99 percent that the paired differences between each of the two-factor HJM models and with the one-factor spot-rate model are zero. In addition, the difference with the absolute one-factor HJM model is different from zero at the 5 percent level. However, no significant difference is found if we compare it with the two-factor spot-rate models.

For the two-factor spot-rate models, which are the models with the second and third lowest absolute deviations, we can also reject the hypothesis, with a probability of at least 99 percent, that the paired differences between each of the two-factor forward-rate models and with the one-factor spot-rate model are zero.

The model with the largest absolute deviation is the two-factor HJM model with proportional volatility. As we have noted, this model performs significantly worse than the linear proportional one-factor HJM model and the two two-factor spot-rate models. We also find a significant difference compared with the one- and two-factor HJM models with absolute volatility.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\beta_5$</th>
<th>$\beta_6$</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolute I</td>
<td>$-0.074$</td>
<td>0.463</td>
<td>0.156</td>
<td>$-0.136$</td>
<td>0.014</td>
<td>0.195</td>
<td>$-0.069$</td>
<td>0.26</td>
</tr>
<tr>
<td></td>
<td>(-2.11)</td>
<td>(6.91)</td>
<td>(2.79)</td>
<td>(-3.35)</td>
<td>(5.51)</td>
<td>(1.27)</td>
<td>(-0.50)</td>
<td></td>
</tr>
<tr>
<td>Linear proportional</td>
<td>0.060</td>
<td>0.364</td>
<td>0.118</td>
<td>0.073</td>
<td>0.016</td>
<td>0.197</td>
<td>$-0.059$</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>(1.24)</td>
<td>(6.18)</td>
<td>(2.09)</td>
<td>(-1.58)</td>
<td>(4.35)</td>
<td>(6.04)</td>
<td>(-1.37)</td>
<td></td>
</tr>
<tr>
<td>Absolute II</td>
<td>0.002</td>
<td>0.038</td>
<td>0.109</td>
<td>0.061</td>
<td>0.019</td>
<td>0.169</td>
<td>0.002</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>(-1.97)</td>
<td>(5.20)</td>
<td>(3.01)</td>
<td>(1.05)</td>
<td>(6.09)</td>
<td>(8.72)</td>
<td>(-1.96)</td>
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</tr>
<tr>
<td>Proportional II</td>
<td>0.003</td>
<td>0.304</td>
<td>0.110</td>
<td>0.070</td>
<td>0.020</td>
<td>0.330</td>
<td>0.009</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td>(-0.91)</td>
<td>(4.92)</td>
<td>(3.95)</td>
<td>(1.17)</td>
<td>(5.34)</td>
<td>(8.93)</td>
<td>(-1.91)</td>
<td></td>
</tr>
<tr>
<td>Short rate</td>
<td>0.216</td>
<td>$-0.299$</td>
<td>$-0.443$</td>
<td>0.166</td>
<td>0.007</td>
<td>-0.031</td>
<td>0.319</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>(5.51)</td>
<td>(-4.13)</td>
<td>(-7.40)</td>
<td>(-3.84)</td>
<td>(1.98)</td>
<td>(1.00)</td>
<td>(-4.41)</td>
<td></td>
</tr>
<tr>
<td>Long rate and spread</td>
<td>0.083</td>
<td>0.312</td>
<td>0.110</td>
<td>0.073</td>
<td>0.007</td>
<td>0.230</td>
<td>-0.071</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>(-2.48)</td>
<td>(4.95)</td>
<td>(1.94)</td>
<td>(-1.39)</td>
<td>(1.80)</td>
<td>(8.41)</td>
<td>(-1.55)</td>
<td></td>
</tr>
<tr>
<td>Short rate and volatility</td>
<td>0.072</td>
<td>0.442</td>
<td>0.021</td>
<td>$-0.262$</td>
<td>0.008</td>
<td>0.096</td>
<td>$-0.169$</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>(1.73)</td>
<td>(4.94)</td>
<td>(4.02)</td>
<td>(-6.07)</td>
<td>(1.64)</td>
<td>(2.05)</td>
<td>(-3.13)</td>
<td></td>
</tr>
</tbody>
</table>
Comparison of Models for Valuing Interest-Rate Options

To analyze the stability of these results, we perform the same comparisons for varying time periods. We successively leave out the beginning of the original four-year time span and consider the periods 1991–1993, 1992–1993, and 1993. A comparison of the results for these three periods shows a remarkable stability of the relative performance of the seven models. In each period, the two-factor forward-rate models, the one-factor spot-rate model, and the two-factor spot-rate model with long rate and spread are significantly dominated by at least one other model. The two one-factor forward-rate models and the second two-factor spot-rate model are never dominated. Since the results are qualitatively identical for the time periods beginning after 1990, Panel B of Table X only shows the model’s relative performance for the last year.

Compared with the results of the whole period, which we present in Panel A of Table X, the time period for the more mature market exhibits two main differences: First, the one-factor forward-rate model with absolute volatility is still dominated by its linear proportional counterpart, but it is no longer significantly dominated. Second, the two-factor spot-rate model with long rate and spread is now significantly outperformed.

In summary, based on the total four-year period, the seven models tested can be grouped into two sets:

**Set One:** One-factor forward-rate model with linear proportional volatility, two-factor spot-rate model with long rate and spread, two-factor spot-rate model with stochastic volatility.

**Set Two:** One-factor forward-rate model with absolute volatility, one-factor spot-rate model, two-factor forward-rate model with absolute volatility, two-factor forward-rate model with proportional volatility.

The first set contains those models that are never significantly outperformed by any of the other models. In contrast, the second set consists of those models that are significantly outperformed by at least one other model. Using the results of the stability analysis, we can further assess our models. From the original three models of the first set, two models remain that are not significantly outperformed by any of the other models in any of the four different periods considered. These are the one-factor forward-rate model with linear proportional volatility and the two-factor spot-rate model with stochastic volatility. The other five models have been significantly outperformed at least once. Taking into consideration the fact that more mature market periods result in smaller pricing errors, these latter results are particularly important.

V. Summary and Conclusions

This study presents an extensive empirical test of those valuation models for interest-rate options that dominate the current theoretical discussion.
We perform empirical tests on four forward-rate models and three spot-rate models. These models are potential candidates for measuring, controlling, and supervising interest-rate risk within a risk management system, and should be able to value very different interest-rate derivatives consistently across different markets. This intended application of a valuation model should be strictly separated from its usage as a fine-tuned trading oriented model.

With this application in mind, we make two important decisions concerning the test methodology. First, we estimate input data from time-series, not implicitly. Second, we select as the dominant assessment criterion the ante predictability of a model and not its ability to identify mispriced options.

Using this evaluation criterion, the one-factor forward rate model with linear proportional volatility and the two spot-rate models with two factors significantly outperform the other four models for the four-year period from 1990 to 1993.

If we take into account the results for the later time periods, we can make an even stronger assessment of the models. Applying the criterion, "A model is not significantly dominated by any of the other models in any of the four time periods," two of the three models remain, the one-factor forward-rate model with linear proportional volatility and the two-factor spot-rate model with stochastic volatility.

Valuation models for interest-rate derivatives, which could be used in a risk management system, must satisfy additional criteria. Besides the ante predictability, differences in estimating the input data, in fitting the model to the current market information, and in numerically valuing the warrants, should be reflected in the overall assessment. When we consider the robustness of the estimation procedure for the input data, the one-factor forward-rate model outperforms the two-factor spot-rate model. In fitting a model to the current term structure of interest rates and volatilities, the forward-rate model is also superior to the other model. In terms of computing the option values numerically, none of the models are especially cumbersome. If we take into account these additional assessment criteria, we can conclude that the one-factor forward-rate model with linear proportional volatility outperforms all other models.
Table AIII
Parameter Estimates of the GARCH Model

Maximum likelihood estimates of the generalized autoregressive conditional heteroskedastic (GARCH) model

\[ r_t - r_{t-1} = \alpha_0 + \alpha_1 r_{t-1} + \beta_1 V_{t-1} + \epsilon_t \]

\[ \epsilon_t \sim N(0, \sigma^2_t) \]

\[ V_t = \alpha_0 + \alpha_1 r_{t-1} + \beta_1 V_{t-1} + \gamma^2 \epsilon_t^2 \]

This table summarizes the parameter estimates for the volatility parameters of the GARCH model. \( \sigma^2_t \) is the short rate at time \( t \), and the error term \( \epsilon_t \) is (conditional) normally distributed with zero mean and variance \( \sigma^2_t \). The remaining 1,031 rates are used as input for the GARCH estimation.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Estimate</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_0 )</td>
<td>0.00037295</td>
<td>0.00016744</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-0.00816464</td>
<td>0.00324843</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>10.84576386</td>
<td>2.35996069</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.01200000</td>
<td>0.00009605</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.72183780</td>
<td>0.18557610</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.27047252</td>
<td>0.03351277</td>
</tr>
</tbody>
</table>

Table AIV
Parameter Estimates for the Volatility Parameters within the Forward-Rate Models

This table summarizes the parameter estimates for the volatility parameters of the forward-rate models. This table shows the parameter estimates for the forward-rate model with constant volatility (Absolute I), \( \sigma^2_t \) and \( \alpha_\sigma \) are two positive parameters for the one-factor model with constant volatility. \( \alpha_\sigma \) and \( \alpha_\epsilon \) are two positive parameters for the one-factor model with constant volatility. The positive parameter \( \kappa \) is determined (implicitly) such that the endogenous volatility of the short rate equals the historically estimated volatility. Panel A summarizes the results for the two-factor spot-rate model, using the long rate \( r \) and the spread \( s \) as stochastic factors. The volatility parameters \( \sigma_\epsilon \) and \( \sigma_\sigma \) are estimated for the long-rate process \( \sigma_\sigma \) and the spread process \( \sigma_\sigma + \sigma_\epsilon \). Panel B summarizes the results for the two-factor spot-rate model with stochastic volatility. \( E(V_{r_1}) \) and \( Var(V_{r_1}) \) are the sample mean and variance of the volatility of the short rate. \( min_{ratio} \) and \( max_{ratio} \) are the minimum and maximum of the volatility-spot rate ratio \( \frac{\sigma_{\epsilon}}{\sigma_{\sigma}} \) within the historical time series. All parameters are estimated for 204 Fridays within the valuation period from January 5, 1990, through November 16, 1993.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>0.1167</td>
<td>0.0592</td>
<td>0.1954</td>
<td>0.0382</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>0.0586</td>
<td>0.0092</td>
<td>0.2107</td>
<td>0.0399</td>
</tr>
<tr>
<td>( \sigma_{\epsilon} )</td>
<td>0.0270</td>
<td>0.0450</td>
<td>0.0169</td>
<td>0.0081</td>
</tr>
<tr>
<td>( \sigma_{\sigma} )</td>
<td>0.0124</td>
<td>0.0175</td>
<td>0.0061</td>
<td>0.0024</td>
</tr>
</tbody>
</table>

Table AV
Parameter Estimates for the Volatility Parameters of the Spot-Rate Models

This table summarizes the parameter estimates for the volatility parameters of the spot-rate models. Panel A represents the estimation results for the mean-reversion parameter \( \kappa \) and the volatility parameter \( \sigma \) of the one-factor spot-rate model \( dr = \kappa (\gamma - r) dt + \sigma dz \). From this equation, the long-term mean \( \gamma \) and the elasticity parameter \( \kappa \) have already been estimated as structural parameters in a previous step. \( \kappa \) is determined (implicitly) such that the endogenous volatility of the long rate equals the historically estimated volatility. Panel B summarizes the results for the two-factor spot-rate model, using the long rate \( r \) and the spread \( s \) as stochastic factors. The volatility parameters \( \sigma_\epsilon \) and \( \sigma_\sigma \) are estimated for the long-rate process \( \sigma_\sigma + \sigma_\epsilon \) and the spread process \( \sigma_\sigma + \sigma_\epsilon \). Panel C summarizes the parameter estimates for the volatility parameters of the two-factor spot-rate model with stochastic volatility. \( E(V_{r_1}) \) and \( Var(V_{r_1}) \) are the sample mean and variance of the volatility of the short rate. \( min_{ratio} \) and \( max_{ratio} \) are the minimum and maximum of the volatility-spot rate ratio \( \frac{\sigma_{\epsilon}}{\sigma_{\sigma}} \) within the historical time series. All parameters are estimated for 204 Fridays within the valuation period from January 5, 1990, through November 16, 1993.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(V_{r_1}) )</td>
<td>0.00335</td>
<td>0.00028</td>
<td>0.00048</td>
<td>0.00014</td>
</tr>
<tr>
<td>( Var(V_{r_1}) )</td>
<td>2 \times 10^{-8}</td>
<td>5 \times 10^{-8}</td>
<td>5 \times 10^{-8}</td>
<td>1 \times 10^{-8}</td>
</tr>
<tr>
<td>( min_{ratio} )</td>
<td>0.045</td>
<td>0.045</td>
<td>0.052</td>
<td>0.002</td>
</tr>
<tr>
<td>( max_{ratio} )</td>
<td>0.045</td>
<td>0.045</td>
<td>0.052</td>
<td>0.002</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>0.106</td>
<td>0.082</td>
<td>0.125</td>
<td>0.015</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.106</td>
<td>0.082</td>
<td>0.125</td>
<td>0.015</td>
</tr>
<tr>
<td>( \sigma_\epsilon )</td>
<td>0.082</td>
<td>0.054</td>
<td>0.113</td>
<td>0.015</td>
</tr>
<tr>
<td>( \sigma_\sigma )</td>
<td>0.040</td>
<td>0.027</td>
<td>0.043</td>
<td>0.006</td>
</tr>
<tr>
<td>( \sigma_{\epsilon} )</td>
<td>0.045</td>
<td>0.045</td>
<td>0.052</td>
<td>0.002</td>
</tr>
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REFERENCES


