“ASSET PRICING WHEN RISK SHARING IS LIMITED BY DEFAULT: A THEORETICAL FRAMEWORK”

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Asset Pricing when Risk Sharing is Limited by Default: A Theoretical Framework

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Abstract

We study the asset pricing implications of an endowment economy when agents can default on contracts that would leave them otherwise worse off. We specialize and extend the environment studied by Kocherlakota (1995) and Kehoe and Levine (1993) to make it comparable to standard studies of asset pricing. We completely characterize efficient allocations for several special cases. We introduce a competitive equilibrium with complete markets and with endogenous solvency constraints. These solvency constraints are such as to prevent default—at the cost of reduced risk sharing. We show a version of the classical welfare theorems for this equilibrium definition. We characterize the pricing kernel, and compare it with the one for economies without participation constraints: interest rates are lower and risk premia can be bigger depending on the covariance of the idiosyncratic and aggregate shocks. Quantitative examples show that for reasonable parameter values the relevant marginal rates of substitution fall within the Hansen-Jagannathan bounds.

1. Introduction

Standard equilibrium asset pricing models have problems reproducing some of the basic facts in the data.¹ A promising direction for improvement in this dimension has been to maintain standard preferences and allow for incomplete risk sharing across

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¹The shortcomings of standard model versions are widely documented in studies such as Mehra and Prescott (1985), Hansen and Cochrane (1992).
agents. Generally, in this class of models, financial markets are exogenously considered as incomplete. With agents having to carry some idiosyncratic risk, pricing kernels are more volatile than those of representative agent economies with the same aggregate consumption. One drawback of studies following this approach, when compared with complete markets economies, is that they require more or less arbitrary assumptions about which set of securities is available. Model conclusions may in turn crucially depend on these assumptions. Another possible problem is tractability. Because finding equilibria of these models involves solving a difficult fixed point problem, it has been very difficult to analyze the case of many assets or many agents. Additionally, previous quantitative studies with incomplete market models required highly persistent relative shocks, a feature not necessarily seen in the data. In this paper, we study a class of models whose equilibrium, in general, entail limited risk sharing but that do not have some of these potential drawbacks.

Our approach for limiting risk sharing builds on work by Kehoe and Levine (1993) and Kocherlakota (1995). These authors present and study efficient allocations in endowment economies where participation constraints ensure that agents would at no time be better off reverting permanently to autarchy. These participation constraints are motivated by an extreme form of limited liability. We imagine a world, where, if agents default on some debt, they can be punished by seizing some assets that they may own, but they can not be punished by garnishing their labor income. In such an environment, risk sharing may be effectively reduced because agents with low income realizations can only borrow up to the amount they are willing to pay back in the future. It is assumed, not modelled, that there is commitment to carry out these permanent punishments.

We adapt this framework in various ways to make it more suitable for quantitative asset pricing evaluation. In particular, we start by extending some results of Kocherlakota (1995) to environments where relative shocks and the growth rate of aggregate endowments are serially correlated. We proceed by characterizing efficient allocations. For the cases where full risk sharing is not possible, we show the existence and uniqueness of the invariant distribution of the relevant statistics. The main characteristics of efficient allocations are: (i) when transiting to a particular state where both agents will be better off than in autarchy, they must equate their marginal rates of substitution, (ii) but if they, transit to a state where only one agent will be better off than in autarchy, this agents marginal rate of substitution will be higher. These properties of the efficient allocations are key for understanding asset pricing, since we show that in equilibrium the highest marginal rate of substitution plays a central role in pricing. Our characterization of the efficient allocations is simple enough to suggest computational algorithms that can be very fast and easy to implement. In fact, for simple cases we can completely characterize efficient allocations analytically. Our parametrization of the aggregate shocks has two desirable properties. First, it

2See Teh mer (1993), and Heaton and Lucas (1995) for numerical simulations with transitory shocks in two agents economies and Constantinides and Duffie (1996) for a structure with permanent relative shocks with a continuum of agents. Heaton and Lucas (1995) found a first order serial correlation for relative incomes around 0.5.

3Earlier work in the sovereign debt literature by Eaton and Gersovitz (1983) first formalized the main idea of this approach.
follows the asset pricing literature and thus makes results comparable. Second, it makes clear that aggregate shocks will have effects on the marginal rates of substitution, over and above the effect that they have in economies without participation constraint, only if they are correlated with idiosyncratic shocks. We also characterize the type of parameters that will lead to very little risk sharing, by making autarchy relative attractive. They involve low discount factors, low risk aversion, and persistent idiosyncratic shocks. We think that this is interesting because these values are at the opposite end of the ones required in variations of the representative agent model to account for high premia and low interest rates.

We introduce a new equilibrium concept that emphasizes the portfolio constraints: a competitive equilibrium with solvency constraints. In particular, we focus on constraints that are tight enough to prevent default but allow as much risk sharing as possible. These endogenously determined solvency constraints, are agent and state specific and ensure that the participation constraints are fulfilled. This means that the amount of wealth that agents can carry to any particular date and event will never be small enough to make them choose to default and revert to autarchy.

We show versions of the classical first and second welfare theorems for our equilibrium concept. Our decentralization differs from the one in Kehoe and Levine (1993) where the participation constraints are included in the consumption possibility sets. We think that our decentralization relates better with the existing literature in asset pricing, by focusing on portfolio restrictions. In any event, we show that both equilibrium notions are closely related. We also study properties of the pricing kernel. One period contingent claims are priced by the agent with the highest marginal rate of substitution, which is the agent that is not constrained with respect to his holding of this asset. Thus the price of a contingent claim (an Arrow security) is equal to the highest marginal valuation across agents. Pricing of an arbitrary asset is accomplished by adding up the prices of the corresponding contingent claims (which, interestingly, does not need to coincide with the highest marginal valuation across agents). For the purpose of asset pricing, this framework has two advantages over the standard incomplete markets specifications: first, allocations do not depend on a particular arbitrary set of assets that is considered to be available. And second, with markets being complete any security can be priced. Finally, we find that interest rates are smaller, independently of the precautionary motive emphasized in the literature.

We also engage in a simple numerical exercises for a first quantitative evaluation of this model. In particular, we analyze the variability of the relevant marginal rates of substitution for asset pricing. One motivation for conducting this exercise is that He and Modest (1995) have shown that among the most commonly considered frictions in asset pricing models, solvency constraints could potentially reconcile the variability of aggregate consumption with data on asset prices for reasonable values of risk aversion. More precisely, they use a version of the Hansen-Jagannathan bounds for complete market economies with solvency constraints following Luttmer (1991). In our study, we go one step further and present a model with endogenous solvency constraints and we find that for some reasonable parameters our economy passes the standard Hansen-Jagannathan test for low values of risk aversion. In particular, individual

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4 See for example, He and Modest (1995), Luttmer (1991) among others.
income processes are calibrated following Heaton and Lucas (1995) based on the PSID. Relative risk aversion is required to be around 2 we need very low values of the discount factor. We also experiment with more persistent processes for the idiosyncratic shocks, motivated by the evidence in Storesletten, Telmer and Yaron (1997), finding similar results for low risk aversion and higher discount factors. Our companion paper Alvarez and Jermann (1997) contains a more detailed analysis of the quantitative properties.

The remainder of this paper is organized as follows. In section 2 we present the model environment. Section 3 characterizes the constrained efficient allocations by introducing a recursive representation, analyzing the extent of risk sharing, the long run dynamics and the case of aggregate uncertainty. Section 4 introduces the competitive equilibrium with solvency constraints, shows versions of the classical welfare theorems, relates the equilibrium concept with the one by Kehoe and Levine and analyzes the pricing kernel. In section 5, we present some quantitative asset pricing implications by solving for efficient (and equilibrium) allocations for a simple case.

2. The environment

We consider a pure exchange economy with two agents. Agents' endowments follow a finite state markov process, agents' preferences are identical and given by time-separable expected discounted utility. We add to this simple environment participation constraints of the following form: the continuation utility implied by any allocation should be at least as high as the one implied by autarchy at any time and for any history.

Formally, we use the following notation, $i = 1, 2$ to denote each agent and we use \{z\} to denote a finite state markov process $z \in Z = \{z_1, ..., z_N\}$ with transition matrix $\Pi$. We use $z^t = (z_1, z_2, ..., z_t)$ to denote a length $t$ history of $z$. The matrix $\Pi$ generates conditional probabilities for all histories $\pi(z^t|z_0 = z)$. We use the symbol $\succeq$ for the partial order $z' \succeq z^t$ for $t' \geq t$ to indicate that $z'$ is a possible continuation of $z^t$, that is, that there exists a history $z^s$ such that $z^t = (z^t, z^s)$ for $s = t' - t$. We assume that all the entries of $\Pi$ are strictly positive. We denote \{c\}, \{e_i\} as the stochastic process for consumption and endowment of each agents, hence \{c\} = \{c_t(z^t) : \forall t \geq 0, z^t \in Z^t\}. Household preferences are given by discounted expected utility. Thus, an agent's utility corresponding to the consumption process \{c\} starting at time $t$ at history $z^t$ is denoted by $U(c)(z^t)$ and is given by:

$$U(c)(z^t) \equiv \sum_{s=0}^{\infty} \sum_{z^t+s \in Z} \beta^s u (c_{t+s}(z^{t+s})) \pi(z^{t+s}|z_t).$$

We assume the following properties, $u(\cdot)$ is strictly increasing, strictly concave, $C^1$ and $u'(0) = +\infty$. The participation constraints that we study consist in restricting the consumption paths so that under no history the associated utility is lower than

\footnote{Throughout the paper we concentrate on the 2 agent case for the sake of lighter and clearer exposition. Except for the result on the uniqueness of the invariant distribution of the endogenous variables it is straightforward to show that our results extend to the more general $N$-agent case.}
the one corresponding to autarchy. Formally, we restrict attention to consumption processes \(\{c_t\}\) that satisfy the following Participation Constraints:

\[
U(c_t)(z^t) \geq U(c_t)(z^t) \equiv U^i(z_t) \quad \forall \ t \geq 0, \ z^t \in Z^t,
\]

where we define \(U^i\) as the utility of consuming the endowment forever starting today when the shock is \(z_t\). Notice that \(U(c_t)(z^t)\) depends only on \(z_t\) because \(c_t(z^t)\) is a function of \(z_t\), which is first order markov.

For the subsequent analysis we further specialize the environment with the following assumptions:

(i) symmetry, for any \(z \in Z\) there is a \(z^* \in Z\) such that \(e_i(z) = e_j(z^*)\), \(j \neq i\) and \(\Pi\) is such that \(\Pr(z_{t+1} = z^* | z_t = z) = \Pr(z_{t+1} = z^* | z_t = z^*)\) for \(j, i = 1, 2\),

(ii) homothetic period utility, \(u(c) = \frac{c^{\gamma-1}}{\gamma-1}\)

(iii) no aggregate uncertainty, \(e_1(z) + e_2(z) = e\), for all \(z \in Z\); this assumption is relaxed later in the paper.

3. Constrained optimal allocations

Constrained optimal allocations are defined as the processes \(\{c_t\}\) \(i = 1, 2\) that maximize period zero expected lifetime utility for agent 1, subject to resource balance and the participation constraints for both agents, given some initial (time zero) promised expected lifetime utility to agent 2. For simplicity, from now on, we drop the qualification of “constrained” optimal allocations and we refer to them simply as “optimal allocations”. Thus, optimal allocations solve the following maximization problem:

\[
V^*(w_0, z_0) \equiv \max \{U(c)(z_0)\}
\]

\[
c_{1,t}(z^t) + c_{2,t}(z^t) = e,
\]

\[
U(c_1)(z^t) \geq U^i(z_t) \forall \ t \geq 0, \ z^t \in Z^t
\]

\[
U(c_2)(z_0) \geq w_0
\]

The function \(V^*(w_0, z_0)\) can be thought as the time zero utility possibility frontier. Now we turn to restate the previous problem recursively. We first present a functional equation and then relate the fixed points of this functional equation to the function \(V^*\).

\[
TV(w, z) = \max \left\{ u(c_1) + \beta \sum_{z' \in Z} V(w(z'), z') \pi(z'|z) \right\}
\]

\[
c_{1,2}, \{w(z')\} \ z' \in Z \leq \epsilon
\]

\[
u(c_2) + \beta \sum_{z' \in Z} w(z') \pi(z'|z) \geq w
\]

\[
w(z') \geq U^2(z') \text{ all } z' \in Z
\]
where the domain of the operator $T V(w, z)$ are the pairs $(w, z)$ such that

$$TV(w, z) \geq U^1(z)$$

$(3.6)$

Notice that, as a difference with a more standard "Bellman equation" the domain of $TV$—and any of its fixed points—depends on the function on the R.H.S. $V$ and hence it is not known a priori, that is, it has to be determined simultaneously with the function itself.

It is straightforward to show that the function $V^*$ defined previously solves this functional equation. However, the functional equation has more than one solution, and hence it can not be a contraction. In particular, autarchy is always a fixed point. To see this take any function $v(w, z)$ with the following properties: for each $z \in Z$, $v(\cdot, z)$ is decreasing in $w$ and $v(U^2(z), z) = U^1(z)$. It is immediate to verify that $v$ is a fixed of $T$ with the domain of the fixed point for each $z$ given by a single point, i.e., $v(w, z)$ is defined only on $[U^2(z), U^2(z)]$. But, if in this case, $V^*(U^1(z), z) > U^2(z)$ then $T$ has at least one other fixed point, namely $V^*$. This observation will "reappear" when we define the decentralized equilibrium and find that autarchy is always an equilibrium.

Even though the $T$ operator is not a contraction it can be useful in computing $V^*$. Its usefulness comes from the following result. Considering the operator $T$ defined exactly like $T$ in 3.1 except that the participations constraints for both agents 3.5 and 3.4 are removed, and consider its fixed point $V$. The function $V(\cdot, z)$ is the full risk sharing frontier when $z_i = z$. The following proposition states the sense in which $T$ is useful for computational purposes.

**Proposition 3.1.** $\lim_{n \to \infty} T^n V = V^*$ pointwise.

The proof uses standard arguments from Stokey and Lucas with Prescott (1989). Because $V$ is easily computed, the previous proposition makes $T$ useful for computing the fixed point $V^*$ and its associated policies.

### 3.1. Risk sharing regimes

Now we turn to the analysis of the stochastic process for $\{w_{t+1}\}$ and $\{c_{i,t}\}$. Depending on parameter values there are three possible "regimes" for the process for $\{w_{t+1}\}$. In all of these cases, independently of the initial condition $(w, z)$ with positive probability, after finite time, one of the following situations arises:

1. There is full risk sharing for ever,
2. Only limited risk sharing is possible,
3. Only autarchy is possible.

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6 It is easy to check that $T$ does not satisfy one of the Blackwell sufficient conditions for a contraction, namely discounting. That is $T(V + a)$ could be bigger than $TV + \beta a$ for a constant $a$, since the feasible set of choices for $(w(\cdot))_{\delta \in Z}$ is bigger for $V + a$ than for $V$.

7 Detailed proofs are in the appendix.
By full risk sharing we mean that the allocation is Pareto Efficient, in the standard sense ignoring the participation constraints, for some initial condition \((w_0, z_0)\). Notice that there cannot be full risk sharing for all initial \((w, z)\) since the participation constraint has to be violated for some, for instance, for \((w, z)\), such that \(w < U^2(z)\). In this simple environment, full risk sharing is characterized by resource feasibility and constancy of the ratio of the marginal utilities across agents for any time and history. Parameter values that produce the first case are not interesting for us since, for the purposes of asset pricing, its implications are the same as the representative agent economy.

We discuss briefly how one can check for each case. Kocherlakota (1995) presents sufficient conditions for each case when the shocks are i.i.d. and symmetric across agents. We consider a slightly different case in the following proposition.

**Proposition 3.2.** If there is no aggregate uncertainty and the endowments are symmetric, then full risk sharing is possible if and only if

\[
\frac{u(c/2)}{1 - \beta} \geq U^2(z) \text{ for all } z.
\]

Figure 1 illustrates the case when full risk sharing is possible for a range of \(w\). When full risk sharing is not possible, there are two cases, one case is where autarchy is the only feasible allocation that satisfies the participation constraints and the other is the one in which some other allocations satisfy the participation constraints. These are the cases that we are interested in, since they are not equivalent to a representative agent economy. Figure 2 illustrates the case when full risk sharing is not possible.

Which case applies depends on how attractive autarchy is relative to some form of risk sharing, which in turn depends on the parameter values. These parameters are time preference, the degree of risk aversion, and the persistence and variability of the individual endowment. Full risk sharing is not possible if agents are very impatient, if agents are very risk tolerant, or if shocks are very persistent. More formally,

**Proposition 3.3.** Consider a case where there is no aggregate uncertainty, and there is symmetry in the endowments; let

\[
\Pi_\delta = \delta I + (1 - \delta) \Pi
\]

for \(\delta \in (0, 1)\), then full risk sharing is not possible in any of the following cases:

(a) The time preference parameter, \(\beta\) is sufficiently small;
(b) The relative risk aversion \(\gamma\), is sufficiently small; and
(c) The persistence of \(\Pi_\delta\), \(\delta\) is sufficiently close to one.

**Remark 1.** Using results that we derive (a), (b) and (c) can be strengthened to say that as \(\beta \downarrow 0, \gamma \downarrow 0, \) or \(\delta \uparrow 1\), not only full risk sharing is not feasible, but autarchy is the only feasible allocation.
3.2. Optimal decision rules

We will now characterize the optimal decision rules for \( V \), a fixed point of the \( T \) operator. Let \( W_z'(w,z) \) and \( C_i(w,z) \) denote the optimal decision rules for \( w(z') \) and \( c_i \) respectively when the state is \( (w,z) \). The following properties of \( V \) and its associated policies are useful for the subsequent analysis.

**Proposition 3.4.** \( V \) is strictly decreasing and strictly concave on \( w \), furthermore it is differentiable in its interior. The optimal policy rules are single-valued and continuous. Finally, consumption is strictly monotone on \( z \), i.e., \( \forall z, \ C_2(w,z) \) (resp. \( C_1(w,z) \) is strictly increasing (strictly decreasing resp.) in \( w \).

Next we examine the first order conditions for the maximization problem (which are reproduced in the appendix). From these conditions it follows that the optimal decision rules are such that if one agent will be constrained in a particular state in the future, then the other will be unconstrained. Thus we have,

**Proposition 3.5.** If autarchy is not the only allocation satisfying the participation constraints, then if \( w = U^2(z) \) we have \( V(w,z) > U^1(z) \), and if \( V(w,z) = U^1(z) \) we have \( w > U^2(z) \).

Given our interest on the implications for asset pricing in decentralizations of these allocations we want to summarize some of the implications for the dynamics of the intertemporal marginal rate of substitution of the different agents.

**Proposition 3.6.** If both agents are unconstrained for \( z_{t+1} \) then their marginal rate of substitution is equalized. Furthermore, if \( z_{t+1} = z \) then we have \( w_{z}'(w,z) = w \) (the “45°-rule”). If one agent will be constrained in \( z_{t+1} \) then his marginal rate of substitution will be strictly smaller than the one for the other agent.

Finally we present a result for optimal decision rules when \( z_{t+1} \neq z_t \). We show that the decision rule \( W_{z}'(w,z) \) is strictly increasing in the interior of its range. Hence these decision rules are globally increasing, and may have (at most) two flat segments, one for low values of its domain and other for high values of it. Formally,

**Proposition 3.7.** If \( (\bar{w},w,z,z') \) are such that (i) \( z \neq z' \) (ii) \( w < \bar{w} \) and (iii) \( W_z'((\bar{w},z) \) and \( W_z'(w,z) \) are in the interior of the range of \( W_z'(z') \) i.e., \( W_z'((\bar{w},z), W_z'(w,z) \in (U^2(z'), V^{-1}(z')(U^1(z))) \). Then \( W_z'((\bar{w},z) > W_z'(w,z) \).

3.3. Long run dynamics : Invariant distributions of \( w' \)s

Now we consider the invariant distribution of the \( \{w_{t+1},z_{t+1}\} \) process. To analyze it we add to the previous assumptions of no aggregate uncertainty and symmetric endowments a requirement that will be satisfied, roughly speaking, if the process for individual endowments is not negatively autocorrelated. Under these circumstances we show that if there is only limited risk sharing, then there is a unique ergodic distribution for the \( \{w_{t+1},z_{t+1}\} \) process.
Let's denote the transition function for the process \( \{W_{t+1}, Z_{t+1}\} \) generated by the probabilities \( \Pi \) and the optimal decision rule \( W(\cdot) \) by \( \Pr\{W_{t-1} \in G, Z_{t+1} = z' \mid w_t = w, z_t = z; \Pi, W\} \) for suitable chosen sets \( G \subseteq D \equiv [\min_1 U^2(z), \max_1 U^2(z)] \). We will assume that the matrix \( \Pi \) and the functions \( e_i(\cdot) \) are such that \( U'_i(\cdot) \) is increasing in \( e_i \) with this extra assumption and the previous result that the optimal decision rule \( W^*(\cdot, z) \) is increasing and continuous on \( w \), we show in the appendix that the transition function for \( \{W_{t+1}, Z_{t+1}\} \) is monotone and satisfies the Feller property. Furthermore we will show that for the no aggregate uncertainty, symmetric endowment case, if there is only limited risk sharing, the domains of the function \( V^*(\cdot, z) \) for different \( z \) are such that the Markov process for \( \{W_{t+1}, Z_{t+1}\} \) satisfies a mixing condition which is sufficient to ensure the uniqueness of the invariant distribution.

The next definition is one of the monotonicity requirements that we mentioned.

**Definition 3.8.** \( \Pi \) is monotone means that \( \pi(e_{2,t+1} \geq e'| e_{2,t} = e) \) is increasing in \( e \) for all \( e' \).

The following lemma, together with the compactness of \( D \), establish the existence of an invariant distribution for the process \( \{W_{t+1}, Z_{t+1}\} \).

**Lemma 3.9. Monotonicity and Feller Property:** Assume that \( \Pi \) is monotone. Then the transition function for the process \( \{W_{t+1}, Z_{t+1}\} \) is monotone and satisfies the Feller property.

Now we turn to show that the process for \( \{W_{t+1}, Z_{t+1}\} \) mixes high and low values enough, so that it has a unique ergodic set. To do so we show a series of lemmas that make the connection between the domain of \( V(\cdot, z) \) for different values of \( z \) and monotonicity properties. We introduce the following assumption.

**Assumption:** Monotonicity of \( U^1 \): the values of \( Z, \pi, \) and the function \( e_i(\cdot) \) are such that:

\[
e_2(z_i) < e_2(z_{i+1}) \quad \text{and} \quad U^2(z_i) < U^2(z_{i+1}) \quad \text{for} \quad i = 1, 2, \ldots, N - 1
\]

Notice that this assumption is implied by monotonicity of \( \Pi \).

We denote the lower and upper bounds of the domains of the function \( V(\cdot, z) \) as \( L(z) \) and \( H(z) \) respectively for each \( z \), that is

\[
L(z) \equiv U^2(z) \quad \text{and} \quad H(z) \equiv V^{-1}(\cdot, z)(U^1(z)) \quad \text{for} \quad z \in Z.
\]

The next lemma states that we can order the lower and upper bounds of the domains of \( V \) for different \( z \)'s.

**Lemma 3.10. Monotonicity on \( U^1 \).** If there is monotonicity on \( U^1 \) then \( L(z_i) < L(z_{i+1}) \) and \( H(z_i) < H(z_{i+1}) \) for all \( i = 1, 2, \ldots, N - 1 \)

The next lemma makes the connection between the domain of \( V(\cdot, z) \) and the extent of risk sharing. This is the key insight of why there is enough recurrence so that there is a unique ergodic set.
Lemma 3.11. **Domains of $V$ and risk sharing.** Assume that there is monotonicity on $U_i$, symmetry on the endowments and that there is no aggregate uncertainty. Then efficient allocations have only limited risk sharing if and only if $H(z_1) < L(z_N)$.

Now we are ready to state a formal definition of mixing that will ensure the uniqueness of the invariant distribution.

**Definition 3.12. Mixing:** there is an $c \in [L(z_1), H(z_N)]$ and $\varepsilon > 0$ such that

$$P([c, H(z_N)], z_N \mid L(z_1), z_1) \geq \varepsilon,$$

and

$$P([L(z_1), c], z_1 \mid H(z_N), z_N) \geq \varepsilon.$$

Finally the next lemma verifies that this property is satisfied.

**Lemma 3.13. Mixing.** If there is monotonicity on $U_i$ and there is no aggregate uncertainty then the transition function satisfy mixing.

Collecting the result from the previous five lemmas we have that by Theorem 12.12 in Stokey and Lucas with Prescott (1989):

**Proposition 3.14.** If there is monotonicity on $U^i$, monotonicity of $\Pi$, symmetry on the endowments and no aggregate uncertainty, then there is a unique invariant distribution for the process $\{w, z\}$.

Kocherlakota (1995) proves a result similar to this last proposition. The difference here is that we allow individual endowments to be autocorrelated. In addition, our characterization of the optimal decision rules, that differs from Kocherlakota (1995), allows us to solve simple examples almost completely analytically. On the other hand, relative to Kocherlakota (1995), we have restricted ourselves in this section to the case of no aggregate uncertainty. However, in the next section, we argue that all the previous results hold for a specification of aggregate uncertainty that is different from the one in Kocherlakota (1995). Namely, the aggregate growth rate is assumed to follow a markov chain. This specification while simplifying the analysis enormously, is also the most commonly used in quantitative asset pricing exercises, which is the focus of this paper.

### 3.4. Bounds on individual consumption

The purpose of this subsection is to show that consumption of each agent is always bounded by the highest and lowest of the individual endowments of that agent. This is a very natural property of the optimal allocation that it will be used to obtain bounds for the marginal rate of substitution. We consider situations where there is some, but only incomplete risk sharing, and we maintain our earlier assumptions of no-aggregate uncertainty, homothetic utility and symmetry.

Let's fix the following notation,

$$\xi \equiv \min_{i=1,2, z \in Z} \frac{e_i(z)}{e(z)},$$

$$C \equiv \min_{z \in Z} C_2(U^2(z), z) \text{ and}$$

$$\bar{z} \equiv \arg \min_{z \in Z} C_2(U^2(z), z).$$
Proposition 3.15. Consumption is bounded by individual endowments,
\[ \epsilon \leq C_i(w, z) \leq \bar{\epsilon} \]
for all \( w, z \) and \( i = 1, 2 \).

To show this proposition we go to a series of simple lemmas, that are in the appendix. The intuition of the result is straightforward from the insurance aspect of the problem. In an efficient allocations in the "worst" possible case consumption of an agent can not be lower than the lowest individual endowment, because for this allocation to satisfy the participation constraint it will have to be followed by some future very high consumption, but this will require a pattern of consumption even more variable than autarchy.

Remark 2. Notice that proposition ?? gives a sufficient condition for bound in the marginal rate of substitution, i.e.
\[ \beta \pi (z' | z) \left( \frac{C_i(W_t'(w, z), z')}{C_i(w, z)} \right)^{-\gamma} \leq \beta \pi (z' | z) \cdot \left( \frac{\epsilon}{\bar{\epsilon}} \right)^{-\gamma} \]
for any \( i = 1, 2 \), and any \( w, z, z' \).

3.5. Optimal allocations and growth

We introduce aggregate growth in the same fashion as the specification of the aggregate endowment process in Mehra-Prescott (1985) and in much of the quantitative consumption based asset pricing literature. In particular, the growth rate of aggregate endowment, \( \lambda(z_{t+1}) \), follows a finite state markov process described as follows:
\[ e_{t+1} = e_t \cdot \lambda(z_{t+1}) \quad \text{and} \quad e_{t+1} = e_t \cdot \alpha_t(z_t) \quad \text{for} \quad i = 1, 2. \]

For this case, we write \( e' = \lambda(z') \cdot e \) and the state for the corresponding functional equation—and hence value function and policies—is \((w, z, e)\). Also notice that the value of autarchy is expressed as \( V(t, z) \). In the appendix we present the functional equation for this case.

When the period utility function is homothetic of the form \( u(c) = \frac{c^{1-\gamma}}{1-\gamma} \) for some positive \( \gamma \) (for simplicity lets assume that \( \gamma \neq 1 \)), then the value function \( V(\cdot) \) and the policies \( \{C_1(\cdot), C_2(\cdot), W(\cdot)\} \) satisfy the following homogeneity property:

Proposition 3.16. for any \( y > 0 \) and any \((w, z, e)\):
\[
\begin{align*}
V(y^{1-\gamma}w, z, y \cdot e) &= y^{1-\gamma} \cdot V(w, z, e) \\
C_i(y^{1-\gamma}w, z, y \cdot e) &= y \cdot C_i(w, z, e) \quad \text{for} \quad i = 1, 2 \\
W_t'(y^{1-\gamma}w, z, y \cdot e) &= y^{1-\gamma} \cdot W_t'(w, z, e).
\end{align*}
\]

This simply means that in the growing economy, there is some homogeneity with respect to the (stochastic) level of aggregate income. We can redefine the economy so
that it has a constant aggregate endowment, but with common shocks to the discount factors of both agents.

By using the homogeneity properties shown in the previous proposition the functional equation for the economy with growth can be rewritten as:

\[
TV(\hat{\omega}, z, 1) = \max \left\{ u(\hat{\varepsilon}_1) + \beta \sum_{z' \in Z} V(\hat{w}(z'), z', 1) \cdot \lambda(z')^{1-\gamma} \pi(z'|z) \right\}
\]

\[
\hat{\varepsilon}_1, \hat{\varepsilon}_2, \{\hat{w}(z')\} \quad z' \in Z
\]

\[
\hat{\varepsilon}_1 + \hat{\varepsilon}_2 \leq 1
\]

\[
u(\hat{\varepsilon}_2) + \beta \sum_{z' \in Z} \hat{w}(z') \cdot (\lambda(z'))^{1-\gamma} \pi(z'|z) \geq \hat{\omega}
\]

\[
\hat{w}(z') \geq U^2(z', 1) \text{ all } z' \in Z
\]

\[
V(\hat{\omega}(z'), z', 1) \geq U^1(z', 1) \text{ all } z' \in Z
\]

where the "hat" variables are defined in the appendix.

Notice that the \(\lambda(\cdot), \pi(\cdot)\) and \(\gamma\) have to satisfy certain constraints so that in the economy with growth utility is finite. It suffices to assume that:

\[
\max \left\{ \beta \sum_{z' \in Z} (\lambda(z'))^{1-\gamma} \pi(z'|z) : z \in Z \right\} < 1.
\]

Hence the functional equation for a growing economy is identical to the previous one except for the terms \((\lambda(z'))^{1-\gamma}\) multiplying the transition probabilities. In fact, by replacing the common constant discount factor \(\beta\) by a common discount factor that follows a markov process \(\beta(z, z') \equiv \beta (\lambda(z'))^{1-\gamma} \pi(z'|z)\), the two problems are exactly identical. Without loss of generality one can then omit the argument \(e\) from the value function \(V\) and the decision rules \(\{C, W\}\), since it is identically equal to one (it should be clear that this adjustment could have been made to the probabilities \(\pi\), in which case the two problems will be exactly identical). This transformation is similar to the standard practice of "detrending" variables for the neoclassical growth model following King, Plosser and Rebelo (1988).

All the propositions shown in previous sections, appropriately restated, with the new stochastic discount factor, will still be true. Because the proofs will be identical, except for details in the notation, we will not repeat them. This should not be surprising given that the shocks to the discount factor affect both agents identically. Clearly, a major benefit from this representation is that we will be able to compute this transformed stationary economy in exactly the same way as the economy with constant aggregate endowment.

4. Decentralized equilibrium and welfare theorems

In order to meaningfully analyze asset prices we propose a definition of a decentralized market equilibrium for these economies. We define a competitive equilibrium as a Radner equilibrium with complete markets and with solvency constraints. The
securities are all Arrow securities and each agent faces a constraint on the minimum amount of the Arrow security holding. The idea behind these constraints is that by not allowing agents to take large amounts of debt, default will be prevented. In general, these solvency constraints restrict the wealth that is carried to each state differently, since the relative value of staying in the economy or going to autarchy should be different.

4.1. Competitive equilibrium with solvency constraints

The period $t$ state $z'$ price of one consumption good delivered at $t + 1$ contingent on the realization of $z_{t+1} = z'$ in terms of period $t$ consumption goods is denoted by $q_t(z', z')$. The corresponding holdings for agent $i$ at $t$ of the Arrow security are denoted by $a_{i,t+1}(z', z')$. The lower limit on the holdings of the corresponding Arrow securities at time $t$ is denoted by $B_{i,t+1}(z', z')$. A plan for the agent $i$ is defined by a consumption process and a security process $\{a_{i,t+1}, c_{i,t}\}$ that satisfy the following budget constraint for an initial wealth $a_{i,0}$

$$\sum_{z' \in Z} a_{i,t+1}(z', z') q_t(z', z') + c_{i,t}(z') \leq a_{i,t}(z') + c_{i,t}(z_t)$$

(4.1)

for all $t = 0, 1, \ldots$ and for all $z' \in Z$, $z' \in Z$.

In each period agents are subject to the following agent and event specific solvency constraint,

$$a_{i,t+1}(z', z') \geq B_{i,t+1}(z', z')$$

(4.2)

for all $t = 0, 1, \ldots$ and for all $z' \in Z$, $z' \in Z$.

**Definition 4.1.** An equilibrium with Solvency Constraints $\{B_{i,t+1}\}$—given the initial wealth of each agent $a_{i,0}$ and given the initial value of $z_0$—is a plan $\{c_{i,t}, a_{i,t+1}\}$, and a price process $\{q_t\}$, such that (i) $\{c_{i,t}, a_{i,t+1}\}$ maximizes utility subject to 4.1 and 4.2 given $\{B_{i,t+1}\}$ and $\{q_t\}$, and (ii) markets clear, i.e.

$$c_{i,t}(z') + c_{2,t}(z') = e(z_t)$$

for all $t = 0, 1, \ldots$ and for all $z' \in Z$, and

$$a_{1,t+1}(z', z') + a_{2,t+1}(z', z') = 0$$

for all $t = 0, 1, \ldots$ and for all $z' \in Z$ and $z' \in Z$.

It is easy to see that for given solvency constraints the sufficient first order conditions for the maximization of the agent $i$ problem are given by:

**Euler equations :**

$$-q_t(z', z') u'(c_{i,t}(z')) + \beta \pi(z'|z) u'(c_{i,t+1}(z', z')) \leq 0$$

(4.3)

with equality if $a_{i,t+1}(z', z') > B_{i,t+1}(z', z')$, for all $t = 0, 1, \ldots$ and for all $z' \in Z$, and $z' \in Z$, and
Transversality Condition (see Stokey and Lucas (1989):)

\[
\lim_{t \to \infty} \sum_{z_t \in Z_t} \beta^t u' \left( c_{i,t} (z_t) \right) \cdot \left[ a_{i,t} (z_t) - B_{i,t} (z_t) \right] \cdot \pi (z_t | z_0) = 0. \tag{4.4}
\]

Now we examine the classical welfare theorems and the relationship with the equilibrium concept proposed by Kehoe and Levine.

4.2. Decentralizing optimal allocations : Second welfare theorem

In this section we characterize the allocations that can be supported as a competitive equilibrium with solvency constraints. In particular, we decentralize (constrained) optimal allocations as an equilibrium with some solvency constraints, i.e. we state a version of the second welfare theorem.

Given an allocation \( \{c_{i,t}^*\} \) for \( i = 1, 2, ..., I \) we start by defining candidate Arrow prices \( \{q_{i,t}^*\} \), Arrow-Debreu prices \( \{Q_{i,t}^*\} \), initial wealth \( a_{i,0} \) and solvency constraints \( \{B_{i,t}\} \) for decentralization of \( \{c_{i,t}^*\} \).

**Definition 4.2.** Given an allocation \( \{c_{i,t}^*\} \) the period \( t \), event \( z_t \) candidate Arrow prices of a security that pays one unit of consumption at \( t+1 \) contingent on \( z_{t+1} = z' \). in terms of consumption at \( t \) are defined as

\[
q_{i,t}^* (z', z_t) = \beta^{t} \pi (z'|z) \max_{i=1,2} \left[ \frac{\pi (c_{i,t+1}^* (z_t'))}{\pi (c_{i,t}^* (z_t'))} \right].
\]

\[
(4.5)
\]

for all \( t, z_t \in Z^t \) and \( z' \in Z_t \).

These Arrow prices imply the following Arrow Debreu prices:

**Definition 4.3.** Given Arrow prices \( \{q_{i,t}^*\} \) the candidate time 0 A-D price of a unit of consumption contingent on the realization of \( z^t \) at time \( t \) are defined as follows,

\[
Q_{0,t} (z_t^t | z_0) = q_{0,t} (z_0, z_1) \cdot q_{1,t} (z_0, z_1, z_2) \cdots q_{t-1} (z_t, z_t). \tag{4.6}
\]

Because of our interest in computing prices of assets that pay dividends in every period, such as consols and stocks, we are going to be looking for equilibria where the interest rate are high. In order to do so, we restrict attention to the allocations that satisfy the following property:

**Definition 4.4.** We say that the implied interest rates for the allocation \( \{c_{i,t}^*\} \) are high if the value of the aggregate endowment, and thus of aggregate consumption, computed using the A-D candidate prices for the allocation \( \{c_{i,t}^*\} \) is finite:

\[
\sum_{t \geq 0} \sum_{z_t \in Z_t} Q_{0,t} (z_t | z_0) (c_{i,t}^* (z_t) + c_{2,t}^* (z_t)) < +\infty. \tag{4.7}
\]
The following proposition states the conditions under which an allocation can be decentralized as a competitive equilibrium with solvency constraints. These solvency constraints can be chosen such as not to be too tight, so that they bind if and only if the continuation utility is equal to the value of autarchy. This is a natural condition to ask for the solvency constraints, since the motivation for the constraints was to avoid default. That is, if the solvency constraint binds and the continuation utility would be strictly higher than the value of autarchy, the solvency constraint could be relaxed a bit, without "causing" the agent to default. Having relaxed the constraint, then most likely the agent will take a bit more debt and will be better off. We imagine that if financial intermediaries are competing against each other by taking prices as given but by being able to set the solvency constraints, then at an equilibrium this property will have to be satisfied. Indeed, the intermediaries that were to lend to a constrained agent would never be reimbursed.

**Definition 4.5.** For an allocation \( \{c_{i,t}\} \), asset holdings \( \{a_{i,t+1}\} \) and solvency constraints \( \{B_{i,t+1}\} \) we say that the solvency constraints are not too tight if they bind only when the participation constraint binds, i.e., if for agent \( i \) at all times \( t \), and events \( z^t \in Z^t \), \( z' \in Z \):

\[
a_{i,t+1} (z', z') = B_{i,t+1} (c_t, z')
\]

if and only if

\[
U (c'_i) (z', z') = U (e_i) (z', z')
\]

Recall that the notation for period \( t \) state \( z^t \) continuation utility for agent \( i \) is \( U (c'_i) (z^t) \).

We can now state the conditions under which an allocation can be decentralized and then we get a version of the second welfare theorem.

**Proposition 4.6.** Given any allocation \( \{c_{i,t}\} \) that satisfies: (a) resource feasibility at any time and event, (b) the participation constraints 2.1 for each agent at each time and event, (c) if the participation constraint of an agent does not bind then this agent has the higher marginal rate of substitution, i.e., for all \( t \geq 0 \) and \( z^t \in Z^t \), \( z' \in Z \) then if agent \( i \)

\[
U (c'_i) (z', z') > U (e_i) (z', z') \implies \beta \frac{u' (c'_{i,t+1} (z', z'))}{u' (c'_{i,t} (z'))} = \max_{j=1 \ldots t} \left\{ \frac{\beta u' (c'_{j,t+1} (z^j, z'))}{u' (c'_{j,t} (z^j))} \pi (z'|z_t) \right\}
\]

and (d) the allocation has high implied interest rates, then (i) there exists a process \( \{B_{i,t}\} \), initial wealth \( a_{i,0} \) and an asset holding process \( \{a_{i,t}\} \) such that the plan \( \{a_{i,t}, c_{i,t}\} \) is a competitive equilibrium for the solvency constraint \( \{B_{i,t}\} \) and the initial wealth \( a_{i,0} \). Moreover (ii) the process for the solvency constraint \( \{B_{i,t}\} \) can be chosen so that the solvency constraints for both agents satisfy 4.8 (are not too tight).

The, almost, converse of the previous proposition is true:
Remark 3. Conditions (a), (b) and (c) of the previous proposition have to be satisfied in any competitive equilibrium given solvency constraints that satisfy 4.8 (not too tight).

As a corollary of the previous proposition we get the second welfare theorem.

Corollary 4.7. Any constrained optimal allocation that has high implied interest rates can be decentralized as a competitive equilibrium with solvency constraints where the constraints are not too tight.

In the previous corollary we have restricted ourselves to the case where the optimal allocation satisfied the assumption that the implied interest rates are high. Now we give a sufficient condition under which that assumption is satisfied.

Lemma 4.8. If preferences are homothetic and the following inequality is satisfied
\[
\beta \lambda_t^* \pi (z' | z) + \beta \sum_{\ell \neq t} \left( \frac{\ell}{\ell} \right)^{-7} \lambda_{\ell}^* \pi (z' | z) < 1
\]
for all \( z \in Z \), then any optimal allocation \( \{c_t^*\} \), generated by the optimal policies, has implied interest rate that are high.

Later on we will develop some results that will allow us to show that if the allocation \( \{c_t^*\} \) is a constrained optimal allocation where there is some risk sharing, then 4.7 has to hold (the implied interest rate are high).

We finally establish a result about autarchy. The result states that autarchy can always be decentralized as an equilibrium with solvency constraints that satisfy 4.8 (that are not too tight).

Lemma 4.9. Define \( \{c_{a,i,t}, q_{a,t}, a_{a,i,t+1}, B_{a,i,t}\} \) as follows: for all \( t \geq 0, z^i \in Z^i, z' \in Z \)
\[
\begin{align*}
c_{a,i,t} (z^i) &= \epsilon_{i,t} (z_i), \\
a_{a,i,t+1} (z^i, z') &= B_{a,i,t} (z^i) = 0, \\
q_{a,t} (z^i, z') &\equiv \max_{i=1,2} \beta u'(\epsilon_{i,t+t+1} (z')) \pi (z' | z_t)
\end{align*}
\]
and \( \{Q_{a,t}\} \) are the A-D prices implied by \( \{q_{a,t}\} \) as defined by ?? Then \( \{c_{a,i,t}, q_{a,t}, a_{a,i,t}\} \) is a competitive equilibrium with solvency constraints given \( \{B_{a,i,t+1}\} \) and the initial conditions \( a_{1,0} = 0, i = 1,2 \). Moreover \( \{B_{a,i,t+1}\} \) are not too tight.

Remark 4. Even though autarchy allocations can always be decentralized, in general, they are not (constrained) efficient allocations. This result is analogous to the fact stated above about multiple solutions of the functional equation. That is, that the value of autarchy was always a solution of the functional equation characterizing optimal allocations. Also notice that the implied interest rate in \( \{Q_{a,t}\} \) can be very low and hence equation (4.7) may be violated.
4.3. 1st Welfare theorem and comparison with Kehoe and Levine equilibrium

In this section we address two related issues, the comparison of our equilibrium concept with the one proposed by Kehoe and Levine and whether the first welfare theorem holds for our equilibrium concept.

Compared to our equilibrium concept the equilibrium in Kehoe and Levine (1993) differs in that they include the participation constraints in the consumption possibility sets, as opposed to our limits to borrowing (our solvency constraints). We think that our alternative decentralization provides a closer link with the literature on equilibrium asset pricing with borrowing and solvency constraints and that it makes the form of the pricing kernel immediate. In particular it allows us to use the results of Luttmer (1996) and He and Modest (1993).

The problem for agent $i$ in the Kehoe-Levine decentralization is

\[
\max_{c_i} U(c_i) (z_0) \\
\text{s.t.} \quad Q_0 (c_i - e_i) \leq a_{i,0} \\
U(c_i)(z^t) \geq U(e_i)(z^t) \quad \text{for all} \quad t \geq 0 \text{ and } z^t \in Z^t
\]

where $Q_0$ is a non-negative linear function.

Formally given $Q_0$ the dot product representation of the A-D prices is defined as follows. For any $t$ and $z^t \in Z^t$ define

\[
Q_0 (z^t|z_0) \equiv Q(\hat{c})
\]

where $\hat{c}_s(z^s) = 0$ if $s \neq t$ and $z^s \neq z^t$ and otherwise $\hat{c}_t(z^t) = 1$.

**Definition 4.10.** The A-D prices are said to have a dot product representation if

\[
Q_0 (c) = \sum_{t \geq 0} \sum_{z^t \in Z^t} c_t (z^t) Q_0 (z^t|z_0).
\]

With this notation we can write the budget constraint of agent $i$ as

\[
\sum_{t \geq 0} \sum_{z^t \in Z^t} (c_{i,t} (z^t) - e_i(z_t)) Q_0 (z^t|z_0) \leq a_{i,0}
\]

The corresponding Arrow prices are defined as

\[
q_{0,t}(z^t, z') = \frac{Q_0 (z^t, z'|z_0)}{Q_0 (z^t|z_0)}.
\]

We set up this notation to show the following two results. First the implied Arrow prices in the Kehoe-Levine decentralization are equal to the highest marginal rate of substitution across agents, like in our equilibrium with solvency constraints that are not too tight. Second, we can say that the allocations of an equilibrium with solvency
constraints correspond to the allocations of a K-L equilibrium. An immediate consequence of these results is that we can use the relationship between the equilibrium with solvency constraints and the K-L equilibrium together with the first welfare theorem shown by Kehoe and Levine. Then we obtain a (qualified) version of the first welfare theorem for our equilibrium concept. That is, allocations corresponding to an equilibrium for solvency constraints that satisfy 4.8 (solvency constraints not too tight) and 4.7 (that have high implied interest rates) are constrained efficient.

**Proposition 4.11.** Let \( \{c_i\}, i=1,2, \{Q_0\} \) be the allocations and A-D prices corresponding to Kehoe-Levine equilibrium. Let \( q_0 \) be the corresponding Arrow prices. Then

\[
q_{0,t}(z', z') = \max_{i=1,2} \left\{ \beta \frac{u'(c_{i,t+1}(z', z'))}{w'(c_{i,t}(z'))} \pi(z'|z_t) \right\}
\]  

(4.9)

and if

\[
U(c_i)(z', z') > U(e_i)(z', z')
\]

then

\[
q_{0,t}(z', z') = \beta \frac{u'(c_{i,t+1}(z', z'))}{u'(c_{i,t}(z'))} \pi(z'|z_t).
\]

Now we show that for any equilibrium with solvency constraints if the implied interest rate are high and the constraints are not too tight then the implied A-D prices and consumption allocations constitute a K-L equilibrium.

**Proposition 4.12.** Let \( \{q_t, c_t, a_{t+1}\} \) be an equilibrium given the solvency constraints \( \{B_t, t+1\} \) and the initial wealth \( \{a_0\} \). Assume that the A-D prices \( \{Q_t\} \) implied by \( \{q_t\} \) satisfies 4.7 (that the implied interest rates are high) and that the solvency constraints satisfy 4.8 (i.e. they are not too tight). Then the consumption allocations \( \{c_t\} \) and the A-D prices \( \{Q_t\} \) constitute a K-L equilibrium.

**Proof.** Sketch. In an equilibrium for given solvency constraints the consumption allocation is clearly feasible and if the solvency constraints are not too tight the consumption allocations satisfies the participation constraints for each agent at all times and states. It only remains to be shown that given the A-D prices implied in \( \{q_t\} \) the consumption allocation maximizes utility subject to budget and participation constraints. It will suffice to find non-negative shadow values (multipliers) associated with the budget constraints and participation constraints. We give an algorithm to define these multipliers as a function of the consumption allocation, Arrow prices and participation constraints. Finally we verify that these multipliers together with the consumption allocation are indeed a saddle. We accomplish this by verifying the first order conditions. A detailed proof with the technical details is in the appendix.

**Corollary 4.13.** Let \( \{q_{at}\} \) be defined as the "autarchy Arrow Prices", i.e. for all \( t \geq 0, \) and \( z^t \in Z^t \)

\[
q_{at}(z^t, z_{t+1}) = \max_{i=1,2} \beta \frac{u'(c_{i,t+1}(z_{t+1}))}{u'(c_{i,t}(z_t))} \pi(z_{t+1}|z_t)
\]

(4.10)
Let \( \{Q_{at}\} \) be the A-D price process computed using \( \{q_{at}\} \). If autarchy interest rates are high, i.e.,

\[
\sum_{t=0}^{\infty} \sum_{z', z_t \in Z_t} Q_{at}(z') [c_1(z_t) + c_2(z_t)] < +\infty. \tag{4.11}
\]

then autarchy is an efficient allocation, and hence is the only feasible allocation.

**Proof.** Recall that for \( B_{1,t} = 0 \) autarchy is an equilibrium with solvency constraint that are not too tight. If 4.11 is satisfied, by definition, the implied interest rates are high. Then proposition 4.3 implies that the consumption allocation and implied A-D prices of the equilibrium with solvency constrains constitute a K-L equilibrium. Since the first welfare theorem holds for the K-L economy, this allocation is efficient.

**Corollary 4.14.** Autarchy is the only feasible allocation in either of the four cases: (i) the time discount factor \( \beta \) is sufficiently small, (ii) risk aversion \( \gamma \) is sufficiently small, (iii) the transition probability matrix is sufficiently close to identity, and (iv) the variance of the idiosyncratic shock is sufficiently close to zero.

The proof of this corollary consist in showing that under the given condition the autarchy-value of endowment becomes finite, that is, autarchy interest rates are high. Related to these two corollaries we will present in the last section of the paper a simple example where aggregate endowment and interest rates are constant.

For completeness we add, without proof, a simple lemma, characterizing a case where perfect risk sharing is feasible.

**Lemma 4.15.** Let \( \{c_{1,t}\} \) be the consumption allocation for an equilibrium with solvency constraints that are not too tight. Then if the marginal rates of substitution are equalized, i.e. for all times \( t \) and events \( z', z' \)

\[
\beta \frac{u'(c_{1,t+1}(z', z'))}{u'(c_{1,t}(z'))} \pi(z'|z_t) = \beta \frac{u'(c_{2,t+1}(z', z'))}{u'(c_{2,t}(z'))} \pi(z'|z_t),
\]

then the consumption allocation is unconstrained efficient. Moreover if an allocation is unconstrained efficient, it can be decentralized as an equilibrium with solvency constraints where the solvency constraints never binds.

We show that for any (constrained) optimal allocation where some risk sharing is possible, the implied interest rates are high. This result complements our statement of the second welfare theorem, that uses as an assumption that the implied interest rates are high.

**Proposition 4.16.** Let \( \{c_{it}\} \) be a (constrained) efficient allocation and let \( \{Q_t\} \) be its associated A-D price process. Then, if some risk sharing is possible, so that for each \( t, z', \) there is \( z' \in Z \) such that one of the agent \( i' = 1, 2 : \)

\[
U^{i'}(c_{i',t})(z', z') > U^{i'}(c_{i',t})(z', z'), \tag{4.12}
\]

then 4.7 is satisfied (i.e. the implied interest rates are high).
We finish the section with two remarks about the usefulness of these results.

**Remark 5.** As alluded earlier in the paper, it is immediate to see that the results of this section on equilibrium characterization and welfare theorems hold for the case of any finite number of agents, not just two.

**Remark 6.** The characterization of equilibrium, welfare theorems and optimal allocations makes it very easy to construct optimal allocations that can be decentralized by guessing and verifying. For instance suppose that an allocation is (i) resource feasible (ii) satisfies the participation constraints and (iii) every time that for an agent the participation constraint does not bind then this agent marginal rate of substitution is (weakly) higher than the one for the other agent. As a direct application of the previous result, if the implied interest rates for this allocation are high then the allocation is (constrained) efficient. The fact that we restrict ourselves to cases where interest rates are high is, in a sense, without cost. This is because we are interested in analyzing the price of infinitely lived securities.

### 4.4. Properties of asset returns in economies with solvency constraints

In this section we analyze some properties of the pricing kernels in economies with solvency constraints. We start by describing the pricing of securities that are more complex than Arrow securities. We then compare pricing implications in economies with and without participation constraints by considering interest rates, marginal valuations and risk premia.

#### 4.4.1. Pricing complex securities

Instead of allowing agents to trade only Arrow securities (one-period contingent claims) we want to let them trade any security, particularly multiperiod securities such as stocks and bonds. A straightforward extension of our framework allows us to do this.

We assume that at time $t$ there is a set of securities that are traded. These securities may pay dividends at multiple dates. In this case, agents face the following sequence of budget constraints:

$$
\sum_{k' \in K_{t+1}(z^t, z')} a_{t,t+1,k'}(z^t, z') q_{t,k'}(z^t) + c_{t,t}(z^t)
\leq
\sum_{k \in K_t(z^t)} a_{t,t,k}(z^{t-1}) [q_{t,k}(z^t) + d_{t,k}(z^t)] + c_{t,t}(z^t)
$$

for all $t = 0, 1, ...$ and for all $z^t \in Z^t$. Here $q_k$ denotes the price and $d_k$ the dividend processes of a security $k$. We use $K_t(z^t)$ as the set of securities that could have possible been bought (or sold) at dates and states priors to $t$, $z^t$. Analogously the set $\{K_{t+1}(z^t, z') : z' \in Z\}$ contains the set of securities that can be bought (or sold) at time and state $t$, $z^t$. In the following definition of $K_t$ we exclude trivial cases of securities that have already matured or that have zero payout forever.

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To have the same budget set as in the case with Arrow securities, we need two conditions: first, that this set of securities be rich enough so that markets are dynamically complete and second, that the portfolio choice be restricted so that the value of an agent's portfolio be high enough in the different states. Specifically, for each period, a set of solvency constraints that limits the minimum value of next period's portfolio for each state is defined as:

\[
\sum_{k' \in K_{t+1}(z^t, z')} \left[ q_{t+1,k'} (z^t, z') + d_{t+1,k'} (z^t, z') \right] a_{t+1,k'} (z^t, z') \geq B_{t+1}(z^t, z')
\]

for all \( t = 0, 1, \ldots \) and for all \( z^t \in Z^t \). Clearly, every portfolio of securities is just a portfolio of Arrow-Debreu securities, and Arrow-Debreu prices are simple products of the sequence of Arrow prices. Notice that, as of time \( t \), this imposes a set of \#Z linear inequalities on the portfolio choice of each agent.

We want to make two points by introducing securities different than Arrow securities. First, a rather trivial one, that the pricing of any security (under the conditions stated above) can be obtained by pricing the corresponding portfolio of Arrow securities. Second, even though in equilibrium the price of an Arrow security must be equal to some agent marginal rate of substitution, the price of a more complex security may be higher than the marginal valuation of both agents, as it is shown in the next section.

### 4.4.2. Prices and marginal valuations

We find that given the type of friction—solvency constraints—the prices of securities with non-negative payoffs are bigger than the valuation of any of the agents if there is only limited risk sharing. Superficially, this result may seem to imply an arbitrage opportunity, in the sense that the prices are too high for everyone, but recall that agents can short sell only limited amounts of securities.\(^8\)

Assume that \( k \) is a security available at time \( t \), i.e. \( k \in K_t(z^t) \) and that

\[
d_{t+s,k} (z^{t+s}) \geq 0 \quad \text{for all} \quad s > 0.
\]

Let us denote the price of this security for the equilibrium with solvency constraints by \( q_{t,k} \) and the marginal valuation of agent \( i \) by \( MV_{i,t,k} \) were the marginal valuation is defined as

\[
MV_{i,t,k} (z^t) \equiv \sum_{s > 0} \beta^s \sum_{z^s \in Z^s} \left[ d_{t+s,k} (z^t, z^s) \left( \frac{u'(c_{i,t+s}(z^t, z^s))}{u'(c_i(t)(z^t))} \right) \pi (z^s | z_t) \right].
\]

This quantity measures the marginal change in utility, in terms of time \( t \) consumption, produced by an increase in the agent \( i \) consumption proportional to the dividends of security \( k \) at each future date. When agents are never constrained, or equivalently for the case of perfect insurance we have that for all \( i \),

\[
q_{k,t} (z^t) = MV_{i,t,k} (z^t).
\]

In our environment, we have the following result.

\(^8\)We thank George Constantinides for highlighting this point.
Lemma 4.17. In an equilibrium with solvency constraints when there is only limited risk sharing, then

\[ q_{k,t}(z^t) \geq \max_{i=1,2} MV_{i,t}(z^t) \]

for all agents, securities, times period \( t = 0,1, \ldots \), and states \( z^t \in Z^t \). With strict inequality for an agent if he is constrained at least once between \( t \) and \( t + s \).

The proof of this lemma follows directly from the fact that Arrow prices are equal to the larger marginal rate of substitution of the two agents.

Notice that since we assume that the dividends are non-negative, holding a bit more of security \( k \) can never lead to default in the future if the original plan did not already contemplate default. One may therefore be tempted to conclude that in equilibrium \( q_{k,t}(z^t) = \max_{i=1,2} MV_{i,t}(z^t) \). The misleading part of this conjecture is that this equality holds only for Arrow securities and not for general securities that pay in different states, since the pricing kernel is defined by the max across agents of the marginal rate of substitution, state by state. That is, the agent whose marginal rate of substitution equates the price of the Arrow security in a given state may not price the Arrow security in a different state.

4.4.3. Interest rates

As a general property, interest rates are smaller in economies with participation or solvency constraints than in corresponding economies without such constraints. Moreover, as opposed to the findings in some application of incomplete markets economies, this effect does not rely on the precautionary savings motive (convexity of the marginal utility).

Proposition 4.18. Given \( w_t \), \( z_t \), the price of a one-period bond is higher in an economy with participation constraints than in the corresponding economy without these constraints. With constraints binding at least once, the price of a one-period bond is strictly higher.

The proof of this properties follows directly from the fact that for each state \( z^t \) the pricing kernel in a participation constraints economy equals the larger of the two agents' intertemporal marginal rate of substitution. In fact, the price of any Arrow security is weakly higher. In the appendix we include the proof for both cases with and without aggregate uncertainty.

4.4.4. Risk premium: Aggregate and idiosyncratic shocks

A major issue for any “heterogenous-agent” asset pricing framework is the mechanism through which idiosyncratic shocks can generate a risk premium for claims contingent on the aggregate shock, that is, the market risk premium. We provide here a look at this issue by deriving sufficient conditions under which the economy with idiosyncratic shocks generates a risk premium on a one-period risky strip identical to the representative agent economy that does not have idiosyncratic shocks.
We consider the risk premium for one-period risky strips, defined as

$$\frac{E[d_{t+1}]}{P_t} B_t,$$

where $d_{t+1}$ is the payout, (which we restrict to be a fraction of $t+1$ aggregate output), $P_t$ is the price of this strip and $B_t$ is the price of a one-period risk-free bond. This is sometimes called the "multiplicative excess return of the one-period risky strip" over the risk-free rate. We choose this premium as opposed to the equity premium for its tractability. Indeed, the equity premium is a weighted sum of the entire infinite sequence of strips, one for each horizon. Our hope is that this premium behaves at least qualitatively similarly to the entire equity premium in the cases we look at numerically.

Proposition 4.19. If the idiosyncratic and the aggregate shocks are independent, and if the aggregate shock is iid, then the multiplicative premium on a one-period risky strip with payout contingent on the aggregate output is the same in an economy with and without participation constraints.

In the appendix we present the proof of the proposition and a formal definition of idiosyncratic and aggregate risk. The extra assumption that the aggregate shock is iid is due to the fact that the aggregate shock is expressed in rates of change and the idiosyncratic in levels. At any rate, quantitatively, the assumption of iid growth rates of aggregate consumption is not a bad first approximation.

5. Quantitative implications: An example

In order to illustrate model mechanisms at work and in order to get a very first idea about the quantitative potential of our model for explaining asset returns, we present here some calibration results. We consider a model version with two agents and two shocks and no aggregate uncertainty. We completely characterize the optimal allocations and we look at the implications for consumption and Hansen-Jagannathan bounds. This is also an illustration of how we can compute the solutions for such models by solving a simple system of nonlinear equations instead of functional equations. This is not only orders of magnitude faster, but also basically eliminates the numerical approximation error. The main point here is to illustrate model mechanisms, for a more thorough quantitative analysis see Alvarez and Jermann (1997). However, even this example suggests that powerful limitations on risk sharing occur in the reasonable parameter region.

5.1. Characterizing optimal allocations

We will show that with two symmetric agents and two shocks, and with no aggregate uncertainty, the ergodic set for $(w,z)$ and the associated consumption take a very simple form. Namely, after a reversal of $z$, consumption will depend exclusively on the current value of $z$, with $w$ also exclusively a function of the current state $z$. We
derive this result below, and use it to fully characterize optimal allocations in a single equation.

To derive a very simple representation of the decision rules we use two properties. First, if \( z' = z \), then the optimal policy is the 45° line as we have shown before. The second property we use is presented below.

**Definition 5.1.** We say that decision rules are "flat" after a reversal of the shock, \( z' \neq z \), if:

\[
W_{z'}(w, z) = U^2(z_2) = \bar{w}(z_2) \text{ if } z' = z_2 \text{ and } z = z_1
\]

\[
W_{z'}(w, z) = V^{-1}(\cdot, z_1)\left(U^1(z_1)\right) = \bar{w}(z_1) \text{ if } z' = z_1 \text{ and } z = z_2.
\]

In order to demonstrate this proposition we start by showing the following two lemmas:

**Lemma 5.2.** Sufficient conditions for flat decision rules after reversal. If

\[
\partial V \left( V^{-1}(\cdot, z_1)\left(U^1(z_1)\right), z_1 \right)/\partial w > \partial V(U^2(z_2), z_2)/\partial w
\]

then the decision rules after reversal are flat.

**Lemma 5.3.** Implication of flat decision rules after a reversal. If the decision rules after reversal are flat, then \( \partial TV(\bar{w}(z_1), z_1)/\partial w > \partial TV(\bar{w}(z_2), z_2)/\partial w. \)

The result that the optimal decision rules are flat after reversal follows by the combination of the previous two lemmas with the result that \( \lim_{n \to \infty} T^n V = V^* \), that is, in each iteration, the operator \( T \) preserves the properties of the optimal decision rules. Starting from the full risk sharing frontier \( V \), the first lemma applies and we have flat decision rules for reversal. Then in the following iteration derivatives are ordered as required by the second lemma.

Combining these results, Figure 3 plots the decision rules \( W_{z'}(w, z) \) as a function of \( w \) and \( z \). Clearly, whatever the initial starting point for \((w, z)\), after one reversal of the shock, continuation utility and consumption, \( \bar{w}(z) \) and \( \bar{c}(z) \), will move forever between only two values, depending on the given state \( z \).

This characterization suggests that in order to find the optimal allocations under the invariant distribution we can simply solve the following system of the two promise keeping conditions:

\[
\bar{w}(z_1) = u(\bar{c}(z_1)) + \beta \pi \bar{w}(z_1) + \beta (1 - \pi) \bar{w}(z_2), \text{ and}
\]

\[
\bar{w}(z_2) = u(\bar{c}(z_2)) + \beta \pi \bar{w}(z_2) + \beta (1 - \pi) \bar{w}(z_1),
\]

in addition to the known boundary value \( \bar{w}(z_2) = U^2(z_2) \), and the symmetry condition \( \bar{c}(z_1) = e - \bar{c}(z_2) \). This system can be written as a single equation in the single unknown \( u_2 = u(\bar{c}(z_2)) \):

\[
u_2 = h(u_2) \equiv \frac{(1 - \beta)(1 - 2\beta \pi - 1)}{(1 - \beta \pi)} u^2(z_2) - \frac{\beta (1 - \pi) f(u_2)}{1 - \beta \pi}.
\]

(5.1)

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with \( f(u_2) \equiv u(e - u^{-1}(u_2)) = u(\bar{c}(z_1)) \). Equation 5.1 has at most two solutions. To see this, notice that \( h \) is the sum of a constant minus the function \( f \) multiplied by positive constant. \( f \) is the static utility possibility frontier, hence it is strictly decreasing and strictly concave, hence \( h \) is strictly increasing and strictly convex. The solutions of 5.1 are the intersection of the identity with \( h \), so there could be at most two intersections. We also know that there is at least one solution: the autarchy value \( u_2 = u(e_2(z_2)) \) is always a solution of 5.1. This claim follows form the fact that the autarchy values have to satisfy resource feasibility, promise keeping and \( w(z_2) = U^2(z_2) \). Whether the other solution of equation 5.1 characterizes the efficient allocation depends on whether it satisfies

\[
\frac{e}{2} < u_2 \leq u(e_2(z_2)), \text{ that is } \frac{e}{2} < \epsilon(z_2) < e_2(z_2).
\]

If it violates the condition that \( u_2 > u(e_2(z_2)) \), then the corresponding allocation is more variable so it cannot be efficient (it violates the bounds for consumption shown in proposition 3.10). One way to check whether the solution that is different from autarchy implies \( \frac{e}{2} < \epsilon(z_2) < e_2(z_2) \) is to check that the slope of \( h \) at autarchy is larger than 1, i.e. \( h'(u(e_2(z_2))) \geq 1 \). This follows because the convexity of \( h \) implies that the smaller fixed point crosses the identity with a slope less than one. Indeed, if autarchy is the larger fixed point, the lower fixed point implies some risk sharing, because then the agent gets less than autarchy utility in it his good state. Simple calculations show that the derivative of \( h \) is given by

\[
h'(u_2) = \frac{-\beta (1 - \pi) f'(u_2)}{(1 - \beta \pi)} = \frac{\beta (1 - \pi) u'(e - u^{-1}(u_2))}{(1 - \beta \pi) u'(u^{-1}(u_2))}.
\]

We collect these results in the following proposition,

**Proposition 5.4.** Assume that full risk sharing is not feasible. Equation 5.1 has at least one solution and at most two. We denote them by \((u_{2k}, u_{2h})\) where we assume that \( u_{2k} \leq u_{2h} \). The period utility of autarchy is always a solution, i.e. \( u(e_2(z_2)) = u_{2k} \) for either \( k = 1 \) or \( h \). If \( u_{2k} = u(e_2(z_2)) \) then autarchy is the only feasible allocation. If \( u_{2h} = u(e_2(z_2)) \) then the efficient allocation is characterized by \( \epsilon(z_2) = u^{-1}(u_{2h}) \). A necessary and sufficient condition for \( \epsilon(z_2) = u^{-1}(u_{2h}) \) is given by

\[
\frac{\beta (1 - \pi) u'(e_2(z_2))}{(1 - \beta \pi) u'(e_2(z_2))} \geq 1.
\]

We include a result about interest rates and the extent of risk sharing.

**Lemma 5.5.** Autarchy is the only feasible allocation if and only if autarchy interest rates are non-negative.

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9 Except in a knife edge case, equation 5.1 has two solutions. This knife edge case occurs when parameters cross the region where some risk sharing is feasible to the region where only autarchy satisfies the participation constraints.
5.2. Calibration and results: the Hansen-Jagannathan bounds

The 2 by 2 case without aggregate uncertainty allows us to focus entirely on the effects of limited risk sharing. We calibrate individual income following Heaton and Lucas (1994) based on a large sample from the PSID. In particular, the log of an agent's income, relative to the aggregate, that is $\ln(y_t/\sum y_t)$, is stationary with a first order serial correlation of 0.5 and a standard deviation of 0.29 for annual data. Initially, we set $\beta = 0.65$ and explore the effect of risk aversion for consumption and for asset pricing implications, we will explore the quantitative effects of $\beta$ below.\(^{10}\)

Figure 4 presents consumption of agent 2 as a function of risk aversion. For low risk aversion, agents do not share income risk because autarchy is sufficiently attractive to prevent any borrowing and lending. As risk aversion increases, autarchy becomes relatively less attractive, risk sharing starts to take place and consumption itself becomes less volatile. When risk aversion equals 4.5, income risk is fully shared and consumption is constant.

Hansen and Jagannathan's volatility bounds for stochastic discount factors provide a concise and widely used diagnostic device.\(^{11}\) The idea is, that, assuming the existence of an arbitrage free pricing kernel, data on asset returns can be used to construct a lower bound on the unconditional standard deviation of this kernel, for a given unconditional mean. Candidate kernels obtained from theoretical model structures can then be compared to this benchmark. Figure 5 presents test results for kernels generated by our model. It can be taken as a clear success that the model is able to generate kernels that fall inside the H.J-bound for risk aversion coefficients around 2, whereas the representative agent economy fails this test for such values of risk aversion.\(^{12}\) This positive finding also confirms empirical results by He and Modest (1991) that show that solvency constraints can pass a similar type of volatility bound test for aggregate consumption data. A close look at this picture reveals the two-sided role of the risk aversion coefficient. As in most of the asset pricing literature, increasing risk aversion increases the volatility of the pricing kernel, because for a given consumption process marginal utility is more volatile. In our framework however, as seen above, the extend of risk sharing and thus the consumption process is endogenous. The highest volatility for the pricing kernel is achieved with very limited risk sharing.

If we were to choose $\beta > 0.65$ as this is usually done in the literature, risk sharing would be limited for lower coefficients of risk aversions and the volatility of the pricing kernel would no longer be above the minimum bound imposed by the given H.J-bound. An alternative calibration, with $\beta > 0.65$ and for which pricing kernels are sufficiently volatile to be inside the admissible HJ-region could be achieved with higher serial correlation of the individual endowments. For instance, setting the first order serial correlation

\[\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}, \sigma^2 = 0.041, \sigma^2 = 0.359, 0.641, 0.641\]

\(^{10}\)The parameters for this case are the following: $\Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$, $\sigma^1 = \begin{bmatrix} 0.641 \\ 0.359 \end{bmatrix}$, $\sigma^2 = \begin{bmatrix} 0.359 \\ 0.641 \end{bmatrix}$.

\(^{11}\)For a detailed survey of applications of this test see, for example, Cochrane and Hansen (1992).

\(^{12}\)Introducing aggregate uncertainty would of course generally give further volatility to the pricing kernel and help it pass the test.
correlation at 0.95, we have now that for a more common $\beta = 0.90$, risk sharing is limited for risk aversion coefficients sufficiently large to generate volatility in the pricing kernel as shown in Figure 6 and 7.

An alternative way of thinking about our model framework is to take it as a two-country open economy, where relative income shocks are rather small. This seems of some interest given that the literature on sovereign debt, in particular the widely cited paper by Eaton and Gersovitz (1981), have studied the same mechanism of lack of commitment of the debtor country. Figure 8 presents consumption for the case where the log of relative income volatility is a mere 3% annually. This can be taken as the order of magnitude of country specific shocks to GDP. The endogenous limitations in risk sharing do appear to be even more powerful in this case given that they occur for a much wider range of risk aversion coefficients. In this last example we can clearly see that the fact that autarchy is less of a punishment with low income volatility ends up reducing risk sharing in equilibrium.

6. Conclusions

We have presented a framework for analyzing asset prices where endogenous solvency constraints may end up limiting risk sharing. After characterizing allocations with participation constraints in a recursive framework we proposed a concept of market equilibrium with explicit endogenous portfolio constraints. We derived the classical first and second welfare theorems and studied the pricing kernel that emerges in this setup. Among the general properties we show that interest rates in the environment with solvency constraints are lower than in the corresponding representative agent economy without the usual need for precautionary saving. Finally, a simple quantitative example shows that powerful limitations to risk sharing occur in the relevant region of the parameter space.

We view this paper as a first step in exploring asset pricing relationships in economy where the possibility of defaults limits risk sharing. In a companion paper we have start examining the quantitative side. We are interested in the first moments such as the mean risk free rate and the equity premium that have attracted so much attention in the recent literature on equilibrium asset pricing. We are also interested in the business cycle behavior of excess returns and the term structure that have been documented empirically.

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13 Eaton and Gersovitz (1981) present a partial equilibrium framework with a given world interest rate where the borrower can default if continuation utility is below the autarchy value.

14 The parameters for this case are the following: $\Pi = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$, $\beta = 0.65$, $c^1 = \begin{bmatrix} 0.515 \\ 0.485 \end{bmatrix}$, $e^2 = \begin{bmatrix} 0.485 \\ 0.515 \end{bmatrix}$. 

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References


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