"Empirical Measures, Learning and Sunspot Equilibrium"

Aloisio Pessoa de Araújo

(EPGE - FGV / IMPA)

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Coordenação: Prof. Pedro Cavalcanti Gomes Ferreira
Email: ferreira@fgv.br - ☎ (021) 536-9250
ABSTRACT. In this paper we construct sunspot equilibria that arise from chaotic deterministic dynamics. These equilibria are robust and therefore observables. We prove that they may be learned by a simple rule based on the histograms of past state variables. This work gives the theoretical justification of deterministic models that might compete with stochastic models to explain real data.

1. Introduction

In this paper we show the existence of stationary sunspot equilibrium (SSE) when the backward policy function has a complex dynamics. Models with this complex behavior were studied by Azariadis (1981, 1986), Grandmont (1985, 1986), Matsuyama (1991) and more recently by de Vilder (1996) and Christiano and Harrison (1996). Here we use these complex structure to construct stochastic equilibria based on sunspot observations. One nice feature of our construction with respect to the others in the literature (see Chiappori and Guesnerie (1991)) is that we get the existence of an invariant measure for the equilibrium Markov process which is absolutely continuous with respect to the Lebesgue measure. This allows us to construct a simple learning rule based on the histogram of past observations which will converge to this SSE.

Another type of learning rule for SSE was proposed by Woodford (1990); however that learning rule is not robust in the sense that it would not be empirically observable since the support of the sunspot equilibrium studied by him is finite and therefore with Lebesgue measure zero. Also, these kind of sunspot equilibria are local equilibria (around the steady state) and therefore they can not explain global phenomena. Moreover this type of linear
procedure (see Ljung and Soderstrom (1983)) describe local behavior which can not be used for explaining "large" oscillations around the steady state. In this work we show that the learning rule we use is robust (in the sense mentioned above) and it converges globally to the rational expectation equilibrium given by the sunspots.

For obtaining this result, it is sufficient that the backward policy function presents a behavior as in the case of a unimodal map with negative Schwarzian derivative (see theorem 3.3). As an example we apply this method to an overlapping generations model and we show explicitly the SSE. In this case we determine the values of the parameters where this happens and compute numerically the histograms corresponding to the invariant measure. The main mathematical technique we use is differentiable ergodic theory.

Results in this vein have been in some sense anticipated by Duffie, Geanakoplos, Mas-Collel and McLennan (1994; pp 755-756). We remark that the sunspot equilibria obtained in this way are not just randomization of deterministic equilibria paths.

As shown by Hommes, van Strien and de Vilder (1994), Medio (1992) and de Vilder (1996) complex dynamics can be used to mimic stochastic economic models. Also, Christiano and Harrison (1996) use sunspot models to explain U.S. economic macro data. Our work can be seen as a step in the direction of giving the theoretical foundation for the construction of complex deterministic models which has stochastic characteristics.

We also prove here the stochastic stability of the intertemporal stationary equilibrium when it is indeterminate. It means that any small perturbation of the stationary equilibrium is also stationary and it is not far from the indeterminate equilibrium.

More formally, in section 2 we present the general framework and the main theorem. In section 3 we show OLG models with this type of sunspot equilibrium. In section 4 we give an adaptative learning process which converges to this kind of equilibria. In section 5 we prove the stochastic stability of the (strongly) indeterminate equilibrium and we conclude that in the linear case it can be interpreted as a SSE. Section 6 is left for the conclusions. Appendix is devoted to review some results of the one-dimensional ergodic theory which we use in sections 3 and 4.
2. General Framework, Examples and the Main Theorem

Let \( X \subseteq \mathbb{R}^n \), borelian be the state variable set. Let \( \mathcal{B}(X) \) denote the borelians of \( X \). The equilibrium condition of our model is represented by the zeros of the function:

\[
\tilde{Z} : X \times \mathcal{P}(X) \to \mathbb{R}^n,
\]

where \( \mathcal{P}(X) = \{ \mu : B(X) \to [0, 1] / \mu \text{ probability measure} \} \). We will call this map the **stochastic excess demand function** because in some models \( \tilde{Z}(x_0, \mu) \) will be the excess demand when the present state variable is \( x_0 \) and the probability measure for the future state variable is \( \mu \).

**Def.2.1.-** The **deterministic excess demand function** is:

\[
Z : X \times X \to \mathbb{R}^n
\]

defined by \( Z(x_0, x_1) = \tilde{Z}(x_0, \delta_{x_1}) \).

In models where we admit representative agent, the rationalizing measures form a convex set, for these cases we have the next

**Def.2.2.-** The stochastic excess demand function \( \tilde{Z} \) has the **convex valuedness of rationalizing measures (CVR) property** if \( \forall x \in X, \forall \mu_1, \mu_2 \in \mathcal{P}(X) \) such that \( \tilde{Z}(x, \mu_1) = \tilde{Z}(x, \mu_2) = 0 \) and \( \forall \alpha \in [0, 1] \) we have \( \tilde{Z}(x, \alpha \mu_1 + (1 - \alpha) \mu_2) = 0 \).

Let us remember that a transition function defined on \( X \) is a function \( Q : X \times \mathcal{B}(X) \to [0, 1] \), such that: i) \( \forall x \in X \ Q(x, \cdot) \in \mathcal{P}(X) \), and ii) \( \forall A \in \mathcal{B}(X) \ Q(\cdot, A) \) is measurable.

**Def.2.3.-** A **sunspot equilibrium (SE)**, is a pair \((X_0, Q), X_0 \subseteq X, Q \text{ a transition function on } X_0, \text{ such that:} \)

i) \( \exists x_0 \in X_0 \text{ such that } Q(x_0, \cdot) \text{ is not a Dirac measure (truly stochastic).} \)

ii) \( \forall x \in X_0 \tilde{Z}(x, Q(x, \cdot)) = 0. \)

We are following the Chiappori and Guesnerie (1991) structure. They do the comparison between this definition and the standard version of the sunspot equilibrium concept. Woodford (1986a) presents another form for the sunspot equilibrium since his excess demand depends on “theories” for the state variables.
Def. 2.4.- A sunspot equilibrium \((X_0, Q)\) is stationary (SSE) if there exists \(\mu \in \mathcal{P}(X)\) with support equal to \(X_0\) such that

\[
\mu(A) = \int_{X_0} Q(x, A) \mu(dx) \quad \forall A \in \mathcal{B}(X_0).
\]

The definition above says that the stochastic process generated by the measure \(\mu\) and the transition function \(Q\) is a stationary Markov process.

Def. 2.5.- A backward perfect foresight (bpf) map is a function \(\phi : X \to X\), such that:

\[Z(\phi(x), x) = 0, \quad \forall x \in X.\]

It is easy to see that if \((x_t)_{t \geq 0}\) is a sequence such that \(x_t = \phi(x_{t+1})\) \(\forall t \geq 0\) then it is a perfect foresight equilibrium.

Examples:

For showing how the SSE can be constructed when the backward policy is complex let us consider the simple overlapping generations (OLG) model where the technology is linear and in each period two kind of agents (young and old) coexist in the same proportion. Each young agent supplies "\(y\)" units of labor to produce the unique perishable good in the same quantity (since the technology is linear) and each old agent consumes "\(c\)" units of the good. The agents hold "\(M\)" units of fiat money and the stock of money in circulation is constant for all periods. The utility function is represented by \(U(c, y)\), where \(y\) is the labor supply when the agent is young and \(c\) is the consumption when the agent is old. If \((p_t)_{t \geq 0}\) is a sequence of prices for the good in this economy, each agent will solve:

\[
\max U(c_{t+1}, y_t)
\]
such that:

\[p_{t+1} c_{t+1} = p_t y_t.\]

Now if in period \(t\) the next period price is a random variable, the agent's problem will be:

\[
\max E[U\left(\frac{p_t}{p_{t+1}}, y_t, y_t\right)]
\]
where the expected value is taken with respect to $\mu_{t+1}$, the probability measure of $p_{t+1}$. In this case the first order condition and the equilibrium equation $\frac{M}{p_t} = y_t$ give us the following excess demand function:

$$\hat{Z}(p_t, \mu_{t+1}) = E_{\mu_{t+1}}[U_c(M/p_{t+1}, M/p_t) + U_y(M/p_{t+1}, M/p_t)].$$

Let us consider the following cases:

**Example 1:** Suppose that the labor supply $y \in X = [0, 1]$ and the preferences of the agents are given by $U(c, y) = u(c) - y$ where:

$$u(c) = \begin{cases} 
2c, & \text{for } 0 \leq c \leq 1/2 \\
2 \ln c - 2c + 2 \ln 2 + 2, & \text{for } 1/2 \leq c \leq 1.
\end{cases}$$

Then the solution of the agent's problem with perfect foresight gives us the following dynamical system:

$$y_t = \begin{cases} 
2y_{t+1}, & \text{for } 0 < y_{t+1} \leq 1/2 \\
2 - 2y_{t+1}, & \text{for } 1/2 \leq y_{t+1} \leq 1,
\end{cases}$$
or $y_t = \phi(y_{t+1})$ where $\phi$ is the tent map.

**Example 2:** Again the labor supply $y \in X = [0, 1]$ and the agents have the utility function: $U(c, y) = 4c - 2c^2 - y$, defined on $[0, 1] \times [0, 1]$. From the first order conditions and the equilibrium equation we obtain the following dynamical system:

$$y_t = 4y_{t+1}(1 - y_{t+1}),$$
or $y_t = \phi(y_{t+1})$, where $\phi$ is the logistic map.

For both examples it is well known that $\phi$ admits an invariant probability measure which is ergodic and absolutely continuous with respect to the Lebesgue measure, i.e. $\exists \mu \in \mathcal{P}(X), \mu << \lambda$ such that $\mu(\phi^{-1}(A)) = \mu(A) \forall A \in \mathcal{B}(X)$ and if $A \in \mathcal{B}(X)$ is such that $\phi^{-1}(A) = A$ then $\mu(A) = 0$ or $\mu(A) = 1$.

Let us consider $f : [0, 1) \to [0, 1/2)$ and $g : [0, 1] \to [1/2, 1]$ local inverse functions of $\phi$. Since $Z(x, f(x)) = Z(x, g(x)) = 0 \forall x$ and the CVR property holds it is easy to see that:

$$Q(x, \cdot) = \frac{1}{2} \delta_{f(x)}(\cdot) + \frac{1}{2} \delta_{g(x)}(\cdot) \forall x \in \text{Supp}(\mu).$$
is a SE. Also we can prove that $\int Q(x,A)\mu(dx) = \mu(A)$ for all $A \in B([0,1])$ then the SE is stationary.

We can think in more general dynamics than the tent or logistic maps where we can make analogous constructions of SSE, for example as in fig.1.

**Fig.1**

More precisely, when the bpf map $\phi : I \rightarrow I$ (where $I = [0,a]$) is a unimodal map with $\phi(0) = 0$ and $\exists \mu \in \mathcal{P}[0,1], \mu << \lambda$ such that $\mu$ is $\phi$-invariant. Let us consider $I_1 (I_2)$ the interval where $\phi$ is strictly increasing (decreasing) and the maps $f : I \rightarrow I_1$ and $g : I \rightarrow I_2$ the local inverses of $\phi$. If $C = \text{supp}(\mu)$ then we have that:

$$Q(x,A) = \frac{d\mu \circ f}{d\mu}(x)\delta_f(x)(A) + \frac{d\mu \circ g}{d\mu}(x)\delta_g(x)(A), \text{ for } x \in C$$

is a SSE. The proof of this is a consequence of theorem 2.6

**Main Theorem:**

**Theorem 2.6.** Let $\phi : K \rightarrow K (K \subset \mathbb{R}^n_+)$ be the bpf function associated to the stochastic excess demand function $\tilde{Z}$ which has the CVR property, if:

(i) There exists a partition $(A_i)_{i=1}^N$ of $K$ with non-empty interior and $\phi : A_i \rightarrow \phi(A_i)$ is a diffeomorphism $\forall i = 1, \ldots, N$.

(ii) There exists $\mu \in \mathcal{P}(K), \mu << \lambda$ and $\phi$-invariant.

Then there exists SSE with invariant measure $\mu$.

**Proof.** Let us introduce some notations: $\varphi_i = (\phi|_{A_i})^{-1}, \mu^i = \mu \circ \varphi_i$ and all sums are from $i = 1$ to $N$.

**First step:** $\mu^i << \mu$ and $\sum(d\mu^i/d\mu)(x) = 1$ for all $x \mu - a.e.$

Let $C$ be a borelian in $K$ such that $\mu(C) = 0$ then $\mu(\phi^{-1}(C)) = 0$.

But $\phi^{-1}(C) = \bigcup_{i=1}^N \varphi_i(C)$ then $\mu(\varphi_i(C)) = \mu^i(C) = 0$.

Now, for all $A \in B(K)$ we have $\mu(A) = \sum \mu^i(A)$ and:

$$\mu^i(A) = \int_A \frac{d\mu^i}{d\mu}(x) \mu(dx)$$
\[ \mu(A) = \int_A \sum \frac{d\mu^i}{d\mu}(x) \mu(dx) \]

denote \( \sum (d\mu^i/d\mu)(x) = 1 \) for all \( x \) \( \mu \)-a.e. (i.e. there exists \( C \in \mathcal{B}(K) \) with \( \mu(C) = 1 \) such that the last equation holds for all \( x \in C \).

**Second step:** \( Q(x,.) = \sum \frac{d\mu^i}{d\mu}(x) \delta_{\varphi_i(x)}(.) \) is a SSE on the set \( C \).

Because \( \phi \) is the bpf function and \( \varphi_i \) is a local inverse of \( \phi \) then \( \tilde{Z}(x, \delta_{\varphi_i(x)}) = 0 \) for all \( i \) and for all \( x \in \phi(K) \), so \( \tilde{Z}(x, Q(x,.)) = 0 \) for all \( x \in C \) because \( \tilde{Z} \) has the CVR property.

Now let us prove that for almost every \( x \), \( \frac{d\mu^i}{d\mu}(x) > 0 \) \( \forall i \). For proving this it is sufficient that \( \mu << \mu^i \) \( \forall i \), because by the first step these measures will be equivalent. Fix \( i \) and let \( A \in \mathcal{B}(K) \) such that \( \mu^i(A) = 0 \) then \( \mu(\varphi_i(A)) = 0 \) and therefore \( \lambda(\varphi_i(A)) = 0 \) because \( \mu \) is equivalent to \( \lambda \) restricted to the support of \( \mu \) (which is \( \phi \)-invariant) and we can consider \( A \subset \text{supp}(\mu) \). By (i)

\[ \lambda(\varphi_i(A)) = \int_A |\text{det}(\varphi_i'(x))| \lambda(dx), \]

hence \( \lambda(A) = 0 \). Then \( \lambda(\varphi_j(A)) = 0 \) \( \forall j \), so by (ii) \( \mu(A) = \sum \mu^i(A) = 0 \), therefore \( \mu << \mu^i \).

Finally, for proving the stationarity let us calculate:

\[
\int_K Q(x, A) \mu(dx) = \int_K \sum \frac{d\mu^i}{d\mu}(x) \delta_{\varphi_i(x)}(A) \mu(dx)
\]

\[
= \sum \int_K 1_{\varphi_i^{-1}(A)}(x) \mu^i(dx) = \sum \mu^i(\varphi_i^{-1}(A))
\]

\[
= \sum \mu(\varphi_i(\varphi_i^{-1}(A))) = \sum \mu(A \cap A_i) = \mu(A).\]

**Remarks:** If the measure \( \mu \) is ergodic, we have at least two important results. The Birkhoff theorem holds:

\[ \forall F \in \mathcal{L}^1(\mu), \frac{1}{N} \sum_{n=0}^{N-1} F(\phi^n(x)) \xrightarrow{N \to \infty} \int_X F(x) \mu(dx), \mu - a.e. x \]

in particular, since \( X \) has finite measure:

\[ \nu_N = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{\phi^n(x)} \xrightarrow{N \to \infty} \mu, \mu - a.e. x, \]
(the convergence is in the weak topology). Therefore, \( \mu - \text{a.e.} \) we have:

i) If we know a backward trajectory \( x_t = x, x_{t-1} = \phi(x), x_{t-2} = \phi^2(x), \ldots \), we may estimate the expected value of any \( F \in L^1(\mu) \) with respect to the stationary measure \( \mu \) as:

\[
\frac{F(x_t) + F(x_{t-1}) + \ldots + F(x_{t-N+1})}{N}
\]

in this way, we may estimate the expected value, variance, and any moment of the random variable associated to \( \mu \).

ii) If we know the map \( \phi \), the histograms corresponding to the measures \( \nu_N \) (the empirical measures) will approximate the density (Radon-Nikodym derivative) of \( \mu \) with respect to the Lebesgue measure (when \( \mu << \lambda \)). These histograms are constructed by taking partitions \( (I_i)_{0 \leq i \leq n} \) and making the approximation of \( \mu(I_i) \) by:

\[
\frac{\{ j; \phi^j(x) \in I_i, 0 \leq j \leq N \}}{N+1}
\]

In section 4 we will see how these properties can be used for learning these types of SSE.

3. Application to the OLG Models

In this section we will apply the results given above for constructing stationary sunspot equilibria with support having positive Lebesgue measure. We will consider the overlapping generations model with money transfer, subsidies and public expenditures treated in Grandmont (1986). Analogous models were studied by Azariadis and Guesnerie (1986), Benhabib and Nishimura (1985), Grandmont (1985), Woodford (1984) and Matsuyama (1991). The agents live two periods and there exists representative agent with separable utility function \( V_1(c_1) + V_2(c_2) \) where \( c_t \) is the consumption of the unique good in period \( t = 1, 2 \). Suppose that one unit of the good is produced with one unit of the unique productive factor (labor). The agent’s endowments at each age \( t = 1, 2 \) are \( l^*_t > 0 \) and \( l^*_t > 0 \).

Assumption 1: For each \( t = 1, 2 \), \( V_t \) is twice continuously differentiable on \((0, +\infty)\), strictly increasing and strictly concave. Furthermore:

\[
\lim_{c \to 0} V'_t(c) = +\infty, \quad \lim_{c \to +\infty} V'_t(c) = 0, \quad \bar{\theta} = \frac{V'_1(l^*_1)}{V'_2(l^*_2)} < 1.
\]
All these hypotheses are usual except the last one which says that autarchy is inefficient (see Grandmont (1985)). Define:

\[ s_t = \frac{M_{t-1}z_t + S_t}{M_{t-1}z_t}, \quad d_t = \frac{M_{t-1}z_t + S_t + G_t}{M_{t-1}z_t}. \]

where \( M_{t-1} > 0 \) represents the money stock at the end of the period \( t - 1 \), \( z_t \) is the money transfer factor (\( z_t - 1 \) is the nominal interest rate), \( S_t \) is the subsidy and \( G_t \) is the amount of money issued when the government purchases (or sells) some quantity of the good. The dynamic of the money supply is given by the equation:

\[ M_t = M_{t-1}z_t + S_t + G_t \quad \text{or} \quad M_t = M_{t-1}z_t d_t, \quad M_0 \text{ given.} \]

\( s_t \) and \( d_t \) are exogenous variables and we suppose \( s_t = s, \ d_t = d \) for all \( t \).

Now, let us analyze the agent’s decision problem. Let \( p_t \) be the price of the good in period 1 and \( p_{t+1} \) the (random) price of the good in period 2. Then the agent must chose consume plans \( c_t \) (deterministic), \( c_{t+1} \) (random) and its money demand \( m \) so as to maximize:

\[ V_1(c_t) + E[V_2(c_{t+1})] \]

with the budget constraints:

\[ p_tc_t + m = p_t l_t^* \]

\[ p_{t+1}c_{t+1} = p_{t+1} l_{t+1}^* + m z_t + S_{t+1}. \]

The first order condition for this problem (when the money demand is positive) is:

\[ \frac{1}{p_t} V'_1(l_t^* - m/p_t) = E[\frac{z_{t+1}}{p_{t+1}} V'_2(l_{t+1}^* + \frac{m z_{t+1} + S_{t+1}}{p_{t+1}})] \] (1)

the monetary equilibrium condition is \( m = M_t \) and if we denote:

\[ x_t = \frac{M_t}{p_t}, \quad v_1(x) = x V'_1(l_t^* - x), \quad v_2(x) = x V'_2(l_{t+1}^* + x), \]

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we will have the equation (1) equivalent to:

\[ v_1(x_t) = E[s^{-1}v_2(sd^{-1}x_{t+1})] \]

(we have used \( M_t x_{t+1} = d^{-1}M_{t+1} \) and therefore \( M_t x_{t+1} + S_{t+1} = sd^{-1}M_{t+1} \)).

Remember that the expected value is taking with respect to the probability measure of \( x_{t+1} \) (or \( p_{t+1} \)) which is the agent’s expectations of the future prices. In this case, the excess demand function is:

\[ \hat{Z}(x, \mu) = v_1(x) - E_\mu[s^{-1}v_2(sd^{-1}x')] \]

and it has the (CVR) property given in definition 2.2. The backward perfect foresight map is \( \phi(x) = v_1^{-1}(s^{-1}v_2(sd^{-1}x)) \), since \( v_1 \) is strictly increasing by A1. In the following, we will see some hypotheses which guarantee that \( \phi \in \mathcal{F} \) (see Appendix).

**Assumption 2:** \( R_2 \) (The Arrow-Pratt relative degree of risk aversion of the old agent) is a nondecreasing function.

This hypothesis is justified in Arrow (1970) ch.3.

**Lemma 3.1.** If A1 and A2 hold and there exists \( x_0 > 0 \) such that \( R_2(x_0) > 1 \) then \( \phi \) is a unimodal map with \( \phi(0) = 0 \) and \( \phi'(0) = (d\theta)^{-1} \).

**Proof.** Note that in this case \( \phi : [0, +\infty) \to [0, l^*_1) \) because by A1 \( u_1 : [0, l^*_1) \to (0, +\infty) \). It is easy to see that \( \phi(0) = 0 \) and:

\[ v_1'(\phi(x))\phi'(x) = d^{-1}v'_2(sd^{-1}x) \]  \hspace{1cm} (2)

therefore \( \phi'(0) = (d\theta)^{-1} \). From (2) we can observe that every critical point of \( v_2 \) is critical point of \( \phi \), then (putting \( y = sd^{-1}x \)) we need to find \( y^* > 0 \) such that:

\[ v'_2(y^*) = V'_2(l^*_2 + y^*) + y^*V''_2(l^*_2 + y^*) = 0 \]

or what is the same:

\[ \frac{l^*_2 + y^*}{y^*} = R_2(l^*_2 + y^*) \]  \hspace{1cm} (3)

but the righthandside of (3) is a nondecreasing function of \( y^* \) and the lefthandside is an strictly decreasing function of \( y^* \) which tends to 1 when \( y^* \to +\infty \); by hypothesis there
exists $x_0 > 0$ such that $R_2(x_0) > 1$ then there exists a unique $y^*$ which satisfies (3).

Furthermore, from (2):

$$v_2'(y) = V_2''(l_2^2 + y)[1 - \frac{y}{l_2^2 + y}R_2(l_2^2 + y)],$$

since $V_2'' < 0$, the term in brackets is strictly decreasing and it vanishes in $y^*$ then it is a local (in fact global) maximum. Finally, from (2):

$$v_1''(\phi(x))(\phi'(x))^2 + v_1'(\phi(x))\phi''(x) = sd^{-2}v_2''(sd^{-1}x)$$

replacing $x = x^* = s^{-1}dy^*$ results $\phi''(x) < 0$, therefore $\phi$ is unimodal map.\[\]

Figure 2 shows the backward policy $\phi$ under the hypotheses of lemma 3.1

**Remarks:**

1) If we put $s = d = 1$ (laissez-faire case) we obtain that $x = 0$ is repelling ($\phi'(0) > 1$). In general, if $d \in (0, \bar{d}^{-1})$ we will have the same result.

2) Note that if $V_t \in C^\infty, t = 1, 2$, results that $x^*$ is a non-flat critical point of $\phi$ (because $\phi''(x^*) < 0$).

The following results use notations given in the appendix.

**Lemma 3.2.** If $A1$ and $A2$ hold, $V_t \in C^\infty$ for $t = 1, 2, d \in (0, \bar{d}^{-1}), Sv_1 \geq 0$ in $[0, l_1^*], Sv_2 < 0$ in $[0, \phi(x^*))$ and $\sup_x R_2(x) > 1$ then $\phi \in \mathcal{F}$.

In particular, if $V_t(c) = c^{1-\alpha_t}/(1-\alpha_t); t = 1, 2$ and $\alpha_1 \in (0, 1], \alpha_2 \in [2, +\infty)$ then $\phi \in \mathcal{F}$.

**Proof.** From the hypotheses, there exists $x_0 > 0$ such that $R_2(x_0) > 1$, then by lemma 3.1 and remarks above we have that $\phi$ is a unimodal map with $\phi(0) = 0$, $x = 0$ is repelling (because $\phi'(0) = (d\bar{d})^{-1}$) and the critical point $x^*$ is non-flat.

Since $v_1 \circ \phi = s^{-1}v_2 \circ (sd^{-1})$, we obtain from properties of Schwarzian derivative:

$$(Sv_1 \circ \phi)(\phi')^2 + S\phi = (Sv_2 \circ (sd^{-1}))(sd^{-1})^2,$$

hence $S\phi < 0$ and finally $\phi \in \mathcal{F}$.

The second part of the lemma is straightforward from the calculus of the Schwarzian derivatives of $v_1$ and $v_2$ for $V_1$ and $V_2$ given. \[\]
Finally for knowing if there exists an invariant B-R-S measure (see appendix for definition of B-R-S measure) we have to test the conditions given in theorems A.7 and A.8. The following theorem shows how it can be done.

**Theorem 3.3.** If the hypotheses of lemma 3.2 hold and:

1) \( \lambda_\phi = \lim \sup \frac{1}{n} |D\phi^n(x)| > 0, \lambda - a.e \) then there exists SSE whose invariant measure is an absolutely continuous B-R-S measure.

2) \( S = \sum_{n=0}^{\infty} |D\phi^n(x)|^{-1/2} < \infty \) then there exists SSE whose invariant measure is an absolutely continuous ergodic measure.

The proof of this theorem is easy because the existence of an invariant absolutely continuous measure for \( \phi \) implies the existence of SSE (theorem 2.6). Part 1) is application of theorem A.7 and part 2) results from theorem A.8 using that \( x^* \) is a critical point of order 2 \( (D^2\phi(x^*) \neq 0) \).

The table below gives some estimates of \( \lambda_\phi \) for some values of \( \alpha_1 \) and \( \alpha_2 \) (relative degrees of risk aversion).

4. Learning Process

In this section we will prove that a simple rule could be used for learning this kind of sunspot equilibrium. Woodford (1990) proposed an adaptative learning process for the local sunspot equilibrium he worked. Here we will see that an adaptative rule of histograms allows the convergence to the rational expectations equilibrium given by the sunspots.

In the previous section we saw that if \( \phi \) belongs to \( F \) and there exists an invariant measure \( \mu \) which is absolutely continuous then for all \( x \) \( \lambda - a.s. \) the measures \( \{\mu_N\}_{N \geq 1} \) converges weakly to \( \mu \). Since \( \mu_N = \frac{1}{N} \sum_{i=0}^{N-1} \delta_{\phi^i(x)} \), it is easy to see that:

\[
\mu_N \circ f = \frac{1}{N} \sum_{i=0}^{N-1} 1_{I_i}(\phi^i(x)) \delta_{\phi^{i+1}(x)}
\]

here \( I_1 (I_2) \) is the interval where \( \phi \) is increasing (decreasing) and \( f(g) \) is the inverse of \( \phi \) in \( I_1 (I_2) \).

Let us consider the following sequence of regular partitions: \( (\pi_n)_{n \geq 1} \) where \( \pi_n = \{t_0 = 0, t_1 = \frac{1}{2^n}, \ldots t_{2^n} = 1\} \). If we put \( J^n_k = [\frac{k}{2^n}, \frac{k+1}{2^n}] \) then \( \mu_N(J^n_k) \) is the frequency
of the $N$-orbit \{\(x, \varphi(x), \ldots \varphi^{N-1}(x)\)\} in the interval \(J_k^N\) and \(\mu_N \circ f(J_k^N)\) is the frequency of the points in the same $N$-orbit which are in \(J_k^N\) and by the perfect foresight policy are carried to \(I_1\). The learning rule is the following: If \(x_t = z\) is observed in period \(t\) the next period could happen \(f(z)\) ("pesimistic" prevision) or \(g(z)\) ("optimistic" prevision). The chance of the pesimistic prevision is:

\[
\frac{\mu_N \circ f(J_k^N)}{\mu_N(J_k^N)} \quad \text{if } f(z) \in J_k^N \quad \text{and} \quad \mu_N(J_k^N) \neq 0
\]

and zero in other cases. When the number of observations \((N)\) is large we can approximate this chance (using the ergodic theorem) by:

\[
c_k^n = \frac{\mu \circ f(J_k^N)}{\mu(J_k^N)}.
\]

The histogram associated to the \(\pi_n\) partition is \(R_n(y) = \sum_{k=1}^{2^n} c_k^n \chi_k^n(y)\) for all \(y\) in \(I\). Therefore the learning rule is defined by:

\[
Q^n(z, \cdot) = R_n(z)\delta_{f(z)}(\cdot) + (1 - R_n(z))\delta_{g(z)}(\cdot).
\]

(*)

The following theorem shows that this rule converges to the stationary sunspot equilibrium.

**Theorem 4.1.** If \(\phi\) is a unimodal map, \(\mu\) is a \(\phi\)-invariant measure, \(f\) and \(g\) are the local inverses of \(\phi\) and \(R_n\) is defined by (*) then for each \(A \in \mathcal{B}(X)\) \((Q^n(A, A))_{n \geq 1}\) converges to \(Q(A, A)\) \(\mu\)-a.s. and in \(\mathcal{L}^1(\mu)\).

**Proof.** It is sufficient to prove that \((R_n)_{n \geq 1}\) converges to \(R = \frac{d\mu \circ f}{d\mu}\). Let \(\mathcal{F}_n\) the \(\sigma\)-algebra generated by \(\pi_n\). Then \(\mathcal{F}_n \subset \mathcal{F}_{n+1}\ \forall n \geq 1\) and it is easy to see that \(\mathcal{B}(X) = \mathcal{F}_\infty = \sigma(\bigcup_{n \geq 1} \mathcal{F}_n)\).

We have that \(R_n = E[R|\mathcal{F}_n]\). To see this, let \(B \in \mathcal{F}_n\) \((B = \cup J_k^n)\) then:

\[
\int_B R_n \, d\mu = \sum_{k=1}^{2^n} c_k^n \int_B 1_{J_k^n}(z) \, d\mu(z) = \sum_{k=1}^{2^n} c_k^n \mu(J_k^n \cap B)
\]
\[
= \sum_i c_i^n \mu(J_i^n) = \sum_i \mu \circ f(J_i^n)
\]
\[
= \int_B \frac{d\mu \circ f}{d\mu} d\mu = \int_B R(z) d\mu.
\]

Since \( R \in L^1(\mu) \) by Lévy's "upward" theorem (see e.g. Williams (1991)) we have that \( R_n \rightarrow E[R|B(X)] = R \) almost surely and in \( L^1(\mu) \).

The histograms above were constructed from the deterministic past observations of the state variable. However, if the measure \( \mu \) is \( Q \)-ergodic then the histograms provided by the stochastic observations will converge to the histogram constructed throughout the measure \( \mu \), namely: If \( \tilde{x}_t \) is the Markov process generated by \( (Q, \mu) \) the Birkoff theorem in the stochastic version gives us:

\[
\frac{1}{N+1} \sum_{t=0}^N \delta_{\tilde{x}_t}(\cdot) \rightarrow \mu(\cdot) \text{ \( \mu \) - a.e.}
\]

in the weak topology (see Kifer (1986)).

**Theorem 4.2.** With The hypothesis of theorem 4.1, if \( \mu \) is absolutely continuous then \( \mu \) is \( Q \)-ergodic.

**Proof.** Since \( \mu \) is \( Q \)-invariant, it is sufficient to prove that if \( A \in \mathcal{B}(X) \) is such that \( Q(x,A) = 1_A(x) \) for \( x \mu \) - a.e. then \( \mu(A) = 0 \) or \( \mu(A) = 1 \). Let \( \alpha(x) = \frac{d\mu \circ f}{d\mu}(x) \); by theorem 2.6 \( \alpha(x) \in (0,1) \). By hypothesis:

\[
\alpha(x)\delta_{f(x)}(A) + (1-\alpha(x))\delta_{g(x)}(A) = 1_A(x) \quad x \mu \text{ - a.e.}
\]

If \( x \notin A \) then \( f(x) \notin A \) and \( g(x) \notin A \) then \( x \notin f^{-1}(A) \cup g^{-1}(A) \); so \( \phi(A) \subset A \).

If \( x \in A \) then \( f(x) \in A \) and \( g(x) \in A \) then \( x \in f^{-1}(A) \cap g^{-1}(A) \); so

\( A \subset \phi(A \cap I_1) \cap \phi(A \cap I_2) \).

Therefore \( \phi(A) \subset A \subset \phi(A \cap I_1) \cap \phi(A \cap I_2) \); from this it is easy to conclude \( A = \phi^{-1}(A) \), i.e. \( A \) is \( \phi \)-invariant, then \( \mu(A) = 0 \) or 1 because \( \mu \) is \( \phi \)-ergodic.

By theorem 4.2 results that the learning rule could be made from the histograms constructed with the stochastic observations. Note that the support of this learning rule (the
support of the invariant measure) has positive Lebesgue measure. This fact shows this learning process to be a robust rule of learning in contrast with the Woodford's rule which has finite support.

5. Small Random Perturbations of Indeterminate Equilibria

In this section we analyze the effect of small random perturbations of indeterminate equilibria. OLG models with these types of equilibria were studied by Azariadis (1981), Azariádis and Guesnerie (1982), Grandmont (1986), Farmer and Woodford (1984) and Woodford (1986a-b). They proved the connection of these equilibria with the existence of cycles and sunspot equilibria.

Our goal in this section is to prove the stationarity and stochastic stability of small random perturbations of deterministic equilibria, it means that stochastic disequilibria which are small enough are stationaries and they are not far from the deterministic equilibrium. We will consider the same class of excess demand function as defined in section 2 and we will suppose that $Z$ is $C^1$ in both arguments.

Def. 5.1.- $\bar{x} \in X$ is called stationary state if $Z(\bar{x}, \bar{x}) = 0$.

It means that the sequence $(x_t)_{t \geq 0}$, $x_t = \bar{x}$, $\forall t$ is a deterministic equilibrium (with perfect foresight). However, we may have random perturbations in each period which could place it in the instability region. Proposition 5.4 below shows that it does not occur when the equilibrium is (strongly) indeterminate. In one-step models the concept of indeterminate steady state is related to the existence of an infinite set of perfect foresight equilibria close to this steady state. Indeterminacy means that there exists at least one eigenvalue of $(\partial_1 Z(\bar{x}, \bar{x}))^{-1} \partial_0 Z(\bar{x}, \bar{x})$ inside the unit disk. We will prove our results in the case of all eigenvalues are inside the unit disk.

Def. 5.2.- The steady state $\bar{x}$ is called strongly indeterminate if every eigenvalue of the matrix $M = (\partial_1 Z(\bar{x}, \bar{x}))^{-1} \partial_0 Z(\bar{x}, \bar{x})$ is inside the unit disk.

In definition 5.2 is implicit the existence of $(\partial_1 Z(\bar{x}, \bar{x}))^{-1}$. So we can define the forward perfect foresight (fpf) map $\varphi$ as the unique function such that $Z(x, \varphi(x)) = 0$ for all $x$ in some neighborhood of $\bar{x}$. The following lemma is easy to prove.

Lemma 5.3. If $\bar{x}$ is strongly indeterminate steady state then:
i) The fpf map is a local contraction with fixed point $\tilde{z}$.

ii) There exists a compact set $K$ (contraction domain of $\varphi$) with $\tilde{z}$ in its interior such that if $\mu \in \mathcal{P}(K)$ is $\varphi$-invariant then $\mu = \delta_{\tilde{z}}$.

Now, let us consider random perturbations of the deterministic path $x_{t+1} = \varphi(x_t)$. Let $(\tilde{\xi}_t)_{t \geq 1}$ be a sequence of i.i.d. random variables with support in $B(0, \varepsilon)$. Given $x_0 \in K$, we have the perturbed path $\tilde{x}_1 = \varphi(x_0) + \tilde{\xi}_1$, $\tilde{x}_2 = \varphi(\tilde{x}_1) + \tilde{\xi}_2$, ...; this random process could be interpreted as a random error in the prevision of the next state given the present state. Note that this process is not necessarily an equilibrium because $\tilde{Z}(z_t, \text{Prob}[\tilde{x}_{t+1} | \tilde{x}_t = x_t])$ in general is not equal to zero, although we will show that it is not far from the steady state equilibrium.

Let $\psi_\varepsilon$ be the density of $\varepsilon$, then:

$$\psi_\varepsilon \geq 0, \; \text{supp}(\psi_\varepsilon) \subset B(0, \varepsilon), \; \int_{B(0, \varepsilon)} \psi_\varepsilon(z) \, dz = 1.$$ 

Also we will suppose that $\psi_\varepsilon$ is bounded from below by $\rho > 0$ (e.g. truncated Gaussian or uniform distribution). So, we have defined the following transition function:

$$Q^\varepsilon(x, A) = \int_A \psi_\varepsilon(y - \varphi(x)) \, dy,$$

defined in $K \times B(K)$. This transition function must be interpreted as the probability distribution of the next (perturbated) state given the present state. At this point we can note the importance of the steady state being strongly indeterminate because if there exists an eigenvalue with modulus greater than one these perturbations will lead the dynamic to the instability region and it will not have an stationary support (unless that the perturbations are in the stable manifold; in such a case the errors lose generality). An invariant measure for this transition function is any $m \in \mathcal{P}(K)$ such that for all $A \in \mathcal{P}(K)$:

$$m(A) = \int_K Q^\varepsilon(z, A) \, m(dz).$$
Proposition 5.4. There exists a unique invariant measure $m^\epsilon$ for $Q^\epsilon$.

Proof. We will apply the theorem 11.12 of Stokey and Lucas (1989); then it is sufficient to verify the $M$ condition: There exists $\delta > 0$ such that for all $A \in B(K)$ either $Q^\epsilon(x, A) \geq \delta$ for all $x \in K$ or $Q^\epsilon(x, A^c) \geq \delta$ for all $x \in K$.

Let us suppose that for all $\delta > 0$ there exists $A \in B(K)$ such that $Q^\epsilon(x', A) < \delta$ and $Q^\epsilon(x'', A^c) < \delta$ for some $x', x'' \in K$. But we know that $Q^\epsilon(x, A) \geq \rho \lambda(A)$ for all $x \in K$ (where $\psi_\epsilon \geq \rho > 0$), then:

$$\rho \lambda(A) < \delta \text{ and } \rho \lambda(A^c) < \delta,$$

therefore $\rho < \delta$, but this is false if we take $\delta < \rho$. ■

It is important to note that such invariant measure results as a fixed point process of the map $T^* : \mathcal{P}(K) \to \mathcal{P}(K)$ defined by $T^*m_0(A) = \int_K Q^\epsilon(x, A)m_0(dx)$, more exactly $T^*m_0 \longrightarrow m^\epsilon$ in the strong topology (in the total variation norm). It means that given any probability measure for $\tilde{\pi}_0$, the process $\tilde{\pi}_1, \tilde{\pi}_2, \ldots$ tends to a random variable with distribution given by $m^\epsilon$.

Finally, we will prove the stochastic stability of this perturbations, namely if the support of the random error is small enough then the perturbated sequence is close to $\tilde{x}$.

Theorem 5.5. If $m^\epsilon$ is the invariant measure of $Q^\epsilon$ then $m^\epsilon \longrightarrow \delta_\epsilon$ (in the weak topology) when $\epsilon \longrightarrow 0$.

Proof. Let us take any sequence $\epsilon_n \longrightarrow 0$, without loss of generality we suppose $m^{\epsilon_n} \longrightarrow m$ (in fact, we will prove that any convergent subsequence of $(m^{\epsilon_n})_{n \geq 0}$ converges to the same limit). We are going to show that $m$ is $\varphi$-invariant, then by lemma 5.3 results $m = \delta_\epsilon$.

First of all note that for any $g \in C(K)$:

$$\left| \int_K g(y)\psi_\epsilon(y - \varphi(x)) \, dy - g(\varphi(x)) \right| \leq \int_{B(0, \epsilon)} |g(z + \varphi(x)) - g(\varphi(x))| \psi_\epsilon(z) \, dz$$

$$\leq \sup_{z \in B(0, \epsilon)} |g(z + \varphi(x)) - g(\varphi(x))|$$

then:

$$\sup_{x \in K} \left| \int_K g(y)\psi_\epsilon(y - \varphi(x)) \, dy - g(\varphi(x)) \right| \leq \sup_{(x, z) \in K \times B(0, \epsilon)} |g(z + \varphi(x)) - g(\varphi(x))|,$$
hence:

$$\limsup_{\varepsilon \to 0} \left| \int_{K} g(y) \psi_{\varepsilon}(y - \varphi(x)) \, dy - g(\varphi(x)) \right| = 0$$

(1)

Now, let us estimate:

$$\left| \int_{K} g(x) \, m(dx) - \int_{K} g(\varphi(x)) \, m(dx) \right| \leq \left| \int_{K} g(x) \, m(dx) - \int_{K} g(x) \, m^\varepsilon(dx) \right|$$

$$\quad + \left| \int_{K} \left( \int_{K} g(y) \, Q^\varepsilon(x, dx) - g(\varphi(x)) \right) \, m^\varepsilon(dx) \right|$$

$$\quad + \left| \int_{K} g(\varphi(x)) \, m^\varepsilon(dx) - \int_{K} g(\varphi(x)) \, m(dx) \right|;$$

here we used: $\int_{K} g(x) \, m(dx) = \int_{K} (\int_{K} g(y) \, Q^\varepsilon(x, dy)) \, m(dx)$. Letting $\varepsilon \to 0$ along the sequence $(\varepsilon_n)_{n \geq 0}$ and using (1) we obtain:

$$\int_{K} g(x) \, m(dx) = \int_{K} g(\varphi(x)) \, m(dx),$$

therefore $m$ is $\varphi$-invariant. ■

Remark: Note that in the linear case $\tilde{Z}(x, \mu) = A x + \int_{X} A^t \mu(\mu')$, the perturbations given here could be taken as extrinsics and the sequence $(\tilde{x}_i)_{i \geq 0}$ will be an SSE with invariant measure $m^\varepsilon$ (Blanchard and Kahn (1980)). Then if the support of the extrinsic noise is small enough, the stochastic equilibrium is close to the deterministic one. In the non-linear case, in general $Q^\varepsilon$ is not an exact equilibrium of the perturbed system.

6. Conclusions

In this work we have exploited chaotic properties that can be present in the deterministic backward perfect foresight path to construct stationary sunspot equilibria with stationary probability measures which are absolutely continuous with respect to the Lebesgue measure which therefore they give zero probability to isolated points. The technique shows that the chaotic properties of the backward policy function can be used to construct sunspot equilibria using some results of the ergodic theory and dynamical systems. Furthermore we prove this sunspot equilibrium can be learned by an adaptative rule based on the
histograms of past state values and "pesimistic" and "optimistic" previsions. We showed that the empirical measure of the backward perfect foresight policy serves to construct the invariant probability measure and we gave conditions for the existence of such a measure. This was made in one-step forward models but we conjecture that this construction can be done in models with memory.

Finally we showed the stochastic stability of the steady state when it is indeterminate which means that small stochastic disequilibria are not far from the deterministic stationary equilibrium.
APPENDIX

Ergodic Theory for Unimodal Maps

Now we will give some definitions and results of the one-dimensional ergodic theory.

Let $I = [0, a]$ be a non-trivial interval.

**Def. A.1.** $f : I \to I$ is called **unimodal** if $f$ has only one interior local extremum and $f(\partial I) \subset \partial I$.

The last condition is not restrictive because any endomorphism of a compact interval can be extended to a bigger interval so that the boundary of the larger interval is mapped into itself.

**Def. A.2.** If $c$ is the local extremum of the unimodal map $f$, we will call it **non-flat** if there exists a $C^2$ local diffeomorphism $h$ such that $h(c) = 0$ and $f(x) = f(c) \pm |h(x)|^\alpha$, for some $\alpha \geq 2$.

For example, if $f$ is $C^\infty$ and some derivative is non-zero at $c$ then $c$ is a non-flat critical point.

**Def. A.3.** Let $f$ a $C^3$ function, the **Schwarzian derivative** of $f$ is:

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2, \text{ if } f'(x) \neq 0.$$

We will consider the following set of functions:

$$\mathcal{F} = \{ f \in C^3(I); \text{ unimodal map with non-flat extremum } c, Sf < 0, f(0) = 0, f'(0) > 1 \}$$

Let $\lambda$ represent the Lebesgue measure. Given a function $f : I \to I$, we define the $\omega$-**limit set** of the orbit of $x \in I$ as:

$$\omega(x) = \{ y \in I; \text{ there exists a subsequence } n_i \to \infty \text{ with } f^{n_i}(x) \to y \}. $$

In other words, it is the set of accumulation points of the sequence $\{f^n(x)\}_{n \geq 0}$. The next theorem characterizes the $\omega$-limit set of Lebesgue almost all points of the interval and it is proved in Blokh-Lyubich (1986-1990).
Theorem A.4. If \( f \in \mathcal{F} \) then there exists a unique set \( A \subset I \) such that \( \omega(x) = A \) \( \lambda \)-a.e. \( x \) and \( A \) either consists of intervals or has Lebesgue measure zero. Furthermore, if \( f \) has an attracting periodic orbit then \( A \) is this periodic orbit.

In the last remark of the section 2 we show the importance of the invariant measure being ergodic, although we will interested in measures which are "visibles", i.e. in measures satisfying the ergodic theorem for all initial value in some set with positive Lebesgue measure. This types of measures are called Bowen-Ruelle-Sinai measures.

Def. A.5.- Let \( f : I \to I \) and \( \mu \) \( f \)-invariant probability measure. We say that \( \mu \) is a Bowen-Ruelle-Sinai (B-R-S) measure if there exists \( B \subset I \) with \( \lambda(B) > 0 \) such that for all \( x \in B \):

\[
\frac{1}{N} \sum_{n=0}^{N-1} \delta_{f^n(x)} \rightharpoonup_{N \to \infty} \mu \quad \text{in the weak topology}
\]

i.e. for all \( F \in C^0(I) : \frac{1}{N} \sum_{n=0}^{N-1} F(f^n(x)) \rightharpoonup_{N \to \infty} \int_I F(x) \mu(dx). \)

The next theorem gives conditions for the existence of B-R-S measures of functions we are working. It is proved in de Melo-van Strien (1993).

Theorem A.6. if \( f \in \mathcal{F} \), then:

1) There is at most one B-R-S measure.

2) If \( \nu \ll \lambda \) and \( \nu \) is \( f \)-invariant then:

i) \( \nu \) is a B-R-S measure for \( f \) and \( \lambda(B) = \lambda(I) \) (\( B \) as in definition A.5).

ii) The unique set \( A \) given in theorem A.4 consists in transitive intervals (\( J \) is a transitive interval if there exists \( N > 0 \) such that \( f^N(J) \cap J \neq \emptyset \)).

iii) The support of \( \nu \) is equal to \( A \) and \( \nu \) is equivalent to \( \lambda|_A \).

The second part of the theorem above says that it is sufficient to find an invariant measure absolutely continuous (with respect to the Lebesgue measure) for obtaining B-R-S measure which is supported in a union of transitive intervals. The following theorems state conditions for existence of absolutely continuous invariant measures.

Keller (1990) proved that there exists an absolutely continuous invariant probability measure if and only if \( f \) has a positive Liapunov exponent in almost every point, and in this way, relating the existence of such measures with the "chaoticity" of \( f \).
Theorem A.7. If $f \in \mathcal{F}$ then there exists $\lambda_f \in \mathbb{R}$ such that for almost every $x$:

$$\lambda_f = \limsup_n \frac{1}{n} \log |Df^n(x)|$$

Furthermore:

1) $\lambda_f > 0 \iff f$ has an absolutely continuous invariant probability measure and:

$$\lambda_f = \lim \frac{1}{n} \log |Df^n(x)|.$$

2) $\lambda_f < 0 \iff f$ has an hyperbolic periodic attractor.

Finally, we will give a result from Nowicki and van Strien (1991). They showed that a much weaker growth rate of $|Df^n(c_1)|$ is sufficient for the existence of invariant measures.

Theorem A.8. If $f \in \mathcal{F}$, $D^1 f(c) \neq 0$ and

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/1} < \infty$$

then $f$ has an absolutely continuous invariant probability measure which is ergodic.

REFERENCES


Est. Dona Castorina 110, Jardim Botânico. Rio de Janeiro. CEP 22460-320
E-mail address: willy@impa.br
Autor: Araújo, Aloisio Pessoa de,
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