Money distribution with intermediation
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Orientador: Ricardo de Oliveira Cavalcanti

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MONEY DISTRIBUTION WITH INTERMEDIATION.

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ASSINATURA DOS MEMBROS DA BANCA EXAMINADORA

Ricardo de Oliveira Cavalcanti
Orientador (a)

Paulo Klinger Monteiro

Jefferson Dónizeti Peraire Bertolai
Abstract

This paper analyzes the distribution of money holdings in a commodity money search-based model with intermediation. Introducing heterogeneity of costs to the Kiyotaki e Wright (1989) model, Cavalcanti e Puzzello (2010) gives rise to a non-degenerated distribution of money. We extend further this model introducing intermediation in the trading process. We show that the distribution of money matters for savings decisions. This gives rises to a fixed point problem for the saving function that difficults finding the optimal solution. Through some examples, we show that this friction shrinks the distribution of money. In contrast to the Cavalcanti e Puzzello (2010) model, the optimal solution may not present the entire surplus going to the consumer. At the end of the paper, we present a strong result, for a sufficient large number of intermediaries the distribution of money is degenerated.

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1 Introduction

This paper analyzes the distribution of money holdings in a commodity money search-based model with intermediation. Introducing heterogeneity of costs to the Kiyotaki e Wright (1989) model, Cavalcanti e Puzzello (2010) gives rise to a non-degenerated distribution of money. We extend further this model introducing intermediation in the trading process.

Intermediation is a phenomenon well established in the real economy. Several trades are made through a chain of intermediaries - e.g., from farmer to broker to distributor to retailer to consumer. This paper provides a monetary search model that supports this trading process and shows that it shapes the money holdings distribution of the economy.

In this paper, intermediation will act as a friction in the economy. Intermediary money holdings will impose an extra upper bound to production precluding some transactions that would occur if the trade was direct between producer and consumer. When the intermediary has less money than the consumer, the production demanded by the latter in an economy with direct trades is not possible to be carried due to this new constraint. Thereby, money holdings dispersion means that intermediation is harmful.

This intermediation friction gives rises to a concern about the money distribution in the economy. Due to the inclusion of this extra restriction, the money return will depends on the money holdings of others. Economies with low levels of money holdings would reduce the marginal benefit of savings. This way, money distribution matters for savings decision.

We use mechanism design to obtain the optimal allocation. The optimal allocation maximizes welfare subject to feasibility, and, trade and savings individual incentives. Besides, in order to have a benchmark, we define a concept of first-best optimality allowing the central planner to force individuals to follow the savings plan. Thereby, the optimal allocation is a second best solution.

We present a first-best optimal allocation with the traditional terms of trade: production is maximum given cash-in-advance constraints and all surplus goes to consumer. In addition, we show that a positive externality of money preclude the implementation of this allocation. Under this trade terms, the liquidity effect of money is a positive externality. Therefore, people would save less than the suggested by the first-best optimal.

In order to take into account the positive implications of the liquidity effect, we also discuss about some distortions in the trade terms that would provide better allocations. The strategies are based in a main feature of economies with intermediation: due to the liquidity effect, it is socially better have one unit of money on the poor hands than on the wealthier hands. Heterogeneity of costs rewards money dispersion, while the liquidity effect punishes

\footnote{See Wong e Wright (2011).}
it. Thus, we distort trade terms in order to increase the low levels of savings.

Finally, we generalize the model to support several intermediaries. Through some numerical examples, we show that as the chain of intermediaries increase, more concentrated is the money holdings distribution. In the limit, for a very large chain, the money distribution degenerates.

2 Basic model with intermediation

This section will describe an extension of the Cavalcanti e Puzzello (2010) model to support the intermediation friction. Instead of direct trading, trades are necessarily made through an intermediary. Now, producers and consumers have no more direct contact.

This new trading process gives rise to an additional trading restriction. In the model without intermediation the upper bound constraint to production is the cash-in-advanced constraint faced by consumer. This time we have an extra restriction, the intermediary cash-in-advance constraint. This additional restriction leads to a concern about liquidity that will shape the money distribution in the economy.

In this economy, the money return depends on the level of trade surplus, which in turn depends on the amount of production. But, in an economy with intermediation, the latter is bounded by the quantity of money from both, intermediary and consumer. This implies that the return of the money, and, consequently, the savings decisions will always depends on the money amount of another individual. So, with intermediation, money holdings distribution matters to savings decision.

For example, an economy with many illiquid individuals will impose a low upper bound to production. Therefore, the marginal benefit of money will decrease rapidly implying that people will not save much.

2.1 Environment

Time is discrete and the horizon is infinite. There is a continuum [0, 1] of agents. There are two types of divisible goods: commodity money and specialized good. Commodity money is durable and anyone can produce or consume it. The net consumption of \( m \) units of this good gives a period utility of \( sm \). In turn, specialized good is perishable, and, production and consumption are constrained. Produce and consume \( y \) units of specialized good gives, respectively, period utility of \( -v(y) \) and \( u(y) \).

At each period, networks are formed. Each network are composed by 3 agents (a producer, a consumer, and, an intermediary) chosen randomly. The producer can produce the
specialized good, and, is linked only to intermediary. The consumer desires consume the specialized good produced by the producer, and, is linked only to intermediary. Finally, the intermediary cannot neither produce nor consume the specialized good, but is linked to both, the producer and the consumer. The probability of an individual be a producer, a consumer or an intermediary of some network is symmetric, and, is \( \pi \). So, the probability of an individual not participate of a network is \( 1 - 3\pi \).

Each period is further divided in two stages: trading stage and saving stage. The first stage, is divided in two substages. In the fist substage networks are formed, and, producer and intermediary trade. In the second substage, intermediary and consumer trade, and, networks are broken. In turn, in the saving stage, agents are hit by an idiosyncratic preference/productivity shock, \( s \), and, in isolation, adjust their money holdings.

The preference/productivity shock \( s \) is taken from a probability measure \( \lambda \) in \( S \subset [\bar{s}, \underline{s}] \subset \mathbb{R} \). And, its expected value is \( \int s \, ds = 1^2 \). The shock is distributed independently and identically across individuals and over time. There is a discount factor \( \beta \), \( \beta < \bar{s} \), between periods. The functions \( u \) and \( v \) are defined on \( \mathbb{R}_+ \) and assumed increasing, continuous and differentiable. In addition, \( u \) is strictly concave, \( v \) is convex, \( u - v \) is bounded from above, and \( u'(0) = +\infty \). Following Cavalcanti e Puzzello (2010), we also choose to normalize units so that \( u(0) = v(0) = 0 \).

There is no commitment and personal histories are private. Agents money holdings are observable but types are not.

### 2.2 Allocation

We restrict our attention to stationary allocations. An allocation is a triple \((f, g, p)\) composed by a saving plan, a production rule and a surplus division rule. A saving plan says how much will be saved by agents in the saving stage and is defined by \( f : S \rightarrow \mathbb{R}_+ \). A production rule and a surplus division rule are defined, respectively, by \( g : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+ \) and \( p = [p_1, p_2] \), \( p_1 : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+ \). The production function says how much will be produced in a meeting. The surplus division rule says how the trade surplus will be divided among agents in a network. It is composed by the payment functions \( p_1 \) and \( p_2 \) where the former says how much money will be transferred by intermediary to producer and the latter says how much money will be transferred by consumer to intermediary. All these functions depends on money holdings of the three agents of the network. Their first, second and third entries are, respectively, the amount of money hold by the producer, by the intermediary and by the consumer. The pair, production rule and surplus division rule, will be called trade plan. An allocation is

\[ ^2 \text{If it’s not, it is just normalize.} \]
said feasible if \(-m_1 \leq p_1(m_1, m_2, m_3) \leq m_2\) and \(-m_2 \leq p_2(m_1, m_2, m_3) \leq m_3\).

### 2.2.1 Implementability

To define which allocations are implementable, we are going to use the concept of strong implementability\(^3\). An allocation is said to be strongly implementable if it is feasible and immune to individual and cooperative defection.\(^4\)

Thereby, in order to be implementable an allocation must respect two incentive constraints: the trade incentives constraints and the savings incentive constraint. The trade incentives constraints are given by

\[
\begin{align*}
-v(g(m_1, m_2, m_3)) + p_1(m_1, m_2, m_3) &\geq 0 \\
p_2(m_1, m_2, m_3) - p_1(m_1, m_2, m_3) &\geq 0 \\
u(g(m_1, m_2, m_3)) - p_2(m_1, m_2, m_3) &\geq 0
\end{align*}
\]

The inequalities say that consumer, intermediary and producer trade surplus must be positive. The savings incentive constraint is given by \(f(s) \in \arg\max_{m'} R(m', s; (f, g, p))\), where

\[
R(m', s; (f, g, p)) = -(s - \beta)m' + \beta \pi T R(m'; (f, g, p))
\]

is the net return of money, and,

\[
TR(m'; (f, g, p)) = \left\{ \int \int [u(g(m_1, m_2, m')) - p_2(m_1, m_2, m')]d\mu_f(m_1)d\mu_f(m_2) + \int \int [p_2(m_1, m', m_3) - p_1(m_1, m', m_3)]d\mu_f(m_1)d\mu_f(m_3) + \int \int [-v(g(m', m_2, m_3)) + p_1(m', m_2, m_3)]d\mu_f(m_2)d\mu_f(m_3) \right\}
\]

is the trade gains of money.\(^5\)

Given this description, we construct three allocations sets: the set of feasible allocation \((F)\), the set of allocations that respect the trade incentives \((I_1)\), and the set of allocations

\(^3\)See Hu et al. (2007).

\(^4\)Once savings decision are taken in isolation, cooperative defection is possible just in the trading stage. This assumption is made to guarantee that all the trade opportunity is explored. However, this assumption doesn’t have much importance to the results achieved and we’re going to omit this constraint in most part of the article.

\(^5\)\(\mu_f\) is the distribution of money implied by the savings rule \(f\) and the shock distribution \(\lambda\).
that respect the savings incentive ($I_2$).

Besides, given the fixed point problem from savings decision we can anticipate a feature of the savings function.

**Lemma 1.** The savings function of an implementable allocation is non-increasing.

*Proof.* See Appendix \[\square\]

### 2.2.2 Welfare

The welfare criteria is the utility of an ex-ante representative agent. Thus,

\[
W(f, g, p) = -\int (s - \beta)f(s)d\lambda(s) + \beta \pi \int (u - v) \circ g(f(s_1), f(s_2), f(s_3)) d\lambda(s_1)d\lambda(s_2)d\lambda(s_3)
\]

Note that the welfare function doesn’t depend on the surplus division rule. So, sometimes we will denote the welfare function just by $W(f, g)$.

### 2.3 First-best

In order to obtain a benchmark allocation, we will allow the central planner to force individuals to follow the suggested saving plan. This way, we expand the allocations set of the maximization problem. In this section, the optimization problem will be carried in the set of allocations that satisfies the factibility and trade incentives constraints but may not respect the savings incentive constraint. The solutions of this problem will be denoted by first-best optimal allocation.

**Definition 1.** *(Optimal first-best)* Let $(f^*, g^*, p^*)$ be solution of

\[
\max_{(f,g,p)} W(f, g)
\]

s.t.

\[(f, g, p) \in F \cap I_1\]

Thus, $(f^*, g^*, p^*)$ is a first-best optimal allocation.

Note that it may exist multiple first-best optimal allocations. The next proposition presents one that we will use as benchmark and characterizes the savings and production functions of any first-best allocation.
First we will define some objects. Let \( y^* = \arg\max_y (u−v)(y); v(y) \leq m_2 \land m_3 \), \( p^*_i(m_1, m_2, m_3) = v(g^*(m_1, m_2, m_3)) \) and \( f^* = \arg\max_{f} W(f, g^*) \).

Also, take \( \xi^*(s) = \frac{s−\beta}{2F_\lambda(s)−\lambda[s]} \).

**Proposition 2.1.** (First-best allocation) The allocation plan \((f^*, g^*, p^*)\) is first-best optimal, \(g^*(., m_2, m_3) = v^{-1}(m_2 \land m_3) \land y^*\), and, if \( \xi^*(.) \) is increasing, \( f^*(s) = \varphi'^{-1} \left[ \frac{s−\beta}{\beta \pi [2F_\lambda(s)−\lambda[s]]} \right] \).

Where \( \varphi(m) = (u−v) \circ g^*(., m, m) \) and \( F_\lambda(.) \) is the CDF of \( s \). Also, any first-best allocation has savings function \( f^* \), and, its production function equals \( g^* \) in the relevant part of domain.

**Proof.** It is immediate to verify that the allocation satisfies factibility and the trade incentives constraints.

For all \( g \) that respects the trade incentives constraints, we have that \((u−v) \circ g \leq (u−v) \circ g^* \Rightarrow \int (u−v) \circ g \, d\tau \leq \int (u−v) \circ g^* \, d\tau \Rightarrow W(f, g) \leq W(f, g^*) \leq W(f^*, g^*) \). The rest of the proof is in appendix.

### 2.4 Mechanism design

With all the main concepts of the model developed, we can define the mechanism design problem for this economy.

**Definition 2.** (Second-best allocation) Let \((f, g, p)\) be solution of

\[
\max_{(f,g,p)} W(f, g)
\]

s.t.

\[(f, g, p) \in F \cap I_1 \cap I_2\]

Thus, \((f, g, p)\) is a second best allocation.

The inclusion of the savings incentive constraint diminishes drastically the set of implementable allocations for a given trade terms. This way, the choice of the savings function becomes almost residual.

The savings incentive constraint says that individuals will follow the suggested saving plan only if it’s individually desirable. Thereby, many socially desirable allocations will not be implementable. That’s the case for our first-best benchmark allocation.

\[\text{where } F_\lambda(s) = \lambda[s; \hat{s} \leq s].\]

\[\text{and } g|_{f(S)} = g|_{f(S)}.\]
An accumulation of a marginal unit of money raises the trade surplus when the saver is the consumer and meets a wealthier intermediary, but also, it raises the trade surplus when the saver is the intermediary and meets a wealthier consumer. The trade gain obtained due to the slack of the intermediary cash-in-advance constraint is called by liquidity effect of money. Wealthier intermediaries enhance the liquidity of the economy alleviating the extra cash-in-advance constraint introduced by the intermediation, increasing the trade volume and consequently the trade surplus.

The first-best savings plan takes into account this liquidity effect. However, when all surplus goes to consumer, this liquidity effect is a positive externality. If all the surplus goes to the consumer, the private marginal benefit from saving is lower than the social marginal benefit. The increasing surplus generated by the slack of the intermediary cash-in-advance constraint is not captured by the saver. Therefore, under this surplus division rule, the saver will save less than suggested by the first-best saving plan.

Heterogeneity has a central role to promote the liquidity effect. In order to the slack of the intermediary cash-in-advance constraint provides an increasing in the trade surplus, the intermediary must hold less money than the consumer. The intermediary constraint must be binding and the consumer constraint not. Therefore, the liquidity effect of money is only effective when there is heterogeneity.

The next result shows these effects.

**Lemma 2.** If $S$ is degenerated, the first-best allocation $(f^*, g^*, p^*)$ is implementable.

**Proof.** Let $T f^*$ be a best response on savings to the allocation $(f^*, g^*, p^*)$. So, it must respect the FOC given by

$$\beta \pi \varphi'(T f^*(s)) \lambda \{ \tilde{s}; f^*(\tilde{s}) > T f^*(s) \} \leq (s - \beta) \leq \beta \pi \varphi'(T f^*(s)) \lambda \{ \tilde{s}; f^*(\tilde{s}) \geq T f^*(s) \}$$

$S$ not degenerated $\Rightarrow 0 \leq (s - \beta) \leq \beta \pi \varphi'(T f^*(s))$

But we know that $(s - \beta) = \beta \pi \varphi'(f^*(s))$.

Therefore, the allocation $(f^*, g^*, p^*)$ is implementable.

\[\Box\]

**Lemma 3.** If $S$ is not degenerated, the first-best allocation $(f^*, g^*, p^*)$ is not implementable.

**Proof.** Let $T f^*$ be a best response on savings to the allocation $(f^*, g^*, p^*)$. So, it must respect the FOC given by

$$\beta \pi \varphi'(T f^*(s)) \lambda \{ \tilde{s}; f^*(\tilde{s}) > T f^*(s) \} \leq (s - \beta) \leq \beta \pi \varphi'(T f^*(s)) \lambda \{ \tilde{s}; f^*(\tilde{s}) \geq T f^*(s) \}$$
But we know that
\[ \beta \pi \varphi'(f^*(s)) \lambda \{ \bar{s}; f^*(\bar{s}) > f^*(s) \} + \beta \pi \varphi'(f^*(s)) \lambda \{ \bar{s}; f^*(\bar{s}) \geq f^*(s) \} = (s - \beta) \]

\( S \) is not degenerated \( \Rightarrow \beta \pi \varphi'(f^*(\bar{s})) \lambda \{ \bar{s}; f^*(\bar{s}) \geq f^*(\bar{s}) \} < (\bar{s} - \beta) \)

Therefore, the allocation \((f^*, g^*, p^*)\) is not implementable.

It’s important to stress that the second result doesn’t mean that the optimal first-best is never achieved in the non-degenerated case. We could have another terms of trade that give arise to the desired savings function.

Given this last result, some questions emerges. Which allocations are implementable with the trade terms \((g^*, p^*)\)? Are they second-best allocations?

The spaces where the optimization is made and the fixed point restriction makes the mechanism design problem very difficult to be solved. The next result presents a savings function compatible with the terms of trade \((g^*, p^*)\).

**Lemma 4.** Consider the savings rule \(x^*(s) = \varphi'^{-1} \left[ \frac{s - \beta}{\beta \pi F_{\lambda}(s)} \right] \). If \( \xi(s) = \frac{s - \beta}{F_{\lambda}(s)} \) is increasing, the allocation \((x^*, g^*, p^*)\) is implementable. Also, \(x^*\) is the decreasing savings rule compatible with \((g^*, p^*)\) that generates more savings.

**Proof.**

\[ x^*(s) = \arg \max_{x(s)} - (s - \beta) x(s) + \beta \pi \int \varphi(x(s) \land x^*(\bar{s})) \, d\lambda(\bar{s}) \]

The first order condition is given by

\[ \beta \pi \varphi'(x^*(s)) \lambda \{ \bar{s}; x^*(\bar{s}) > x^*(s) \} \leq (s - \beta) \leq \beta \pi \varphi'(x^*(s)) \lambda \{ \bar{s}; x^*(\bar{s}) \geq x^*(s) \} \]

Taking \(x^*\) decreasing

\[ \beta \pi \varphi'(x^*(s)) \lambda \{ \bar{s}; \bar{s} < s \} \leq (s - \beta) \leq \beta \pi \varphi'(x^*(s)) \lambda \{ \bar{s}; \bar{s} \leq s \} \]

\[ \Rightarrow \varphi'^{-1} \left[ \frac{s - \beta}{\beta \pi [F_{\lambda}(s) - \lambda \{ s \}]} \right] \leq x^*(s) \leq \varphi'^{-1} \left[ \frac{s - \beta}{\beta \pi F_{\lambda}(s)} \right] \]

We call \((x^*, g^*, p^*)\) by standard implementable allocation.

Note that the savings in the standard implementable allocation is lower than the first-best optimal savings. Thus, this allocation is not first-best optimal. Actually, it may not
even be second-best optimal but, due to the difficulty of finding the solution, we will use it as a benchmark for the implementable allocations.

The next section provides other candidates to second-best allocation.

3 Distortions

Due to the positive externality caused by the liquidity effect, the traditional trading protocol that gives the entire surplus to the consumer does not achieve the first-best. Since the intermediary is not rewarded by saving more, the savers don’t have incentives to save the socially optimal quantity.

Therefore, changes in the trade terms that distort the money returns could bring up a welfare gain. Compensate the less productive agents to save more seems to be a good strategy. One unit of money on the poor hands is socially better than one unit of money on the wealthier hands. Although the opportunity cost from saving of the former be higher than the opportunity cost of the latter, the positive externality caused by the latter compensate this difference of costs.\(^8\)

In this section, we analyze two main types of distortions: constrained and unconstrained transferences.

3.1 Constrained transferences

In constrained transferences distortion, we maintain the production function and alters only the surplus division rule. This strategy reduces the surplus received by the consumer seeking an increase in the lower levels of savings. For this, the transference must goes from the wealthier consumers to the poor, increasing the marginal benefit from savings for low levels of money. Nevertheless, once the resources used for this purpose is from the wealthier consumer, it comes at the cost of a decreasing in the marginal benefit from savings for higher levels of money.

This strategy of constrained transferences has limited scope. Without distortions in the production function, for reasonable levels of production, the producer can never receive any surplus and the intermediary only can receive some surplus if he has less money than the producer. This happens because the cash-in-advance constraint let part of the trade surplus not transferable.

Remark 1. If an allocation with maximum production function \((x, g^*, p)\) is implementable,\(^8\)

\(^8\)Another positive effect is caused by the concavity of the utility function. But this effect was already captured by the original model.
producer receives no surplus for production below $y^*$. Also for production below $y^*$, intermediary receives some positive surplus only if it holds less money than consumer.

Given the last result, we will characterize the constrained transferences strategy by a rent over the transferable trade surplus. This way, we take the transference from consumer to intermediary as $p_2^*(m_1, m_2, m_3) = \theta(m_2, m_3)[m_3 - m_2 \land m_3] + v(g^*(m_1, m_2, m_3))$, where, $0 \leq \theta(m_2, m_3) \leq 1$ is the rent rate.\(^9\) So, the net surplus received by the intermediary is $\theta(m_2, m_3)[m_3 - m_2 \land m_3]$.

Thus, we are interested in allocations $(x^\theta, g^\theta, p^\theta)$, such that, $g^\theta(m_1, m_2, m_3) = g^*(m_1, m_2, m_3)$, $p_1^\theta(m_1, m_2, m_3) = v(g^*(m_1, m_2, m_3))$, $p_2^\theta(m_1, m_2, m_3) = \theta(m_2, m_3)[m_3 - m_2 \land m_3] + v(g^*(m_1, m_2, m_3))$, and, $x^\theta(s) \in \arg\max_{m'} R(m', s; (x^\theta, g^\theta, p^\theta))$, where

$$R(m', s; (x^\theta, g^\theta, p^\theta)) = -(s - \beta)m' + \beta \pi \left\{ \int \varphi(m \land m') \, d\mu_{x^\theta}(m) - \int \theta(m, m')[m' - m \land m'] \, d\mu_{x^\theta}(m) + \int \theta(m', m)\left[m' - m \land m\right] \, d\mu_{z^\theta}(m) \right\}$$

Yet, in order to respect the consumer incentive constraint, we must have $\varphi(m_2 \land m_3) \geq \theta(m_2, m_3)[m_3 - m_2 \land m_3]$.

### 3.2 Unconstrained transferences

In order to slack the incentives constraints to allow more transferences, we could distort the production function. Producing below the maximum permits create transferences from a poor consumer to a wealthier individual. This way, we can punish individuals with very low levels of savings increasing the marginal benefit from savings for low levels of money.

A distortion in the production function that punish very low levels of savings combined with a surplus division rule that reward low levels would raise the savings of the less productive agents to an intermediary level.

Analogously the former section, we characterize this strategy defining a rent rate over the transferable surplus, $0 \leq \theta(m_2, m_3) \leq 1$, and a production rate, $0 \leq \eta(m_2, m_3) \leq 1$.\(^{10}\) This characterization generalizes the former. The constrained transferences distortion is the case that $\eta(m_2, m_3) = 1$.

Thus, an allocations $(x_1^\eta, g_1^\eta, p_1^\eta)$ that follow this strategy is such that, $g_1^\eta(m_1, m_2, m_3) = \eta(m_2, m_3)g^*(m_1, m_2, m_3)$, $p_1^\eta(m_1, m_2, m_3) = v(\eta(m_1, m_2)g^*(m_1, m_2, m_3))$, $p_2^\eta(m_1, m_2, m_3) =$\(^{10}\) Again, it could be function of the producer money holdings.

---

\(^9\)Note that it could also be function of the producer money holdings but we rule out for ease the tractability.

\(^{10}\)Again, it could be function of the producer money holdings.
\[ \theta(m_2, m_3)[m_3 - m_2 \wedge m_3] + v(\eta(m_1, m_2)g^*(m_1, m_2, m_3)), \text{ and, } x^\varphi(s) \in \argmax_{m'} R(m', s; (x^\varphi, g^\varphi, p^\varphi)), \]

where

\[
R(m', s; (x^\varphi, g^\varphi, p^\varphi)) = -(s - \beta)m' + \beta \pi \left\{ \int \varphi(\eta(m, m)m \wedge m') \, d\mu_{\varphi}(m) - \int \theta(m, m')[m' - \eta(m, m)m \wedge m'] \, d\mu_{\varphi}(m) + \int \theta(m', m)[m - \eta(m', m)m' \wedge m] \, d\mu_{\varphi}(m) \right\}
\]

And also, to respect the consumer incentive constraint, \( \varphi(\eta(m_2, m_3)m_2 \wedge m_3) \geq \theta(m_2, m_3)[m_3 - \eta(m_2, m_3)m_2 \wedge m_3] \).

### 3.2.1 An interesting strategy

An extreme use of unconstrained transferences strategy would give arise to a not so intuitive but efficient type of distortion that simply restricts the savings set. This distortion eases the implementability of an allocation changing slightly the terms of trade but maintaining the savings rule, the production rule in the relevant part and the welfare achieved.

This strategy consists in reducing the set of alternatives to the saving plan suggested. For this, it punishes as harder as possible savings choices outside the set of suggested savings.\(^{11}\)

The punishment consists in selecting the terms of trade in a way that if some saver choose a not desirable amount of money in the saving stage this one will receive no surplus. That said, the savings choice outside the suggested savings set becomes less attractive than choose not to save.

The strategy works as follow. In a network if the one who doesn’t carry an amount of money suggested is the producer, production is maximum given the cash-in-advance constraint and all the surplus goes to the consumer. On the other hand, if the one who doesn’t carry an amount of money suggested is the intermediary or the consumer, production is distorted and all surplus goes to producer (The distortion is necessary to all surplus be transferable). This way, if a person saves a quantity of money not suggested by the plan he receives no surplus and is better not to save.

Nevertheless, the set of suggested savings contains the savings designed to every type of agent. So, deviations within this set remains possible. This implies that the incentive savings constraint can be resumed to concerns about a participations constraint (i.e. individuals must prefer save than not to save) and about individuals do not prefer the savings options suggested to the others type.

\(^{11}\)The set of suggested savings is the image of the saving plan suggested, \( f(S) \).
The next result eases the mechanism design problem. It says that for any implementable allocation there is another allocation that has the same saving plan and the same trade terms in the relevant part, and, it gives zero surplus for individual who saves a not suggested amount. Therefore, we could restrict attention to a reduced set of implementable allocations in which its easier to verify if the incentive savings constraint is respected.

**Lemma 5.** Let \((x, g, p)\) be an implementable allocation and consider the allocation

\[
g'(m_1, m_2, m_3) = \begin{cases} 
    u^{-1}(m_2 \land m_3) \land y^*, & \text{if } m_1 \in f(S), \text{and, } m_2 \notin f(S) \text{ or } m_3 \notin f(S) \\
    g^*(m_1, m_2, m_3), & \text{if } m_1 \notin f(S) \\
    g(m_1, m_2, m_3), & \text{c.c.}
\end{cases}
\]

\[
p'_i(m_1, m_2, m_3) = \begin{cases} 
    m_2 \land m_3 \land u(y^*), & \text{if } m_1 \in f(S), \text{and, } m_2 \notin f(S) \text{ or } m_3 \notin f(S) \\
    v(g(m_1, m_2, m_3)), & \text{if } m_1 \notin f(S) \\
    p_i(m_1, m_2, m_3), & \text{c.c.}
\end{cases}
\]

Thus, the allocation \((x, g', p')\) is also implementable and achieves the same welfare level of \((x, g, p)\).

**Proof.** See appendix.

It’s important to stress that the punishment included in the new allocation of the previous lemma will not be effective. It is just a threat. Also, we don’t worry about the cases in which more than one person carries the wrong quantity of money because at the savings stage we are looking for Nash Equilibriums. There is no room for cooperative defection.

Thereby, once the allocation remains the same in the relevant set, we could say that we replicate the previous allocation.

A particular important set of allocations that explores this strategy is the set of degenerated allocations. These allocations have the feature of all agents carry the same amount of money. Thereby, the illiquidity problem in these economies is fully addressed. In an economy in which this intermediation friction is very severe it would be a relief.

Next, we define a degenerated allocation and present a result to guide the use of this type of allocations.
Definition 3. Let \((\bar{m}, g_{\bar{m}}, p_{\bar{m}})\) be an allocation, such that, \(\bar{m}(s) = \bar{m}\),
\[
\begin{align*}
g_{\bar{m}}(., m_2, m_3) &= \begin{cases} 
  u^{-1}(m_2 \land m_3) \land y^*, & \text{if } m_3 \neq \bar{m} \\
  g^*(., m_2, m_3) & \text{c.c.}
\end{cases} \\
p_i(., m_2, m_3) &= \begin{cases} 
  m_2 \land m_3 \land u(y^*), & \text{if } m_3 \neq \bar{m} \\
  p_{i,\bar{m}}^*(., m_2, m_3) & \text{c.c.}
\end{cases}
\end{align*}
\]

\((\bar{m}, g_{\bar{m}}, p_{\bar{m}})\) is a degenerated allocation.

Lemma 6. Any degenerated allocation is implementable if, only if,
\[-(s - \beta)\bar{m} + \beta \pi \varphi(\bar{m}) \geq 0.\]
Yet, take \(m^* = \arg\max_{m} \{-(1 - \beta)m + \beta \pi \varphi(m); -(1 - \bar{s})m + \beta \pi \varphi(m) \geq 0\}.\) Thus, \((m^*, g_{m^*}, p_{m^*})\) is the best degenerated allocation.

Proof. For the first part, we have
\[
R(\bar{m}, s; (\bar{m}, g_{\bar{m}}, p_{\bar{m}})) = -(s - \beta)\bar{m} + \beta \pi \varphi(\bar{m}) \\
\geq -(s - \beta)\bar{m} + \beta \pi \varphi(\bar{m}) \geq 0 \\
\geq R(m', s; (\bar{m}, g_{\bar{m}}, p_{\bar{m}})) \text{ for } m' \neq \bar{m}
\]
The part 'only if' is straightforward.

For the second part, note that the welfare function of a degenerated allocations is given by \(W(\bar{m}, g_{\bar{m}}) = -(1 - \beta) + \beta \pi \varphi(\bar{m})\), the result is immediate.

4 General model \((n\) intermediaries)

In this section, we generalize the model to support a larger chain of intermediaries. More precisely, we study economies in which trade is carried through a chain of \(n\) intermediaries and analyzes its effect to the shape of the money holdings distribution.

We briefly describe the environment and some aspects of the allocation for this general model, and, define the mechanism design problem. Then, we conclude the section presenting a strong result about the behavior of the optimal savings function for economies with a very large chain of intermediaries.

4.1 Environment

The unique change in the environment is the trading process. Instead of the networks be composed by three agents, it will be composed by \(n + 2\) agents: a producer, a consumer and \(n\) intermediaries. This way, the trading stage will be divided in \(n + 1\) substages.
4.2 Allocation

An allocation remains been a triple \((f, g, p)\) composed by a saving plan, a production rule and a surplus division rule. The saving plan is defined in the same way before \(f : S \rightarrow \mathbb{R}_+\). Nevertheless, the production rule and the surplus division rule are defined slightly different. Now these rules are functions of the money holdings of the producer, the consumer and the \(n\) intermediaries of the network, and, the surplus division rule is defined by \(n + 1\) payments functions for the \(n + 1\) trade substages. Thus, \(g : \mathbb{R}_+^{n+2} \rightarrow \mathbb{R}_+\) and \(p = [p_1, \ldots, p_{n+1}]\), \(p_i : \mathbb{R}_+^{n+2} \rightarrow \mathbb{R}\) where \(p_i\) is the payment of the \((i + 1)\)-th agent to the \(i\)-th agent, where the first agent is the producer, \((n+2)\)-th agent is the consumer and the others are the ordered intermediaries. The \(i\)-th entry of these functions is the money hold by the \(i\)-th agent of the network. An allocation is said feasible if \(-m_i \leq p_i(m_1, \ldots, m_{n+2}) \leq m_{i+1}\).

4.2.1 Implementability

The criteria of strong implementability is maintained. Thus, in order to be implementable an allocation must be feasible and immune to individual and cooperative defection.

To be immune to individual defection it must respects the trade incentives constraints and the savings incentives constraints. Now, the trade incentives constraints are given by

\[
-v(g(m_1, \ldots, m_{n+2})) + p_1(m_1, \ldots, m_{n+2}) \geq 0
\]

\[
p_{i+1}(m_1, \ldots, m_{n+2}) - p_i(m_1, \ldots, m_{n+2}) \geq 0, \quad i = 1, \ldots, n
\]

\[
u(g(m_1, \ldots, m_{n+2})) - p_{n+1}(m_1, \ldots, m_{n+2}) \geq 0
\]

The savings incentive constraint is given by \(f(s) \in \arg\max_{m'} R(m', s; (f, g, p))\), where

\[
R(m', s; (f, g, p)) = -(s - \beta)m' + \beta \pi TR(m'; (f, g, p))
\]

and,

\[
TR(m'; (f, g, p)) = \left\{ \int \int \cdots \int [u(g(m_1, \ldots, m')) - p_{n+1}(m_1, \ldots, m')]d\mu_f(m_1) \cdots d\mu_f(m_{n+1}) + 
\sum_{i=2}^{n+1} \int \int \cdots \int [p_i(m_1, \ldots, m', \ldots, m_{n+2}) - p_{i-1}(m_1, \ldots, m', \ldots, m_{n+2})]d\mu_f(m_1) \cdots d\mu_f(m_{n+2}) + 
\int \int [-v(g(m', \ldots, m_{n+2})) + p_1(m', \ldots, m_{n+2})]d\mu_f(m_2) \cdots d\mu_f(m_{n+2}) \right\}
\]
The same way before, we can define the three sets: the set of feasible allocation \((F)\), the set of allocations that respect the trade incentives \((I_1)\), and the set of allocations that respect the savings incentive \((I_2)\).

### 4.2.2 Welfare

The natural extension of the welfare criteria is given by

\[
W_n(f, g, p) = -\int (s-\beta)f(s)d\lambda(s)+\beta\pi \int \ldots \int (u-v)\circ g(f(s_1), \ldots, f(s_{n+2})) d\lambda(s_1) \ldots d\lambda(s_{n+2})
\]

Again, the welfare function doesn’t depend on the surplus division rule, and so, we will also denote the welfare function by \(W_n(f, g)\).

### 4.3 Mechanism Design

The mechanism design problem of the general model remains very similar. The optimization is carried in the set of implementable allocations and the savings function is almost residual given the terms of trade.

**Definition 4.** *(Second-best allocation)* Let \((f, g, p)\) be solution of

\[
\max_{(f,g,p)} W_n(f, g)
\]

s.t.

\[
(f, g, p) \in F \cap I_1 \cap I_2
\]

Thus, \((f, g, p)\) is a second best allocation.

Again, due to the difficulty to find the solution we will work with numerical examples. As in the basic model, we can take an allocation analytically to use as benchmark.

Take the trade terms in which the production is maximum given the cash-in-advanced constraint and all the surplus goes to consumer, then

**Lemma 7.** Let \(x^*_n(s) = \phi'^{-1} \left[ \frac{s-\beta}{\beta F^*_{\lambda}(s)} \right] \), \(g^*(m_1, \ldots, m_n) = \arg\max \{(u-v)(y); v(y) \land_{i=2}^n m_i\} \)

and \(p^*_1(m_1, \ldots, m_n) = v(g^*(m_1, \ldots, m_n))\). If \(\xi_n(s) = \frac{s-\beta}{F^*_{\lambda}(s)}\) is increasing, the allocation

\[{}^{12}\text{For each different } n, \text{ the spaces where the sets are defined are different. The context will always make clear which space we are talking about.}\]
\((x^*, g^*, p^*)\) is implementable. Also, \(x^*\) is the decreasing savings rule that generates more savings.\(^{13}\)

**Proof.** Analogous to lemma 4.\(^ \Box \)

Note that the condition of \(\xi_n(s)\) be increasing is very restrictive for large \(n\). So, though we have an analytical result, its use will be very limited.

As before, we call \((x^*_n, g^*, p^*)\) by standard implementable allocation.

### 4.4 No intermediation

The description of the general model made above is also valid for the case \(n = 0\), the case without intermediation. In this case, we would have a welfare function given by

\[
W_0(f, g, p) = -\int (s - \beta)f(s)d\lambda(s) + \beta\pi \int\int (u - v) \circ g(f(s), f(\tilde{s}))d\lambda(s)d\lambda(s)
\]

implying that the standard implementable allocation for \(n = 0\), \(x^*_0(s) = \varphi^{-1}\left[\frac{s - \beta}{\beta\pi}\right]\), is optimal first-best for this economy.\(^{14}\)

We will use this allocation to comparison.

### 4.5 Optimum

In an economy with so many intermediaries the cash-in-advance constraint would be very severe. The lowest level of money would frequently impose an upper bound in production. This would happen because greater the number of intermediaries, greater the probability of having the poorest individuals among the intermediaries. Thereby, increasing the intermediary chain would raise the concern about liquidity in this economy shrinking its money holdings distribution.

**Proposition 4.1.** For \(n\) sufficiently large, the optimal distribution of money degenerates to \(m^*\).\(^{15}\)

\(^{13}\)We use the same notation \(g^*\) and \(p^*\) for economies with different number of intermediaries. The context will always make clear which function we are using.

\(^{14}\)See Cavalcanti e Puzzello (2010).

\(^{15}\)\(m^*\) is defined in lemma 6.
Proof. Let \((x, g, p)\) be an implementable allocation. Thus,

\[
W_n(x, g) = -\int (s - \beta) x(s) d\lambda(s) + \beta \pi \int \ldots \int (u - v) \circ g(x(s_1), x(s_2), \ldots, x(s_n)) d\lambda(s_1) d\lambda(s_2) \ldots d\lambda(s_n)
\]

\[
\leq -\int (s - \beta) x(s) d\lambda(s) + \beta \pi \int \ldots \int \varphi(x(s_1)) d\lambda(s_2) \ldots d\lambda(s_n)
\]

\[
\approx -\int (s - \beta) x(s) d\lambda(s) + \beta \pi \varphi(x(\bar{s})) \leq -(1 - \beta) x(\bar{s}) + \beta \pi \varphi(x(\bar{s})).
\]

Now, we will show that the degenerated allocation \((x(\bar{s}), g_{x(\bar{s})}, p_{x(\bar{s})})\) is implementable. (i.e. \(-(\bar{s} - \beta) x(\bar{s}) + \beta \pi \varphi(x(\bar{s})) \geq 0)\)

We have that \(\int R(x(s), s; (f, g, p)) d\lambda(s) = W_n(x, g) \leq -\int (s - \beta) x(s) d\lambda(s) + \beta \pi \varphi(x(\bar{s})).\) Thus, \(\int [R(x(s); (x, g, p)) - \varphi(x(\bar{s}))] d\lambda(s) \leq 0 \Rightarrow R(x(\overline{s}); (x, g, p)) \leq \varphi(x(\bar{s})).\) This implies that, \(0 \leq R(x(\bar{s}); (x, g, p)) \leq -(\bar{s} - \beta) x(\bar{s}) + \beta \pi \varphi(x(\bar{s})).\)

Therefore,

\[
W_n(x, g) \leq W_n(x(\bar{s}), g_{x(\bar{s})}) \leq W_n(m^*, g_{m^*})
\]

\[
\square
\]

5 Results

This section analyzes the effects of the intermediation in the money holdings distribution of an economy. For this, we will compare the distributions generated by the allocations obtained above in some numerical examples.

Intermediation would reduce the return of the money for high levels of savings, reducing the marginal benefit of money that would imply a decreasing in savings level of the economy. In other words, it would shift the money holdings distribution to the left.

In addition, intermediation would have impact in the dispersion of this distribution. In an economy with intermediation, the concern about the others money holdings to carry trades would tend to concentrate the distribution of money. Once the reduction in marginal benefit of money imposed by intermediation is increasing in the money amount, the wealthier individuals are the ones that would reduce more the savings amount. This in turn would concentrate the savings in a low level.

Besides, due to the positive externality of the money caused by the liquidity effect, it is socially desirable to have a more concentrated distribution of money. Thus, the trading protocol chosen in the mechanism design would reinforce this tendency to shrinks the money
holdings distribution.

We develop three distinct scenarios to the shocks to study the behavior of the money distribution. A case with degenerated shock distribution, a case with 5 types of shocks and a case with 10 types of shocks. We take shocks evenly distributed and the set $S$ is chosen in order to conditions of proposition 2.1 and lemma 4 for standard and first-best allocations be respected. For all three cases, we take $u(y) = \sqrt{y}$, $v(y) = y^2$, $\beta = 0.59$ and $\pi = 0.2$.

In the case with a degenerated shock, the intermediation friction is not a problem and the optimal first-best is always achieved. In addition, at optimum, there is no difference between economies with and without intermediation. The optimal savings functions are equal and the welfare achieved is the same. Since every agent carries the same amount of money, the intermediation problem is fully addressed.

For the non-degenerated cases, we can assess some features of the money distribution behavior analyzing the analytical form of the savings functions. First note that as mentioned before, the savings for the standard allocation in an economy with intermediation is lower than the optimal savings in an economy without intermediation, i.e. $x_1(s) \leq x_0(s)$, for all $s$. (See figure 3)

However, also note that the lowest levels of savings from the first-best allocation in an economy with intermediation are higher than the lowest levels of savings from an economy without intermediation. This captures the concern about the externality caused by the liquidity effect.

Besides, note that since $x_0(\bar{s}) = x_1(\bar{s}) \leq f^*(\bar{s})$ and $x_1(\underline{s}) = f^*(\underline{s}) \leq x_0(\underline{s})$, the support of the savings distribution of both, the standard allocation and first-best, are contained in the support of the savings distribution for economies without intermediation. Actually, if the shock is not degenerated, it is strictly shorter. This is consistent with the conjecture that the distribution of money shrinks with intermediation.

We can see by figures 1 and 2 that the distributions of money generated by $x_1(.)$ and $f^*(.)$ are more concentrated than the money distribution generated by $x_0(.)$. We compute the variance of the money distribution for these economies (See table 1). The results support the conjecture that the money holdings distribution shrinks with the introduction of intermediation.

Until now, we use the standard and the first-best allocations to assess the behavior of the savings in economies with intermediation. Nevertheless, the allocation we use to this should be the solution of the mechanism design problem. As we know, the standard allocation is not necessarily the mechanism design solution and the first-best allocation cannot be even implementable.

We develop two alternatives allocations using the distortion ideas mentioned in section
Table 1: Welfare and money distribution variance

<table>
<thead>
<tr>
<th>Allocations</th>
<th>5 shocks</th>
<th>10 shocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Welfare*</td>
<td>Money Distribution Variance**</td>
</tr>
<tr>
<td>without intermediation</td>
<td>-</td>
<td>21.2390</td>
</tr>
<tr>
<td>first-best allocation</td>
<td>3.4759</td>
<td>4.9732</td>
</tr>
<tr>
<td>standard allocation</td>
<td>3.3665</td>
<td>6.0508</td>
</tr>
<tr>
<td>constrained transferences</td>
<td>3.3742</td>
<td>4.6925</td>
</tr>
<tr>
<td>unconstrained transferences</td>
<td>3.4424</td>
<td>5.1379</td>
</tr>
<tr>
<td>2 intermediaries</td>
<td>-</td>
<td>0.2151</td>
</tr>
</tbody>
</table>

(*) and (**) Welfare and money distribution variances are multiplied, respectively, by 10 and 10,000.

3 and compute their welfare. For this, we define the allocations trade terms and compute, numerically, a compatible savings function. It works like that: Given the trade terms, we guess an initial savings function, $f(.)$, forming an allocation. Then, given this allocation, we compute the savings function generated by the optimization of the savings problem faced by individuals, $Tf(.)$. After, we repeat the process using this new savings function as the guess, until we find a fixed point. \(^\text{16}\)

The first alternative allocation is made through a constrained transferences distortion. We maintain the production function and distort the surplus division rule, giving some positive surplus to the intermediary in certain occasions. Since our goal is make the poor save more, we begin the transference in a money level near the lowest money level of the standard allocation. Also, the rent rate paid by to the intermediary must be increasing in the money amount. Finally, we also let the rent rate to be concave in order to the benefit of the transferences goes to the less productive agents.

The second alternative allocation is made through unconstrained transferences strategy. We define a threshold level for savings in which savings below this level implies in zero surplus to the saver. This way, we punish as hard as possible the ones very illiquid, forcing them to save at least the threshold level. We use the first-best allocation as reference to define this threshold level of savings. The threshold level of savings chosen is the lowest savings level of the first-best allocation.

In table 1, we can see that both distorted allocations obtained a higher welfare than the standard allocation. In both cases, the poorest save more and the wealthier save less than in the standard allocation (See figures 4 and 5). Therefore, the money distribution

\(^{16}\)There is no theory that ensures that we will find the fixed point, but at least for these examples, we find it.
supports generated by these allocations are shorter than the one generated by the standard allocation. Also, looking at figures 1 and 2 and table 1, we can conclude that these allocations provides more concentrated money distribution than the one generated by the economy without intermediation. This reinforces the conjecture that intermediation would shrinks the distribution of money holdings.

These examples shows that some distorted allocations achieve higher welfare levels than the standard allocation (See table 1). It looks like that allocations which the savings function resembles the one generated by the first-best allocation provides higher welfare. So, the fact that the first-best allocation provides more concentrated distribution is relevant to supports the conjecture made. Besides, these results reinforce the idea that optimal allocation may not have the entire surplus going to the consumer.

Finally, we assess the money holdings distribution behavior as the intermediation chain increases. For this, for the economy with 2 intermediaries, we will compute numerically an allocation generated by the trade terms \((g^*, p^*)\). The procedure to obtain this allocation is the same as the used in the distortion cases.\(^{17}\)

Figures 1 and 2, and, 6 provide respectively the money holdings distribution and the savings functions for economies with \(n = 0, 1, 2\), while, table 1 provides their money distribution variances. Note that as expected, larger the intermediation chain more concentrated is the money distribution. Moreover, the proposition 4.1 says that for a very large chain of intermediaries, the money holdings distribution degenerates.

6 Conclusion

This work extends Cavalcanti e Puzzello (2010) commodity money model introducing intermediation to the trading process. Instead of direct bilateral tradings, exchanges must occur through an intermediary chain. This new trading process provides some implications about the money holdings distribution of the economy. It shrinks and shifts to the left this distribution.

The trading process with intermediation presents an extra cash-in-advance constraint that gives rises to a concern about money distribution for both, central planner and savers. The former desires to shrink the money holdings distribution of the economy to exploit a positive externality, the liquidity effect of money. The latter has less incentive to save much more than the rest of the economy, reinforcing these effects in the money distribution.

We also show that due to the liquidity effect of money, the optimal allocation may not have

\(^{17}\)For several examples, when the standard allocation exist, giving a good initial guess, this procedure provided the standard allocation.
the traditional terms of trade in which production is maximum given the cash-in-advance constraint and all surplus goes to consumer. Changing the trade terms, it is possible to distort the money return. This way we could transfer money from wealthier to the poor, exploiting the money liquidity effect to obtain an improvement. Through some numerical examples we show that this improvement is possible.

At last, we show that more severe is the intermediation friction, more concentrated is the money holdings distribution of the economy. In particular, we show that for a sufficiently large chain of intermediaries, the money holdings distribution degenerates.
References


A Proofs

Proof. (The saving function is non-increasing) Let \((x, g, p)\) be an implementable allocation. By the saving incentive constraints, we have

\[
R(x(s), s; (x, g, p)) - R(x(\tilde{s}), s; (x, g, p)) = [s - \beta][x(\tilde{s}) - x(s)] + \beta \pi [TR(x(s)) - TR(x(\tilde{s}))] \geq 0
\]

\[
R(x(\tilde{s}), \tilde{s}; (x, g, p)) - R(x(s), \tilde{s}; (x, g, p)) = [\tilde{s} - \beta][x(s) - x(\tilde{s})] + \beta \pi [TR(x(\tilde{s})) - TR(x(s))] \geq 0
\]

Summing the two inequalities, we have

\[
[s - \tilde{s}][x(\tilde{s}) - x(s)] \geq 0
\]

Thus, \(s \leq \tilde{s} \Rightarrow x(s) \geq x(\tilde{s})\). The saving function is non-increasing. \(\square\)

Proof. (First-best allocation)

To finish the first part of the proof, we will find the solution to \(f^* = \arg\max_{f} W(f, g^*)\).

But first, we will prove that \(f^*(.)\) is decreasing.

\[
W(f, g^*) = - \int (s - \beta) f(s) d\lambda(s) + \beta \pi \int \int \varphi(f(s) \wedge f(\tilde{s})) d\lambda(s) d\lambda(\tilde{s})
\]

\[
= \int \left\{ -(s - \beta) f(s) + \beta \pi \varphi(f(s)) \lambda\{ \tilde{s}; f(\tilde{s}) \geq f(s) \} + \beta \pi \int_{f(\tilde{s}) < f(s)} \varphi(f(\tilde{s})) d\lambda(\tilde{s}) \right\} d\lambda(s)
\]

\[
= \int \left\{ -(s - \beta) f(s) + \beta \pi \varphi(f(s)) \lambda\{ \tilde{s}; f(\tilde{s}) \geq f(s) \} + \beta \pi \varphi(f(s)) \lambda\{ \tilde{s}; f(\tilde{s}) > f(s) \} \right\} d\lambda(s)
\]

Take partition of the set \(S\) induced by the function \(f^*(.)\).\(^{18}\) Thus, the first order condition would say that\(^{19}\)

\[
\int_{f(\tilde{s}) = f^*(s)} (\tilde{s} - \beta) d\lambda(\tilde{s}) = \beta \pi \varphi'(f^*(s)) [\lambda\{ \tilde{s}; f^*(\tilde{s}) \geq f^*(s) \} + \lambda\{ \tilde{s}; f^*(\tilde{s}) > f^*(s) \}]
\]

and

\[
\beta \pi \varphi'(f^*(s)) [\lambda\{ \tilde{s}; f^*(\tilde{s}) \geq f^*(s) \} + \lambda\{ \tilde{s}; f^*(\tilde{s}) > f^*(s) \}]
\leq s - \beta
\]

\[
\leq \beta \pi \varphi'(f^*(s)) [\lambda\{ \tilde{s}; f^*(\tilde{s}) \geq f^*(s) \} + \lambda\{ \tilde{s}; f^*(\tilde{s}) > f^*(s) \}]
\]

\(^{18}\)We want sets like \( \{ \tilde{s}; f^*(\tilde{s}) = f^*(s) \} \).
\(^{19}\)\(\varphi'(.)\) is continuos.
This implies that \( s \neq s' \Rightarrow f^*(s) \neq f^*(s') \) and then the first order condition is given by

\[
s = \beta\pi' f^*(s) \left[ \lambda\{s; f^*(s) \geq f^*(s)\} + \lambda\{s; f^*(s) > f^*(s)\} \right]
\]

Now, suppose that \( f^* \) is not decreasing. Thus, there is a \( s' > s \) such that \( f^*(s') > f^*(s) \). This would imply that,

\[
\lambda\{s; f^*(s) \geq f^*(s')\} + \lambda\{s; f^*(s) > f^*(s')\} > \lambda\{s; f^*(s) \geq f^*(s)\} + \lambda\{s; f^*(s) > f^*(s)\}
\]

We would have a contradiction.

Once \( f^*(\cdot) \) is decreasing, we have

\[
s - \beta = \beta\pi' f^*(s) \left[ \lambda\{s; f^*(s) \geq f^*(s)\} + \lambda\{s; f^*(s) > f^*(s)\} \right]
\]

\[
s - \beta = \beta\pi' f^*(s) \left[ \lambda\{s; \tilde{s} < s\} + \lambda\{s; \tilde{s} \leq s\} \right]
\]

\[
(s - \beta) = \beta\pi' (2F_\lambda(s) - \lambda\{s\})
\]

\[
\Rightarrow f^*(s) = \varphi^{-1} \left[ \frac{s - \beta}{\beta\pi(2F_\lambda(s) - \lambda\{s\})} \right]
\]

Note that \( W(f, g) \) is concave in \( f \), then, since \( f^* \) is the unique solution to the FOC, it is the unique solution to the problem.

For the second part of the proof, suppose that \((f', g', p')\) is a first-best solution. If \((f', g', p')\) satisfies the trade incentives constraints, so do \((f', g^*, p^*)\). But then, \( W(f', g') \leq W(f', g^*) \leq W(f^*, g^*) \) with the last inequality strict if \( f' \neq f^* \). Also, note that if \( g'| f^*(s) \neq g^*| f^*(s) \) then \( W(f', g') < W(f', g^*) \).

Proof. (lemma 5) In order to verify that \((x, g', p')\) is implementable, we need to verify the trade and savings incentives. Once the trade incentives for \((x, g, p)\) are respected, and, \( v(u^{-1}(m_2 \wedge m_3) \wedge y^*) \leq m_2 \wedge m_3 \wedge u(y^*) \), then, trade incentives for \((x, g', p')\) are respected trivially. To verify that \((x, g', p')\) satisfies the savings incentive, just note that for \( m \neq x(s), R(m, s; (x, g', p')) = -(s - \beta)m \). The rest of the proof is straightforward.\(^{20}\)

\(^{20}\)Note that an allocation with zero production when someone deviates also satisfies the trade and savings incentives, and, would achieve the same objective. Though, it is not immune to cooperative defection. The verification of this property for the allocation presented in the lemma is straightforward.
B Figures

Figure 1: Money holdings distribution
Figure 2: Money holdings distribution
Figure 3: Savings function (5 shocks - 10 shocks)
Figure 4: Savings function (5 shocks - 10 shocks - 10 shocks (zoom))
Figure 5: Savings function (5 shocks - 10 shocks - 10 shocks (zoom))
Figure 6: Savings function (5 shocks - 10 shocks)